GRADED HECKE ALGEBRAS AND EQUIVARIANT CONSTRUCTIBLE SHEAVES ON THE NILPOTENT CONE

Maarten Solleveld
IMAPP, Radboud Universiteit Nijmegen
Heyendaalseweg 135, 6525AJ Nijmegen, the Netherlands
email: m.solleveld@science.ru.nl

Abstract. Graded Hecke algebras can be constructed geometrically, with constructible sheaves and equivariant cohomology. The input consists of a complex reductive group G (possibly disconnected) and a cuspidal local system on a nilpotent orbit for a Levi subgroup of G. We prove that every such "geometric" graded Hecke algebra is naturally isomorphic to the endomorphism algebra of a certain $G \times \mathbb{C}^\times$-equivariant semisimple complex of sheaves on the nilpotent cone $\mathfrak{g}_N$ in the Lie algebra of G.

From there we provide an algebraic description of the $G \times \mathbb{C}^\times$-equivariant bounded derived category of constructible sheaves on $\mathfrak{g}_N$. Namely, it is equivalent with the bounded derived category of finitely generated differential graded modules of a suitable direct sum of graded Hecke algebras. This can be regarded as a categorification of graded Hecke algebras.

Contents

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Introduction

The story behind this paper started with with the seminal work of Kazhdan and Lusztig [KaLu]. They showed that an affine Hecke algebra $\mathcal{H}$ is naturally isomorphic with a K-group of equivariant coherent sheaves on the Steinberg variety of a complex reductive group. (Here $\mathcal{H}$ has a formal variable $\mathbf{q}$ as single parameter and the reductive group must have simply connected derived group.) This isomorphism enables one to regard the category of equivariant coherent sheaves on that particular variety as a categorification of an affine Hecke algebra. Later that became quite an important theme in the geometric Langlands program, see for instance [Bez, ChGi, MiVi].

Our paper is inspired by the quest for a generalization of such a categorification of $\mathcal{H}$ to affine Hecke algebras with more than one q-parameter. That is relevant because such algebras arise in abundance from reductive p-adic groups and types [ABPS, §2.4]. However, up to today it is unclear how several independent q-parameters can be incorporated in a setup with equivariant K-theory or K-homology. The situation improves when one formally completes an affine Hecke algebra with respect to (the kernel of) a central character, as in [Lus4]. Such a completion is Morita equivalent with a completion of a graded Hecke algebra with respect to a central character.

Graded Hecke algebras $\mathbb{H}$ with several parameters (now typically called k) do admit a geometric interpretation [Lus3, Lus6]. (Not all combinations of parameters occur though, there are conditions on the ratios between the different k-parameters.) For this reason graded Hecke algebras, instead of affine Hecke algebras, play the main role in this paper.

Such algebras, and minor generalizations called twisted graded Hecke algebras, appear in several independent ways. Consider a connected reductive group $\mathcal{G}$ defined over a non-archimedean local field F. Let $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ be any Bernstein block in the category of (complex, smooth) $\mathcal{G}(F)$-representations. Locally on the space of characters of the Bernstein centre of $\mathcal{G}(F)$, $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ is always equivalent with the module category of some twisted graded Hecke algebra [Sol4, §7]. This is derived from an equivalence of $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ with the module category of an algebra which is almost an affine Hecke algebra, established in full generality in [Sol4].

The same kind of algebras arise from enhanced Langlands parameters for $\mathcal{G}(F)$ [AMS2]. That construction involves complex geometry and the cuspidal support map for enhanced L-parameters from [AMS1]. It matches specific sets of enhanced L-parameters for $\mathcal{G}(F)$ with specific sets of irreducible representations of twisted graded Hecke algebras. For more background we refer to [AMS3], and for applications of the current paper in that direction we point out the sequel [Sol6].

Like affine Hecke algebras, graded Hecke algebras are related to equivariant sheaves on varieties associated to complex reductive groups. However, here the sheaves must be constructible and one uses equivariant cohomology instead of equivariant K-theory. Just equivariant sheaves do not suffice to capture all the structure of graded Hecke algebras, one rather needs differential complexes of those. Thus we arrive at the (bounded) equivariant derived categories of constructible sheaves from [BeLu]. Via intersection cohomology, such objects have many applications in representation theory, see for instance [Lus5].

Main results

Let G be a complex reductive group and let M be a Levi subgroup of G. To cover all enhanced Langlands parameters for p-adic groups and all instances of twisted graded Hecke algebras mentioned above, we must allow disconnected reductive groups. Let $q\mathcal{E}$ be an irreducible M-equivariant cuspidal local system on a nilpotent orbit in the Lie algebra of M. From these data a twisted graded Hecke algebra $\mathbb{H}(G, M, q\mathcal{E})$ can be constructed [AMS2, §4]. As a graded vector space, it is the tensor product of:

- the algebra $\mathcal{O}(\mathfrak{t})$ of polynomial functions on $\mathrm{Lie}(Z(M^\circ)) = \mathfrak{t}$, with grading 2 times the standard grading,
- $\mathbb{C}[\mathbf{r}]$, where $\mathbf{r}$ is a formal variable of degree 2,
- the (twisted) group algebra of a finite "Weyl-like" group $W_{q\mathcal{E}}$ (in degree 0).

We will work in $\mathcal{D}^b_{G\times\mathbb{C}^\times}(X)$, the $G \times \mathbb{C}^\times$-equivariant bounded derived category of constructible sheaves on a complex variety X [BeLu]. We let G act on its Lie algebra $\mathfrak{g}$ via the adjoint representation, and we let $\lambda \in \mathbb{C}^\times$ act on $\mathfrak{g}$ as multiplication by λ^{-2}. In [Lus3, Lus6, AMS2] an important object $K \in \mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})$ was constructed from $q\mathcal{E}$, by a process that bears some similarity with parabolic induction. With G° instead of $G \times \mathbb{C}^\times$, K would be a character sheaf as in [Lus2]. In general it does not fit entirely with Lusztig's notion of character sheaves on disconnected reductive groups, because those are only G°-equivariant.

Let $\mathfrak{g}_N$ be the variety of nilpotent elements in the Lie algebra $\mathfrak{g}$ of G and let K_N be the pullback of K to $\mathfrak{g}_N$. Up to degree shifts, both K and K_N are direct sums of simple perverse sheaves. This K_N generalizes the equivariant perverse sheaves used to establish the (generalized) Springer correspondence [Lus1]. The following result was already known for connected G, from [Lus3, Lus6].

Theorem A. *(see Theorem 2.2)*
There exist natural isomorphisms of graded algebras

$$\mathbb{H}(G, M, q\mathcal{E}) \longrightarrow \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K) \longrightarrow \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)}(K_N).$$

We point out that here the additional $\mathbb{C}^\times$-action makes things much more interesting (like in [KaLu] for K-theory and affine Hecke algebras). Indeed, the simpler version $\mathrm{End}^*_{\mathcal{D}^b_G(\mathfrak{g})}(K)$ is isomorphic to the crossed product of $\mathcal{O}(\mathfrak{t})$ with a twisted group algebra of $W_{q\mathcal{E}}$, and that does not involve any Hecke type relations.

Let $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N, K_N)$ be the full triangulated subcategory of $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)$ generated by K_N. By analogy with progenerators of module categories, Theorem A indicates that $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N, K_N)$ should be equivalent to some category of right $\mathbb{H}(G, M, q\mathcal{E})$-modules. Our geometric objects are differential complexes of sheaves (up to equivalences), and accordingly we need (equivalence classes of complexes of) differential graded $\mathbb{H}(G, M, q\mathcal{E})$-modules.

Theorem B. *(see Theorem 3.3)*
There exists an equivalence of triangulated categories between $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N, K_N)$ and $\mathcal{D}^b(\mathbb{H}(G, M, q\mathcal{E}) - \mathrm{Mod}_{\mathrm{fgdg}})$, the bounded derived category of finitely generated differential graded right $\mathbb{H}(G, M, q\mathcal{E})$-modules.

This is a geometric categorification of $\mathbb{H}(G, M, q\mathcal{E})$, albeit of a different kind than in [KaLu, Bez]. It is a variation (with $G \times \mathbb{C}^\times$ instead of G°) on the derived version of the generalized Springer correspondence from [Rid, RiRu2]. In that setting, the algebra is $\mathcal{O}(\mathfrak{t}) \rtimes W(G,T)$, which can also be considered as a graded Hecke algebra with parameters $k = 0$. Further, one may regard Theorem B as "formality" of the graded algebra $\mathbb{H}(G,M,q\mathcal{E})$, in the following sense. There exists a differential graded algebra $\mathcal{R}$ (with nonzero differential) such that $H^*(\mathcal{R}) \cong \mathbb{H}(G,M,q\mathcal{E})$ and $\mathcal{R}$ is formal, that is, quasi-isomorphic with $H^*(\mathcal{R})$. The equivalence in Theorem B maps $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N, K_N)$ to $\mathcal{D}^b(\mathcal{R} - \mathrm{Mod}_{\mathrm{fgdg}})$ via some Hom-functor, and from there to $\mathcal{D}^b(\mathbb{H}(G,M,q\mathcal{E}) - \mathrm{Mod}_{\mathrm{fgdg}})$ by taking cohomology.

From a geometric point of view, it is more natural to consider the entire category $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)$ in Theorem B. It turns out that this category decomposes, like in a related setting in [RiRu1]:

Theorem C. *There exists an orthogonal decomposition*

$$\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N) = \bigoplus\nolimits_{[M,q\mathcal{E}]_G} \mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N, K_N).$$

Here K_N is constructed from an M-equivariant cuspidal local system $q\mathcal{E}$ on a nilpotent orbit in $\mathrm{Lie}(M)$*, and the direct sum runs over G-conjugacy classes of such pairs $(M, q\mathcal{E})$.*

Theorems B and C describes $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)$ as a derived module category. Let us point out that the category $\mathbb{H}(G,M,q\mathcal{E}) - \mathrm{Mod}_{\mathrm{fgdg}}$ in Theorem B is much smaller than the category of ungraded finitely generated right $\mathbb{H}(G,M,q\mathcal{E})$-modules (which relates to categories of smooth representations of reductive p-adic groups). In the sequel to this paper [Sol6] we focus on standard and irreducible $\mathbb{H}(G,M,q\mathcal{E})$-modules, and we develop further techniques to study those with equivariant derived constructible sheaves. That will enable us to prove a version of the Kazhdan–Lusztig conjecture for reductive p-adic groups.

In words, Theorem C says that the $G \times \mathbb{C}^\times$-equivariant derived category of constructible sheaves on the nilpotent cone $\mathfrak{g}_N$ decomposes as a direct sum of subcategories associated to the various involved cuspidal supports $(M, q\mathcal{E})$. Let us speculate on how this relates to the Langlands program. An element $N \in \mathfrak{g}_N$ and a semisimple element $g \in G$ with $\mathrm{Ad}(g)N = q_F N$ can be used to define a Langlands parameter for a reductive group over a non-archimedean local field F. It may be an unramified L-parameter like in [KaLu]: trivial on the inertia group $\mathbf{I}_F$ and with g the image of an arithmetic Frobenius element Frob. But one can also start with a more complicated L-parameter ϕ, let G° be the connected centralizer of $\phi(\mathbf{I}_F)$ in the complex dual group and let G/G° be generated by $\phi(\mathrm{Frob})$.

One may hope that an analogue of Theorem C holds for equivariant sheaves on varieties (or stacks) of Langlands parameters, as defined in [DHKM, Zhu]. It would say that the relevant sheaves on Langlands parameters can be decomposed as orthogonal direct sums of pieces associated to suitable cuspidal supports. That would be similar to the Bernstein decomposition of the category of the smooth complex representations of a reductive p-adic group. A result of this kind is already known for enhanced L-parameters [AMS1, §8], that covers the cases of simple equivariant constructible sheaves. To fit with the (conjectural) framework for geometrization of the local Langlands correspondence from [FaSc, Zhu], a version of Theorem C with equivariant coherent sheaves on varieties of L-parameters is desirable.

Structure of the paper

We start with recalling twisted graded Hecke algebras in terms of generators and relations. We generalize a few results from [Sol3], which say that the set of irreducible representations of a graded Hecke algebra is essentially independent of the parameters k and r. Then we prove a generally useful result:

Theorem D. *The global dimension of* $\mathbb{H}(G, M, q\mathcal{E})$ *equals* $\dim(Z(M^\circ)) + 1$.

In Paragraph 2.1 we describe the geometric construction of $\mathbb{H}(G, M, q\mathcal{E})$ in detail, and we establish Theorem A. Next we check that K_N is a semisimple object of $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)$ and we relate it to parabolic induction for perverse sheaves – which is needed for Theorems B and C. Paragraph 2.3 is mainly preparation for an argument with localization to $\exp(\mathbb{C}\sigma)$-invariants in the sequel to this paper. We include it already here because it is closely related to Paragraph 2.1 and because our analysis of $(G/P)^\sigma = (G/P)^{\exp(\mathbb{C}\sigma)}$ for $\sigma \in \mathfrak{t}$ is of independent interest.

Section 3 is dedicated to Theorems B and C. We prove them by reduction to the setting of [Rid, RiRu1, RiRu2], where sheaves of $\overline{\mathbb{Q}_\ell}$-modules on varieties over fields of positive characteristic are considered. This involves checking many things, among others that $\mathbb{H}(G, M, q\mathcal{E})$ is Koszul as differential graded algebra.

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1. Graded Hecke algebras

Let $\mathfrak{a}$ be a finite dimensional Euclidean space and let W be a finite Coxeter group acting isometrically on $\mathfrak{a}$, and hence also on the linear dual space $\mathfrak{a}^\vee$. Let $R \subset \mathfrak{a}^\vee$ be a reduced integral root system, stable under the action of W, such that the reflections s_α with $\alpha \in R$ generate W. These conditions imply that W acts trivially on the orthogonal complement of $\mathbb{R}R$ in $\mathfrak{a}^\vee$.

Write $\mathfrak{t} = \mathfrak{a} \otimes_\mathbb{R} \mathbb{C}$ and let $S(\mathfrak{t}^\vee) = \mathcal{O}(\mathfrak{t})$ be the algebra of polynomial functions on $\mathfrak{t}$. We also fix a base Δ of R. Let Γ be a finite group which acts faithfully and orthogonally on $\mathfrak{a}$ and stabilizes R and Δ. Then Γ normalizes W and $W \rtimes \Gamma$ is a group of automorphisms of $(\mathfrak{a}, R)$. We choose a $W \rtimes \Gamma$-invariant parameter function $k : R \to \mathbb{C}$. Let $\mathbf{r}$ be a formal variable, identified with the coordinate function on $\mathbb{C}$ (so $\mathcal{O}(\mathbb{C}) = \mathbb{C}[\mathbf{r}]$).

Let $\natural : \Gamma^2 \to \mathbb{C}^\times$ be a 2-cocycle and inflate it to a 2-cocycle of $W \rtimes \Gamma$. Recall that the twisted group algebra $\mathbb{C}[W \rtimes \Gamma, \natural]$ has a $\mathbb{C}$-basis $\{N_w : w \in W \rtimes \Gamma\}$ and multiplication rules

$$N_w \cdot N_{w'} = \natural(w, w')N_{ww'}.$$

In particular it contains the group algebra of W.

Proposition 1.1. [AMS2, Proposition 2.2]
There exists a unique associative algebra structure on $\mathbb{C}[W \rtimes \Gamma, \natural] \otimes \mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[\mathbf{r}]$ *such that:*

- *the twisted group algebra* $\mathbb{C}[W \rtimes \Gamma, \natural]$ *is embedded as subalgebra;*
- *the algebra* $\mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[\mathbf{r}]$ *of polynomial functions on* $\mathfrak{t} \oplus \mathbb{C}$ *is embedded as a subalgebra;*
- $\mathbb{C}[\mathbf{r}]$ *is central;*
- *the braid relation* $N_{s_\alpha}\xi - {}^{s_\alpha}\xi N_{s_\alpha} = k(\alpha)\mathbf{r}(\xi - {}^{s_\alpha}\xi)/\alpha$ *holds for all* $\xi \in \mathcal{O}(\mathfrak{t})$ *and all simple roots* α*;*

- $N_w \xi N_w^{-1} = {}^w\xi$ *for all* $\xi \in \mathcal{O}(\mathfrak{t})$ *and* $w \in \Gamma$.

We denote the algebra from Proposition 1.1 by $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)$ and we call it a twisted graded Hecke algebra. It is graded by putting $\mathbb{C}[W \rtimes \Gamma, \natural]$ in degree 0 and $\mathfrak{t}^\vee \setminus \{0\}$ and $\mathbf{r}$ in degree 2. When Γ is trivial, we omit $\natural$ from the notation, and we obtain the usual notion of a graded Hecke algebra $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$.

Notice that for $k = 0$ Proposition 1.1 yields the crossed product algebra

$$\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, 0, \mathbf{r}, \natural) = \mathbb{C}[\mathbf{r}] \otimes_{\mathbb{C}} \mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W \rtimes \Gamma, \natural], \tag{1.1}$$

with multiplication rule

$$N_w \xi N_w^{-1} = {}^w\xi \qquad w \in W \rtimes \Gamma, \xi \in \mathcal{O}(\mathfrak{t}).$$

It is possible to scale all parameters $k(\alpha)$ simultaneously. Namely, scalar multiplication with $z \in \mathbb{C}^\times$ defines a bijection $m_z : \mathfrak{t}^\vee \to \mathfrak{t}^\vee$, which clearly extends to an algebra automorphism of $S(\mathfrak{t}^\vee)$. From Proposition 1.1 we see that it extends even further, to an algebra isomorphism

$$m_z : \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, zk, \mathbf{r}, \natural) \to \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural) \tag{1.2}$$

which is the identity on $\mathbb{C}[W \rtimes \Gamma, \natural] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}]$. Notice that for $z = 0$ the map m_z is well-defined, but no longer bijective. It is the canonical surjection

$$\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, 0, \mathbf{r}, \natural) \to \mathbb{C}[W \rtimes \Gamma, \natural] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}].$$

One also encounters versions of $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)$ with $\mathbf{r}$ specialized to a nonzero complex number. In view of (1.2) it hardly matters which specialization, so it suffices to look at $\mathbf{r} \mapsto 1$. The resulting algebra $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)$ has underlying vector space $\mathbb{C}[W \rtimes \Gamma, \natural] \otimes_{\mathbb{C}} \mathcal{O}(\mathfrak{t})$ and cross relations

$$\xi \cdot s_\alpha - s_\alpha \cdot s_\alpha(\xi) = k(\alpha)(\xi - s_\alpha(\xi))/\alpha \qquad \alpha \in \Delta, \xi \in S(\mathfrak{t}^\vee). \tag{1.3}$$

Since Γ acts faithfully on $(\mathfrak{a}, \Delta)$, and W acts simply transitively on the collection of bases of R, $W \rtimes \Gamma$ acts faithfully on $\mathfrak{a}$. From (1.3) we see that the centre of $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)$ is

$$Z(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)) = S(\mathfrak{t}^\vee)^{W \rtimes \Gamma} = \mathcal{O}(\mathfrak{t}/W \rtimes \Gamma). \tag{1.4}$$

As a vector space, $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)$ is still graded by $\deg(w) = 0$ for $w \in W \rtimes \Gamma$ and $\deg(x) = 2$ for $x \in \mathfrak{t}^\vee \setminus \{0\}$. However, it is not a graded algebra any more, because (1.3) is not homogeneous in the case $\xi = \alpha$. Instead, the above grading merely makes $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)$ into a filtered algebra. The graded algebra associated to this filtration is obtained by setting the right hand side of (1.3) equal to 0. In other words, the associated graded object of $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)$ is the crossed product algebra (1.1).

Graded Hecke algebras can be decomposed like root systems and reductive Lie algebras. Let $R_1, \ldots, R_d$ be the irreducible components of R. Write $\mathfrak{a}_i^\vee = \mathrm{span}(R_i) \subset \mathfrak{a}^\vee$, $\mathfrak{t}_i = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_i^\vee, \mathbb{C})$ and $\mathfrak{z} = R^\perp \subset \mathfrak{t}$. Then

$$\mathfrak{t} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_d \oplus \mathfrak{z}. \tag{1.5}$$

The inclusions $W(R_i) \to W(R), \mathfrak{t}_i^\vee \to \mathfrak{t}^\vee$ and $\mathfrak{z}^\vee \to \mathfrak{t}^\vee$ induce an algebra isomorphism

$$\mathbb{H}(\mathfrak{t}_1, W(R_1), k) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{H}(\mathfrak{t}_d, W(R_d), k) \otimes_{\mathbb{C}} \mathcal{O}(\mathfrak{z}) \longrightarrow \mathbb{H}(\mathfrak{t}, W, k). \tag{1.6}$$

The central subalgebra $\mathcal{O}(\mathfrak{z}) \cong S(\mathfrak{z}^\vee)$ is of course very simple, so the study of graded Hecke algebras can be reduced to the case where the root system R is irreducible.

1.1. **Some representation theory.**

We list some isomorphisms of (twisted) graded Hecke algebras that will be useful later on. For any $z \in \mathbb{C}^\times$, $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)$ admits a "scaling by degree" automorphism

$$x \mapsto z^n x \qquad \text{if } x \in \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural) \text{ has degree } 2n. \tag{1.7}$$

Extend the sign representation to a character sgn of $W \rtimes \Gamma$, trivial on Γ. That yields the sign involution

$$\begin{aligned} &\mathrm{sgn} : \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural) \to \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural) \\ &\mathrm{sgn}(N_w) = \mathrm{sgn}(w) N_w, \quad \mathrm{sgn}(\mathbf{r}) = -\mathbf{r}, \quad \mathrm{sgn}(\xi) = \xi \qquad w \in W \rtimes \Gamma, \xi \in \mathfrak{t}^\vee. \end{aligned} \tag{1.8}$$

Upon specializing $\mathbf{r} = 1$, it induces an algebra isomorphism

$$\mathrm{sgn} : \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural) \to \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, -k, \natural).$$

More generally, we can pick a sign $\epsilon(s_\alpha)$ for every simple reflection $s_\alpha \in W$, such that $\epsilon(s_\alpha) = \epsilon(s_\beta)$ if s_α and s_β are conjugate in $W \rtimes \Gamma$. Then ϵ extends uniquely to a character of $W \rtimes \Gamma$ trivial on Γ (and every character of $W \rtimes \Gamma$ which is trivial on Γ has this form). Define a new parameter function ϵk by

$$\epsilon k(\alpha) = \epsilon(s_\alpha) k(\alpha).$$

Then there are algebra isomorphisms

$$\begin{array}{lll} \phi_\epsilon : & \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural) & \to \quad \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, \epsilon k, \mathbf{r}, \natural), \\ \phi_\epsilon : & \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural) & \to \quad \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, \epsilon k, \natural), \\ \multicolumn{3}{l}{\phi_\epsilon(N_w) = \epsilon(w) N_w, \quad \phi_\epsilon(\mathbf{r}) = \mathbf{r}, \quad \phi_\epsilon(\xi) = \xi, \qquad w \in W \rtimes \Gamma, \xi \in \mathcal{O}(\mathfrak{t}).} \end{array} \tag{1.9}$$

Notice that for ϵ equal to the sign character of W, ϕ_ϵ agrees with sgn from (1.8) on $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)$ but not on $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)$.

For R irreducible of type B_n, C_n, F_4 or G_2, there are two further nontrivial possible ϵ's. Consider the characters ϵ_s, ϵ_l of W with

$$\epsilon_s(s_\alpha) = \left\{ \begin{array}{ll} 1 & \alpha \text{ long} \\ -1 & \alpha \text{ short} \end{array} \right. , \qquad \epsilon_l(s_\alpha) = \left\{ \begin{array}{ll} 1 & \alpha \text{ short} \\ -1 & \alpha \text{ long} \end{array} \right. .$$

Since Γ acts isometrically on $\mathfrak{a}$, ϵ_l and ϵ_s are Γ-invariant. Thus we obtain algebra isomorphisms

$$\phi_{\epsilon_s} : \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural) \to \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, \epsilon_s k, \natural), \ \phi_{\epsilon_l} : \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural) \to \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, \epsilon_l k, \natural).$$

Lemma 1.2. *Let $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)$ be a twisted graded Hecke algebra with a real-valued parameter function k. Then it is isomorphic to a twisted graded Hecke algebra $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, \epsilon k, \natural)$ with $\epsilon k : R \to \mathbb{R}_{\geq 0}$, via an isomorphism ϕ_ϵ that is the identity on $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$.*

Proof. Define

$$\epsilon(s_\alpha) = \left\{ \begin{array}{ll} 1 & k(\alpha) \geq 0 \\ -1 & k(\alpha) < 0 \end{array} \right. .$$

Since k is Γ-invariant, this extends to a Γ-invariant quadratic character of W. Then ϕ_ϵ has the required properties. $\square$

With the above isomorphisms we will generalize the results of [Sol3, §6.2], from graded Hecke algebras with positive parameters to twisted graded Hecke algebras with real parameters.

For the moment, we let $\mathbb{H}$ stand for either $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)$ or $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)$. Every finite dimensional $\mathbb{H}$-module V is the direct sum of its generalized $\mathcal{O}(\mathfrak{t})$-weight spaces

$$V_\lambda := \{v \in V : (\xi - \xi(\lambda))^{\dim V} v = 0 \; \forall \xi \in \mathcal{O}(\mathfrak{t})\} \qquad \lambda \in \mathfrak{t}.$$

We denote the set of $\mathcal{O}(\mathfrak{t})$-weights of V by

$$\mathrm{Wt}(V) = \{\lambda \in \mathfrak{t} : V_\lambda \neq 0\}.$$

Let $\mathfrak{a}^-$ be the obtuse negative cone in $\mathbb{R}R \subset \mathfrak{a}$ determined by (R, Δ). We denote the interior of $\mathfrak{a}^-$ in $\mathbb{R}R$ by $\mathfrak{a}^{--}$. We recall that a finite dimensional $\mathbb{H}$-module V is tempered if

$$\mathrm{Wt}(V) \subset \mathfrak{a}^- \oplus i\mathfrak{a}$$

and that V is essentially discrete series if, with $\mathfrak{z}$ as in (1.5):

$$\mathrm{Wt}(V) \subset \mathfrak{a}^{--} \oplus (\mathfrak{z} \cap \mathfrak{a}) \oplus i\mathfrak{a}.$$

For a subset U of $\mathfrak{t}$ we let $\mathrm{Mod}_{\mathrm{fl},U}(\mathbb{H})$ be the category of finite dimensional $\mathbb{H}$-modules V with $\mathrm{Wt}(V) \subset U$. For example, we have the category of $\mathbb{H}$-modules with "real" weights $\mathrm{Mod}_{\mathrm{fl},\mathfrak{a}}(\mathbb{H})$. We indicate a subcategory/subset of tempered modules by a subscript "temp". In particular, we have the category of finite dimensional tempered $\mathbb{H}$-modules $\mathrm{Mod}_{\mathrm{fl}}(\mathbb{H})_{\mathrm{temp}}$.

We want to compare the irreducible representations of

$$\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural) = \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)/(\mathbf{r} - 1)$$

with those of

$$\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, 0, \natural) = \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)/(\mathbf{r}).$$

The latter algebra has $\mathrm{Irr}(\mathbb{C}[W \rtimes \Gamma, \natural])$ as the set of irreducible representations on which $\mathcal{O}(\mathfrak{t})$ acts via evaluation at $0 \in \mathfrak{t}$. The correct analogue of this for $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)$, at least with k real-valued, is

$$\mathrm{Irr}_{\mathfrak{a}}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural))_{\mathrm{temp}} := \mathrm{Irr}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural))_{\mathrm{temp}} \cap \mathrm{Mod}_{\mathrm{fl},\mathfrak{a}}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)).$$

As $\mathbb{C}[W \rtimes \Gamma, \natural]$ is a subalgebra of $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)$, there is a natural restriction map

$$\mathrm{Res}_{W \rtimes \Gamma} : \mathrm{Mod}_{\mathrm{fl}}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)) \to \mathrm{Mod}_{\mathrm{f}}(\mathbb{C}[W \rtimes \Gamma, \natural]).$$

However, when $k \neq 0$ this map usually does not preserve irreducibility, not even on $\mathrm{Irr}_{\mathfrak{a}}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural))_{\mathrm{temp}}$.

In the remainder of this paragraph we assume that the parameter function k only takes real values. Let ϵ be as in Lemma 1.2. Since ϕ_ϵ is the identity on $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$, it induces equivalences of categories

$$\begin{array}{lll} \mathrm{Mod}_{\mathrm{fl},U}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, \epsilon k, \natural)) & \longrightarrow & \mathrm{Mod}_{\mathrm{fl},U}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)) \qquad U \subset \mathfrak{t}, \\ \mathrm{Mod}_{\mathrm{fl}}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, \epsilon k, \natural))_{\mathrm{temp}} & \longrightarrow & \mathrm{Mod}_{\mathrm{fl}}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural))_{\mathrm{temp}} \end{array}$$

and a bijection

$$\mathrm{Irr}_{\mathfrak{a}}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, \epsilon k, \natural))_{\mathrm{temp}} \longrightarrow \mathrm{Irr}_{\mathfrak{a}}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural))_{\mathrm{temp}}.$$

Theorem 1.3. *Let $k : R \to \mathbb{R}$ be a Γ-invariant parameter function.*

(a) The set $\mathrm{Res}_{W \rtimes \Gamma}(\mathrm{Irr}_{\mathfrak{a}} \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \natural)_{\mathrm{temp}})$ is a $\mathbb{Z}$-basis of $\mathbb{Z}\,\mathrm{Irr}(\mathbb{C}[W \rtimes \Gamma, \natural])$.

Suppose that the restriction of k to any type F_4 component of R has $k(\alpha) = 0$ for a root α in that component or is the form $\epsilon k'$ for a character $\epsilon : W(F_4) \to \{\pm 1\}$ and a geometric $k' : F_4 \to \mathbb{R}_{>0}$.

(*b*) *There exist total orders on* $\mathrm{Irr}_{\mathfrak a}(\mathbb H(\mathfrak t, W \rtimes \Gamma, k, \natural)_{\mathrm{temp}})$ *and on* $\mathrm{Irr}(\mathbb C[W \rtimes \Gamma, \natural])$, *such that the matrix of the* $\mathbb Z$*-linear map*

$$\mathrm{Res}_{W \rtimes \Gamma} : \mathbb Z \, \mathrm{Irr}_{\mathfrak a}(\mathbb H(\mathfrak t, W \rtimes \Gamma, k, \natural))_{\mathrm{temp}} \to \mathbb Z \, \mathrm{Irr}(\mathbb C[W \rtimes \Gamma, \natural])$$

is upper triangular and unipotent.

(*c*) *There exists a unique bijection*

$$\zeta_{\mathbb H(\mathfrak t, W \rtimes \Gamma, k, \natural))} : \mathrm{Irr}_{\mathfrak a}(\mathbb H(\mathfrak t, W \rtimes \Gamma, k, \natural))_{\mathrm{temp}} \to \mathrm{Irr}(\mathbb C[W \rtimes \Gamma, \natural])$$

such that $\zeta_{\mathbb H(\mathfrak t, W \rtimes \Gamma, \epsilon k, \natural)}(\pi)$ *always occurs in* $\mathrm{Res}_{W \rtimes \Gamma}(\pi)$.

Proof. (a) is known from [Sol2, Proposition 1.7]. The proof of that shows we can reduce the entire theorem to the case where $\natural$ is trivial. We assume that from now on, and omit $\natural$ from the notations.

Parts (b) and (c) were shown in [Sol3, Theorem 6.2], provided that $k(\alpha) \geq 0$ for all $\alpha \in R$. Choose ϵ as in Lemma 1.2, so that $\epsilon k : R \to \mathbb R_{\geq 0}$. For $V \in \mathrm{Mod}_{\mathrm{fl}}(\mathbb H(\mathfrak t, W, \epsilon k))$ we have

$$\mathrm{Res}_W(\phi_\epsilon^* V) = \mathrm{Res}_W(V) \otimes \epsilon,$$

so we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb Z \, \mathrm{Irr}_{\mathfrak a}(\mathbb H(\mathfrak t, W, \epsilon k))_{\mathrm{temp}} & \xrightarrow{\mathrm{Res}_W} & \mathbb Z \, \mathrm{Irr}(W) \\ \downarrow \phi_\epsilon^* & & \downarrow \otimes \epsilon \\ \mathbb Z \, \mathrm{Irr}_{\mathfrak a}(\mathbb H(\mathfrak t, W, k))_{\mathrm{temp}} & \xrightarrow{\mathrm{Res}_W} & \mathbb Z \, \mathrm{Irr}(W) \end{array} \tag{1.10}$$

All the maps in this diagram are bijective and the vertical maps preserve irreducibility. Thus the theorem for $\mathbb H(\mathfrak t, W, \epsilon k)$ implies it for $\mathbb H(\mathfrak t, W, k)$.

The commutative diagram (1.10) also allows us to extend [Sol3, Lemma 6.5] from $\mathbb H(\mathfrak t, W, \epsilon k)$ to $\mathbb H(\mathfrak t, W, k)$. Then we can finish our proof for $\mathbb H(\mathfrak t, W \rtimes \Gamma, k)$ by applying [Sol3, Lemma 6.6]. $\square$

Remark 1.4. Geometric parameter functions will appear in Section 2. Let us make the allowed parameter functions for a type F_4 root system explicit here. Write $k = (k(\alpha), k(\beta))$ where α is short root and β is a long root. The possibilities are

$$(0,0), (c,0), (0,c), (c,c), (2c,c), (c/2,c), (4c,c), (-c,c), (-2c,c), (-c/2,c), (-4c,c),$$

where $c \in \mathbb R^\times$ is arbitrary. We expect that Theorem 1.3 also holds without extra conditions for type F_4.

Theorem 1.5. *Let* $\mathbb H(\mathfrak t, W \rtimes \Gamma, k, \natural)$ *be as in Theorem 1.3.b. There exists a canonical bijection*

$$\zeta_{\mathbb H(\mathfrak t, W \rtimes \Gamma, k, \natural))} : \mathrm{Irr}(\mathbb H(\mathfrak t, W \rtimes \Gamma, k, \natural)) \to \mathrm{Irr}(\mathbb H(\mathfrak t, W \rtimes \Gamma, 0, \natural))$$

which (as well as its inverse)

- *respects temperedness,*
- *preserves the intersections with* $\mathrm{Mod}_{\mathrm{fl},\mathfrak a}$,
- *generalizes Theorem 1.3.c, via the identification*

$$\mathrm{Irr}_{\mathfrak a}(\mathbb H(\mathfrak t, W \rtimes \Gamma, 0, \natural))_{\mathrm{temp}} = \mathrm{Irr}(\mathbb C[W \rtimes \Gamma, \natural]).$$

Proof. As discussed in the proof of Theorem 1.3.a, we can easily reduce to the case where $\natural$ is trivial. In [Sol3, Proposition 6.8], that case is derived from [Sol3, Theorem 6.2] (under more strict conditions on the parameters k). Using Theorem 1.3 instead of [Sol3, Theorem 6.2], this works for all parameters allowed in Theorem 1.3.

Although [Sol3, Proposition 6.8] is only formulated for irreducible representations in $\mathrm{Mod}_{\mathfrak{fl},\mathfrak{a}}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k))$, the argument applies to all of $\mathrm{Irr}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k))$. $\square$

1.2. **Global dimension.**

We want to determine the global dimension of $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)$. For $\mathbb{H}(\mathfrak{t}, W, kr)$ that has already been done in [Sol1], and our argument is based on reduction to that case. A lower bound for the global dimension is easily obtained:

Lemma 1.6. $\mathrm{gl.\,dim}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)) \geq \dim_{\mathbb{C}}(\mathfrak{t} \oplus \mathbb{C})$.

Proof. We abbreviate $\mathbb{H} = \mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)$. Pick $\lambda \in \mathfrak{t}$ such that $w\lambda \neq \lambda$ for all $w \in W \rtimes \Gamma \setminus \{1\}$. Fix any $r \in \mathbb{C}$ and let $\mathbb{C}_{\lambda,r}$ be the onedimensional $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$-module with character (λ, r). By [BaMo, Theorem 6.4], which generalizes readily to include Γ, the $\mathcal{O}(\mathfrak{t})$-weights of

$$\mathrm{Res}^{\mathbb{H}}_{\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})} \mathrm{ind}^{\mathbb{H}}_{\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})} \mathbb{C}_{\lambda,r} \tag{1.11}$$

are precisely the $w\lambda$ with $w \in W \rtimes \Gamma$. These are all different and the dimension of (1.11) is $|W \rtimes \Gamma|$, so (1.11) must be isomorphic with $\bigoplus_{w \in W \rtimes \Gamma} \mathbb{C}_{w\lambda,r}$. By Frobenius reciprocity

$$\begin{aligned} \mathrm{Ext}^n_{\mathbb{H}}\big(\mathrm{ind}^{\mathbb{H}}_{\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})} \mathbb{C}_{\lambda,r}, \mathrm{ind}^{\mathbb{H}}_{\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})} \mathbb{C}_{\lambda,r}\big) &\cong \mathrm{Ext}^n_{\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})}\big(\mathbb{C}_{\lambda,r}, \mathrm{Res}^{\mathbb{H}}_{\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})} \mathrm{ind}^{\mathbb{H}}_{\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})} \mathbb{C}_{\lambda,r}\big) \\ &\cong \bigoplus_{w \in W \rtimes \Gamma} \mathrm{Ext}^n_{\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})}\big(\mathbb{C}_{\lambda,r}, \mathbb{C}_{w\lambda r}\big) = \mathrm{Ext}^n_{\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})}\big(\mathbb{C}_{\lambda,r}, \mathbb{C}_{\lambda r}\big). \end{aligned}$$

It is well-known (and can be computed with a Koszul resolution) that the last expression equals (with $\mathfrak{T}$ for tangent space)

$$\bigwedge\nolimits^n \big(\mathfrak{T}_{\lambda,r}(\mathfrak{t} \oplus \mathbb{C})\big) = \bigwedge\nolimits^n (\mathfrak{t} \oplus \mathbb{C}).$$

This is nonzero when $0 \leq n \leq \dim_{\mathbb{C}}(\mathfrak{t} \oplus \mathbb{C})$, so the global dimension must be at least $\dim_{\mathbb{C}}(\mathfrak{t} \oplus \mathbb{C})$. $\square$

With a general argument, the computation of the global dimension of $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)$ can be reduced to the cases with $\Gamma = \{1\}$.

Lemma 1.7. *Let Γ be a finite group acting by automorphisms on a complex algebra A. Let $\natural : \Gamma^2 \to \mathbb{C}^\times$ be a 2-cocycle and build the twisted crossed product $A \rtimes \mathbb{C}[\Gamma, \natural]$ with multiplication relations as in Proposition 1.1 – the role of $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$ is played by A. Then*

$$\mathrm{gl.\,dim}(A \rtimes \mathbb{C}[\Gamma, \natural]) = \mathrm{gl.\,dim}(A).$$

Proof. For any A-module M

$$\mathrm{Res}^{A \rtimes \mathbb{C}[\Gamma,\natural]}_A \mathrm{ind}^{A \rtimes \mathbb{C}[\Gamma,\natural]}_A M \cong \bigoplus_{\gamma \in \Gamma} \gamma^*(M).$$

Hence $\mathrm{Ext}^n_A(M', M)$ is a direct summand of

$$\mathrm{Ext}^n_A\big(M', \mathrm{Res}^{A \rtimes \mathbb{C}[\Gamma,\natural]}_A \mathrm{ind}^{A \rtimes \mathbb{C}[\Gamma,\natural]}_A M\big) \cong \mathrm{Ext}^n_A\big(\mathrm{ind}^{A \rtimes \mathbb{C}[\Gamma,\natural]}_A M', \mathrm{ind}^{A \rtimes \mathbb{C}[\Gamma,\natural]}_A M\big).$$

In particular gl. dim $(A) \leq$ gl. dim $(A \rtimes \mathbb{C}[\Gamma, \natural])$.

For any $A \rtimes \mathbb{C}[\Gamma, \natural]$-module V there is a surjective module homomorphism

$$\begin{array}{cccc} p: & \mathrm{ind}^{A \rtimes \mathbb{C}[\Gamma,\natural]}_A \mathrm{Res}^{A \rtimes \mathbb{C}[\Gamma,\natural]}_A V & \to & V \\ & x \otimes v & \mapsto & xv \end{array}.$$

On the other hand, there is a natural injection

$$\begin{array}{cccc} \imath : & V & \to & \mathrm{ind}_A^{A \rtimes \mathbb{C}[\Gamma,\natural]} \mathrm{Res}_A^{A \rtimes \mathbb{C}[\Gamma,\natural]} V \\ & v & \mapsto & \sum_{\gamma \in \Gamma} N_\gamma^{-1} \otimes N_\gamma v \end{array} .$$

This in fact a module homomorphism. Namely, for $a \in A$:

$$\begin{aligned} \imath(av) &= \sum_{\gamma \in \Gamma} N_\gamma^{-1} \otimes N_\gamma a v = \sum_{\gamma \in \Gamma} N_\gamma^{-1} \otimes \gamma(a) N_\gamma v \\ &= \sum_{\gamma \in \Gamma} N_\gamma^{-1} \gamma(a) \otimes N_\gamma v = \sum_{\gamma \in \Gamma} a N_\gamma^{-1} \otimes N_\gamma v = a \imath(v). \end{aligned}$$

Similarly, for $g \in \Gamma$:

$$\begin{aligned} \imath(N_g v) &= \sum_{\gamma \in \Gamma} N_\gamma^{-1} \otimes N_\gamma N_g v = \sum_{\gamma \in \Gamma} N_g N_g^{-1} N_\gamma^{-1} \otimes N_\gamma N_g v \\ &= \sum_{\gamma \in \Gamma} N_g N_{\gamma g}^{-1} \otimes N_{\gamma g} v = \sum_{h \in \Gamma} N_g N_h^{-1} \otimes N_h v = N_g \imath(v). \end{aligned}$$

Clearly $p \circ \imath = |\Gamma| \, \mathrm{id}_V$, so

$$\mathrm{ind}_A^{A \rtimes \mathbb{C}[\Gamma,\natural]} \mathrm{Res}_A^{A \rtimes \mathbb{C}[\Gamma,\natural]} V \cong V \oplus \ker p \qquad \text{as } A \rtimes \mathbb{C}[\Gamma,\natural]\text{-modules.}$$

For any second $A \rtimes \mathbb{C}[\Gamma,\natural]$-module V', $\mathrm{Ext}^n_{A \rtimes \mathbb{C}[\Gamma,\natural]}(V, V')$ is a direct summand of

$$\mathrm{Ext}^n_{A \rtimes \mathbb{C}[\Gamma,\natural]}\big(V \oplus \ker p, V'\big) = \mathrm{Ext}^n_{A \rtimes \mathbb{C}[\Gamma,\natural]}\big(\mathrm{ind}_A^{A \rtimes \mathbb{C}[\Gamma,\natural]} \mathrm{Res}_A^{A \rtimes \mathbb{C}[\Gamma,\natural]} V, V'\big).$$

By Frobenius reciprocity the latter is isomorphic with $\mathrm{Ext}^n_A\big(V, \mathrm{Res}_A^{A \rtimes \mathbb{C}[\Gamma,\natural]} V'\big)$. Hence $\mathrm{Ext}^n_{A \rtimes \mathbb{C}[\Gamma,\natural]}(V, M)$ vanishes whenever $n >$gl. $\dim(A)$. □

In view of Lemma 1.7 and the construction of $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)$, it can be expected that it has the same global dimension as $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$. The latter equals $\dim_{\mathbb{C}}(\mathfrak{t} \oplus \mathbb{C})$, see for instance [Wei, Theorem 4.3.7].

While the global dimensions of these algebras do indeed agree, Lemma 1.7 does not suffice to show that. One complication is that a map like $\imath$ above is not a module homomorphism in the setting of the group $W \rtimes \Gamma$ and the algebra $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$, when the parameters of the Hecke algebra are nonzero.

Lemma 1.8. *For $r \in \mathbb{C}$, let $\hat{\mathbb{H}}_r$ be the formal completion of $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$ with respect to the ideal $(\mathbf{r} - r)$ of $\mathbb{C}[\mathbf{r}]$. Then*

$$\mathrm{gl.\,dim}(\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)) = \sup_{r \in \mathbb{C}} \mathrm{gl.\,dim}(\hat{\mathbb{H}}_r).$$

Proof. By Lemma 1.7 we may assume that $\Gamma = \{1\}$, so that $\natural$ disappears. We abbreviate $\mathbb{H} = \mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$. All the algebras in this proof are Noetherian, so by [Wei, Proposition 4.1.5] their global dimensions equal their Tor-dimensions. We will use both the characterization in terms of Ext-groups and that in terms of Tor-groups, whatever we find more convenient.

For an $\mathbb{H}$-module V and $r \in \mathbb{C}$, we consider the formal completion of V with respect to the ideal $(\mathbf{r} - r)$ of $\mathbb{C}[\mathbf{r}]$:

$$\hat{V}_r = V \otimes_{\mathbb{C}[\mathbf{r}]} \mathbb{C}[[\mathbf{r} - r]].$$

Completion is an exact functor, so by [Wei, Corollary 3.2.10]

$$\mathrm{Tor}_m^{\mathbb{H}}(V, M) \otimes_{\mathbb{C}[\mathbf{r}]} \mathbb{C}[[\mathbf{r} - r]] \cong \mathrm{Tor}_m^{\hat{\mathbb{H}}_r}(\hat{V}_r, \hat{M}_r). \tag{1.12}$$

Every $\hat{\mathbb{H}}_r$-module is of the form $\hat{V}_r$ (namely, starting from $\hat{V}_r$ as $\mathbb{H}$-module). Thus (1.12) shows that gl. $\dim(\hat{\mathbb{H}}_r) \leq$ gl. $\dim(\mathbb{H})$.

When V and M are finitely generated (which suffices to compute global dimensions), they have projective resolutions consisting of free modules of finite rank [KNS, Lemma 3]. The centre of $\mathbb{H}$ was identified in [Lus4, Proposition 4.5] as

$$Z(\mathbb{H}) = \mathcal{O}(\mathfrak{t})^W \oplus \mathbb{C}[\mathbf{r}],$$

and we see that $\mathbb{H}$ has finite rank as module over $Z(\mathbb{H})$. It follows that $\mathrm{Tor}_m^{\mathbb{H}}(V,M)$ is finitely generated as $Z(\mathbb{H})$-module. By [Sol5, Lemma 2.9], $\mathrm{Tor}_m^{\mathbb{H}}(V,M)$ is nonzero if and only if its formal completion with respect to some character of $Z(\mathbb{H})$ is nonzero. That happens if and only if (1.12) is nonzero for some $r \in \mathbb{C}$. $\square$

It remains to find a good upper bound for the global dimension of $\hat{\mathbb{H}}_r$.

Theorem 1.9. *The global dimension of* $\mathbb{H}(\mathfrak{t}, W \rtimes \Gamma, k, \mathbf{r}, \natural)$ *equals* $\dim_{\mathbb{C}}(\mathfrak{t}) + 1$.

Proof. By Lemma 1.7 it suffices to consider the cases with $\Gamma = \{1\}$. The crucial point of our proof is that the global dimension of the graded Hecke algebra

$$\mathbb{H}/(\mathbf{r} - r) = \mathbb{H}(\mathfrak{t}, W, rk)$$

has already been computed, and equals $\dim_{\mathbb{C}}(\mathfrak{t})$ [Sol1, Theorem 5.3]. For any $\mathbb{H}/(\mathbf{r}-r)$-module V_1, [Wei, Theorem 4.3.1] provides an equality of projective dimensions

$$\mathrm{pd}_{\mathbb{H}}(V_1) = \mathrm{pd}_{\mathbb{H}/(\mathbf{r}-r)}(V_1) + \mathrm{pd}_{\mathbb{H}}(\mathbb{H}/(\mathbf{r}-r)).$$

From the short exact sequence

$$0 \to \mathbb{H} \xrightarrow{\mathbf{r}-r} \mathbb{H} \to \mathbb{H}/(\mathbf{r}-r) \to 0$$

we see that $\mathrm{pd}_{\mathbb{H}}(\mathbb{H}/(\mathbf{r}-r)) = 1$. Hence

$$\mathrm{pd}_{\mathbb{H}}(V_1) = \mathrm{pd}_{\mathbb{H}/(\mathbf{r}-r)}(V_1) + 1 \leq \dim_{\mathbb{C}}(\mathfrak{t}) + 1. \tag{1.13}$$

In other words, $\mathrm{Tor}_m^{\mathbb{H}}(V_1, M) = 0$ for all $m > \dim_{\mathbb{C}}(\mathfrak{t}) + 1$.

Let V_2 be an $\mathbb{H}$-module on which $(\mathbf{r}-r)^2$ acts as 0. In the short exact sequence

$$0 \to (\mathbf{r}-r)V_2 \to V_2 \to V_2/(\mathbf{r}-r)V_2 \to 0,$$

$\mathbf{r}-r$ annihilates both the outer terms, so (1.13) applies to them. Applying $\mathrm{Tor}_*^{\mathbb{H}}(?, M)$ to this short exact sequence yields a long exact sequence, and taking (1.13) into account we see that $\mathrm{Tor}_m^{\mathbb{H}}(V_2, M) = 0$ for all $m > \dim_{\mathbb{C}}(\mathfrak{t}) + 1$.

This argument can be applied recursively, and then it shows that

$$\mathrm{Tor}_m^{\mathbb{H}}(V_n, M) = 0 \qquad \text{if } m > \dim_{\mathbb{C}}(\mathfrak{t})+1 \text{ and } (\mathbf{r}-r)^n V_n = 0 \text{ for some } n \in \mathbb{N}. \tag{1.14}$$

Assume now that V and M are finitely generated $\hat{\mathbb{H}}_r$-modules. By (1.12) and (1.14)

$$\mathrm{Tor}_m^{\hat{\mathbb{H}}_r}(V/(\mathbf{r}-r)^n V, M) = 0 \qquad \text{for } m > \dim_{\mathbb{C}}(\mathfrak{t}) + 1 \text{ and } n \in \mathbb{N}. \tag{1.15}$$

Let $P_* \to M$ be a resolution by free $\hat{\mathbb{H}}_r$-modules P_i of finite rank μ_i (this is possible because $\hat{\mathbb{H}}_r$ is Noetherian). Then

$$\mathrm{Tor}_m^{\hat{\mathbb{H}}_r}(V/(\mathbf{r}-r)^n V, M) = H_m\big(V/(\mathbf{r}-r)^n V \otimes_{\hat{\mathbb{H}}_r} \hat{\mathbb{H}}_r^{\mu_*}, d_*\big) = H_m\big((V/(\mathbf{r}-r)^n V)^{\mu_*}, d_*\big).$$

Here the sequence of differential complexes $\big((V/(\mathbf{r}-r)^n V)^{\mu_*}, d_*\big)$, indexed by $n \in \mathbb{N}$, satisfies the Mittag–Leffler condition because the transition maps are surjective. The inverse limit of the sequence is (V^{μ_*}, d_*), which computes $\mathrm{Tor}_m^{\hat{\mathbb{H}}_r}(V, M)$. According to [Wei, Theorem 3.5.8] there is a short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{Tor}_{m+1}^{\hat{\mathbb{H}}_r}(V/(\mathbf{r}-r)^n V, M) \to \mathrm{Tor}_m^{\hat{\mathbb{H}}_r}(V, M) \to \varprojlim \mathrm{Tor}_m^{\hat{\mathbb{H}}_r}(V/(\mathbf{r}-r)^n V, M) \to 0.$$

For $m > \dim_{\mathbb{C}}(\mathfrak{t}) + 1$, (1.15) shows that both outer terms vanish, so $\mathrm{Tor}_m^{\hat{\mathbb{H}}_r}(V, M) = 0$ as well. Hence

$$\mathrm{gl.}\ \dim(\hat{\mathbb{H}}_r) \leq \dim_{\mathbb{C}}(\mathfrak{t}) + 1.$$

Together with Lemmas 1.6 and 1.8 that finishes the proof. □

2. Equivariant sheaves and equivariant cohomology

We follow the setup from [Lus3, Lus6, AMS1, AMS2]. In these references a graded Hecke algebra was associated to a cuspidal local system on a nilpotent orbit for a complex reductive group, via equivariant cohomology. For future applications to Langlands parameters we deal not only with connected groups, but also with disconnected reductive groups G.

We work in the G-equivariant bounded derived category $\mathcal{D}^b_G(X)$, as in [BeLu], [Lus3, §1] and [Lus6, §1]. The formalism of [BeLu] entails that this is not exactly the bounded derived category of the category of G-equivariant constructible sheaves on a G-variety X. Morphisms in $\mathcal{D}^b_G(X)$ are defined via a resolution of X by G-varieties Y as in [BeLu], and on each such Y we use morphisms in a (non-equivariant) derived category of sheaves. Equivariant cohomology for objects of $\mathcal{D}^b_G(X)$ is defined via push-forward to a point, representing the result as a complex of sheaves on a classifying space $\mathfrak{B}G$ for G and then taking cohomology in $\mathcal{D}^b(\mathfrak{B}G)$. For more background we refer to [Ach, Chapter 6].

We will use some notations and conventions from [Lus6], in particular functors from or to $\mathcal{D}^b_G(X)$ are by default derived functors. Let $[n]$ be the functor that shifts degrees by n. For objects A, B of $\mathcal{D}^b_G(X)$ and $n \in \mathbb{Z}$, we write

$$\mathrm{Hom}^n_{\mathcal{D}^b_G(X)}(A, B) = \mathrm{Hom}_{\mathcal{D}^b_G(X)}(A, B[n]).$$

In the case $A = B$ one obtains the graded algebra

$$\mathrm{End}^*_{\mathcal{D}^b_G(X)}(A) = \bigoplus\nolimits_{n \in \mathbb{Z}} \mathrm{Hom}^n_{\mathcal{D}^b_G(X)}(A, A).$$

2.1. **Geometric construction of graded Hecke algebras.**

Recall from [AMS1] that a quasi-Levi subgroup of G is a group of the form $M = Z_G(Z(L)^\circ)$, where L is a Levi subgroup of G°. Thus $Z(M)^\circ = Z(L)^\circ$ and $M \longleftrightarrow L = M^\circ$ is a bijection between the quasi-Levi subgroups of G and the Levi subgroups of G°.

Definition 2.1. A cuspidal quasi-support for G is a triple $(M, \mathcal{C}^M_v, q\mathcal{E})$ where:

- M is a quasi-Levi subgroup of G;
- $\mathcal{C}^M_v$ is the $\mathrm{Ad}(M)$-orbit of a nilpotent element $v \in \mathfrak{m} = \mathrm{Lie}(M)$.
- $q\mathcal{E}$ is a M-equivariant cuspidal local system on $\mathcal{C}^M_v$, i.e. as M°-equivariant local system it is a direct sum of cuspidal local systems.

We denote the G-conjugacy class of $(M, \mathcal{C}^M_v, q\mathcal{E})$ by $[M, \mathcal{C}^M_v, q\mathcal{E}]_G$. With this cuspidal quasi-support we associate the groups

$$N_G(q\mathcal{E}) = \mathrm{Stab}_{N_G(M)}(q\mathcal{E}) \quad \text{and} \quad W_{q\mathcal{E}} = N_G(q\mathcal{E})/M. \tag{2.1}$$

Let $\mathfrak{g}_N$ be the variety of nilpotent elements in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Cuspidal quasi-supports are useful to partition the set of G-equivariant local systems on $\mathrm{Ad}(G)$-orbits in $\mathfrak{g}_N$. Let $\mathcal{E}$ be an irreducible constituent of $q\mathcal{E}$ as M°-equivariant local system on $\mathcal{C}_v^M$ (which by the cuspidality of $\mathcal{E}$ equals the $\mathrm{Ad}(M^\circ)$-orbit of v). Then

$$W_{\mathcal{E}}^\circ := N_{G^\circ}(M^\circ)/M^\circ \cong N_{G^\circ}(M^\circ)M/M$$

is a subgroup of $W_{q\mathcal{E}}$. It is normal because G° is normal in G. Write $T = Z(L)^\circ$ and $\mathfrak{t} = \mathrm{Lie}(T)$. It is known from [Lus3, Proposition 2.2] that $R(G^\circ, T) \subset \mathfrak{t}^\vee$ is a root system with Weyl goup $W_{\mathcal{E}}^\circ$.

Let P° be a parabolic subgroup of G° with Levi decomposition $P^\circ = M^\circ \ltimes U$. The definition of M entails that it normalizes U, so

$$P := M \ltimes U$$

is a again a group, a "quasi-parabolic" subgroup of G. We put

$$\begin{aligned} N_G(P, q\mathcal{E}) &= N_G(P, M) \cap N_G(q\mathcal{E}), \\ \Gamma_{q\mathcal{E}} &= N_G(P, q\mathcal{E})/M. \end{aligned}$$

The same proof as for [AMS2, Lemma 2.1.b] shows that

$$W_{q\mathcal{E}} = W_{\mathcal{E}}^\circ \rtimes \Gamma_{q\mathcal{E}}. \tag{2.2}$$

The $W_{q\mathcal{E}}$-action on T gives rise to an action of $W_{q\mathcal{E}}$ on $\mathcal{O}(\mathfrak{t}) = S(\mathfrak{t}^\vee)$.

We specify our parameters $k(\alpha)$. For α in the root system $R(G^\circ, T)$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the associated eigenspace for the T-action. Let Δ_P be the set of roots in $R(G^\circ, T)$ which are simple with respect to P. For $\alpha \in \Delta_P$ we define $k(\alpha) \in \mathbb{Z}_{\geq 2}$ by

$$\begin{array}{llll} \mathrm{ad}(v)^{k(\alpha)-2} : & \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \to \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} & \text{is nonzero,} \\ \mathrm{ad}(v)^{k(\alpha)-1} : & \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \to \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} & \text{is zero.} \end{array} \tag{2.3}$$

Then $(k(\alpha))_{\alpha \in \Delta_P}$ extends to a $W_{q\mathcal{E}}$-invariant function $k : R(G^\circ, T)_{\mathrm{red}} \to \mathbb{C}$, where the subscript "red" indicates the set of reduced (or indivisible) roots. Let $\natural : (W_{q\mathcal{E}}/W_{\mathcal{E}}^\circ)^2 \to \mathbb{C}^\times$ be a 2-cocycle (to be specified later). To these data we associate the twisted graded Hecke algebra $\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, k, \mathbf{r}, \natural)$, as in Proposition 1.1.

To make the connection of the above twisted graded Hecke algebra with the cuspidal local system $q\mathcal{E}$ complete, we involve the geometry of G and $\mathfrak{g}$. Write

$$\mathfrak{t}_{\mathrm{reg}} = \{x \in \mathfrak{t} : Z_{\mathfrak{g}}(x) = \mathfrak{l}\} \quad \text{and} \quad \mathfrak{g}_{RS} = \mathrm{Ad}(G)(\mathcal{C}_v^M \oplus \mathfrak{t}_{\mathrm{reg}} \oplus \mathfrak{u}).$$

Consider the varieties

$$\begin{aligned} \dot{\mathfrak{g}} &= \{(X, gP) \in \mathfrak{g} \times G/P : \mathrm{Ad}(g^{-1})X \in \mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}\}, \\ \dot{\mathfrak{g}}^\circ &= \{(X, gP) \in \mathfrak{g} \times G^\circ/P^\circ : \mathrm{Ad}(g^{-1})X \in \mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}\}, \\ \dot{\mathfrak{g}}_{RS} &= \dot{\mathfrak{g}} \cap (\mathfrak{g}_{RS} \times G/P), \\ \dot{\mathfrak{g}}_N &= \dot{\mathfrak{g}} \cap (\mathfrak{g}_N \times G/P). \end{aligned}$$

We let $G \times \mathbb{C}^\times$ act on these varieties by

$$(g_1, \lambda) \cdot (X, gP) = (\lambda^{-2}\mathrm{Ad}(g_1)X, g_1 gP).$$

By [Lus3, Proposition 4.2] there is a natural isomorphism

$$H^*_{G \times \mathbb{C}^\times}(\dot{\mathfrak{g}}) \cong \mathcal{O}(\mathfrak{t}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}]. \tag{2.4}$$

The same calculation (omitting $\mathfrak{t}$ from the definition of $\dot{\mathfrak{g}}$) shows that

$$H^*_{G \times \mathbb{C}^\times}(\dot{\mathfrak{g}}_N) \cong \mathcal{O}(\mathfrak{t}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}]. \tag{2.5}$$

Consider the maps

$$\begin{aligned} &\mathcal{C}_v^M \xleftarrow{f_1} \{(X,g) \in \mathfrak{g} \times G : \mathrm{Ad}(g^{-1})X \in \mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}\} \xrightarrow{f_2} \dot{\mathfrak{g}}, \\ &f_1(X,g) = \mathrm{pr}_{\mathcal{C}_v^M}(\mathrm{Ad}(g^{-1})X), \qquad\qquad f_2(X,g) = (X, gP). \end{aligned} \tag{2.6}$$

The group $G \times \mathbb{C}^\times \times P$ acts on $\{(X,g) \in \mathfrak{g} \times G : \mathrm{Ad}(g^{-1})X \in \mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}\}$ by

$$(g_1, \lambda, p) \cdot (X, g) = (\lambda^{-2}\mathrm{Ad}(g_1)X, g_1 g p).$$

Notice that the local system $q\mathcal{E}$ on $\mathcal{C}_v^M$ is $M \times \mathbb{C}^\times$-equivariant, because $\mathbb{C}^\times$ is connected and stabilizes nilpotent M-orbits. Further f_1 is constant on G-orbits, so $f_1^* q\mathcal{E}$ is naturally a $G \times \mathbb{C}^\times$-equivariant local system. Let $q\dot{\mathcal{E}}$ be the unique $G \times \mathbb{C}^\times$-equivariant local system on $\dot{\mathfrak{g}}$ such that $f_2^* q\dot{\mathcal{E}} = f_1^* q\mathcal{E}$. Let $\mathrm{pr}_1 : \dot{\mathfrak{g}} \to \mathfrak{g}$ be the projection on the first coordinate. When G is connected, Lusztig [Lus6] has constructed graded Hecke algebras from

$$K := \mathrm{pr}_{1,!} q\dot{\mathcal{E}} \qquad \in \mathcal{D}^b_{G \times \mathbb{C}^\times}(\mathfrak{g}).$$

For our purposes the pullback K_N of K to the nilpotent variety $\mathfrak{g}_N \subset \mathfrak{g}$ will be more suitable than K itself.

We can relate $\dot{\mathfrak{g}}$ and K to their versions for G°, as follows. Write

$$G = \bigsqcup_{\gamma \in N_G(P,M)/M} G^\circ \gamma M/M \quad \text{and} \quad G/P = \bigsqcup_{\gamma \in N_G(P,M)/M} G^\circ \gamma P/P. \tag{2.7}$$

Then we can decompose

$$\begin{aligned} \dot{\mathfrak{g}} &= \bigsqcup\nolimits_{\gamma \in N_G(P,M)/M} \{(X, g\gamma P) \in \dot{\mathfrak{g}} : g \in G^\circ\} = \\ &\bigsqcup_{\gamma \in N_G(P,M)/M} \{(X, g\gamma P\gamma^{-1}) : X \in \mathfrak{g}, g \in G^\circ/\gamma P^\circ \gamma^{-1}, \mathrm{Ad}(g^{-1})X \in \mathrm{Ad}(\gamma)(\mathcal{C}_v^M + \mathfrak{t} + \mathfrak{u})\} \\ &= \bigsqcup\nolimits_{\gamma \in N_G(P,M)/M} \dot{\mathfrak{g}}^\circ_\gamma. \end{aligned} \tag{2.8}$$

Here each term $\dot{\mathfrak{g}}^\circ_\gamma$ is a twisted version of $\dot{\mathfrak{g}}^\circ$. Consequently K is a direct sum of $G^\circ \times \mathbb{C}^\times$-equivariant subobjects, each of which is a twist of the K for $(G^\circ M, \mathcal{C}_v^M, q\mathcal{E})$ by an element of $N_G(M)/M$.

Let $q\dot{\mathcal{E}}_{RS}$ be the pullback of $q\dot{\mathcal{E}}$ to $\dot{\mathfrak{g}}_{RS}$. Let $\mathrm{IC}_{G \times \mathbb{C}^\times}(\mathfrak{g} \times G/P, q\dot{\mathcal{E}}_{RS})$ be the equivariant intersection complex determined by $q\dot{\mathcal{E}}_{RS}$, supported on the closure of $\dot{\mathfrak{g}}_{RS}$ in $\mathfrak{g} \times G/P$. Notice that pr_1 becomes proper on this domain. The map

$$\mathrm{pr}_{1,RS} : \dot{\mathfrak{g}}_{RS} \to \mathfrak{g}_{RS}$$

is a fibration with fibre $N_G(M)/M$, so $(\mathrm{pr}_{1,RS})_! q\dot{\mathcal{E}}_{RS}$ is a local system on $\mathfrak{g}_{RS}$. It is shown in [Lus6, Proposition 7.12.c] that

$$K \cong \mathrm{pr}_{1,!} \mathrm{IC}_{G \times \mathbb{C}^\times}(\mathfrak{g} \times G/P, q\dot{\mathcal{E}}_{RS}) \cong \mathrm{IC}_{G \times \mathbb{C}^\times}(\mathfrak{g}, (\mathrm{pr}_{1,RS})_! q\dot{\mathcal{E}}_{RS}). \tag{2.9}$$

The last expression shows that K is a direct sum of simple perverse sheaves with support $\overline{\mathfrak{g}_{RS}}$. Further, [AMS1, Lemma 5.4] and [Lus6, Proposition 7.14] say that

$$\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}] \cong \mathrm{End}^0_{\mathcal{D}^b_{G \times \mathbb{C}^\times}(\mathfrak{g}_{RS})}\big((\mathrm{pr}_{1,RS})_! q\dot{\mathcal{E}}_{RS}\big) \cong \mathrm{End}^0_{\mathcal{D}^b_{G \times \mathbb{C}^\times}(\mathfrak{g})}(K), \tag{2.10}$$

where $\natural_{q\mathcal{E}} : (W_{q\mathcal{E}}/W_{\mathcal{E}}^\circ)^2 \to \mathbb{C}^\times$ is a suitable 2-cocycle. As in [AMS2, (8)], we record the subalgebra of endomorphisms that stabilize Lie(P):

$$\mathrm{End}^0_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}\big(\mathrm{pr}_{1,!}\dot{q\mathcal{E}}\big)_P \cong \mathbb{C}[\Gamma_{q\mathcal{E}}, \natural_{q\mathcal{E}}]. \tag{2.11}$$

Now we associate to $(M, \mathcal{C}_v^M, q\mathcal{E})$ the twisted graded Hecke algebra

$$\mathbb{H}(G, M, q\mathcal{E}) := \mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, k, \mathbf{r}, \natural_{q\mathcal{E}}),$$

where the parameters $c(\alpha)$ come from (2.3). As in [AMS2, Lemma 2.8], we can regard it as

$$\mathbb{H}(G, M, q\mathcal{E}) = \mathbb{H}(\mathfrak{t}, W_{\mathcal{E}}^\circ, k, \mathbf{r}) \rtimes \mathrm{End}^0_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}\big(\mathrm{pr}_{1,!}\dot{q\mathcal{E}}\big)_P,$$

and then it depends canonically on $(G, M, q\mathcal{E})$. We note that (2.2) implies

$$\mathbb{H}(G^\circ N_G(P, q\mathcal{E}), M, q\mathcal{E}) = \mathbb{H}(G, M, q\mathcal{E}). \tag{2.12}$$

There is also a purely geometric realization of this algebra. For $\mathrm{Ad}(G) \times \mathbb{C}^\times$-stable subvarieties $\mathcal{V}$ of $\mathfrak{g}$, we define, as in [Lus3, §3],

$$\begin{aligned} \dot{\mathcal{V}} &= \{(X, gP) \in \dot{\mathfrak{g}} : X \in \mathcal{V}\}, \\ \ddot{\mathcal{V}} &= \{(X, gP, g'P) : (X, gP) \in \dot{\mathcal{V}}, (X, g'P) \in \dot{\mathcal{V}}\}. \end{aligned} \tag{2.13}$$

Let $q\mathcal{E}^\vee$ be the dual equivariant local system on $\mathcal{C}_v^M$, which is also cuspidal. It gives rise to $K^\vee = \mathrm{pr}_{1,!}\dot{q\mathcal{E}}^\vee$, another equivariant intersection cohomology complex on $\mathfrak{g}$. The two projections $\pi_{12}, \pi_{13} : \ddot{\mathcal{V}} \to \dot{\mathcal{V}}$ give rise to a $G \times \mathbb{C}^\times$-equivariant local system

$$\ddot{q\mathcal{E}} = (\pi_{12} \times \pi_{13})^*\big(\dot{q\mathcal{E}} \boxtimes \dot{q\mathcal{E}}^\vee\big) \quad \text{on} \quad \ddot{\mathcal{V}},$$

which carries a natural action of (2.4). As in [Lus3], the action of $\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}^{-1}]$ on $K^\vee$ leads to

$$\text{actions of } \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}] \otimes \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}^{-1}] \text{ on } \ddot{q\mathcal{E}} \text{ and on } H_j^{G^\circ\times\mathbb{C}^\times}\big(\ddot{\mathcal{V}}, \ddot{q\mathcal{E}}\big). \tag{2.14}$$

In [Lus3] and [AMS2, §2] a left action Δ and a right action Δ' of $\mathbb{H}(G, M, q\mathcal{E})$ on $H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N, \ddot{q\mathcal{E}})$ are constructed.

Theorem 2.2. (*a*) *The actions Δ and Δ' identify $H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}, \ddot{q\mathcal{E}})$ and $H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N, \ddot{q\mathcal{E}})$ with the biregular representation of $\mathbb{H}(G, M, q\mathcal{E})$.*

(*b*) *Methods from equivariant cohomology provide natural isomorphisms of graded vector spaces*

$$\begin{aligned} \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K) &\cong H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}, \ddot{q\mathcal{E}}), \\ \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)}(K_N) &\cong H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N, \ddot{q\mathcal{E}}). \end{aligned}$$

(*c*) *Parts (a) and (b) induce canonical isomorphisms of graded algebras*

$$\mathbb{H}(G, M, q\mathcal{E}) \to \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K) \to \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)}(K_N).$$

Proof. (a) When G is connected, this is shown for $\ddot{\mathfrak{g}}_N$ in [Lus3, Corollary 6.4] and for $\ddot{\mathfrak{g}}$ in the proof of [Lus6, Theorem 8.11], based on [Lus3]. In [AMS2, Corollary 2.9 and §4] both are generalized to possibly disconnected G.
(b) For $(\mathfrak{g}, K)$ with G connected this is the beginning of the proof of [Lus6, Theorem 8.11]. The same argument applies when G is disconnected, and with $(\mathfrak{g}_N, K_N)$ instead of $(\mathfrak{g}, K)$.

(c) In [Lus6, Theorem 8.11] the first isomorphism is shown when G is connected. Using parts (a,b) the same argument applies when G is disconnected. Similarly we obtain

$$\mathbb{H}(G, M, q\mathcal{E}) \cong \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)}(K_N).$$

These two graded algebra isomorphisms are linked via parts (a,b) and functoriality for the inclusion $\mathfrak{g}_N \to \mathfrak{g}$. $\square$

2.2. **Semisimplicity of some complexes of sheaves.**

For an alternative construction of $q\dot{\mathcal{E}}$ and K, we consider the isomorphism of $G \times \mathbb{C}^\times$-varieties

$$\begin{array}{lcc} G \times^P (\mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}) & \to & \dot{\mathfrak{g}} \\ (g, X) & \mapsto & (\mathrm{Ad}(g)X, gP) \end{array} . \tag{2.15}$$

We note that the middle term in (2.6) is isomorphic to $G \times (\mathcal{C}_v^M \oplus \mathfrak{t} \oplus u)$ via the map $(X, g) \mapsto (g, \mathrm{Ad}(g^{-1})X)$. In these terms, (2.6) becomes

$$\mathcal{C}_v^M \xleftarrow{f_1'} G \times (\mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}) \xrightarrow{f_2'} G \times^P (\mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}), \tag{2.16}$$

with the natural maps f_1' and f_2'. We get $q\dot{\mathcal{E}}$ as $G \times \mathbb{C}^\times$-equivariant local system on $G \times^P (\mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u})$, satisfying $f_2'^* q\dot{\mathcal{E}} = f_1'^* q\mathcal{E}$. In this setup pr_1 is replaced by

$$\begin{array}{rlcc} \mu : & G \times^P (\mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}) & \to & \mathfrak{g} \\ & (g, X) & \mapsto & \mathrm{Ad}(g)X \end{array} \tag{2.17}$$

and then

$$K = \mu_! \, q\dot{\mathcal{E}}. \tag{2.18}$$

Recall that we defined K_N as the pullback K_N of K to the variety $\mathfrak{g}_N$, and that K is a semisimple complex (that is, isomorphic to a direct sum of simple perverse sheaves, maybe with degree shifts). We will prove that K_N is also semisimple complex of sheaves. We write

$$\dot{\mathfrak{g}}_N = \dot{\mathfrak{g}} \cap (\mathfrak{g}_N \times G/P).$$

The maps (2.6) restrict to

$$\mathcal{C}_v^M \xleftarrow{f_{1,N}} \{(X, g) \in \mathfrak{g}_N \times G : \mathrm{Ad}(g^{-1})X \in \mathcal{C}_v^M \oplus \mathfrak{u}\} \xrightarrow{f_{2,N}} \dot{\mathfrak{g}}_N, \tag{2.19}$$

which allows us to define a local system $q\dot{\mathcal{E}}_N$ on $\dot{\mathfrak{g}}_N$ by $f_{2,N}^* q\dot{\mathcal{E}} = f_{1,N}^* q\mathcal{E}_N$. Then $q\dot{\mathcal{E}}_N$ is the pullback of $q\dot{\mathcal{E}}$ to $\mathfrak{g}_N$, because $f_{1,N}^* q\mathcal{E}_N$ is the pullback of $f_1^* q\mathcal{E}$. Let $\mathrm{pr}_{1,N}$ be the restriction of pr_1 to $\dot{\mathfrak{g}}_N$. From the Cartesian diagram

$$\begin{array}{ccc} \dot{\mathfrak{g}}_N & \longrightarrow & \dot{\mathfrak{g}} \\ \downarrow{\scriptstyle \mathrm{pr}_{1,N}} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathfrak{g}_N & \longrightarrow & \mathfrak{g} \end{array} \tag{2.20}$$

we see with base change [BeLu, Theorem 3.4.3] that

$$\mathrm{pr}_{1,N,!} \, q\dot{\mathcal{E}}_N \text{ equals the pullback } K_N \text{ of } K \text{ to } \mathfrak{g}_N. \tag{2.21}$$

Proposition 2.3. *There is a natural isomorphism*

$$K_N \cong \mathrm{pr}_{1,N,!} \, \mathrm{IC}_{G\times\mathbb{C}^\times}(\mathfrak{g}_N \times G/P, q\dot{\mathcal{E}}_N).$$

Proof. Notice that the middle term in (2.19) is isomorphic with $G \times \mathcal{C}_v^M \oplus \mathfrak{u}$ and that (2.15) provides an isomorphism

$$\dot{\mathfrak{g}}_N \cong G \times^P (\mathcal{C}_v^M \oplus \mathfrak{u}).$$

With the commutative diagram

$$\begin{tikzcd} \mathcal{C}_v^M \arrow[d] & \mathcal{C}_v^M \oplus \mathfrak{u} \arrow[l, "\mathrm{pr}_{\mathcal{C}_v^M}"'] \arrow[d] \\ G \times^P \mathcal{C}_v^M & G \times^P (\mathcal{C}_v^M \oplus \mathfrak{u}) \arrow[l, "\mathrm{id}_G \times \mathrm{pr}_{\mathcal{C}_v^M}"'] \end{tikzcd} \tag{2.22}$$

we can construct $q\dot{\mathcal{E}}_N \in \mathcal{D}^b_{G\times\mathbb{C}^\times}(G \times^P (\mathcal{C}_v^M \oplus \mathfrak{u}))$ in two equivalent ways:

- pullback of $q\mathcal{E}$ to $\mathcal{C}_v^m \oplus \mathfrak{u}$ (as $P \times \mathbb{C}^\times$-equivariant local system) and then equivariant induction $\mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times}$ as in [BeLu, §2.6.3];
- equivariant induction $\mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times}$ of $q\mathcal{E}$ to $G \times^P \mathcal{C}_v^M$ and the pullback to $G \times^P (\mathcal{C}_v^M \oplus \mathfrak{u})$.

In these terms

$$K_N = \mu_{N,!}\, q\dot{\mathcal{E}}_N, \tag{2.23}$$

where $\mu_N : G \times^P (\mathcal{C}_v^M \oplus \mathfrak{u}) \to \mathfrak{g}_N$ is the restriction of (2.17). Let $j_{\mathfrak{m}_N} : \mathcal{C}_v^M \to \mathfrak{m}_N$ be the inclusion. Then

$$K_N = \mu_{N,!}(\mathrm{id}_G \times j_{\mathfrak{m}_N} \times \mathrm{id}_{\mathfrak{u}})_! q\dot{\mathcal{E}}_N, \tag{2.24}$$

where now the domain of μ_N is $\mathfrak{m}_N \oplus \mathfrak{u}$.

Regarded as $M^\circ \times \mathbb{C}^\times$-equivariant local system on $\mathcal{C}_v^M$, $q\mathcal{E}$ is a direct sum of irreducible cuspidal local systems $\mathcal{E}$. Each of those $\mathcal{E}$ is clean [Lus1, Theorem 23.1], which means that

$$j_{\mathfrak{m}_N,!}\mathcal{E} = \mathrm{IC}(\mathfrak{m}_N, \mathcal{E}) = j_{\mathfrak{m}_N,*}\mathcal{E}.$$

Taking direct sums over the appropriate $\mathcal{E}$, we find that $q\mathcal{E}$ is clean as well:

$$j_{\mathfrak{m}_N,!} q\mathcal{E} = \mathrm{IC}(\mathfrak{m}_N, q\mathcal{E}) = j_{\mathfrak{m}_N,*} q\mathcal{E}. \tag{2.25}$$

In the diagram (2.22) the map $\mathrm{pr}_{\mathcal{C}_v^M}$ extends naturally to

$$\mathrm{pr}_{\mathfrak{m}_N} : \mathfrak{m}_N \oplus \mathfrak{u} \to \mathfrak{m}_N,$$

and both are trivial vector bundles. Hence (up to degree shifts)

$$\begin{aligned} \mathrm{pr}^*_{\mathfrak{m}_N} j_{\mathfrak{m}_N,!} q\mathcal{E} = \mathrm{pr}^*_{\mathfrak{m}_N} \mathrm{IC}_{P\times\mathbb{C}^\times}(\mathfrak{m}_N, q\mathcal{E}) = \mathrm{pr}^*_{\mathfrak{m}_N} j_{\mathfrak{m}_N,*} q\mathcal{E} = \\ (j_{\mathfrak{m}_N} \times \mathrm{id}_{\mathfrak{u}})_* \mathrm{pr}^*_{\mathcal{C}_v^M} q\mathcal{E} = \mathrm{IC}_{P\times\mathbb{C}^\times}(\mathfrak{m}_N \oplus \mathfrak{u}, \mathrm{pr}^*_{\mathcal{C}_v^M} q\mathcal{E}) = (j_{\mathfrak{m}_N} \times \mathrm{id}_{\mathfrak{u}})_! \mathrm{pr}^*_{\mathcal{C}_v^M} q\mathcal{E}. \end{aligned} \tag{2.26}$$

The vertical maps in (2.22) induce equivalences of categories $\mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times}$, which commute with the relevant functors induced by the horizontal maps in (2.22), so

$$
\begin{aligned}
(\mathrm{id}_G \times j_{\mathfrak{m}_N} \times \mathrm{id}_{\mathfrak{u}})_! \dot{q\mathcal{E}}_N &= (\mathrm{id}_G \times j_{\mathfrak{m}_N} \times \mathrm{id}_{\mathfrak{u}})_! \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{pr}^*_{\mathcal{C}_v^M} q\mathcal{E} \\
&= \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} (j_{\mathfrak{m}_N} \times \mathrm{id}_{\mathfrak{u}})_! \mathrm{pr}^*_{\mathcal{C}_v^M} q\mathcal{E} \\
&= \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{IC}_{P\times\mathbb{C}^\times}(\mathfrak{m}_N \oplus \mathfrak{u}, \mathrm{pr}^*_{\mathcal{C}_v^M} q\mathcal{E}) \\
&= \mathrm{IC}_{G\times\mathbb{C}^\times}\big(G \times^P (\mathfrak{m}_N \oplus \mathfrak{u}), \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{pr}^*_{\mathcal{C}_v^M} q\mathcal{E}\big) \\
&= \mathrm{IC}_{G\times\mathbb{C}^\times}\big(G \times^P (\mathfrak{m}_N \oplus \mathfrak{u}), \dot{q\mathcal{E}}_N\big).
\end{aligned} \tag{2.27}
$$

Since $G \times^P (\mathfrak{m}_N \oplus \mathfrak{u})$ is closed in $G \times^P \mathfrak{g}_N$, the last expression is isomorphic with

$$\mathrm{IC}_{G\times\mathbb{C}^\times}(G \times^P \mathfrak{g}_N, \dot{q\mathcal{E}}_N). \tag{2.28}$$

Via the isomorphism

$$G \times^P \mathfrak{g}_N \cong \mathfrak{g}_N \times G/P \tag{2.29}$$

obtained from (2.15) by restriction, (2.28) becomes $\mathrm{IC}_{G\times\mathbb{C}^\times}(\mathfrak{g}_N \times G/P, \dot{q\mathcal{E}}_N)$. Combine that with (2.24) and (2.27). $\square$

The following method to prove semisimplicity of K_N is based on the decomposition theorem for perverse sheaves of algebraic origin [BBD, Théorème 6.2.5]. The same method can be applied to K, using the first isomorphism in (2.9).

Lemma 2.4. *K_N is a semisimple object of $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)$.*

Proof. By construction every M°-equivariant (cuspidal) local system on a $\mathrm{Ad}(M^\circ)$-orbit in $\mathfrak{m}_N$ is algebraic. The automorphism group $\mathrm{Aut}(M^\circ_{\mathrm{der}})$ of the derived subgroup of M° is algebraic and defined over $\mathbb{Z}$. The action of M on $\mathfrak{m}_N$ factors through $\mathrm{Aut}(M^\circ_{\mathrm{der}})$, and hence the cuspidal local system $q\mathcal{E}$ on $\mathcal{C}_v^M$ is of algebraic origin.

Like for M, the automorphism group of G°_{der} is algebraic and defined over $\mathbb{Z}$, and the adjoints actions of G and P on $\mathfrak{g}$ factor via that group. Therefore not only $\mathrm{pr}^*_{\mathcal{C}_v^M} q\mathcal{E}$ but also

$$\dot{q\mathcal{E}}_N = \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{pr}^*_{\mathcal{C}_v^M} q\mathcal{E} \in \mathcal{D}^b_{G\times\mathbb{C}^\times}(G \times^P \mathfrak{m}_N \oplus \mathfrak{u})$$

is of algebraic origin. As the isomorphism (2.35) only involves G via the adjoint action, it follows that $\mathrm{IC}_{G\times\mathbb{C}^\times}(\mathfrak{g}_N \times G/P, \dot{q\mathcal{E}}_N)$ is of algebraic origin as well. Since

$$\mathrm{pr}_{1,N} : \mathfrak{g}_N \times G/P \to \mathfrak{g}_N$$

is proper, we can apply the decomposition theorem for equivariant perverse sheaves [BeLu, §5.3.1] to Proposition 2.3. This is based on the non-equivariant version from [BBD, §6], and therefore requires objects of algebraic origin. $\square$

For compatibility with other papers we record that, by (2.24), (2.26) and (2.27):

$$
\begin{aligned}
K_N &\cong \mu_{N,!} \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{IC}_{P\times\mathbb{C}^\times}(\mathfrak{m}_N \oplus \mathfrak{u}, \mathrm{pr}^*_{\mathcal{C}_v^M} q\mathcal{E}) \\
&\cong \mu_{N,!} \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{pr}^*_{\mathfrak{m}_N} \mathrm{IC}_{P\times\mathbb{C}^\times}(\mathfrak{m}_N, q\mathcal{E}) \\
&\cong \mu_{N,!} (\mathrm{id}_G \times \mathrm{pr}_{\mathfrak{m}_N})^* \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{IC}_{P\times\mathbb{C}^\times}(\mathfrak{m}_N, q\mathcal{E}).
\end{aligned} \tag{2.30}
$$

Like in [Ach, §8.4], the diagram

$$\mathfrak{m}_N \to G \times^P \mathfrak{m}_N \xleftarrow{\mathrm{id}_G \times \mathrm{pr}_{\mathfrak{m}_N}} G \times^P (\mathfrak{m}_N \oplus \mathfrak{u}) \xrightarrow{\mu_N} \mathfrak{g}_N \tag{2.31}$$

gives rise to a parabolic induction functor

$$\mathcal{I}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} = \mu_{N,!}(\mathrm{id}_G \times \mathrm{pr}_{\mathfrak{m}_N})^* \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} : \mathcal{D}^b_{P\times\mathbb{C}^\times}(\mathfrak{m}_N) \to \mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N). \tag{2.32}$$

Since $U \subset P$ is contractible and acts trivially on $\mathfrak{m}_N$, inflation along the quotient map $P \to M$ induces an equivalence of categories

$$\mathcal{D}^b_{P\times\mathbb{C}^\times}(\mathfrak{m}_N) \cong \mathcal{D}^b_{M\times\mathbb{C}^\times}(\mathfrak{m}_N).$$

With these notions (2.30) says precisely that

$$K_N \cong \mathcal{I}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{IC}_{M\times\mathbb{C}^\times}(\mathfrak{m}_N, q\mathcal{E}). \tag{2.33}$$

For later use we also mention the parabolic restriction functor

$$\mathcal{R}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} = \big(\mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times}\big)^{-1} (\mathrm{id}_G \times \mathrm{pr}_{\mathfrak{m}_N})_* \mu_N^! : \mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N) \to \mathcal{D}^b_{P\times\mathbb{C}^\times}(\mathfrak{m}_N). \tag{2.34}$$

The arguments in Proposition 2.3 and (2.30) admit natural analogues for K. Namely, with the diagram

$$\mathfrak{m}_N \to G \times^P \mathfrak{m}_N \xleftarrow{\mathrm{id}_G \times \mathrm{pr}_{\mathfrak{m}_N}} G \times^P (\mathfrak{m}_N \oplus \mathfrak{t} \oplus \mathfrak{u}) \xrightarrow{\mu} \mathfrak{g}$$

instead of (2.31), we get a functor similar to (2.32). That yields an isomorphism

$$K \cong \mu_!(\mathrm{id}_G \times \mathrm{pr}_{\mathfrak{m}_N})^* \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{IC}_{P\times\mathbb{C}^\times}(\mathfrak{m}_N, q\mathcal{E}). \tag{2.35}$$

This also follows from [Lus6, Proposition 7.12], at least when G is connected.

2.3. **Variations for centralizer subgroups.**

Let $\sigma \in \mathfrak{t}$, so that $M = Z_G(T) \subset Z_G(\sigma)$. We would like to compare Theorem 2.2 with its version for $(Z_G(\sigma), M, q\mathcal{E})$. First we analyse the variety

$$(G/P)^\sigma := \{gP \in G/P : \sigma \in \mathrm{Lie}(gPg^{-1})\}.$$

This is also the fixed point set of $\exp(\mathbb{C}\sigma)$ in G/P. Let $Z_G^\circ(\sigma)$ be the connected component of $Z_G(\sigma)$.

Lemma 2.5. *For any $gP \in (G/P)^\sigma$, the subgroup $gP^\circ g^{-1} \cap Z_G^\circ(\sigma)$ of $Z_G^\circ(\sigma)$ is parabolic.*

Proof. Consider the parabolic subgroup $P' := gP^\circ g^{-1}$ of G°. Its Lie algebra $\mathfrak{p}'$ contains the semisimple element σ, so there exists a maximal torus T' of P' with $\sigma \in \mathfrak{t}'$. Let M' be the unique Levi factor of P' containing T'. The unipotent radical U' of P' and the opposite parabolic $M'\bar{U}'$ give rise to decompositions of $Z(\mathfrak{m}')$-modules

$$\mathfrak{g} = \bar{\mathfrak{u}}' \oplus \mathfrak{p}', \quad \mathfrak{p}' = Z(\mathfrak{m}') \oplus \mathfrak{m}'_{\mathrm{der}} \oplus \mathfrak{u}'.$$

Since $Z(\mathfrak{m}') \subset \mathfrak{t}' \subset Z_{\mathfrak{g}}(\sigma)$, these decompositions are preserved by intersecting with $Z_{\mathfrak{g}}(\sigma)$:

$$Z_{\mathfrak{g}}(\sigma) = Z_{\bar{\mathfrak{u}}'}(\sigma) \oplus Z_{\mathfrak{p}'}(\sigma), \quad Z_{\mathfrak{p}'}(\sigma) = Z(\mathfrak{m}') \oplus Z_{\mathfrak{m}'_{\mathrm{der}}}(\sigma) \oplus Z_{\mathfrak{u}'}(\sigma).$$

This shows that $Z_{\mathfrak{g}}(\sigma) \cap \mathfrak{p}'$ is a parabolic subalgebra of $Z_{\mathfrak{g}}(\sigma)$. Hence $Z_G^\circ(\sigma) \cap P'$ is a parabolic subgroup of $Z_G^\circ(\sigma)$. □

The subgroup $Z_G(\sigma) \subset G$ stabilizes $(G/P)^\sigma$, so the latter is a union of $Z_G(\sigma)$-orbits.

Lemma 2.6. *The connected components of $(G/P)^\sigma$ are precisely its $Z_G^\circ(\sigma)$-orbits.*

Proof. Clearly every $Z_G^\circ(\sigma)$-orbit is connected. From (2.7) we get an isomorphism of varieties

$$G/P = \bigsqcup\nolimits_{\gamma \in N_G(M)/M} \gamma G^\circ P/P \cong \bigsqcup\nolimits_\gamma \gamma G^\circ/P^\circ. \tag{2.36}$$

Here $Z_G^\circ(\sigma)$ acts on $\gamma G^\circ/P^\circ$ by

$$z \cdot \gamma g P^\circ = \gamma(\gamma^{-1} z \gamma) g P^\circ,$$

so via conjugation by γ^{-1} and the natural action of $\gamma^{-1} Z_G^\circ(\sigma)\gamma = Z_G^\circ(\mathrm{Ad}(\gamma^{-1})\sigma)$ on G°/P°. Taking $\exp(\mathbb{C}\sigma)$-fixed points in (2.36) gives

$$\begin{aligned}(G/P)^\sigma &\cong \bigsqcup\nolimits_\gamma (\gamma G^\circ/P^\circ)^\sigma \\ &= \bigsqcup\nolimits_\gamma \{\gamma g P^\circ : g \in G^\circ, \sigma \in \mathrm{Lie}(\gamma g P^\circ g^{-1}\gamma^{-1})\} \\ &= \bigsqcup\nolimits_\gamma \gamma\{g P^\circ : g \in G^\circ, \mathrm{Ad}(\gamma^{-1})\sigma \in \mathrm{Lie}(g P^\circ g^{-1})\} \\ &= \bigsqcup\nolimits_\gamma \gamma (G^\circ/P^\circ)^{\mathrm{Ad}(\gamma^{-1})\sigma}.\end{aligned}$$

This reduces the lemma to the case G°/P°, so to the connected group G°. For that we refer to [ChGi, Proposition 8.8.7.ii]. That reference is written for Borel subgroups, but with Lemma 2.5 the proof also applies to other conjugacy classes of parabolic subgroups. $\square$

It is also shown in [ChGi, Proposition 8.8.7.ii] that every $Z_G^\circ(\sigma)$-orbit in $(G/P)^\sigma$ is a submanifold and an irreducible component.

Lemma 2.7. *There are isomorphisms of $Z_G(\sigma)$-varieties*

$$\bigsqcup_{w \in N_{Z_G(\sigma)}(M)\backslash N_G(M)} Z_G(\sigma)/Z_{wPw^{-1}}(\sigma) \cong \bigsqcup_{w \in N_{Z_G(\sigma)}(M)\backslash N_G(M)} Z_G(\sigma)\cdot wP = (G/P)^\sigma.$$

Proof. By Lemma 2.6 there exist finitely many $\gamma \in G$ such that

$$(G/P)^\sigma = \sqcup_\gamma Z_G^\circ(\sigma)\cdot \gamma P. \tag{2.37}$$

Then the same holds with $Z_G(\sigma)$ instead of $Z_G^\circ(\sigma)$, and fewer γ's. The $Z_G(\sigma)$-stabilizer of γP is

$$\{z \in Z_G(\sigma) : z\gamma P \gamma^{-1} = \gamma P \gamma^{-1}\} = Z_G(\sigma) \cap \gamma P \gamma^{-1} = Z_{\gamma P \gamma^{-1}}(\sigma).$$

That proves the lemma, except for the precise index set.

Fix a maximal torus T' of $Z_G^\circ(\sigma)$ with $T \subset T'$. Every parabolic subgroup of G° or $Z_G^\circ(\sigma)$ is conjugate to one containing T'. The G°-conjugates of P° that contain T' are the $wP^\circ w^{-1}$ with $w \in N_{G^\circ}(T')$, or equivalently with

$$w \in N_{G^\circ}(T')/N_{P^\circ}(T') = N_{G^\circ}(T')/N_{M^\circ}(T') \cong N_{G^\circ}(M^\circ)/M^\circ.$$

For $w, w' \in N_{G^\circ}(M^\circ)$, wP° and $w'P^\circ$ are in the same $Z_G^\circ(\sigma)$-orbit if and only if $w'w^{-1} \in N_{Z_G^\circ(\sigma)}(M^\circ)$. We find that

$$(G^\circ/P^\circ)^\sigma = \bigsqcup\nolimits_{w \in N_{Z_G^\circ(\sigma)}(M^\circ)\backslash N_{G^\circ}(M^\circ)} Z_G^\circ(\sigma)\cdot wP^\circ. \tag{2.38}$$

We note that the group

$$N_{G^\circ}(M^\circ)/M^\circ = N_{G^\circ}(T)/Z_{G^\circ}(T) = N_{G^\circ}(M)/M^\circ \cong N_{G^\circ}(M)M/M$$

normalises P. When we replace G°/P° by G/P in (2.38), the options for w need to be enlarged to $N_G(M)/M$. Next we replace $Z_G^\circ(\sigma)$ by $Z_G(\sigma)$, so that wP and $w'P$ are in the same $Z_G(\sigma)$-orbit if and only if $w'w^{-1} \in N_{Z_G(\sigma)}(M)/M$. Notice that $wP \in (G/P)^\sigma$ because

$$\sigma \in \mathfrak{m} = \mathrm{Lie}(wMw^{-1}) \subset \mathrm{Lie}(wPw^{-1}).$$

We conclude that

$$(G/P)^\sigma = \bigsqcup_{w \in N_{Z_G^\circ(\sigma)}(M) \backslash N_G(M)} Z_G^\circ(\sigma) \cdot wP = \bigsqcup_{w \in N_{Z_G(\sigma)}(M) \backslash N_G(M)} Z_G(\sigma) \cdot wP. \quad \square$$

The fixed point set of $\exp(\mathbb{C}\sigma)$ in $\dot{\mathfrak{g}}$ is

$$\dot{\mathfrak{g}}^\sigma = \dot{\mathfrak{g}} \cap (Z_{\mathfrak{g}}(\sigma) \times (G/P)^\sigma) = \{(X, gP) \in Z_{\mathfrak{g}}(\sigma) \times (G/P)^\sigma : \mathrm{Ad}(g^{-1})X \in \mathcal{C}_v^M + \mathfrak{t} + \mathfrak{u}\}.$$

Clearly $\dot{\mathfrak{g}}^\sigma$ is related to $Z_{\mathfrak{g}}\dot{(}\sigma)$ and to $Z_{\mathfrak{g}}\dot{(}\sigma)^\circ$. With (2.38) and (2.8) we can make that precise:

$$\dot{\mathfrak{g}}^\sigma = \bigsqcup_{w \in N_{Z_G^\circ(\sigma)}(M) \backslash N_G(M)} Z_{\mathfrak{g}}\dot{(}\sigma)^\circ_w = \bigsqcup_{w \in N_{Z_G(\sigma)}(M) \backslash N_G(M)} Z_{\mathfrak{g}}\dot{(}\sigma)_w \tag{2.39}$$

$$\begin{aligned} Z_{\mathfrak{g}}\dot{(}\sigma)_w = \{(X, gZ_{wPw^{-1}}(\sigma)) \in Z_{\mathfrak{g}}(\sigma) \times Z_G(\sigma)/Z_{wPw^{-1}}(\sigma) :& \\ \mathrm{Ad}(g^{-1})X \in \mathrm{Ad}(w)(\mathcal{C}_v^M + \mathfrak{t} + \mathfrak{u})\}.& \end{aligned}$$

Let $j' : \dot{\mathfrak{g}}^\sigma \to \dot{\mathfrak{g}}$ be the inclusion and let pr_1^σ be the restriction of pr_1 to $\dot{\mathfrak{g}}^\sigma$. We define

$$K_\sigma = (\mathrm{pr}_1^\sigma)_! j'^* q\dot{\mathcal{E}} \in \mathcal{D}^b_{Z_G(\sigma) \times \mathbb{C}^\times}(Z_{\mathfrak{g}}(\sigma)).$$

From (2.39) we infer that K_σ is a direct sum of the parts $K_{\sigma,w}$ (resp. $K^\circ_{\sigma,w}$) coming from $Z_{\mathfrak{g}}\dot{(}\sigma)_w$ (resp. from $Z_{\mathfrak{g}}\dot{(}\sigma)^\circ_w$), and each such part is a version of the K for $Z_G(\sigma)$ (resp. for $Z_G^\circ(\sigma)$), twisted by $w \in N_G(M)/M$.

These objects admit versions restricted to subvarieties of nilpotent elements, which we indicate by a subscript N. In particular

$$K_{N,\sigma} = (\mathrm{pr}^\sigma_{1,N})_! j'^*_N q\dot{\mathcal{E}}_N \in \mathcal{D}^b_{Z_G(\sigma) \times \mathbb{C}^\times}(Z_{\mathfrak{g}}(\sigma)_N)$$

can be decomposed as a direct sum of subobjects $K_{N,\sigma,w}$ or $K^\circ_{N,\sigma,w}$.

Lemma 2.8. *The objects $K_\sigma, K_{\sigma,w} \in \mathcal{D}^b_{Z_G(\sigma) \times \mathbb{C}^\times}(Z_{\mathfrak{g}}(\sigma))$ and $K_{N,\sigma}, K_{N,\sigma,w} \in \mathcal{D}^b_{Z_G(\sigma) \times \mathbb{C}^\times}(Z_{\mathfrak{g}}(\sigma)_N)$ are semisimple.*

Proof. We note that, like in (2.15), there is an isomorphism of $Z_G(\sigma) \times \mathbb{C}^\times$-varieties

$$Z_G\dot{(}\sigma)_w \cong Z_G(\sigma) \times^{Z_{wPw^{-1}}(\sigma)} \big(\mathrm{Ad}(w)(\mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}) \cap Z_{\mathfrak{g}}(\sigma)\big).$$

Here $Z_{wPw^{-1}}(\sigma)$ is a quasi-parabolic subgroup of G with quasi-Levi factor M and

$$\mathrm{Ad}(w)(\mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}) \cap Z_{\mathfrak{g}}(\sigma) = \mathcal{C}_v^M \oplus \mathfrak{t} \oplus \big(\mathrm{Ad}(w)\mathfrak{u} \cap Z_{\mathfrak{g}}(\sigma)\big)$$

with $\mathrm{Ad}(w)\mathfrak{u} \cap Z_{\mathfrak{g}}(\sigma)$ the Lie algebra of the unipotent radical of $Z_{wPw^{-1}}(\sigma)$. Comparing that with the construction of K in (2.17)–(2.18), we deduce that $K_{\sigma,w}$ is the K for the group $Z_G(\sigma)$ and the cuspidal local system $\mathrm{Ad}(w)_* q\mathcal{E}$. As K is semisimple, see (2.9), so is the current $K_{\sigma,w}$.

The same reasoning, now using (2.23), shows that $K_{N,\sigma,w}$ is the K_N for $Z_G(\sigma)$ and $\mathrm{Ad}(w)_* q\mathcal{E}$. By Proposition 2.3.b, $K_{N,\sigma,w}$ is semisimple.

The objects K_σ and $K_{N,\sigma}$ are direct sums of objects $K_{\sigma,w}$ and $K_{N,\sigma,w}$, so these are also semisimple. $\square$

The above decompositions of K_σ and $K_{N,\sigma}$ are the key to analogues of parts of Paragraph 2.1 for $Z_G(\sigma)$.

Lemma 2.9. *Let $w, w' \in N_G(M)/M$. The inclusion $Z_\mathfrak{g}(\sigma)_N \to Z_\mathfrak{g}(\sigma)$ induces an isomorphism of graded $H^*_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})$-modules*

$$\mathrm{Hom}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(Z_\mathfrak{g}(\sigma))}(K^\circ_{\sigma,w}, K^\circ_{\sigma,w'}) \longrightarrow \mathrm{Hom}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(Z_\mathfrak{g}(\sigma)_N)}(K^\circ_{N,\sigma,w}, K^\circ_{N,\sigma,w'}).$$

Proof. Decompose $q\dot{\mathcal{E}}|_{Z_\mathfrak{g}\dot{(\sigma)}^\circ_w}$ as direct sum of irreducible $Z_G^\circ(\sigma)\times\mathbb{C}^\times$-equivariant local systems. Each summand is of the form $\mathrm{Ad}(w)_*\dot{\mathcal{E}}$, for an irreducible summand $\mathcal{E}$ of $q\mathcal{E}$ as M°-equivariant local system. Similarly we decompose $q\dot{\mathcal{E}}|_{Z_\mathfrak{g}\dot{(\sigma)}^\circ_{w'}}$ as direct sum of terms $\mathrm{Ad}(w')_*\dot{\mathcal{E}}$. Like in the proof of Lemma 2.8:

$$K^\circ_{\sigma,w} = \bigoplus\nolimits_{\mathcal{E}} (\mathrm{pr}_{1,Z_\mathfrak{g}(\sigma)})_! \mathrm{Ad}(w)_*\dot{\mathcal{E}}, \tag{2.40}$$

and similarly for $K^\circ_{N,\sigma,w}, K^\circ_{\sigma,w'}$ and $K^\circ_{N,\sigma,w'}$. A computation like the start of the proof of [Lus6, Theorem 8.11] (already used in Theorem 2.2.b) shows that

$$\begin{aligned}(2.41)\quad & \mathrm{Hom}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(Z_\mathfrak{g}(\sigma))}\big((\mathrm{pr}_{1,Z_\mathfrak{g}(\sigma)})_!\mathrm{Ad}(w)_*\dot{\mathcal{E}}, (\mathrm{pr}_{1,Z_\mathfrak{g}(\sigma)})_!\mathrm{Ad}(w')_*\dot{\mathcal{E}}'\big) \\ & \qquad \cong H_*^{Z_G^\circ(\sigma)\times\mathbb{C}^\times}\big(Z_\mathfrak{g}\ddot{(\sigma)}^\circ, i_\sigma^*\big(\mathrm{Ad}(w)_*\dot{\mathcal{E}}\boxtimes \mathrm{Ad}(w')_*\dot{\mathcal{E}}'^\vee\big)\big).\end{aligned}$$

Here $Z_\mathfrak{g}\ddot{(\sigma)}^\circ = Z_\mathfrak{g}\dot{(\sigma)}^\circ \times^{Z_\mathfrak{g}(\sigma)} Z_\mathfrak{g}\dot{(\sigma)}^\circ$ and

$$i_\sigma : Z_\mathfrak{g}\ddot{(\sigma)}^\circ \to Z_\mathfrak{g}\dot{(\sigma)}^\circ \times Z_\mathfrak{g}\dot{(\sigma)}^\circ$$

denotes the inclusion. The same applies with subscripts N:

$$\begin{aligned}(2.42)\quad & \mathrm{Hom}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(Z_\mathfrak{g}(\sigma)_N)}\big((\mathrm{pr}_{1,Z_\mathfrak{g}(\sigma)_N})_!\mathrm{Ad}(w)_*\dot{\mathcal{E}}_N, (\mathrm{pr}_{1,Z_\mathfrak{g}(\sigma)_N})_!\mathrm{Ad}(w')_*\dot{\mathcal{E}}'_N\big) \\ & \qquad \cong H_*^{Z_G^\circ(\sigma)\times\mathbb{C}^\times}\big(Z_\mathfrak{g}\ddot{(\sigma)}^\circ_N, i_{N,\sigma}^*\big(\mathrm{Ad}(w)_*\dot{\mathcal{E}}\boxtimes \mathrm{Ad}(w')_*\dot{\mathcal{E}}'^\vee\big)\big).\end{aligned}$$

When $w = w'$ and $\mathcal{E} = \mathcal{E}'$, (2.41) and (2.42) are computed in [Lus3, Proposition 4.7]. In fact [Lus3, Proposition 4.7] also applies in our more general setting, with different $\mathrm{Ad}(w)_*\dot{\mathcal{E}}$ and $\mathrm{Ad}(w')\dot{\mathcal{E}}'$. Namely, to handle those we add the argument from the proof of [AMS2, Proposition 2.6], especially [AMS2, (11)]. That works for both $Z_\mathfrak{g}(\sigma)$ and for $Z_\mathfrak{g}(\sigma)_N$, and entails that there are natural isomorphisms of graded $H^*_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})$-modules

(2.43)

$$\begin{aligned} H^*_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(Z_\mathfrak{g}\dot{(\sigma)}^\circ)\otimes_\mathbb{C} H_0\big(Z_\mathfrak{g}\ddot{(\sigma)}^\circ, i_\sigma^*\big(\mathrm{Ad}(w)_*\dot{\mathcal{E}}\boxtimes\mathrm{Ad}(w')_*\dot{\mathcal{E}}'^\vee\big)\big) \quad &\cong \quad (2.41), \\ H^*_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(Z_\mathfrak{g}\dot{(\sigma)}^\circ)\otimes_\mathbb{C} H_0\big(Z_\mathfrak{g}\ddot{(\sigma)}^\circ_N, i_{N,\sigma}^*\big(\mathrm{Ad}(w)_*\dot{\mathcal{E}}\boxtimes\mathrm{Ad}(w')_*\dot{\mathcal{E}}'^\vee\big)\big) \quad &\cong \quad (2.42).\end{aligned}$$

Moreover, the proof of [Lus3, Proposition 4.7] shows that the two lines of (2.43) are isomorphic via the inclusion $Z_\mathfrak{g}(\sigma)_N \to Z_\mathfrak{g}(\sigma)$. $\square$

Finally, we can generalize the second isomorphism in Theorem 2.2.c.

Proposition 2.10. *The inclusion $Z_{\mathfrak{g}}(\sigma)_N \to Z_{\mathfrak{g}}(\sigma)$ induces a graded algebra isomorphism*

$$\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(Z_{\mathfrak{g}}(\sigma))}(K_\sigma) \longrightarrow \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(Z_{\mathfrak{g}}(\sigma)_N)}(K_{N,\sigma}).$$

Proof. Take the direct sum of the instances of Lemma 2.9, over all $w, w' \in N_{Z_G^\circ(\sigma)}(M)\backslash N_G(M)$. By (2.39), that yields a natural isomorphism

$$\mathrm{End}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(Z_{\mathfrak{g}}(\sigma))}(K_\sigma) \longrightarrow \mathrm{End}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(Z_{\mathfrak{g}}(\sigma)_N)}(K_{N,\sigma}).$$

Now we take $\pi_0(Z_G(\sigma))$-invariants on both sides, that replaces $\mathrm{End}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(?)}$ by $\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(?)}$. □

3. Description of $\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_N)$ with Hecke algebras

We want to make a (right) module category of $\mathbb{H} = \mathbb{H}(G, M, q\mathcal{E})$ equivalent with a category of equivariant constructible sheaves. Since we work with complexes of sheaves, we have to look at differential graded $\mathbb{H}$-modules. Recall that $\mathbb{H}$ has no terms in odd degrees, so that its differential can only be zero. Hence a differential graded $\mathbb{H}$-module M is just a graded $\mathbb{H}$-module $\bigoplus_{n\in\mathbb{Z}} M_n$ with a differential d_M of degree 1.

Further, as our previous results were formulated with (equivariant) derived categories, we also have to involve derived categories of $\mathbb{H}$-modules. Thus we arrive at $\mathcal{D}(\mathbb{H} - \mathrm{Mod}_{dg})$, the derived category of differential graded right $\mathbb{H}$-modules. Its bounded version is $\mathcal{D}^b(\mathbb{H} - \mathrm{Mod}_{\mathrm{fgdg}})$, where the subscript stands for "finitely generated differential graded". We note that $\mathbb{H} - \mathrm{Mod}_{dg}$ is much smaller than $\mathbb{H} - \mathrm{Mod}$, for instance the only irreducible $\mathbb{H}$-modules it contains are those on which $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$ acts via evaluation at $(0, 0)$. In fact the triangulated category $\mathcal{D}^b(\mathbb{H} - \mathrm{Mod}_{\mathrm{fgdg}})$ is already generated by a single object, namely $\mathbb{H}$ [BeLu, Corollary 11.1.5].

The isomorphism $\mathbb{H}(G, M, q\mathcal{E}) \cong \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)}(K_N)$ from Theorem 2.2 gives rise to an additive functor

$$\begin{array}{ccc} \mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N) & \to & \mathcal{D}^b(\mathbb{H} - \mathrm{Mod}_{\mathrm{fgdg}}) \\ S & \mapsto & \mathrm{Hom}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)}(K_N, S) \end{array} . \tag{3.1}$$

However, it is not clear whether this functor is triangulated or fully faithful (on an appropriate subcategory). One problem is that $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)$ is not exactly a (bounded) derived category, another that $\mathrm{Hom}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)}$ is defined rather indirectly.

3.1. **Equivalence of triangulated categories.**

We will overcome the above problems by constructing a more subtle functor instead of (3.1), which will lead to an equivalence of categories. Let $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N, K_N)$ be the triangulated subcategory of $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)$ generated by the simple summands of the semisimple object K_N. We aim to show that it is equivalent with $\mathcal{D}^b(\mathbb{H} - \mathrm{Mod}_{\mathrm{fgdg}})$. We follow the strategy outlined in [RiRu2, §4], based on [Rid], but with $G \times GL_1$ instead of G. We need the following objects as substitutes for objects appearing in the derived generalized Springer correspondence from [Rid, RiRu2]:

our setting	setting from [RiRu2]	setting from [Rid]
$\mathfrak{g}_N$	$\mathcal{N}$	$\mathcal{N}$
$\mathrm{IC}(\mathfrak{m}_N, q\mathcal{E})$	$\mathrm{IC}_{\mathbf{c}}$	$\mathrm{IC}(\{0\}, \overline{\mathbb{Q}_\ell})$
K_N	$\mathbb{A}_{\mathbf{c}}$	$\mathbf{A}$
$\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$	$\overline{\mathbb{Q}_\ell}[W(L)]$	$\overline{\mathbb{Q}_\ell}[W]$
$H^*_{G\times\mathbb{C}^\times}(\dot{\mathfrak{g}_N}) \cong \mathcal{O}(\mathfrak{t}\oplus\mathbb{C})$	$H_L(\mathcal{O}_L) \cong S\mathfrak{z}^*$	$H^*_G(G/B) \cong S\mathfrak{h}^*$
$\mathbb{H}(G, M, q\mathcal{E})$	$\overline{\mathbb{Q}_\ell}[W(L)] \ltimes S\mathfrak{z}^*$	$\mathcal{A}_G = \overline{\mathbb{Q}_\ell}[W(L)] \ltimes S\mathfrak{h}^*$
$\mathcal{I}^{G\times\mathbb{C}^\times}_{P\times\mathbb{C}^\times}$	$\mathcal{I}^G_P$	Ψ
$\mathcal{R}^{G\times\mathbb{C}^\times}_{P\times\mathbb{C}^\times}$	$\mathcal{R}^G_P$	Φ
$\mathcal{D}^b_{M\times\mathbb{C}^\times}(\mathfrak{m}_N, \mathrm{IC}(\mathfrak{m}_N, q\mathcal{E}))$	$\mathcal{D}^b_L(\mathcal{N}_L, \mathrm{IC}_{\mathbf{c}}) \cong \mathcal{D}^b_Z(\mathrm{pt})$	$\mathcal{D}^b_G(G/B) \cong \mathcal{D}^b_T(\mathrm{pt})$

For the third till sixth lines of the table we refer to, respectively, (2.33), (2.10), (2.5) and Theorem 2.2.c. To justify the last line of the table, we note that the proof of [RiRu2, Lemma 2.3 and Proposition 2.4] shows that

$$\mathcal{D}^b_{M\times\mathbb{C}^\times}(\mathfrak{m}_N, \mathrm{IC}(\mathfrak{m}_N, q\mathcal{E})) \cong \mathcal{D}^b_{Z(M)^\circ\times\mathbb{C}^\times}(\mathrm{pt}) = \mathcal{D}^b_{T\times\mathbb{C}^\times}(\mathrm{pt}). \tag{3.2}$$

In our setting, the topology of the coefficient field $\mathbb{C}$ of our sheaves does not play a role. Since $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$ as fields, we may just as well look at sheaves of $\overline{\mathbb{Q}_\ell}$-vector spaces everywhere.

Our varieties, algebraic groups and (complexes of) sheaves may also be considered over any algebraically closed ground field instead of $\mathbb{C}$, see [BeLu, §4.3]. In particular we can take an algebraically closed field k_s whose characteristic p is good for G, like in [Rid, RiRu2]. As we do not require that G is connected, we decree that "good" also means that p does not divide the order of $\pi_0(G)$. For consistency, we replace the variety $\mathbb{C}$ (on which $\mathbf{r}$ is the standard coordinate) by the affine space $\mathbb{A}^1$.

This setup has the advantage that one can pass to varieties over finite fields, and to mixed (equivariant) sheaves. To emphasize that we consider an object with ground field k_s we will sometimes add a subscript s (which comes from [BBD, §6], where k_s arises as the residue field for some discrete valuation ring, which relates k_s to special fibres). In the remainder of this section we will regard G as an algebraic group, and for an action of G or $G \times GL_1$ we tacitly assume that these groups are considered over the same field as the varieties on which they act. To that end, and to get semisimplicty of K_N from Lemma 2.4, we assume that G can be defined over a finite extension of $\mathbb{Z}$. That is hardly a restriction, since by Chevalley's construction it holds for any connected complex reductive group.

As explained in [RiRu2, §3.2], the cuspidal local system $q\mathcal{E}$ on $\mathcal{C}_v^M \subset \mathfrak{m}_N$ admits a version over a finite field $\mathbb{F}_q$, such that a Frobenius element of $\mathrm{Gal}(\mathbb{F}_q)$ acts trivially (after extension of scalars to $\overline{\mathbb{F}_q}$). Then everything can be set up over $\mathbb{F}_q$ with mixed sheaves, as in [Rid, §4–5]. Like in [Rid, RiRu2], we indicate the analogous objects over $\mathbb{F}_q$ with a subscript $\circ$. Now the computation of $\mathrm{End}^*_{\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,\circ})}(K_{N,\circ})$ in [Rid, §3.3], including the action of Frobenius, can be carried out in the same way. Here we use [AMS2] to generalize the relevant parts of [Lus3] to disconnected G. Like in Theorem 2.2, we obtain

$$\mathrm{End}^*_{\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s})}(K_{N,s}) = \mathbb{H}_{\overline{\mathbb{Q}_\ell}} = \mathbb{H}_{\overline{\mathbb{Q}_\ell}}(G, M, q\mathcal{E}),$$

the version of $\mathbb{H}$ with scalars $\overline{\mathbb{Q}_\ell}$ instead of $\mathbb{C}$. With that settled, the proof of [RiRu2, Theorem 4.1] applies to $(\mathfrak{g}_{N,\circ}, K_{N,\circ})$. It provides a triangulated category

$$K^b \mathrm{Pure}_{G\times GL_1}(\mathfrak{g}_{N,\circ}, K_{N,\circ}),$$

which is a mixed version of $\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s}, K_{N,s})$ in the sense of [Rid, Definition 4.2]. Next [RiRu2, Theorem 4.2] and [Rid, §6] generalize readily to our setting (but with objects over the ground field k). In particular these entail an equivalence of triangulated categories

$$K^b \mathrm{Pure}_{G\times GL_1}(\mathfrak{g}_{N,\circ}, K_{N,\circ}) \cong \mathcal{D}^b(\mathbb{H}_{\overline{\mathbb{Q}_\ell}} - \mathrm{Mod}_{\mathrm{fgdg}}). \tag{3.3}$$

Recall the notion of Koszulity for differential graded algebras from [BGS].

Lemma 3.1. *(a) The algebra $\mathbb{H}_{\overline{\mathbb{Q}_\ell}}$ is Koszul.*
(b) The Koszul dual $E(\mathbb{H}_{\overline{\mathbb{Q}_\ell}})$ of $\mathbb{H}_{\overline{\mathbb{Q}_\ell}}$ is a finite dimensional graded algebra.

Proof. (a) Consider the degree zero part $\mathbb{H}_{\overline{\mathbb{Q}_\ell},0} = \overline{\mathbb{Q}_\ell}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$ as $\mathbb{H}_{\overline{\mathbb{Q}_\ell}}$-module, annihilated by all terms of positive degree. We have to find a resolution of $\mathbb{H}_{\overline{\mathbb{Q}_\ell},0}$ by projective graded modules P^n, such that each P^n is generated by its part in degree n. We will use that the multiplication map

$$\mathbb{H}_{\overline{\mathbb{Q}_\ell},0} \otimes_{\overline{\mathbb{Q}_\ell}} \mathcal{O}(\mathfrak{t} \oplus \mathbb{A}^1) \to \mathbb{H}_{\overline{\mathbb{Q}_\ell}}$$

is an isomorphism of graded vector spaces. Start with the standard Koszul resolution for $\mathcal{O}(\mathfrak{t} \oplus \mathbb{A}^1)$:

$$\overline{\mathbb{Q}_\ell} \leftarrow \mathcal{O}(\mathfrak{t} \oplus \mathbb{A}^1) \leftarrow \mathcal{O}(\mathfrak{t} \oplus \mathbb{A}^1) \otimes_{\overline{\mathbb{Q}_\ell}} \bigwedge^1 (\mathfrak{t} \oplus \mathbb{A}^1) \leftarrow \mathcal{O}(\mathfrak{t} \oplus \mathbb{A}^1) \otimes_{\overline{\mathbb{Q}_\ell}} \bigwedge^2 (\mathfrak{t} \oplus \mathbb{A}^1) \leftarrow \cdots$$

It is graded so that $\mathcal{O}(\mathfrak{t} \oplus \mathbb{A}^1)_d \otimes_{\overline{\mathbb{Q}_\ell}} \bigwedge^n (\mathfrak{t} \oplus \mathbb{A}^1)$ sits in degree $d+n$. Define

$$P^n = \mathrm{ind}_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{A}^1)}^{\mathbb{H}_{\overline{\mathbb{Q}_\ell}}} \big(\mathcal{O}(\mathfrak{t} \oplus \mathbb{A}^1) \otimes_{\overline{\mathbb{Q}_\ell}} \bigwedge^n (\mathfrak{t} \oplus \mathbb{A}^1) \big) = \mathbb{H}_{\overline{\mathbb{Q}_\ell}} \otimes_{\overline{\mathbb{Q}_\ell}} \bigwedge^n (\mathfrak{t} \oplus \mathbb{A}^1).$$

Then $P^n = \mathbb{H}_{\overline{\mathbb{Q}_\ell}} P^n_n$ and we have a graded projective resolution

$$P^* \to \mathrm{ind}_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{A}^1)}^{\mathbb{H}_{\overline{\mathbb{Q}_\ell}}} (\overline{\mathbb{Q}_\ell}) = \mathbb{H}_{\overline{\mathbb{Q}_\ell},0}.$$

Thus $\mathbb{H}_{\overline{\mathbb{Q}_\ell}}$ fulfills [BGS, Definition 1.1.2] and is Koszul.
(b) In [BGS, §1.2], the Koszul dual $E(\mathbb{H}_{\overline{\mathbb{Q}_\ell}})$ is defined as $\mathrm{Ext}^*_{\mathbb{H}_{\overline{\mathbb{Q}_\ell}}}(\mathbb{H}_{\overline{\mathbb{Q}_\ell},0}, \mathbb{H}_{\overline{\mathbb{Q}_\ell},0})$. This is easily computed as graded vector space:

$$\begin{aligned} E(\mathbb{H}_{\overline{\mathbb{Q}_\ell}}) &= \mathrm{Ext}^*_{\mathbb{H}_{\overline{\mathbb{Q}_\ell}}} \big(\mathrm{ind}_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{A}^1)}^{\mathbb{H}_{\overline{\mathbb{Q}_\ell}}} \overline{\mathbb{Q}_\ell}, \mathbb{H}_{\overline{\mathbb{Q}_\ell},0} \big) \\ &= \mathrm{Ext}^*_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{A}^1)} \big(\overline{\mathbb{Q}_\ell}, \mathbb{H}_{\overline{\mathbb{Q}_\ell},0} \big) \\ &= \mathrm{Ext}^*_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{A}^1)} \big(\overline{\mathbb{Q}_\ell}, \overline{\mathbb{Q}_\ell} \big) \otimes_{\overline{\mathbb{Q}_\ell}} \mathbb{H}_{\overline{\mathbb{Q}_\ell},0} \\ &= \bigwedge^* (\mathfrak{t} \oplus \mathbb{A}^1) \otimes_{\overline{\mathbb{Q}_\ell}} \mathbb{H}_{\overline{\mathbb{Q}_\ell},0}. \end{aligned}$$

Note that both $\bigwedge^* (\mathfrak{t} \oplus \mathbb{A}^1)$ and $\mathbb{H}_{\overline{\mathbb{Q}_\ell},0}$ have finite dimension. $\square$

The opposite algebra of $\mathbb{H}_{\overline{\mathbb{Q}_\ell}}$ is of the same kind, namely $\mathbb{H}_{\overline{\mathbb{Q}_\ell}}(G, M, q\mathcal{E}^\vee)$ for the dual local system $q\mathcal{E}^\vee$. Hence Lemma 3.1 also holds for $\mathbb{H}^{op}_{\overline{\mathbb{Q}_\ell}}$, which means that we may use the results of [BGS] with right modules instead of left modules.

Lemma 3.1 entails that $\mathcal{D}^b(\mathbb{H}_{\overline{\mathbb{Q}_\ell}} - \mathrm{Mod}_{\mathrm{fgdg}})$ admits the "geometric t-structure" from [BGS, §2.13]. Its heart is equivalent with $E(\mathbb{H}_{\overline{\mathbb{Q}_\ell}}) - \mathrm{Mod}_g$, the abelian category of graded right $E(\mathbb{H}_{\overline{\mathbb{Q}_\ell}})$-modules. Next [Rid, Theorem 7.1] shows that (3.3) sends this t-structure to the "second t-structure" on $K^b\mathrm{Pure}_{G\times GL_1}(\mathfrak{g}_{N,\circ}, K_{N,\circ})$ from [Rid, §4.2]. In particular the heart of the second t-structure is equivalent with the heart of the geometric t-structure:

$$\mathrm{Perv}_{KD}(\mathfrak{g}_{N,\circ}, K_{N,\circ}) \cong E(\mathbb{H}_{\overline{\mathbb{Q}_\ell}}) - \mathrm{Mod}_g. \tag{3.4}$$

Choose a resolution of $E(\mathbb{H}_{\overline{\mathbb{Q}_\ell}})$ by free (graded right) modules of finite rank, that is possible by Lemma 3.1.b. Via (3.4), that yields a projective resolution

$$\cdots \to P_\circ^{-2} \to P_\circ^{-1} \to P_\circ^0 \to K_{N,\circ}$$

in $\mathrm{Perv}_{KD}(\mathfrak{g}_{N,\circ}, K_{N,\circ})$. Let $F_\circ \in \mathrm{Perv}_{KD}(\mathfrak{g}_{N,\circ}, K_{N,\circ})$ be the image of $E(\mathbb{H}_{\overline{\mathbb{Q}_\ell}})$ and let F_s be the image of $F_\circ$ in $\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s}, K_{N,s})$ via [RiRu2, Theorem 4.1]. Then each $P_\circ^n$ is a direct sum of finitely many copies of $F_\circ$. Let P_s^n be the image of $P_\circ^n$ in $\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s}, K_{N,s})$. If $I \subset \mathbb{Z}$ is a segment such that F_s lies in $\mathcal{D}^I_{G\times GL_1}(\mathfrak{g}_{N,s}, K_{N,s})$, then all P_s^n belong to $\mathcal{D}^I_{G\times GL_1}(\mathfrak{g}_{N,s}, K_{N,s})$. This yields a chain complex

$$\cdots \to P_s^{-2} \to P_s^{-1} \to P_s^0 \to K_{N,s}, \tag{3.5}$$

where all objects and all morphisms come from $\mathcal{D}^I_{G\times GL_1}(\mathfrak{g}_{N,s}, K_{N,s})$. However, the entire complex is usually unbounded, because it is likely that $P_\circ^n$ and P_s^n are nonzero for all $n \in \mathbb{Z}_{\leq 0}$. We define a graded algebra $\mathcal{R} = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{R}^n$ with

$$\mathcal{R}^n = \bigoplus\nolimits_{k,j\in\mathbb{Z}_{\leq 0}} \mathrm{Hom}_{\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s},K_{N,s})}(P_s^k, P_s^j[n+k-j]).$$

The product in $\mathcal{R}$ comes from composition in $\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s}, K_{N,s})$. In [Rid] a similar $\mathcal{R}$ appears with a direct product over k, j, but the arguments (especially the proof of [Rid, Theorem 7.4]) work better with our direct sum. For $M \in \mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s}, K_{N,s})$ and $n \in \mathbb{Z}_{\geq 0}$ we put

$$\mathcal{H}om^n(P_s^*, M) = \bigoplus\nolimits_{j\in\mathbb{Z}_{\leq 0}} \mathrm{Hom}_{\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s})}(P_s^j, M[j+n]),$$

so that we obtain a functor

$$\mathcal{H}om^*(P_s^*, ?) = \bigoplus\nolimits_{n\geq 0} \mathcal{H}om^n(P_s^*, ?) : \mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s}, K_{N,s}) \to \mathcal{D}^b(\mathcal{R} - \mathrm{Mod}_{\mathrm{fgdg}}).$$

By [Rid, Theorem 7.4] and [Sch, Proposition 4], $\mathcal{R}$ is quasi-isomorphic to its own cohomology ring and

$$H^*(\mathcal{R}) \cong \mathrm{End}^*_{\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s})}(K_{N,s}) \cong \mathbb{H}_{\overline{\mathbb{Q}_\ell}}.$$

Moreover, by [Rid, Remark 7.5] there exists a quasi-isomorphism $\mathcal{R} \to \mathbb{H}_{\overline{\mathbb{Q}_\ell}}$. According to [BeLu, Theorem 10.12.5.1 and §11.1], that induces an equivalence of categories

$$\otimes^L_{\mathcal{R}} \mathbb{H}_{\overline{\mathbb{Q}_\ell}} : \mathcal{D}^b(\mathcal{R} - \mathrm{Mod}_{\mathrm{fgdg}}) \to \mathcal{D}^b(\mathbb{H}_{\overline{\mathbb{Q}_\ell}} - \mathrm{Mod}_{\mathrm{fgdg}}).$$

Combining all the above, we get an additive functor

$$\otimes^L_{\mathcal{R}} \mathbb{H}_{\overline{\mathbb{Q}_\ell}} \circ \mathcal{H}om^*(P_s^*, ?) : \mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s}, K_{N,s}) \to \mathcal{D}^b(\mathbb{H}_{\overline{\mathbb{Q}_\ell}} - \mathrm{Mod}_{\mathrm{fgdg}}). \tag{3.6}$$

The proof of [RiRu2, Theorem 4.3] explains why the arguments from [Rid, §7 and Appendix] generalize to our setting. These results show that (3.6) is triangulated, commutes with the shift operator and sends $K_{N,s}$ to $\mathbb{H}_{\overline{\mathbb{Q}_\ell}}$. Finally, an application of Beilinson's lemma (in the version from [Sch, Lemma 6]) proves:

Theorem 3.2. *Transfer the setup of Paragraph 2.1 to groups and varieties over an algebraically closed field of good characteristic for G, and use $\overline{\mathbb{Q}_\ell}$ as coefficient field for all sheaves and representations. Then the functor* (3.6) *is an equivalence of triangulated categories.*

Next we want to transfer Theorem 3.2 back to our original setting with algebraic varieties over $\mathbb{C}$. We follow the strategy that was used to derive the decomposition theorem for equivariant perverse sheaves [BeLu, §5.3] from its non-equivariant version [BBD, Théorème 6.2.5], which in turn relied on an analogue for varieties over finite fields. To apply the techniques from [BBD, §6.1], it seems necessary that G can be defined over a finite extension of $\mathbb{Z}$.

Fix a segment $I \subset \mathbb{Z}$ and assume that $G \times GL_1$ is embedded in GL_r. It was noted in [BeLu, §3.1] that the variety $M_{|I|}$ of k-frames in the affine space $\mathbb{A}^{|I|+k}$ is an acyclic $G \times GL_1$-space. Then $G \times GL_1$ acts freely on $Q := M_{|I|} \times \mathfrak{g}_N$ and the projection $p : Q \to \mathfrak{g}_N$ is an $|I|$-acyclic resolution of $G \times GL_1$-varieties. Let $\overline{Q} = Q/(G \times GL_1)$ be the quotient variety. By [BeLu, §2.3.2], $\mathcal{D}^I_{G\times GL_1}(\mathfrak{g}_N)$ is naturally equivalent with $\mathcal{D}^I(\overline{Q}|p)$, the full subcategory of $\mathcal{D}^I(\overline{Q})$ made from all the objects that come from $\mathfrak{g}_N$ via p.

For a variety X defined over some finite extension of $\mathbb{Z}$, we denote by X_s the base change to a suitable algebraically closed field of positive characteristic. According to [BBD, §6.1.10] there is an equivalence of categories

$$\mathcal{D}^b_{\mathcal{T},L}(\overline{Q}, \mathbb{Z}_\ell) \longleftrightarrow \mathcal{D}^b_{\mathcal{T},L}(\overline{Q}_s, \mathbb{Z}_\ell). \tag{3.7}$$

Here $(\mathcal{T}, L)$ means essentially that only finitely many irreducible objects are used. Moreover, unlike the rest of the paper, in (3.7) we must allow sheaves whose stalks have infinite rank as $\mathbb{Z}_\ell$-modules, for otherwise we could never get sheaves of $\overline{\mathbb{Q}_\ell}$-modules. We can restrict (3.7) to an equivalence of categories

$$\mathcal{D}^b_{\mathcal{T},L}(\overline{Q}|p, \mathbb{Z}_\ell) \longleftrightarrow \mathcal{D}^b_{\mathcal{T},L}(\overline{Q}_s|p_s, \mathbb{Z}_\ell). \tag{3.8}$$

We note that, in contrast with [BBD, §6.1.10], constructibility is not an issue here, because we need only one stratification of $\overline{Q}$ and $\overline{Q}_s$, namely that coming from the $G \times GL_1$-orbits on $\mathfrak{g}_N$. Since there are only finitely many such orbits, for sufficiently large $(\mathcal{T}, L)$ (3.8) becomes an equivalence

$$\mathcal{D}^I(\overline{Q}|p, \mathbb{Z}_\ell) \longleftrightarrow \mathcal{D}^I(\overline{Q}_s|p_s, \mathbb{Z}_\ell).$$

With another application of [BeLu, §2.3.2], we find equivalences of triangulated categories

$$\mathcal{D}^I_{G\times GL_1}(\mathfrak{g}_N, \mathbb{Z}_\ell) \leftrightarrow \mathcal{D}^I(\overline{Q}|p, \mathbb{Z}_\ell) \leftrightarrow \mathcal{D}^I(\overline{Q}_s|p_s, \mathbb{Z}_\ell) \leftrightarrow \mathcal{D}^I_{G\times GL_1}(\mathfrak{g}_{N,s}, \mathbb{Z}_\ell). \tag{3.9}$$

This works for any segment $I \subset \mathbb{Z}$, so also with $\mathcal{D}^b$ instead of $\mathcal{D}^I$. The composition of the maps from left to right in (3.9) sends K_N to $K_{N,s}$, so it restricts to an equivalence of triangulated categories

$$\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_N, K_N) \longleftrightarrow \mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_{N,s}, K_{N,s}).$$

Combining that with Theorem 3.2, we have proven:

Theorem 3.3. *Assume that G can be defined over a finite extension of $\mathbb{Z}$. There exists an equivalence of triangulated categories*

$$\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_N, K_N) \longrightarrow \mathcal{D}^b\big(\mathbb{H}_{\overline{\mathbb{Q}_\ell}}(G, M, q\mathcal{E}) - \mathrm{Mod}_{\mathrm{fgdg}}\big),$$

which sends K_N *to* $\mathbb{H}_{\overline{\mathbb{Q}_\ell}}(G, M, q\mathcal{E})$.

The same holds with the coefficient field $\mathbb{C}$ *instead of* $\overline{\mathbb{Q}_\ell}$.

We note that replacing $\overline{\mathbb{Q}_\ell}$ by the isomorphic field $\mathbb{C}$ is allowed because the topology of $\overline{\mathbb{Q}_\ell}$ does not play a role any more (it did when we looked at sheaves of $\mathbb{Z}_\ell$-modules). This categorifies $\mathbb{H}(G, M, q\mathcal{E})$ as differential graded algebra.

3.2. **Orthogonal decomposition.**

The goal of this paragraph is a description of the entire category $\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_N)$ in terms like Theorem 3.3. From [RiRu1] it can be expected that it decomposes as an orthogonal direct sum of full subcategories of the form $\mathcal{D}^b_{G\times GL_1}(\mathfrak{g}_N, K_N)$. Here orthogonality means that there are no nonzero morphisms between objects from different summands.

We start with an orthogonality statement on the cuspidal level. Let $\mathcal{C}^M_v, \mathcal{C}^M_{v'}$ be nilpotent $\mathrm{Ad}(M)$-orbits in $\mathfrak{m}$ and let $q\mathcal{E}, q\mathcal{E}$ be M-equivariant irreducible cuspidal local systems on respectively $\mathcal{C}^M_v$ and $\mathcal{C}^M_{v'}$. As noted in [Lus3, §2.1.f], $q\mathcal{E}$ and $q\mathcal{E}'$ are automatically $M \times GL_1$-equivariant. Let $\mathrm{IC}(\mathfrak{m}_N, q\mathcal{E}), \mathrm{IC}(\mathfrak{m}_N, q\mathcal{E}')$ be the associated ($M \times GL_1$-equivariant) intersection cohomology complexes. Notice that $\mathrm{IC}_{M\times GL_1}(\mathfrak{m}_N, q\mathcal{E})$ is the version of K_N for M.

Lemma 3.4. *Suppose that* $\mathcal{C}^M_v \neq \mathcal{C}^M_{v'}$ *or that* $\mathcal{C}^M_v = \mathcal{C}^M_{v'}$ *and* $q\mathcal{E}, q\mathcal{E}'$ *are not isomorphic in* $\mathcal{D}^b_{M\times GL_1}(\mathcal{C}^M_v)$. *Then*

$$\mathrm{Hom}^*_{\mathcal{D}^b_{M\times GL_1}(\mathfrak{m}_N)}\big(\mathrm{IC}(\mathfrak{m}_N, q\mathcal{E}), \mathrm{IC}(\mathfrak{m}_N, q\mathcal{E}')\big) = 0.$$

Proof. Suppose that the given Hom-space is nonzero. As M°-equivariant local systems, we can decompose $q\mathcal{E} = \bigoplus_i \mathcal{E}_i$ and $q\mathcal{E}' = \bigoplus_j \mathcal{E}'_j$, where the $\mathcal{E}_i$ and the $\mathcal{E}'_j$ are irreducible and cuspidal. Then

$$\bigoplus_{i,j} \mathrm{Hom}^*_{\mathcal{D}^b_{M\times GL_1}(\mathfrak{m}_N)}\big(\mathrm{IC}(\mathfrak{m}_N, \mathcal{E}_i), \mathrm{IC}(\mathfrak{m}_N, \mathcal{E}'_j)\big) \neq 0.$$

By [RiRu1, Theorem 3.5 and Proposition A.8], $\mathcal{E}_i$ is isomorphic to $\mathcal{E}'_j$ for some i, j. Hence $\mathcal{C}^{M^\circ}_v = \mathcal{C}^{M^\circ}_{v'}$, so we may assume that $v = v'$. Recall from (2.25) that $q\mathcal{E}$ and $q\mathcal{E}'$ are clean (on $\mathfrak{m}_N$). With adjunction we compute

$$\begin{aligned}
&\mathrm{Hom}^*_{\mathcal{D}^b_{M\times GL_1}(\mathfrak{m}_N)}\big(\mathrm{IC}(\mathfrak{m}_N, q\mathcal{E}), \mathrm{IC}(\mathfrak{m}_N, q\mathcal{E}')\big) = \\
&\mathrm{Hom}^*_{\mathcal{D}^b_{M\times GL_1}(\mathfrak{m}_N)}\big(\mathrm{IC}(\mathfrak{m}_N, q\mathcal{E}), j_{\mathfrak{m}_N,*} q\mathcal{E}'\big) = \\
&\mathrm{Hom}^*_{\mathcal{D}^b_{M\times GL_1}(\mathcal{C}^M_v)}\big(j^*_{\mathfrak{m}_N}\mathrm{IC}(\mathfrak{m}_N, q\mathcal{E}), q\mathcal{E}'\big) = \mathrm{Hom}^*_{\mathcal{D}^b_{M\times GL_1}(\mathcal{C}^M_v)}\big(q\mathcal{E}, q\mathcal{E}'\big).
\end{aligned} \tag{3.10}$$

Let $\rho, \rho' \in \mathrm{Irr}\big(\pi_0(Z_{M\times GL_1}(v))\big)$ be the images of $q\mathcal{E}$ and $q\mathcal{E}'$ under the equivalence of categories

$$\mathcal{D}^b_{M\times GL_1}(\mathcal{C}^M_v) \cong \mathcal{D}^b_{Z_{M\times GL_1}(v)}(\{v\}).$$

Then (3.10) reduces to

$$\mathrm{Hom}^*_{\mathcal{D}^b_{Z_{M\times GL_1}(v)}(\{v\})}\big(\rho, \rho'\big) = \mathrm{Hom}_{\pi_0(Z_{M\times GL_1}(v))}(\rho, \rho').$$

Since $q\mathcal{E}$ and $q\mathcal{E}'$ are not isomorphic, ρ and ρ' are not isomorphic, and this expression vanishes. That contradicts the assumption at the start of the proof. $\square$

Consider the collection of all cuspidal quasi-supports $(M, \mathcal{C}_v^M, q\mathcal{E})$ for G. Since each $\mathfrak{m}_N$ admits only very few irreducible M-equivariant cuspidal local systems [Lus1, Introduction], there are only finitely many G-conjugacy classes of cuspidal quasi-supports for G. Each such conjugacy class $[M, \mathcal{C}_v^M, q\mathcal{E}]_G$ gives rise to a full triangulated subcategory

$$\mathcal{D}^b_{G \times GL_1}(\mathfrak{g}_N, K_N) = \mathcal{D}^b_{G \times GL_1}\big(\mathfrak{g}_N, \mathcal{I}_P^G \, \mathrm{IC}_{M \times GL_1}(\mathfrak{m}_N, q\mathcal{E})\big),$$

see (2.33) for the equality.

Theorem 3.5. *There is an orthogonal decomposition*

$$\mathcal{D}^b_{G \times GL_1}(\mathfrak{g}_N) = \bigoplus\nolimits_{[M, \mathcal{C}_v^M, q\mathcal{E}]_G} \mathcal{D}^b_{G \times GL_1}\big(\mathfrak{g}_N, \mathcal{I}_P^G \, \mathrm{IC}_{M \times GL_1}(\mathfrak{m}_N, q\mathcal{E})\big).$$

Proof. This is the translation of [RiRu1, Theorem 3.5] to our setting. Almost the entire proof in [RiRu1, §2–3] is valid in our generality, only the argument with central characters (near the end of the proof of [RiRu1, Theorem 3.5]) does not work any more. We extend that to our setting with Lemma 3.4. $\square$

Let us formulate the combination of Theorems 3.3 and 3.5 explicitly.

Corollary 3.6. *Assume that G can be defined over a finite extension of $\mathbb{Z}$. There exists an equivalence of triangulated categories*

$$\begin{aligned} \mathcal{D}^b_{G \times GL_1}(\mathfrak{g}_N) \longrightarrow & \bigoplus\nolimits_{[M, \mathcal{C}_v^M, q\mathcal{E}]_G} \mathcal{D}^b\big(\mathbb{H}(G, M, q\mathcal{E}) - \mathrm{Mod}_{\mathrm{fgdg}}\big) \\ & = \mathcal{D}^b\big(\bigoplus\nolimits_{[M, \mathcal{C}_v^M, q\mathcal{E}]_G} \mathbb{H}(G, M, q\mathcal{E}) - \mathrm{Mod}_{\mathrm{fgdg}}\big). \end{aligned}$$

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