HOCHSCHILD HOMOLOGY OF TWISTED CROSSED PRODUCTS AND TWISTED GRADED HECKE ALGEBRAS

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Abstract. Let A be a $\mathbb{C}$-algebra with an action of a finite group G, and consider a twisted crossed product $A \rtimes \mathbb{C}[G, \natural]$. We determine the Hochschild homology of $A \rtimes \mathbb{C}[G, \natural]$ for two classes of algebras A:

- rings of regular functions on nonsingular affine varieties,
- graded Hecke algebras.

The results are achieved via algebraic families of (virtual) representations and include a description of the Hochschild homology as a module over the centre of $A \rtimes \mathbb{C}[G, \natural]$. In noncommutative geometric terms, our results describe the differential forms on the space of irreducible representations of $A \rtimes \mathbb{C}[G, \natural]$.

This paper prepares for a computation of the Hochschild homology of the Hecke algebra of a reductive p-adic group.

Contents

Introduction 2
1. Twisted crossed product algebras 6
1.1. Hochschild homology via differential forms 6
1.2. Algebraic families of representations 12
1.3. Hochschild homology via families of representations 17
2. Twisted graded Hecke algebras 22
2.1. Representation theory 24
2.2. Hochschild homology 28
2.3. A Morita equivalent algebra 34
References 36

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Introduction

Consider a finite group G and a 2-cocycle $\natural : G \times G \to \mathbb{C}^\times$. The twisted group algebra $\mathbb{C}[G,\natural]$ is the vector space with basis $\{T_g : g \in G\}$ and multiplication

$$T_g \cdot T_{g'} = \natural(g,g') T_{gg'} \qquad g, g' \in G.$$

Such algebras arise for instance from a projective representation $\pi : G \to PGL_n(\mathbb{C})$. Even if π cannot be linearized, one can always regard π as a representation of a suitable twisted group algebra of G. The general aim of this paper is to make certain results for algebras involving $\mathbb{C}[G]$ available for similar algebras that involve $\mathbb{C}[G,\natural]$. In other words, we want to treat $\mathbb{C}[G,\natural]$ on the same footing as the group algebra $\mathbb{C}[G]$. Although $\mathbb{C}[G,\natural]$ is always semisimple, this is not so trivial, already because the dimensions of irreducible $\mathbb{C}[G,\natural]$-representations depend on the image of $\natural$ in $H^2(G,\mathbb{C}^\times)$.

Let A be a unital $\mathbb{C}$-algebra on which G acts by algebra automorphisms. The twisted crossed product algebra $A \rtimes \mathbb{C}[G,\natural]$ is the vector space $A \otimes_{\mathbb{C}} \mathbb{C}[G,\natural]$ with multiplication rules

- A and $\mathbb{C}[G,\natural]$ are embedded as subalgebras,
- $T_g a T_g^{-1} = g(a)$ for $g \in G$ and $a \in A$.

Twisted crossed products with rings of regular functions.
Interesting examples of the above algebras arise when V is a complex affine variety endowed with a G-action and $A = \mathcal{O}(V)$, the ring of regular functions on V. Our motivation to study algebras like $\mathcal{O}(V) \rtimes \mathbb{C}[G,\natural]$ stems from reductive p-adic groups. There twisted versions of Hecke algebras appear in several ways, see e.g. [AMS2, Mor, Sol7]. If one manually sets the q-parameters of such Hecke algebras equal to 1, one obtains an algebra of the form $\mathcal{O}(V) \rtimes \mathbb{C}[G,\natural]$.

It is well-known that the irreducible representations of $\mathcal{O}(V) \rtimes G = \mathcal{O}(V) \rtimes \mathbb{C}[G]$ are naturally parametrized by

$$(V/\!/G)_2 = \{(v,\pi_v) : v \in V, \pi_v \in \mathrm{Irr}(G_v)\}/G, \tag{1}$$

where $g \cdot (v,\pi_v) = (gv, \pi_v \circ \mathrm{Ad}(g)^{-1})$. The parametrization map is simple:

$$(v,\pi_v) \mapsto \mathrm{ind}_{\mathcal{O}(V) \rtimes G_v}^{\mathcal{O}(V) \rtimes G} (\mathbb{C}_v \otimes \pi_v).$$

Many invariants of $\mathcal{O}(V) \rtimes G$ are related to the space

$$(V/\!/G)_1 = \{(v,g) : v \in V, g \in G_v\}/G, \tag{2}$$

where $g \cdot (v,g') = (gv, gg'g^{-1})$. Indeed, for nonsingular V the Hochschild homology was computed by Brylinski and Nistor [Bry, Nis]:

$$HH_n(\mathcal{O}(V) \rtimes G) = \Big(\bigoplus\nolimits_{g \in G} \Omega^n(V^g) \Big)^G = \Omega^n \big(\{(v,g) : v \in V, g \in G_v\} \big)^G. \tag{3}$$

Now we discuss our analogues with twisting by $\natural$. The same arguments as for (1) show that $\mathrm{Irr}(\mathcal{O}(V) \rtimes \mathbb{C}[G,\natural])$ is naturally parametrized by

$$(V/\!/G)_\natural = \{(v,\pi_v) : v \in V, \pi_v \in \mathrm{Irr}(\mathbb{C}[G_v,\natural])\}/G, \tag{4}$$

where $g \cdot (v, \pi_v) = (gv, \pi_v \circ \mathrm{Ad}(T_g)^{-1})$. However, there is no direct generalization of (2). To get around that, we define (for $g \in G$)

$$\begin{array}{cccc} \natural^g : & G & \to & \mathbb{C}^\times \\ & h & \mapsto & T_h T_g T_h^{-1} T_{hgh^{-1}}^{-1} \end{array} .$$

Then $\natural^g \big|_{Z_G(g)}$ is a character, and the $\natural^g$ measure how far away from a group algebra $\mathbb{C}[G, \natural]$ is. Indeed, we check in Lemma 1.1 that

$$\mathrm{Irr}(\mathbb{C}[G,\natural]) \quad \text{and} \quad \{g \in G : \natural^g \big|_{Z_G(g)} = 1\}/G\text{-conjugation}$$

have the same cardinality. This generalizes the well-known equality between the number of conjugacy classes and the number of inequivalent irreducible representations of G.

Notice that $\mathcal{O}(V)^G$ is contained in the centre of $\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]$, so that it acts naturally on the Hochschild homology of that algebra.

Theorem A. (see Theorem 1.2 and (1.17))
Let V be a nonsingular complex affine variety with a G-action. There exists an isomorphism of $\mathcal{O}(V)^G$-modules

$$HH_n\big(\mathcal{O}(V) \rtimes \mathbb{C}[G,\natural]\big) \cong \big(\bigoplus\nolimits_{g \in G} \Omega^n(V^g) \otimes \natural^g \big)^G .$$

We can interpret (3) as $HH_0(\mathcal{O}(V) \rtimes G) = \mathcal{O}((V//G)_1)$. In contrast, it is not clear whether $HH_0(\mathcal{O}(V) \rtimes \mathbb{C}[G,\natural])$ is naturally isomorphic to the coordinate ring of a complex affine variety. It is preferable to phrase this in noncommutative geometric terms. Then $\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]$ is the ring of "functions on the space $(V//G)_\natural$" and Theorem A describes the "differential forms on $(V//G)_\natural$".

Twisted crossed products with graded Hecke algebras.
Another class of algebras that we want to investigate is intrinsically non-commutative. Let $\mathbb{H}(\mathfrak{t}, W, k)$ be a graded Hecke algebra, where W is a Weyl group acting on a complex vector space $\mathfrak{t}$ and k is a real-valued parameter function. Let Γ be a finite group acting on $\mathbb{H}(\mathfrak{t}, W, k)$, such that all structure used to define $\mathbb{H}(\mathfrak{t}, W, k)$ is preserved by the action. Given a 2-cocycle $\natural$ of Γ, we build the twisted graded Hecke algebra

$$\mathbb{H} = \mathbb{H}(\mathfrak{t}, W, k) \rtimes \mathbb{C}[\Gamma, \natural]. \tag{5}$$

For $k = 0$, this algebra is just $\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W \rtimes \Gamma, \natural]$, where we inflate $\natural$ to a 2-cocycle of $W \rtimes \Gamma$. Algebras of the form (5), sometimes with a nontrivial $\natural$, play an important role in the study of parabolically induced representations of reductive p-adic groups [Sol7]. That motivated us to determine their Hochschild homology.

It follows quickly from [Sol1] and Theorem A that as vector spaces

$$HH_n(\mathbb{H}) \cong \big(\bigoplus\nolimits_{w \in W\Gamma} \Omega^n(\mathfrak{t}^w) \otimes \natural^w \big)^{W\Gamma}, \tag{6}$$

see (2.6). The nontrivial content of this statement is that for every element on the right hand side, a particular representative in a differential complex computing $HH_n(\mathbb{H})$ is exhibited. However, usually (6) is not an isomorphism of $Z(\mathbb{H})$-modules, or even of modules over the central subalgebra $\mathcal{O}(\mathfrak{t})^{W\Gamma}$. To work well with $HH_n(\mathbb{H})$, we need to understand the isomorphism (6) better and to realize it with maps induced by algebra homomorphisms.

In [Sol4] this is achieved (without twisting by $\natural$) with families of representations. For every $w \in W\Gamma$ a family $\mathfrak{F}_w$ of $\mathbb{H}(\mathfrak{t}, W, k) \rtimes \Gamma$-representations parametrized by $\mathfrak{t}^w$ is chosen, such that:

(i) in the Grothendieck group $R(\mathbb{H}(\mathfrak{t}, W, k) \rtimes \Gamma)$ of finite dimensional $\mathbb{H}(\mathfrak{t}, W, k) \rtimes \Gamma$-representations, the span of $\mathfrak{F}_w$ is linearly independent from the span of the union of the $\mathfrak{F}_{w'}$ with w' not conjugate to w in $W\Gamma$,
(ii) the union of all the $\mathfrak{F}_w$ spans $\mathbb{Q} \otimes_{\mathbb{Z}} R(\mathbb{H}(\mathfrak{t}, W, k) \times \Gamma)$.

(It has to be mentioned that small problems with the construction of the families $\mathcal{F}_w$ in [Sol4] have surfaced, but it still works in the large majority of cases.) Each $\mathfrak{F}_w$ induces an algebra homomorphism

$$\mathcal{F}_w : \mathbb{H}(\mathfrak{t}, W, k) \rtimes \Gamma \to \mathcal{O}(\mathfrak{t}^w) \otimes \mathrm{End}_{\mathbb{C}}(V_w), \tag{7}$$

where V_w is the vector space underlying all representations in $\mathfrak{F}_w$. Recall that by the Hochschild–Kostant–Rosenberg theorem

$$HH_n\big(\mathcal{O}(\mathfrak{t}^w) \otimes \mathrm{End}_{\mathbb{C}}(V_w)\big) \cong \Omega^n(\mathfrak{t}^w).$$

It is shown in [Sol4] that the maps $HH_n(\mathcal{F}_w)$ together induce an isomorphism of $Z(\mathbb{H}(\mathfrak{t}, W, k) \rtimes \Gamma)$-modules

$$HH_n(\mathbb{H}(\mathfrak{t}, W, k) \rtimes \Gamma) \cong \big(\bigoplus\nolimits_{w \in W\Gamma} \Omega^n(\mathfrak{t}^w)\big)^{W\Gamma}.$$

For the twisted graded Hecke algebra $\mathbb{H} = \mathbb{H}(\mathfrak{t}, W, k) \rtimes \mathbb{C}[\Gamma, \natural]$, the situation is less favorable: it may be impossible to find families of representations with the above properties. A counterexample is provided by Example 1.15, which shows that for $\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \natural]$ property (i) is problematic for representations with central character $W\Gamma v \in \mathfrak{t}/W\Gamma$ such that $\natural$ is nontrivial in $H^2((W\Gamma)_v, \mathbb{C}^\times)$.

To overcome that, we consider not only (algebraic) families of representations, but also families of virtual representations, in $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H})$. In Lemma 1.7 we check that every such family canonically induces a map on Hochschild homology, a linear combination of maps induced by algebra homomorphisms. For each $w \in W\Gamma$ we construct an algebraic family of virtual $\mathbb{H}$-representations $\nu_w^1 = \{\nu_{w,v}^1 : v \in \mathfrak{t}^w\}$, which satisfies (i) and (ii).

Theorem B. (see Theorem 2.8)
(*a*) *The families ν_w^1 with $w \in W\Gamma$ induce an isomorphism of vector spaces*

$$HH_n(\mathbb{H}) \cong \big(\bigoplus\nolimits_{w \in W\Gamma} \Omega^n(\mathfrak{t}^w) \otimes \natural^w\big)^{W\Gamma}.$$

(*b*) $HH_0(\mathbb{H})$ *is naturally isomorphic to the set of f in $\big(\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H})\big)^*$ with the property: for any algebraic family $\mathfrak{F} : \lambda \mapsto \mathfrak{F}_\lambda$ of $\mathbb{H}$-representations, $\lambda \mapsto f(\mathfrak{F}_\lambda)$ is a regular function.*

We note that Theorem B.b is quite similar to the description of the zeroth Hochschild homology for a reductive p-adic group obtained in [BDK].

The above does not yet describe the structure of $HH_n(\mathbb{H})$ as $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-module, because in general the virtual representations $\nu_{w,v}^1$ do not admit a central character. Things can be improved by a canonical decomposition of the category of finite dimensional tempered $\mathbb{H}$-representations, or at least the Grothendieck group thereof (Theorem 2.2). The decomposition is indexed by equivalence classes of pairs (Q, δ), where δ is a discrete series representation of a parabolic subalgebra $\mathbb{H}_Q$ of $\mathbb{H}$. That induces analogous decompositions for many objects associated to $\mathbb{H}$. In particular

$$\nu^1_{v,w} = \sum\nolimits_{[Q,\delta] \in \Delta_{\mathbb{H}}} \nu^{1,[Q,\delta]}_{v,w},$$

where each virtual $\mathbb{H}$-representation $\nu^{1,[Q,\delta]}_{v,w}$ admits the $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-character $W\Gamma(cc(\delta)+v)$, see Lemma 2.6.

Theorem C. (see Corollary 2.13)
There exists a canonical decomposition

$$HH_n(\mathbb{H}) = \bigoplus\nolimits_{[Q,\delta]\in\Delta_{\mathbb{H}}} HH_n(\mathbb{H})^{[Q,\delta]}$$

such that the injection

$$HH_n(\mathbb{H})^{[Q,\delta]} \to \Big(\bigoplus\nolimits_{w\in W\Gamma} \Omega^n(\mathfrak{t}^w) \otimes \natural^w \Big)^{W\Gamma}$$

from Theorem B becomes $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-linear if we let $\mathcal{O}(\mathfrak{t})^{W\Gamma}$ act at the stalk over $w \in W\Gamma, v \in \mathfrak{t}^w$ via evaluation at $W\Gamma(cc(\delta)+v)$.

Applications to p-adic groups.
It has been known for a long time that affine Hecke algebras play a role in the representation theory of reductive p-adic groups. The author has made that precise in full generality [Sol7], although it turned out that (twisted) graded Hecke algebras are involved more naturally. This has opened several new research options, for instance, it can be used to determine homologies of the Hecke algebra $\mathcal{H}(G)$ of an arbitrary reductive p-adic group G. Locally $\mathcal{H}(G)$ is Morita equivalent with (a local part of) a twisted graded Hecke algebra [Sol7, §7–8]. That prompted us to compute the Hochschild homology of such algebras. In Proposition 2.11 we show that there is a canonical isomorphism of vector spaces

$$HH_n\big(\mathbb{H}(\mathfrak{t},W,k) \rtimes \mathbb{C}[\Gamma,\natural]\big) \cong HH_n\big(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma,\natural]\big).$$

We expect that similarly $HH_n(\mathcal{H}(G))$ will be isomorphic to a direct sum of terms

$$HH_n\big(\mathcal{O}(T_\mathfrak{s}) \rtimes \mathbb{C}[W_\mathfrak{s},\natural_\mathfrak{s}]\big).$$

Here the complex torus $T_\mathfrak{s}$, the finite group $W_\mathfrak{s}$ and the 2-cocycle $\natural_\mathfrak{s}$ are canonically associated to a Berstein component $\mathrm{Irr}(G)^\mathfrak{s}$ of $\mathrm{Irr}(G)$. The details will appear in [Sol8]. One substantial complication is that we will have to deal with discontinuous families of twisted graded Hecke algebras. To the end we will employ at least two strategies:

- realize Hochschild homology in terms of algebraic families of representations,
- make such families less discontinuous in the sense that at least the involved vector space $\mathfrak{t}$ and the finite group $W\Gamma$ are locally constant.

The former already permeates this paper, for the latter we can replace the twisted graded Hecke algebras by larger Morita equivalent algebras. In Paragraph 2.3 we generalize our main results to such algebras, which combine features of twisted crossed products with commutative algebras and of graded Hecke algebras.

Structure of the paper.
In Paragraph 1.1 we introduce the characters $\natural^g$ and we prove Theorem A. Some

generalities involving families of representations, valid for many algebras, are discussed in Paragraph 1.2. To prepare for Theorem B, we establish a simpler analogue with algebras of the form $\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]$, in Paragraph 1.3. As an intermediate step we map $HH_n(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural])$ to n-forms on some algebraic varieties, via families of representations.

We start Section 2 with recalling the definition of a (twisted) graded Hecke algebra. Then we generalize some representation theoretic results, which allow to reduce certain issues for $\mathbb{H}(\mathfrak{t}, W, k) \rtimes \mathbb{C}[\Gamma, \natural]$ to its version $\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \natural]$ with $k = 0$. Next we prove Theorem B, in many small steps. Again it goes via differential forms coming from auxiliary algebraic families of representations. After that we wrap up the proof of Theorem C.

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1. Twisted crossed product algebras

1.1. Hochschild homology via differential forms.

Let G be a finite group, let $\natural : G \times G \to \mathbb{C}^\times$ be a 2-cocycle and form the twisted group algebra $\mathbb{C}[G, \natural]$. It has a basis $\{T_g : g \in G\}$ and multiplication rules

$$T_g \cdot T_{g'} = \natural(g, g') T_{gg'}.$$

By the theory of Schur multipliers [CuRe, §53] there exists a finite central extension

$$1 \to Z \to \tilde{G} \to G \to 1 \tag{1.1}$$

such that the corresponding lift of $\natural$ is trivial in $H^2(\tilde{G}, \mathbb{C}^\times)$. Then there exists a minimal central idempotent $p_\natural \in \mathbb{C}[Z]$ and an algebra isomorphism

$$\begin{array}{ccc} p_\natural \mathbb{C}[\tilde{G}] & \to & \mathbb{C}[G, \natural] \\ p_\natural \tilde{g} & \mapsto & c_{\tilde{g}} T_g \end{array} . \tag{1.2}$$

Here $\tilde{g} \in \tilde{G}$ has image g in G, and $c_{\tilde{g}} \in \mathbb{C}$ is a suitable scalar. Notice that $p_\natural \mathbb{C}[\tilde{G}]$ is a direct summand of the semisimple algebra $\mathbb{C}[\tilde{G}]$, so itself semisimple. The minimal idempotent $p_\natural \in \mathbb{C}[Z]$ is associated to some character $\chi_\natural$ of Z, so

$$p_\natural = |Z|^{-1} \sum\nolimits_{z \in Z} \chi_\natural^{-1}(z) z.$$

Let $\natural^g$ be the character

$$\begin{array}{ccc} Z_{\tilde{G}}(g) & \to & \mathbb{C}^\times \\ \tilde{h} & \mapsto & \chi_\natural^{-1}([\tilde{g}, \tilde{h}]) = \chi_\natural([\tilde{h}, \tilde{g}]) \end{array} .$$

Here $\tilde{g} \in \tilde{G}$ is a lift of $g \in G$, and the choice does not matter because any other lift differs from $\tilde{g}$ by a central element. The kernel of $\natural^g$ contains Z, so we can also regard it as a character of $Z_G(g)$. As such, one can also express it as

$$\natural^g(h) = T_g T_h T_g^{-1} T_h^{-1} = T_h T_g T_h^{-1} T_g^{-1}. \tag{1.3}$$

This shows that $\natural^g$ is insensitive to rescaling T_g and T_h, which entails that $\natural^g$ depends only on g and the cohomology class of $\natural$.

Let $\langle G \rangle$ be a set of representatives for the conjugacy classes in G and recall that $HH_0(A)^*$ is the space of trace functions on an algebra A.

Lemma 1.1. (*a*) *For $g \in G$ with $\natural^g = 1$, there exists a unique trace function ν_g on $\mathbb{C}[G, \natural]$ with $\nu_g(T_g) = 1$ and $\nu_g(T_{g'}) = 0$ if g and g' are not conjugate in G.*
(*b*) *The set $\{\nu_g : g \in \langle G \rangle, \natural^g = 1\}$ is a basis of $HH_0(\mathbb{C}[G, \natural])^*$. The number of inequivalent irreducible representation of $\mathbb{C}[G, \natural]$ equals $|\{g \in \langle G \rangle, \natural^g = 1\}|$.*

Proof. (a) Since $\mathbb{C}[G, \natural] \cong p_\natural \mathbb{C}[\tilde{G}]$ is a direct summand of $\mathbb{C}[\tilde{G}]$, every trace function on $p_\natural \mathbb{C}[\tilde{G}]$ can be extended to one on $\mathbb{C}[\tilde{G}]$. A basis of $HH_0(\mathbb{C}[\tilde{G}])^*$ is the set of indicator functions $1_{\tilde{C}}$ for the conjugacy classes $\tilde{C}$ in $\tilde{G}$.

Suppose that $\tilde{g} \in \tilde{C}$ and $\natural^g \neq 1$ (where g is the image of $\tilde{g}$ in G). Then $\tilde{g}$ is $\tilde{G}$-conjugate to $\tilde{g}z$ for some $z \in Z$ with $\chi_\natural(z) \neq 1$. Hence $1_{\tilde{C}}(p_\natural T_{\tilde{g}})$ is a multiple of $\sum_{n=1}^{\mathrm{ord}(z)} \chi_\natural(z^n) = 0$, which implies that $1_{\tilde{C}}$ vanishes on $p_\natural \mathbb{C}[\tilde{G}]$.

On the other hand, suppose that $\tilde{g} \in \tilde{C}$ and $\natural^g = 1$. Then $1_{\tilde{C}}\big|_{p_\natural \mathbb{C}[\tilde{G}]}$ is nonzero and has support

$$p_\natural \operatorname{span}\{T_{\tilde{g}} : \tilde{g} \in \tilde{C}\} = p_\natural \operatorname{span}\{T_{\tilde{g}z} : \tilde{g} \in \tilde{C}, z \in Z\}.$$

Thus $1_{\tilde{C}}$ defines a trace function on $\mathbb{C}[G, \natural] \cong p_\natural \mathbb{C}[\tilde{G}]$ supported on the conjugacy class of g in G. A unique scalar multiple ν_g of $1_{\tilde{C}}$ satisfies $\nu_g(T_g) = 1$.
(b) The above argument also shows that $\{\nu_g : g \in G\}$ spans $HH_0(\mathbb{C}[G, \natural])^*$. If we pick just one g from every conjugacy class, the span does not change and the set becomes linearly independent, so a basis.

As the algebra $\mathbb{C}[G, \natural]$ is semisimple, it number of irreducible representations equals the dimension of $HH_0(\mathbb{C}[G, \natural])$. □

Let V be a nonsingular affine variety over $\mathbb{C}$, with an algebraic G-action. Then G also acts on the algebra of regular functions $\mathcal{O}(V)$. We want to compute the Hochschild homology of $\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]$ (as defined in the introduction). We denote the set of (algebraic) differential n-forms on V by $\Omega^n(V)$. Important background for all our computations is the Hochschild–Kostant–Rosenberg (HKR) theorem, which provides a natural isomorphism of $\mathcal{O}(V)$-modules

$$HH_n(\mathcal{O}(V)) \cong \Omega^n(V). \tag{1.4}$$

We recall, e.g. from [Lod, §1.1], that the Hochschild homology of a unital algebra A can be computed as the homology of the bar complex $(C_*(A), b_*)$, where $C_n(A) = A^{\otimes(n+1)}$ and

$$b_n(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

The isomorphism (1.4) is realized by the map

$$\begin{array}{cccc} \Omega : & C_n(\mathcal{O}(V)) & \to & \Omega^n(\mathcal{O}(V)) \\ & f_0 \otimes f_1 \otimes \cdots \otimes f_n & \mapsto & f_0 \mathrm{d}f_1 \cdots \mathrm{d}f_n / n! \end{array} . \tag{1.5}$$

We let $\tilde{G}$ act on V via its quotient G. Then (1.2) induces an algebra isomorphism

$$\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural] \cong \mathcal{O}(V) \rtimes p_\natural \mathbb{C}[\tilde{G}] = p_\natural(\mathcal{O}(V) \rtimes \tilde{G}),$$

where the right hand side is a direct summand of the crossed product algebra $\mathcal{O}(V) \rtimes \tilde{G}$. Brylinski [Bry] and Nistor [Nis] computed the Hochschild homology of such algebras. It can be done in a few steps:

- For each $\tilde{g} \in \tilde{G}$, $C_*(\mathcal{O}(V) \rtimes \tilde{G})$ contains a subcomplex $\tilde{g}C_*(\mathcal{O}(V))$. It can be identified with the complex that computes the Hochschild homology of $\mathcal{O}(V)$ with coefficients in the bimodule $\tilde{g}\mathcal{O}(V)$, so

$$H_n\big(\tilde{g}C_*(\mathcal{O}(V)), b_*\big) = HH_n\big(\mathcal{O}(V), \tilde{g}\mathcal{O}(V)\big). \tag{1.6}$$

- Varying on the HKR theorem, one computes that

$$HH_n\big(\mathcal{O}(V), \tilde{g}\mathcal{O}(V)\big) \cong \Omega^n(V^{\tilde{g}}). \tag{1.7}$$

- The complex $C_*(\mathcal{O}(V) \rtimes \tilde{G})$ decomposes as a direct sum of subcomplexes indexed by the conjugacy classes $\langle \tilde{g} \rangle$ in $\tilde{G}$. The summand $C_n(\mathcal{O}(V) \rtimes \tilde{G})_{\langle \tilde{g} \rangle}$ is spanned by the elementary tensors such that if you multiply the $n+1$ involved group elements, you end up in $\langle \tilde{g} \rangle$. One shows that the inclusion

$$\Big(\bigoplus\nolimits_{\tilde{h} \in \langle \tilde{g} \rangle} \tilde{h} C_*(\mathcal{O}(V)) \Big)^{\tilde{G}} \to C_*(\mathcal{O}(V) \rtimes \tilde{G})_{\langle \tilde{g} \rangle}$$

is a quasi-isomorphism.
- Let $\langle \tilde{G} \rangle$ be a set of representatives for the conjugacy classes in $\tilde{G}$. From the above one obtains an isomorphism of $\mathcal{O}(V)^{\tilde{G}}$-modules

$$HH_n\big(\mathcal{O}(V) \rtimes \tilde{G}\big) \cong \Big(\bigoplus\nolimits_{\tilde{g} \in \tilde{G}} \Omega^n(V^{\tilde{g}}) \Big)^{\tilde{G}} \cong \bigoplus\nolimits_{\tilde{g} \in \langle \tilde{G} \rangle} \Omega^n(V^{\tilde{g}})^{Z_{\tilde{G}}(\tilde{g})}. \tag{1.8}$$

On the level of the complexes $\tilde{g}C_*(\mathcal{O}(V))$, (1.8) comes from the map (1.5) followed by restriction of differential forms on V to $V^{\tilde{g}}$, and then averaging over $Z_{\tilde{G}}(\tilde{g})$. The isomorphism is made natural in [Nis, Theorem 2.11].

While (1.8) holds for an algebraic action of any finite group on a nonsingular variety, our setting is more specific, with a central subgroup Z that acts trivially. Hence $\Omega^n(V^{\tilde{g}})^{Z_{\tilde{G}}(\tilde{g})}$ depends only the image g of $\tilde{g}$ in G:

$$\Omega^n(V^{\tilde{g}})^{Z_{\tilde{G}}(\tilde{g})} = \Omega^n(V^g)^{Z_G(\tilde{g})}.$$

Notice that $Z_G(\tilde{g})$ makes sense because the conjugation action of $\tilde{G}$ on itself factors through an action of G on $\tilde{G}$. In general $Z_G(\tilde{g})$ is contained in $Z_G(g)$, but they need not be equal. In these circumestances (1.8) specializes to an isomorphism of $\mathcal{O}(V)^G$-modules

$$HH_n\big(\mathcal{O}(V) \rtimes \tilde{G}\big) \cong \Big(\bigoplus\nolimits_{\tilde{g}Z \in G} \bigoplus\nolimits_{z \in Z} \Omega^n(V^g) \Big)^G. \tag{1.9}$$

The difference between the various direct summands $\Omega^n(V^g)$ is that they come from distinct subcomplexes $\tilde{g}zC_*(\mathcal{O}(V))$.

Theorem 1.2. *There exists an isomorphism of $\mathcal{O}(V)^G$-modules*

$$HH_n\big(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]\big) \cong \bigoplus\nolimits_{g \in \langle G \rangle} \big(\Omega^n(V^g) \otimes \natural^g\big)^{Z_G(g)}.$$

Proof. Since $HH_n(A)$ is always a module over $Z(A)$ and $p_\natural$ is a central idempotent:

$$HH_n\big(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]\big) \cong HH_n\big(p_\natural(\mathcal{O}(V) \rtimes \tilde{G})\big) = p_\natural HH_n\big(\mathcal{O}(V) \rtimes \tilde{G}\big). \tag{1.10}$$

By (1.9), (1.6) and (1.7) the right hand side of (1.10) equals

$$\Big(p_\natural \bigoplus\nolimits_{\tilde g Z \in G} \bigoplus\nolimits_{z \in Z} H_n \big(\tilde g C_* (\mathcal O (V)), b_* \big) \Big)^G .$$

This expression decomposes naturally as a direct sum over the conjugacy classes of G, namely

$$HH_n \big(\mathcal O (V) \rtimes \mathbb C [G,\natural] \big) \cong \bigoplus\nolimits_{\tilde g Z \in \langle G \rangle} p_\natural \Big(\bigoplus\nolimits_{z \in Z} H_n \big(\tilde g C_* (\mathcal O (V)), b_* \big) \Big)^{Z_G (g)} . \tag{1.11}$$

The action of $h \in Z_G (g)$ (with a lift $\tilde h \in \tilde G$) sends $\tilde g z \cdot c \in \tilde g z C_* (\mathcal O (V))$ to

$$\tilde h \tilde g \tilde h^{-1} z \cdot h(c) = \tilde g [\tilde h, \tilde g] z \cdot h(c).$$

We find

$$\begin{aligned} \Big(\bigoplus\nolimits_{z \in Z} H_n \big(\tilde g C_* (\mathcal O (V)), b_* \big) \Big)^{Z_G (g)} = & \\ \Big\{ \sum\nolimits_{z \in Z} \omega_z \in \bigoplus\nolimits_{z \in Z} \Omega^n (\mathcal O (V^g)) : \omega_{[\tilde h, \tilde g] z} = h(\omega_z) \; \forall h \in Z_G (g) \Big\} & . \end{aligned} \tag{1.12}$$

The shape of $p_\natural$ entails that the image of this idempotent in (1.12) is

$$\begin{aligned} \Big\{ \sum\nolimits_{z \in Z} \omega_z \in \bigoplus\nolimits_{z \in Z} \Omega^n (\mathcal O (V^g)) : & \\ h(\omega_{[\tilde g, \tilde h] z}) = \omega_z \; \forall h \in Z_G (g), \omega_{z' z} = \chi_\natural^{-1}(z') \omega_z \; \forall z' \in Z \Big\} & . \end{aligned} \tag{1.13}$$

The two conditions in (1.13) are equivalent with

$$\omega_{z'z} = \chi_\natural^{-1} (z') \omega_z \text{ and } \omega_z = \natural^g (h) h(\omega_z) \quad \forall z,z' \in Z, h \in Z_G (g). \tag{1.14}$$

From (1.11)–(1.14) we obtain the required description of $HH_n \big(\mathcal O (V) \rtimes \mathbb C [G,\natural] \big)$ as $\mathcal O (V)^G$-module. $\Box$

Unfortunately this isomorphism does not seem to be natural, unlike (1.8). In the way we constructed it, it depends on a choice of representatives in $\tilde G$ for the conjugacy classes of G and the choice of the algebra isomorphism (1.2). This can be improved a little by a more explicit construction, we can compose (1.5) with averaging over $Z_G (g)$. But the 2-cocycle $\natural$ makes conjugation by $h \in Z_G (g)$ a bit more subtle, namely

$$\begin{aligned} T_h (T_g f_0 \otimes f_1 \otimes \cdots \otimes f_n) T_h^{-1} & = T_h T_g T_h^{-1} \, h \cdot (f_0 \otimes f_1 \otimes \cdots \otimes f_n) \\ & = \natural^g (h) T_g \, h \cdot (f_0 \otimes f_1 \otimes \cdots \otimes f_n). \end{aligned}$$

Hence we can realize Theorem 1.2 for the summand indexed by g as

$$\begin{array}{ccc} T_g C_n (\mathcal O (V)) & \to & (\Omega^n (V^g) \otimes \natural^g)^{Z_G (g)} \\ T_g f_0 \otimes f_1 \otimes \cdots \otimes f_n & \mapsto & \sum_{h \in Z_G (g)} \dfrac{\natural^g (h) \, h \cdot (f_0 \mathrm d f_1 \cdots \mathrm d f_n)}{|Z_G (g)| \; n!} \end{array} . \tag{1.15}$$

This entails that on the summand indexed by g, the isomorphism from Theorem 1.2 is canonical up to a scalar (from the choice of T_g). To analyse the dependence on the choice of representatives of the conjugacy classes, we define

$$\natural^g (h) = T_h T_g T_h^{-1} T_{hgh^{-1}}^{-1} \in \mathbb C^\times \qquad \text{for all } g,h \in G.$$

Lemma 1.3. *Let $g,h,\tilde h \in G$.*

(a) $\natural^g (\tilde h h) = \natural^{hgh^{-1}} (\tilde h) \natural^g (h)$.

(b) $\natural^g(h) : (\natural^g, \mathbb{C}) \to (h^{-1} \cdot \natural^{hgh^{-1}}, \mathbb{C})$ *is an isomorphism of* $Z_G(g)$*-representations.*

Proof. (a) Using $\natural^g(h) = T_h^{-1} T_{hgh^{-1}}^{-1} T_h T_g$ we compute

$$\begin{aligned}\natural^{hgh^{-1}}(\tilde h)\natural^g(h)T_g^{-1} &= \natural^{hgh^{-1}}(\tilde h) T_h^{-1} T_{hgh^{-1}}^{-1} T_h \\ &= T_h^{-1} \natural^{hgh^{-1}}(\tilde h) T_{hgh^{-1}}^{-1} T_h \\ &= T_h^{-1} T_{\tilde h}^{-1} T_{\tilde h hgh^{-1}\tilde h^{-1}}^{-1} T_{\tilde h} T_h \\ &= (\natural(\tilde h, h) T_{h\tilde h})^{-1} T_{\tilde h hgh^{-1}\tilde h^{-1}}^{-1} (\natural(\tilde h, h) T_{h\tilde h}) \\ &= T_{h\tilde h}^{-1} T_{\tilde h hgh^{-1}\tilde h^{-1}}^{-1} T_{h\tilde h} \quad = \quad \natural^g(\tilde h h) T_g^{-1}.\end{aligned}$$

(b) Assume that $\tilde h \in Z_G(hgh^{-1})$. Applying part (a) twice, we find

$$\natural^{hgh^{-1}}(\tilde h)\natural^g(h) = \natural^g(\tilde h h) = \natural^g(h)\natural^g(h^{-1}\tilde h h).$$

Hence $\natural^g(h)$ intertwines the $Z_G(hgh^{-1})$-representations $h \cdot \natural^g$ and $\natural^{hgh^{-1}}$. $\square$

Lemma 1.3 provides a canonical bijection

$$\Omega^n(h^{-1}) \otimes \natural^g(h) : \Omega^n(V^g) \otimes \natural^g \to \Omega^n(V^{hgh^{-1}}) \otimes \natural^{hgh^{-1}}, \tag{1.16}$$

which intertwines the $Z_G(g)$-actions (where $Z_G(g)$ acts on the right hand side via precomposing with conjugation by h). Regarding (1.16) as an action of $h \in G$ on the sum of these spaces, we can reformulate Theorem 1.2 as

$$HH_n\big(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]\big) \cong \bigoplus_{g \in \langle G \rangle} \big(\Omega^n(V^g) \otimes \natural^g\big)^{Z_G(g)} \cong \Big(\bigoplus_{g \in G} \Omega^n(V^g) \otimes \natural^g\Big)^G. \tag{1.17}$$

Consider any

$$\omega = \sum\nolimits_{g \in G} T_g \omega_g \in \Big(\bigoplus\nolimits_{g \in G} \Omega^n(V^g) \otimes \natural^g\Big)^G.$$

By construction

$$T_{hgh^{-1}} \omega_{hgh^{-1}} = T_h T_g \omega_g T_h^{-1} = T_h T_g T_h^{-1} h \cdot \omega_g = \natural^g(h) T_{hgh^{-1}} h \cdot \omega_g. \tag{1.18}$$

We deduce that $\omega_{hgh^{-1}} = \natural^g(h) h \cdot \omega_g$.

Next we relate Theorem 1.2 to $\mathrm{Irr}\big(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]\big)$. For $g \in G$ and $v \in V^g$ with $\natural^g|_{G_v \cap Z_G(g)} = 1$, we define $\nu_{g,v} \in HH_0(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural])^*$ as evaluation at (g, v) in the expression

$$HH_0(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]) \cong \big(\bigoplus\nolimits_{g \in G} \mathcal{O}(V^g) \otimes \natural^g\big)^G \tag{1.19}$$

from Theorem (1.17). By (1.18), for any $h \in G$:

$$\nu_{hgh^{-1}, hv}(\omega) = \nu_{hgh^{-1}, hv}(\natural^g(h)\, h \cdot \omega) = \natural^g(h) \nu_{g,v}(\omega). \tag{1.20}$$

Hence $\nu_{hgh^{-1}, hv} = \natural^g(h) \nu_{g,v}$. From (1.15) we also see how $\nu_{g,v}$ becomes a map $\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural] \to \mathbb{C}$ supported on $T_g \mathcal{O}(V)$.

Lemma 1.4. *The following numbers are equal:*

- (i) *the number of inequivalent irreducible representations of* $\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]$ *with* $\mathcal{O}(V)^G$*-character* Gv*,*
- (ii) $|\{g \in \langle G_v \rangle : \natural^g|_{G_v \cap Z_G(g)} = 1\}|$*, where* $\langle G_v \rangle$ *is a set of representatives for the conjugacy classes in* G_v*,*
- (iii) *the dimension of the specialization of* $HH_0(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural])$ *at* Gv*,*

(iv) $|\{\nu_{g,v'} : g \in \langle G \rangle, v' \in (V^g \cap Gv)/Z_G(g), \natural^g|_{G_v \cap Z_G(g)} = 1\}|$.

In (iv) $v' \in (V^g \cap Gv)/Z_G(g)$ means that from every $Z_G(g)$-orbit we pick one element in $V^g \cap Gv$.

Proof. By Mackey theory there is a bijection from $\mathrm{Irr}(\mathbb{C}[G_v, \natural])$ to the set in (i), namely

$$\rho \mapsto \mathrm{ind}_{\mathcal{O}(V) \rtimes \mathbb{C}[G,\natural]}^{\mathcal{O}(V) \rtimes \mathbb{C}[G_v,\natural]} (\mathbb{C}_v \otimes \rho). \tag{1.21}$$

By Lemma 1.1, $|\mathrm{Irr}(\mathbb{C}[G_v, \natural])|$ equals (ii).

From Theorem 1.2 we see that specializing $HH_0(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural])$ at Gv yields

$$\Big(\bigoplus\nolimits_{g \in G} C(V^g \cap Gv) \otimes \natural^g \Big)^G \cong \Big(\bigoplus\nolimits_{g \in G_v} C(\{v\}) \otimes \natural^g \Big)^{G_v}.$$

The dimension of the right hand side is (ii) and the dimension of the left hand side equals (iv). $\square$

As a consequence of Lemma 1.4, we record that there is a bijection

$$\mathrm{Irr}(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]) \longleftrightarrow \{\nu_{g,v} : g \in \langle G \rangle, v \in V^g/Z_G(g), \natural^g|_{G_v \cap Z_G(g)} = 1\}, \tag{1.22}$$

which preserves the underlying G-orbits in V.

Example 1.5. We illustrate the constructions in this section with an example that exhibits some non-standard behaviour. Let Q_8 be the quaternion group, with centre $Z(Q_8) = \{1, -1\}$. Let G be the quotient

$$Q_8/Z(Q_8) = \{\pm \mathbf{1}, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}.$$

For a nontrivial 2-cocycle on G, let $\chi_\natural$ be the nontrivial character of $Z(Q_8)$ and define $\mathbb{C}[G, \natural] = p_\natural \mathbb{C}[Q_8]$. From calculations like $T_{\pm \mathbf{j}} T_{\pm \mathbf{i}} T_{\pm \mathbf{j}}^{-1} = -T_{\pm \mathbf{i}}$ we obtain

$$\natural^{\pm \mathbf{i}}(g) = \left\{ \begin{array}{cc} 1 & g \in \langle \pm \mathbf{i} \rangle \\ -1 & g \notin \langle \pm \mathbf{i} \rangle \end{array} \right. .$$

Similarly $\natural^{\pm \mathbf{j}}$ and $\natural^{\pm \mathbf{k}}$ are nontrivial characters of G, while $\natural^{\pm \mathbf{1}} = \mathrm{triv}_G$.

The group G acts on $V = \mathbb{C}^2$ by

$$\pm \mathbf{i} \cdot (z_1, z_2) = (-z_1, z_2), \quad \pm \mathbf{j} \cdot (z_1, z_2) = (z_1, -z_2), \quad \pm \mathbf{k} \cdot (z_1, z_2) = (-z_1, -z_2).$$

From Lemma 1.4 we can compute the number of irreducible representations of $\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]$ with a fixed $\mathcal{O}(V)^G$-character:

$(z_1, z_2) \in \mathbb{C}^2/G$	$z_1 \neq 0 \neq z_2$	$z_1 = 0 \neq z_2$	$z_1 \neq 0 = z_2$	$z_1 = 0 = z_2$
# irreps	1	2	2	1

We work out the description of $HH_n(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural])$ from Theorem 1.2:

- $g = \pm \mathbf{1}$: $V^g = V, (\Omega^n(V^g) \otimes \natural^g)^{Z_G(g)} = \Omega^n(V)^G$,
- $g = \pm \mathbf{i}$: $V^g = \{0\} \times \mathbb{C}, (\Omega^0(V^g) \otimes \natural^g)^{Z_G(g)} = \{f \in \mathcal{O}(V^g) : f(-x) = -f(x)\}$, $(\Omega^1(V^g) \otimes \natural^g)^{Z_G(g)} = \{f \mathrm{d}z_2 \in \Omega^1(V^g) : f(-x) = f(x)\}$,
- $g = \pm \mathbf{j}$: $V^g = \mathbb{C} \times \{0\}, (\Omega^0(V^g) \otimes \natural^g)^{Z_G(g)} = \{f \in \mathcal{O}(V^g) : f(-x) = -f(x)\}$, $(\Omega^1(V^g) \otimes \natural^g)^{Z_G(g)} = \{f \mathrm{d}z_1 \in \Omega^1(V^g) : f(-x) = f(x)\}$,
- $g = \pm \mathbf{k}$: $V^g = \{(0, 0)\}$, $(\Omega^0(V^g) \otimes \natural^g)^{Z_G(g)} = 0$.

1.2. **Algebraic families of representations.**

Let A be a $\mathbb{C}$-algebra and let $\mathrm{Mod}_f(A)$ be the category of finite length A-modules. Let $R(A)$ be the Grothendieck group of $\mathrm{Mod}_f(A)$. Assume that, for every $a \in A$ and every $\pi \in \mathrm{Mod}_f(A)$, $\mathrm{tr}\,\pi(a)$ is well-defined. This holds for instance if all finite length A-representations have finite dimension. That is the case for all algebras that we study in this paper, because they have finite rank as modules over their centre. Under this condition, there is a natural pairing

$$\begin{array}{ccc} HH_0(A) \times \mathrm{Mod}_f(A) & \to & \mathbb{C} \\ (a, \pi) & \mapsto & \mathrm{tr}\,\pi(a) \end{array}, \tag{1.23}$$

which induces a $\mathbb{C}$-bilinear map

$$HH_0(A) \times \mathbb{C} \otimes_{\mathbb{Z}} R(A) \to \mathbb{C}.$$

These can also be interpreted as natural linear maps

$$R(A) \to \mathbb{C} \otimes_{\mathbb{Z}} R(A) \to HH_0(A)^*, \tag{1.24}$$

$$HH_0(A) \to (\mathbb{C} \otimes_{\mathbb{Z}} R(A))^*. \tag{1.25}$$

By the linear independence of irreducible characters (1.24) is injective, so this identifies $R(A)$ and $\mathbb{C} \otimes_{\mathbb{Z}} R(A)$ with subgroups of $HH_0(A)^*$.

By an algebraic family $\mathfrak{F}$ of A-representations over a complex affine variety Y we mean a family of A-representations $\mathfrak{F}_y$ ($y \in Y$), all on the same finite dimensional vector space W, which together give an algebra homomorphism

$$\begin{array}{cccc} \mathcal{F}_Y : & A & \to & \mathcal{O}(Y) \otimes \mathrm{End}_{\mathbb{C}}(W) \\ & a & \mapsto & [y \mapsto \mathfrak{F}_y(a)] \end{array}.$$

Here (and later) $\mathcal{O}(Y) \otimes \mathrm{End}_{\mathbb{C}}(W)$ is identified with the algebra of regular $\mathrm{End}_{\mathbb{C}}(W)$-valued functions on Y. For any $a \in A$, the map

$$Y \to \mathbb{C} : y \mapsto \mathrm{tr}\,\mathfrak{F}_y(a)$$

is a regular function. We call a linear function f on $\mathbb{C} \otimes_{\mathbb{Z}} R(A)$ regular if for every algebraic family of A-representations $\mathfrak{F}_Y$,

$$\text{the function } Y \to \mathbb{C} : y \mapsto f(\mathfrak{F}_y) \text{ is regular.}$$

Then the image of $HH_0(A)$ under (1.25) is contained in

$$(\mathbb{C} \otimes_{\mathbb{Z}} R(A))^*_{\mathrm{reg}} = \{ f \in (\mathbb{C} \otimes_{\mathbb{Z}} R(A))^* : f \text{ is regular} \}. \tag{1.26}$$

Assume that Y is nonsingular. By the Hochschild–Kostant–Rosenberg theorem and Morita invariance, the Hochschild homology of $\mathcal{O}(Y) \otimes \mathrm{End}_{\mathbb{C}}(W)$ is $\Omega^*(Y)$. We recall from [Lod, §1.2] that the isomorphism

$$HH_n\big(\mathcal{O}(Y) \otimes \mathrm{End}_{\mathbb{C}}(W)\big) \to HH_n(\mathcal{O}(Y))$$

can be implemented by the generalized trace map

$$\begin{array}{cccc} \mathrm{gtr} : & C_n(\mathcal{O}(Y) \otimes \mathrm{End}_{\mathbb{C}}(W)) & \to & C_n(\mathcal{O}(Y)) \\ & f_0 m_0 \otimes f_1 m_1 \otimes \cdots \otimes f_n m_n & \mapsto & \mathrm{tr}(m_0 m_1 \cdots m_n) f_0 \otimes f_1 \otimes \cdots \otimes f_n \end{array},$$

where $f_i \in \mathcal{O}(Y)$ and $m_i \in \mathrm{End}_{\mathbb{C}}(W)$.

Recall that for any finite length A-module M there exists a unique semisimple A-module M_{ss} with the same image in $R(A)$, called the semisimplication of M.

Lemma 1.6. (*a*) *For $y \in Y$, the map $\mathrm{ev}_y \circ \mathrm{gtr} \circ C_*(\mathcal{F}_Y)$ depends only on the semisimplification of the A-representation $\mathfrak{F}_y$.*

(*b*) *The maps* $\mathrm{gtr} \circ C_*(\mathcal{F}_Y)$ *and*

$$HH_n(\mathcal{F}_Y) : HH_n(A) \to \Omega^n(Y)$$

depend only on the image of the family $\mathfrak{F}$ *in* $R(A)$.

Proof. (a) Let $0 = W_0 \subset W_1 \subset \cdots \subset W_k = W$ be a composition series of the A-representation $(\mathfrak{F}_y, W)$. If $m \in \mathfrak{F}_y(A)$ maps every W_j to W_{j-1}, then $\mathrm{tr}(m m_1 \cdots m_n) = 0$ for all $m_i \in \mathfrak{F}_y(A)$. Hence $\mathrm{ev}_y \circ \mathrm{gtr} \circ C_*(\mathcal{F}_Y)$ factors through

$$C_*\big(\bigoplus\nolimits_j \mathrm{End}_{\mathbb{C}}(W_j / W_{j-1}) \big),$$

and can be computed from the semisimplification $W_{ss} = \bigoplus_j W_j / W_{j-1}$ of $(\mathfrak{F}_y, W)$.
(b) We recall that the HKR isomorphism is realized by the map Ω from (1.5). Hence $HH_n(\mathcal{F}_Y)$ with target $\Omega^n(Y)$ can be realized as $\Omega \circ \mathrm{gtr} \circ C_*(\mathcal{F}_Y)$. Combine that with the first claim. □

We will often use a generalization of Lemma 1.6 to virtual representations:

Lemma 1.7. *Let* $\mathfrak{F}_i$ *be a finite collection of algebraic families of* A*-representations over* Y. *For any* $\lambda_i \in \mathbb{C}$ *there is a well-defined map*

$$\sum\nolimits_i \lambda_i HH_n(\mathcal{F}_i) : HH_n(A) \to \Omega^n(Y).$$

If $\mathfrak{F}'_j, \lambda_j$ *are data of the same kind and*

$$\sum\nolimits_i \lambda_i \mathfrak{F}_{i,y} = \sum\nolimits_j \lambda'_j \mathfrak{F}'_{j,y} \quad \text{in } \mathbb{C} \otimes_{\mathbb{Z}} R(A), \text{ for all } y \in Y,$$

then $\sum_i \lambda_i HH_n(\mathcal{F}_i) = \sum_j \lambda_j HH_n(\mathcal{F}'_j)$.

Proof. All the λ_i and the λ'_j live in one finitely generated subgroup of $\mathbb{C}$, which we can express as $\bigoplus_{b \in B} \mathbb{Z} b$. Accordingly we write

$$\lambda_i = \sum\nolimits_{b \in B} \lambda_{i,b} b, \ \lambda'_j = \sum\nolimits_{b \in B} \lambda'_{j,b} b \text{ with } \lambda_{i,b}, \lambda'_{j,b} \in \mathbb{Z}.$$

Now we have to show that

$$\sum\nolimits_i \lambda_{i,b} HH_n(\mathcal{F}_i) = \sum\nolimits_j \lambda_{j,b} HH_n(\mathcal{F}'_j) \tag{1.27}$$

for every $b \in B$. We note that, by the $\mathbb{Z}$-linear independence of B:

$$\sum\nolimits_i \lambda_{i,b} \mathfrak{F}_{i,y} = \sum\nolimits_j \lambda'_{j,b} \mathfrak{F}'_{j,y} \in R(A). \tag{1.28}$$

Bringing some summands to the other side in (1.28), we can arrange that all the $\lambda_{i,b}$ and all the $\lambda'_{j,b}$ lie in $\mathbb{Z}_{\geq 0}$. Then (1.27) can be rewritten as

$$HH_n\big(\bigoplus\nolimits_i \mathfrak{F}_i^{\oplus \lambda_{i,b}} \big) = HH_n\big(\bigoplus\nolimits_j \mathcal{F'}_j^{\oplus \lambda_{j,b}} \big).$$

This equality is an instance of Lemma 1.6.b. □

One can interpret Lemma 1.7 as: every algebraic family over Y in $\mathbb{C} \otimes_{\mathbb{Z}} R(A)$ gives rise to a well-defined map on Hochschild homology.

We would like to realize the isomorphism from Theorem 1.2 with families of representations. To define the desired families of representations, we specialize to a setup similar to root data. From now on V will be finite dimensional complex vector space on which G acts linearly. We assume that we are given a family of "parabolic" subgroups G_P of G, indexed by the subsets of some finite set Δ, such that

- $G_\emptyset = \{1\}$ and $G_\Delta = G$,
- for every $P \subset \Delta$ there are G_P-stable linear subspaces V_P and $V^P \subset V^{G_P}$, such that $V = V_P \oplus V^P$,
- for $P \supset Q$: $G_P \supset G_Q, V_P \supset V_Q$ and $V^P \subset V^Q$,
- for any $P \subset \Delta$,

$$\mathbb{Q} \otimes_{\mathbb{Z}} \sum\nolimits_{Q \subsetneq P} \mathrm{ind}_{\mathcal{O}(V_P) \rtimes \mathbb{C}[G_Q,\natural]}^{\mathcal{O}(V_P) \rtimes \mathbb{C}[G_P,\natural]} R(\mathcal{O}(V_P) \rtimes \mathbb{C}[G_Q,\natural]) \tag{1.29}$$

has finite codimension in $\mathbb{Q} \otimes_{\mathbb{Z}} R(\mathcal{O}(V_P) \rtimes \mathbb{C}[G_P,\natural])$.

The second bullet entails that $\mathcal{O}(V) \cong \mathcal{O}(V^P) \otimes \mathcal{O}(V_P)$. For a representation δ of $\mathcal{O}(V_P) \rtimes \mathbb{C}[G_P,\natural]$ and $v \in V^P$, we define a representation $\mathbb{C}_v \otimes \delta$ of

$$\mathcal{O}(V^P) \otimes \mathcal{O}(V_P) \rtimes \mathbb{C}[G_P,\natural] = \mathcal{O}(V) \rtimes \mathbb{C}[G_P,\natural] \quad \text{by}$$

$$f_1 \otimes f_2 \otimes T_g \mapsto f_1(v)\delta(f_2 \otimes T_g) \qquad f_1 \in \mathcal{O}(V^P), f_2 \in \mathcal{O}(V_P), g \in G_P.$$

Definition 1.8. We call a finite dimensional representation δ of $\mathcal{O}(V_P) \rtimes \mathbb{C}[G_P,\natural]$ elliptic if it admits a $\mathcal{O}(V_P)^{G_P}$-character and does not belong to (1.29). For such (P,δ), the family of representations

$$\pi(P,\delta,v) := \mathrm{ind}_{\mathcal{O}(V) \rtimes \mathbb{C}[G_P,\natural]}^{\mathcal{O}(V) \rtimes \mathbb{C}[G,\natural]} (\mathbb{C}_v \otimes \delta) \qquad v \in V^P$$

is called the algebraic family $\mathfrak{F}(P,\delta)$. The dimension of $\mathcal{F}_{P,\delta}$ is the dimension of V^P.

The ellipticity condition implies that an algebraic family of this kind can not be extended to a larger parameter space $V^{P'}$. The construction of irreducible $\mathcal{O}(V_P) \rtimes \mathbb{C}[G_P,\natural]$-representations in (1.21) shows that the $\mathcal{O}(V_P)^{G_P}$-character of an elliptic δ is just $0 \in V_P/G_P$. By the assumption on our parabolic subalgebras, there exist only finitely many such algebraic families with δ irreducible. We record the central characters

$$cc(\mathbb{C}_v \otimes \delta) = G_P v = v, \qquad cc(\pi(P,\delta,v)) = Gv. \tag{1.30}$$

In the remainder of this paragraph we abbreviate

$$A = \mathcal{O}(V) \rtimes \mathbb{C}[G,\natural].$$

Lemma 1.9. *Let $\{\mathfrak{F}(P_i,\delta_i)\}_{i=1}^{n_{\mathfrak{F}}}$ be a set of algebraic families of A-representations, such that*

$$\{\pi(P_i,\delta_i,v_i) : v_i \in V^{P_i}, i = 1,\ldots,n_{\mathfrak{F}}\}$$

spans $\mathbb{C} \otimes_{\mathbb{Z}} R(A)$. Let $R^d(A) \subset R(A)$ be the $\mathbb{Z}$-span of the members of the families $\mathfrak{F}(P_i,\delta_i)$ of dimension $\geq d$. Then

$$\mathbb{C} \otimes_{\mathbb{Z}} R^d(A) \subset HH_0(A)^*$$

has a $\mathbb{C}$-basis

$$\{\nu_{g,v} : g \in \langle G \rangle, \dim V^g \geq d, v \in V^g/Z_G(g), \natural^g|_{G_v \cap Z_G(g)} = 1\}.$$

Proof. By (1.22) every $\nu_{g,v}$ belongs to $\mathbb{C} \otimes_{\mathbb{Z}} R^0(A)$. Consider a family $\mathfrak{F}(P_i,\delta_i)$ and an element

$$\omega \in (\mathcal{O}(V^g) \otimes \natural^g)^{Z_G(g)} \subset HH_0(A),$$

such that $\dim V^{P_i} > \dim V^g$. Then

$$v_i \mapsto \mathrm{tr}(\omega, \pi(P_i,\delta_i,v_i)) \tag{1.31}$$

is a regular function on V^{P_i}, like for any element of $HH_0(A)$. But ω is not defined outside V^g, so (1.31) can only be zero there. Since $V^{P_i} \setminus V^g$ is dense in V^{P_i}, (1.31) is zero everywhere.

Consequently every $\nu_{g,v}$ lies in $\mathbb{C} \otimes_{\mathbb{Z}} R^{\dim V^g}(A)$, and the trace functions from $R^d(A)$ are determined by their restrictions to

$$\bigoplus\nolimits_{g \in \langle G \rangle : \dim V^g \geq d} (\mathcal{O}(V^g) \otimes \natural^g)^{Z_G(g)}.$$

From this we see that $\mathbb{C} \otimes_{\mathbb{Z}} R^d(A)$ is exactly the $\mathbb{C}$-span of the $\nu_{g,v}$ with $\dim V^g \geq d$. By Lemma 1.4 these $\nu_{g,v}$ are linearly independent. $\Box$

Next we describe an algorithm to choose, from data as in Lemma 1.9, a minimal set of algebraic families of A-representations. We start with the family $\mathfrak{F}(\emptyset, \mathrm{triv})$ and proceed recursively. Suppose that for every dimension $D > d$ we have chosen a set of D-dimensional algebraic families $\mathfrak{F}(P_i, \delta_i)$, where i runs through some index set I_D, with the following property: for generic $v_i \in V^{P_i}$ the set

$$\big\{ \pi(P_j, \delta_j, v_j) : j \in I_D, D > d, cc(\pi(P_j, \delta_j, v_j)) = cc(\pi(P_i, \delta_i, v_i)) \big\}$$

is linearly independent in $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$[5]. Here cc denotes the $\mathcal{O}(V)^G$-character of an A-representation (if it exists). Moreover we regard $\mathfrak{F}(P_j, \delta_j)$ here as a family in $R(A)$, which means that $\pi(P_j, \delta_j, v_j)$ and $\pi(P_j, \delta_j, v_j')$ are considered as the same element if they have the same trace. This point of view is necessary for the linear independence criterion.

Next we consider the set of d-dimensional algebraic families $\mathfrak{F}(P_i', \delta_i')$. Suppose that for generic $v_i' \in V^{P_i'}$, the representation $\pi(P_i', \delta_i', v_i')$ is $\mathbb{Q}$-linearly independent from

$$\big\{ \pi(P_j, \delta_j, v_j) : j \in I_D, D > d, cc(\pi(P_j, \delta_j, v_j)) = cc(\pi(P_i', \delta_i', v_i')) \big\},$$

were we still regard $\mathfrak{F}(P_j, \delta_j)$ as a family in $R(A)$. Then we add $\mathfrak{F}(P_i', \delta_i')$ to our collection of algebraic families.

Consider the remaining d-dimensional algebraic families. For $\mathfrak{F}(P_j', \delta_j')$ we look at the same condition as for $\mathfrak{F}(P_i', \delta_i')$, but now with respect to the index set $\cup_{D>d} I_D \cup \{i'\}$ instead of $\cup_{D>d} I_D$. If that condition is fulfilled, we add $\mathfrak{F}(P_j', \delta_j')$ to our set of algebraic families. We continue this process until none of the remaining d-dimensional algebraic families is (over generic points of that family) $\mathbb{Q}$-linearly independent from the algebraic families that we chose already. At that point our set of d-dimensional algebraic families is complete, and we move on to families of dimension $d-1$.

In the end, this algorithm yields a collection

$$\{ \mathfrak{F}(P_i, \delta_i) : i \in I_d, 0 \leq d \leq \dim V \} \tag{1.32}$$

such that:

- the representations $\{\pi(P_i, \delta_i, v_i), i \in \cup_d I_d, v_i \in V^{P_i}\}$ span $\mathbb{Q} \otimes_{\mathbb{Z}} R(A)$,
- if we remove any index from $\cup_d I_d$, the previous bullet becomes false,
- for generic $v_i \in V^{P_i}$, $\pi(P_i, \delta_i, v_i)$ does not belong to the span in $\mathbb{Q} \otimes_{\mathbb{Z}} R(A)$ of the other families $\mathfrak{F}(P_j, \delta_j)$.

Notice that each V^g is a vector space, and in particular an irreducible variety. That entails that the $g \in G$ underlie a dichotomy, based the behaviour of the group

$$Z_G(g, V^g) = \{ h \in Z_G(g) : hv = v \ \forall v \in V^g \} :$$

- Suppose that $\natural^g(h) \neq 1$ for some $h \in Z_G(g, V^g)$. Then
$$(\Omega^n(V^g) \otimes \natural^g)^{Z_G(g)} \subset \Omega^n(V^g) \otimes (\natural^g)^{Z_G(g,V^g)} = 0$$
and the summand of $HH_n(A)$ indexed by g is zero.
- Suppose that $Z_G(g, V^g) \subset \ker(\natural^g)$. Notice that $\dim(V^g)^k < \dim(V^g)$ for every $k \in Z_G(g) \setminus Z_G(g, V^g)$. Hence the set $\tilde{V^g}$ of $v \in V^g$ that are not fixed by any such k is Zariski open and dense in V^g. The action of $Z_G(g)$ on $\tilde{V^g}$ factors through a free action of $Z_G(g)/Z_G(g, V^g)$, so it is easy to attain $Z_G(g)$-invariance on $\tilde{V^g}$. Therefore the summand of $HH_n(A)$ indexed by g is nonzero.

To distinguish these cases, we say g is $HH(A)$-irrelevant or $HH(A)$-relevant. Recall from Lemma 1.9 that the set of trace functions
$$\{\nu_{g,v} : g \in \langle G\rangle, \dim V^g = d, v \in V^g/Z_G(g), G_v \cap Z_G(g) \subset \ker(\natural^g)\}$$
forms a $\mathbb{C}$-basis of $\mathbb{C} \otimes_{\mathbb{Z}} (R^d(A)/R^{d+1}(A))$. For a fixed $HH(A)$-relevant g, this gives an algebraic family of trace functions on A, supported on the sum of the linear subspaces $T_{g'}\mathcal{O}(V)$ with g' conjugate to g. Every member of this family factors through A/I_v^G for the appropriate maximal ideal I_v^G of $\mathcal{O}(V)^G$, so corresponds to a unique virtual A-representation with $\mathcal{O}(V)^G$-character Gv.

Lemma 1.10. *For $i = 1, \ldots, n_{\mathfrak{F}}$ and $g \in \langle G\rangle$ there exist $\lambda_{g,i} \in \mathbb{C}$ and a map $\phi_{g,i} : V^g \to V^{P_i}$, given by some element of G, such that*
$$\nu_{g,v} = \sum\nolimits_{i:\dim(V^{P_i}) \geq \dim V^g} \lambda_{g,i} \mathrm{tr}\, \pi(P_i, \delta_i, \phi_{g,i}(v))$$
for all $v \in V^g$ with $G_v \cap Z_G(g) \subset \ker(\natural^g)$.

Proof. Fix $g \in \langle G\rangle$. To make the construction easier, we may omit some of the families $\mathfrak{F}(P_i, \delta_i)$, so that the remaining families minimally span the part of $\mathbb{C} \otimes_{\mathbb{Z}} R(A)$ with $\mathcal{O}(V)^G$-characters in GV^g/G. Only families with $\dim(V^{P_i}) \geq \dim V^g$ can remain. This can be compared with the construction of the families of representations in (1.32).

Let us restrict to the generic part $\tilde{V^g}$ of V^g. There exist $\lambda_{g,i}(v) \in \mathbb{C}$ and $\phi_{g,i}(v)$ such that the equality in the lemma holds for $v \in \tilde{V^g}$. Since $\nu_{g,v}$ admits the $\mathcal{O}(V)^G$-character Gv and the remaining families are minimal in the above sense, every $\phi_{g,i}(v)$ is unique up to applying some element of G that stabilizes $\mathfrak{F}(P_i, \delta_i)$. We fix a generic $\tilde{v} \in \tilde{V^g}$ and we pick maps $\phi_{g,i}$ such that the required property holds for $\nu_{g,\tilde{v}}$. By the uniqueness up to G and the genericity, $\phi_{g,i}$ extends uniquely to continuous map $\tilde{V^g} \to V^{P_i}$.

Every $\phi_{g,i}$ preserves the $\mathcal{O}(V)^G$-characters, so is given by some element of G. In particular it is an injective regular map $V^g \to V^{P_i}$. As a representation of $\mathbb{C}[G, \natural]$, $\pi(P_i, \delta_i, \phi_{g,j}(v))$ does not depend on v. The numbers $\lambda_{g,i}(v)$ are determined by the earlier choices, so they do not depend on $v \in \tilde{V^g}$ either.

Now the definition of the $\phi_{g,i}$ and the $\lambda_{g,i}$ applies to all $v \in V^g$. For all $i \in \{1, \ldots, n_{\mathfrak{F}}\}$ that do not appear in this construction, we set $\lambda_{g,i} = 0$. All items in the statement of the lemma depend algebraically on v, so the validity of the lemma extends from $\tilde{V^g}$ to all $v \in V^g$ for which $\nu_{g,v}$ is defined. $\square$

1.3. **Hochschild homology via families of representations.**

Let $\mathfrak{F}(P,\delta)$ be any algebraic family of representations in the sense of Definition 1.8, and let $W_{P,\delta}$ be the vector space underlying $\pi(P,\delta,v)$ for any $v \in V^P$. Notice that $\mathcal{O}(V^P) \otimes \mathrm{End}_{\mathbb{C}}(W_{P,\delta})$ is an algebra over $\mathcal{O}(V^P) \otimes \mathcal{O}(V_P)^{G_P}$, for the action

$$(f_1 \otimes f_2)(f_3 \otimes T) = f_1 f_3 \otimes f_2(0) T \quad f_1, f_3 \in \mathcal{O}(V^P), f_2 \in \mathcal{O}(V_P)^{G_P}, T \in \mathrm{End}_{\mathbb{C}}(W_{P,\delta}).$$

As $\mathcal{O}(V)^G \subset \mathcal{O}(V^P) \otimes \mathcal{O}(V_P)^{G_P}$, $\mathcal{O}(V^P) \otimes \mathrm{End}_{\mathbb{C}}(W_{P,\delta})$ is also an $\mathcal{O}(V)^G$-algebra. In view of (1.30), $\mathfrak{F}(P,\delta)$ gives rise to a homomorphism of $\mathcal{O}(V)^G$-algebras

$$\begin{array}{cccc} \mathcal{F}_{P,\delta} : & A & \to & \mathcal{O}(V^P) \otimes \mathrm{End}_{\mathbb{C}}(W_{P,\delta}) \\ & f & \mapsto & [v \mapsto \pi(P,\delta,v)(f)] \end{array} .$$

Consider a finite set of algebraic families $\mathfrak{F}(P_i,\delta_i)$ $(i = 1, \ldots, n_{\mathfrak{F}})$ whose members span $\mathbb{Q} \otimes_{\mathbb{Z}} R(A)$, like in Lemma 1.9. All the $\mathcal{F}_{P_i,\delta_i}$ together induce a homomorphism of $\mathcal{O}(V)^G$-modules

$$HH_n(\mathcal{F}_A) = HH_n\big(\bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \mathcal{F}_{P_i,\delta_i}\big) : HH_n(A) \to \bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \Omega^n(V^{P_i}).$$

We want to show that this map is injective and to describe its image. To that end, we set things up so that we can write down an inverse map. For each (g,i) as in Lemma 1.10, $\phi_{g,i}$ yields an algebra homomorphism

$$\begin{array}{cccc} \phi_{g,i}^* : & \mathcal{O}(V^{P_i}) \otimes \mathrm{End}_{\mathbb{C}}(W_{P_i,\delta_i}) & \to & \mathcal{O}(V^g) \otimes \mathrm{End}_{\mathbb{C}}(W_{P_i,\delta_i}) \\ & f \otimes A & \mapsto & f \circ \phi_{g,i} \otimes A \end{array} \tag{1.33}$$

and an induced map on Hochschild homology

$$HH_n(\phi_{g,i}^*) : \Omega^n(V^{P_i}) \to \Omega^n(V^g).$$

Summing over all g, i we obtain a homomorphism of $\mathcal{O}(V)^G$-modules

$$HH_n(\phi^*) = \bigoplus_{g \in \langle G \rangle} \sum_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} HH_n(\phi_{g,i}^*) : \bigoplus_{i=1}^{n_{\mathfrak{F}}} \Omega^n(V^{P_i}) \to \bigoplus_{g \in \langle G \rangle} \Omega^n(V^g), \tag{1.34}$$

where $\lambda_{g,i} = 0$ if g is $HH(A)$-irrelevant or $\dim(V^g) > \dim(V^{P_i})$.

Proposition 1.11. *Fix a $HH(A)$-relevant $g \in \langle G \rangle$. The map*

$$\sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} HH_n(\phi_{g,i}^*) \circ HH_n(\mathcal{F}_A) \ : \ HH_n(A) \longrightarrow \Omega^n(V^g)$$

(*a*) *annihilates the summands $(\Omega^n(V^{g'}) \otimes \natural^{g'})^{Z_G(g')}$ of $HH_n(A)$ with $g' \in \langle G \rangle \setminus \{g\}$,*
(*b*) *equals the identification $\Omega^n(V^g) \otimes_{\mathbb{C}} \mathbb{C} \xrightarrow{\sim} \Omega^n(V^g)$ on the summand $(\Omega^n(V^g) \otimes \natural^g)^{Z_G(g)}$ of $HH_n(A)$.*

Proof. We can express the map as

$$\sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} HH_n(\phi_{g,i}^* \circ \mathcal{F}_{P_i,\delta_i}),$$

where $\phi_{g,i}^* \circ \mathcal{F}_{P_i,\delta_i}$ is an algebraic family of representations. By Lemma 1.10 the members of these families satisfy

$$\nu_{g,v} = \sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} \mathrm{tr}\, \pi(P_i, \delta_i, \phi(g,i)(v)) \tag{1.35}$$

whenever $\nu_{g,v}$ is defined. For $g' \in \langle G \rangle$ we consider the commutative algebra

$$A_{g'} := \mathbb{C}[\{T_{g^n} : n \in \mathbb{Z}\}] \otimes \mathcal{O}(V/(g'-1)V)$$

As each $\pi(P_i, \delta_i, v_i)$ is obtained by induction from an $\mathcal{O}(V_{P_i}) \rtimes \mathbb{C}[G_{P_i}, \natural]$-representation on which $\mathcal{O}(V_{P_i})$ acts via evaluation at 0, it is a semisimple A-representation and $\mathcal{O}(V)$ acts on $T_w V_{\delta_i}$ via evaluation at $w^{-1} v_i$. It follows that $\phi_{g,i}^* \circ \mathcal{F}_{P_i,\delta_i}$ consists of semisimple representations and can be decomposed as a direct sum of families of $\mathcal{O}(V/(g'-1)V)$-representations of the form $f \mapsto f \circ w^{-1}$ for some $w \in G$. Not all $w \in G$ appear here and some w' may give the same family. We record this as an "isotypic" decomposition of $\mathcal{O}(V/(g'-1)V)$-representations:

$$\phi_{g,i}^* \circ \mathcal{F}_{P_i,\delta_i} = \bigoplus\nolimits_{w/\sim} (\phi_{g,i}^* \circ \mathcal{F}_{P_i,\delta_i})_w \tag{1.36}$$

As $T_{g'}$ commutes with $\mathcal{O}(V/(g'-1)V)$, it stabilizes this decomposition.
(a) For $g' \neq g$ and $f \in \mathcal{O}(V/(g'-1)V)$ we have $\nu_{g,v}(T_{g'} f) = 0$. In terms of (1.36) that becomes

$$\sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} \sum\nolimits_{w/\sim} \mathrm{tr}\big(f(w^{-1}v) T_{g'}, (\pi(P_i, \delta_i, \phi_{g,i}(v))_w\big) = 0. \tag{1.37}$$

The subalgebra $\mathbb{C}[\{T_{g^n} : n \in \mathbb{Z}\}]$ has finite dimension and is semisimple, so the restrictions of the above representations to this subalgebra do not depend on $v \in V^{g'}$. For $v \in V^g$ in generic position, we can separate the various $w/\sim$ in (1.37), which leads to

$$\sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} \mathrm{tr}\big(T_{g'}, (\pi(P_i, \delta_i, \phi_{g,i}(v))_w\big) = 0 \tag{1.38}$$

for all $w/\sim$. By continuity that extends from generic v to all $v \in V^{g'}$. From (1.38) we see that

$$\sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} \mathrm{gtr} \circ C_*(\phi_{g,i}^* \circ \mathcal{F}_{P_i,\delta_i}) \quad \text{annihilates} \quad T_{g'} C_*\big(\mathcal{O}(V/(g'-1))\big).$$

Combine that with (1.15).
(b) For $g' = g$, (1.37) becomes

$$\sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} \sum\nolimits_{w/\sim} \mathrm{tr}\big(f(w^{-1}v) T_g, (\pi(P_i, \delta_i, \phi_{g,i}(v))_w\big) = \nu_{g,v}(T_g f). \tag{1.39}$$

From (1.15) and the $HH(A)$-relevance of g we see that

$$\nu_{g,v}(T_g f) = [Z_G(g) : Z_G(g, V^g)]^{-1} \sum\nolimits_{h \in Z_G(g)/Z_G(g,V^g)} \natural^g(h) f(h^{-1}v). \tag{1.40}$$

Comparing (1.39) and (1.40), we deduce that every $w/\sim$ can be rewritten as a unique $h \in Z_G(g)/Z_G(g, V^g)$. Then (1.39) becomes

$$\sum_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} \sum_{h \in Z_G(g)/Z_G(g,V^g)} \mathrm{tr}\big(f(h^{-1}v) T_g, (\pi(P_i, \delta_i, \phi_{g,i}(v))_h\big) = \nu_{g,v}(T_g f).$$

Like in part (a) we can separate the various h, leading to

$$\sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i}\big(T_g, (\pi(P_i, \delta_i, \phi_{g,i}(v))_h\big) = [Z_G(g) : Z_G(g, V^g)]^{-1} \natural^g(h).$$

Initially this holds only for generic v, but by continuity it extends to all $v \in V^g$. We deduce

$$\sum_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} \mathrm{gtr} \circ C_n(\phi_{g,i}^* \circ \mathcal{F}_{P_i,\delta_i})(T_g \omega) = [Z_G(g) : Z_G(g, V^g)]^{-1} \sum_{h \in Z_G(g)/Z_G(g,V^g)} \natural^g(h) h \cdot \omega$$

for all $\omega \in C_n(\mathcal{O}(V/(g-1)V))$. In view of (1.15), this says exactly that the map of the lemma is induced by the identity on $T_g C_*(\mathcal{O}(V/(g-1)V))$. In terms of differential forms, that becomes the identification $\Omega^n(V^g) \otimes \natural^g \xrightarrow{\sim} \Omega^n(V^g)$. $\square$

From (1.32) and Lemmas 1.6 and 1.7 we see that

$$HH_n(\phi^*) \circ HH_n(\mathcal{F}_A) \;:\; HH_n(A) \to \bigoplus\nolimits_{g \in \langle G \rangle} \Omega^n(V^g) \tag{1.41}$$

can be considered as evaluation at the families of virtual A-representations $\nu_{g,v}$ (extended naturally to all $v \in V^g$).

From now on we assume that our collection of algebraic families $\mathfrak{F}(P_i, \delta_i)$ has been chosen in a minimal way, as in (1.32).

Lemma 1.12. *Under the above assumption, the map*

$$HH_n(\phi^*) \;:\; \bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \Omega^n(V^{P_i}) \to \bigoplus\nolimits_{g \in \langle G \rangle} \Omega^n(V^g)$$

is injective.

Proof. Consider a nonzero $\sum_{i=1}^{n_{\mathfrak{F}}} \omega_i \in \bigoplus_{i=1}^{n_{\mathfrak{F}}} \Omega^n(V^{P_i})$. We select an index j and a small open set U (for the analytic topology) in V^{P_j} such that $\omega_j(u) \neq 0$ for all $u \in U$. Since the set of generic points in V^{P_j} (i.e. the points whose G-stabilizer is minimal on V^{P_j}) is open and dense, we may assume that $\{\pi(P_j, \delta_j, u) : u \in U\}$ does not share any $\mathcal{O}(V)^G$-characters with any family $\mathfrak{F}(P_i, \delta_i)$ of lower dimension, and that $wU \cap U$ is empty unless $w \in Z_G(g, V^g)$.

The construction of a minimal set of algebraic families of A-representations entails that none of the representations $\{\pi(P_j, \delta_j, u) : u \in U\}$ belongs to the span in $\mathbb{Q} \otimes_{\mathbb{Z}} R(A)$ of the other families $\mathfrak{F}(P_i, \delta_i)$. The same holds for any linear combination of these representations, because U does not contain two points from any $Z_G(g)$-orbit.

Now Lemma 1.4 shows there must exist a $g \in \langle G \rangle$ with $\lambda_{g,j} \neq 0$. The component of $HH_n(\phi^*)$ indexed by g is $\sum_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} HH_n(\phi^*_{g,i})$, so

$$(HH_n(\phi^*)\omega)\big|_{\phi_{g,j}(U)} = \sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} HH_n(\phi^*_{g,i})\omega\big|_{\phi_{g,i}^{-1}\phi_{g,j}U}.$$

This is nonzero by the above linear independence property of the set $\{\pi(P_j, \delta_j, u) : u \in U\}$. $\square$

With Proposition 1.11 and Lemma 1.12 we can provide a description of $HH_n(A)$ in the style of [BDK]. Recall that V is a finite dimensional complex G-representation, that $A = \mathcal{O}(V) \rtimes \mathbb{C}[G, \natural])$ and that $\mathfrak{F}(P_i, \delta_i)$ $(i = 1, \ldots, n_{\mathfrak{F}})$ are algebraic families of A-representations whose members span $\mathbb{C} \otimes_{\mathbb{Z}} R(A)$ in a minimal way.

Theorem 1.13. (a) *The homomorphism of $\mathcal{O}(V)^G$-modules*

$$HH_n(\mathcal{F}_A) = \bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} HH_n(\mathcal{F}_{P_i,\delta_i}) \;:\; HH_n(A) \to \bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \Omega^n(V^{P_i})$$

is injective. The homomorphism of $\mathcal{O}(V)^G$-modules

$$HH_n(\phi^*) \;:\; HH_n(\mathcal{F}_A)\big(HH_n(A)\big) \to \bigoplus\nolimits_{g \in \langle G \rangle} \big(\Omega^n(V^g) \otimes \natural^g\big)^{Z_G(g)}$$

is bijective.

(b) *In degree $n = 0$, the condition on $\omega = \sum_{i=1}^{n_{\mathfrak{F}}} \omega_i \in \bigoplus_{i=1}^{n_{\mathfrak{F}}} \mathcal{O}(V^{P_i})$ that describes the image of $HH_0(\mathcal{F}_A)$ is: whenever $\lambda_j \in \mathbb{C}, i_j \in \{1, \ldots, n_{\mathfrak{F}}\}, v_{i_j} \in V^{P_{i_j}}$ and $\sum_j \lambda_j \pi(P_{i_j}, \delta_{i_j}, v_{i_j}) = 0$ in $\mathbb{C} \otimes_{\mathbb{Z}} R(A)$, also $\sum_j \lambda_j \omega_{i_j}(v_{i_j}) = 0$. This determines an isomorphism of $Z(A)$-modules*

$$HH_0(A) \cong (\mathbb{C} \otimes_{\mathbb{Z}} R(A))^*_{\mathrm{reg}}.$$

Proof. (a) Proposition 1.11 entails that $HH_n(\phi^*) \circ HH_n(\mathcal{F}_A)$ is the identity on

$$\bigoplus\nolimits_{g \in \langle G \rangle} (\Omega^n(V^g) \otimes \natural^g)^{Z_G(g)},$$

a vector space which by Theorem 1.2 is isomorphic with $HH_n(A)$. Hence $HH_n(\mathcal{F}_A)$ is injective, and from Lemma 1.12 we obtain the desired bijectivity of $HH_n(\phi^*)$.
(b) The image of $HH_0(\phi^*) \circ HH_0(\mathcal{F}_A)$ is

$$\bigoplus\nolimits_{g \in \langle G \rangle} (\mathcal{O}(V^g) \otimes \natural^g)^{Z_G(g)}. \tag{1.42}$$

The map associated to $\omega \in \bigoplus_{i=1}^{n_{\mathfrak{F}}} \mathcal{O}(V^{P_i})$ in the statement sends

$$\nu_{g,v} \quad \text{to} \quad \sum\nolimits_{i=1}^{n_{\mathcal{F}}} \omega_i(\phi_{g,i}(v)) = (HH_0(\phi^*)\omega)(g, v). \tag{1.43}$$

The $\nu_{g,v}$ satisfy the relations (1.20), so ω must respect these in order to descend to a function on $\mathbb{C} \otimes_{\mathbb{Z}} R(A)$. In view of (1.43), that means that $HH_0(\phi^*)\omega$ must be $Z_G(g)$-invariant. By construction the image of $HH_0(\phi^*)$ consists of regular functions, so ω must belong to (1.42).

On the other hand, from Lemma 1.1 we know that the maximal ideal spectrum of (1.42) is in bijection with the set of all $\nu_{g,v}$, modulo the relations (1.20). By (1.22) the resulting quotient set forms a basis of $\mathbb{C} \otimes_{\mathbb{Z}} R(A)$. Hence (1.42) can be considered as a subset of the linear dual space $(\mathbb{C} \otimes_{\mathbb{Z}} R(A))^*$ and $HH_0(\phi^*)^{-1}$ of (1.42) is the set of all elements that satisfy the conditions stated in the theorem.

For $a \in A$ and $\omega = HH_0(\mathcal{F}_A)(a)$, the definition of the generalized trace map gtr shows that

$$\omega(\pi(P_i, \delta_i, v_i)) = \mathrm{tr}\,(\pi(P_i, \delta_i, v_i)a). \tag{1.44}$$

Hence the map

$$HH_0(A) \to HH_0(\mathcal{F}_A)\big(HH_0(A)\big) \to (\mathbb{C} \otimes_{\mathbb{Z}} R(A))^*$$

constructed above is just the $Z(A)$-linear map (1.25). It is injective because $HH_0(\mathcal{F}_A)$ is injective and the values (1.44) can be recovered from the image of ω in $(\mathbb{C} \otimes_{\mathbb{Z}} R(A))^*$. We know from (1.26) that the image of (1.25) is contained in $(\mathbb{C} \otimes_{\mathbb{Z}} R(A))^*_{\mathrm{reg}}$. Conversely every element $f \in (\mathbb{C} \otimes_{\mathbb{Z}} R(A))^*_{\mathrm{reg}}$ yields a regular function on V^{P_i} via pairing with $\mathfrak{F}_{P_i,\delta_i}$, so f comes from an element of $\bigoplus_{i=1}^{n_{\mathfrak{F}}} \Omega^n(V^{P_i})$. □

With the equality $\nu_{hgh^{-1},hv} = \natural^g(h)\nu_{g,v}$ from (1.20) we can extend Lemma 1.10 from $g \in \langle G \rangle$ to all $g \in G$. Namely, for $g \in \langle G \rangle$ and $h \in G$ we define

$$\lambda_{hgh^{-1},i} = \natural^g(h)\lambda_{g,i} \quad \text{and} \quad \phi_{hgh^{-1},i} = \phi_{g,i} \circ h^{-1}. \tag{1.45}$$

That yields a variation on (1.34):

$$HH_n(\tilde{\phi}^*) := \bigoplus_{g \in G} \sum_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} HH_n(\phi^*_{g,i}) \ : \ \bigoplus_{i=1}^{n_{\mathfrak{F}}} \Omega^n(V^{P_i}) \to \bigoplus_{g \in G} \Omega^n(V^g), \tag{1.46}$$

We note that this map hardly differs from $HH_n(\phi^*)$, because it is entirely determined by the components indexed by $g \in \langle G \rangle$ via the actions from (1.16). From Theorem 1.13 and (1.17) we conclude:

Corollary 1.14. *Let V be a finite dimensional complex G-representation. There exists an $\mathcal{O}(V)^G$-linear bijection*

$$HH_n(\tilde{\phi}^*) \circ HH_n(\mathcal{F}_A) \;:\; HH_n(A) \to \Big(\bigoplus\nolimits_{g \in G} \Omega^n(V^g) \otimes \natural^g \Big)^G .$$

This realizes the isomorphism Theorem 1.2 in terms of algebra homomorphisms.

Example 1.15. We continue Example 1.5, so $G = Q_8 / Z(Q_8)$ acts on $V = \mathbb{C}^2$ by reflections. We note that $V^{\pm \mathbf{k}} = \{(0,0)\}, Z_G(\pm \mathbf{k}, V^{\pm \mathbf{k}}) = G$ and $\natural^{\pm \mathbf{k}}$ is nontrivial. Hence $\pm \mathbf{k}$ is $HH(A)$-irrelevant, where $A = \mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]$.

As parabolic subgroups we take

$$G_\emptyset = \{\pm \mathbf{1}\},\ G_{\mathbf{i}} = \{\pm \mathbf{1}, \pm \mathbf{i}\},\ G_{\mathbf{j}} = \{\pm \mathbf{1}, \pm \mathbf{j}\},\ G_{\mathbf{i},\mathbf{j}} = G.$$

In each case $V^P = V^{G_P}$ and V_P is the orthogonal complement to V^P. To span $\mathbb{Q} \otimes_{\mathbb{Z}} R(A)$ we need three algebraic families of representations, for instance:

- $\mathfrak{F}(\emptyset, \mathrm{triv}) = \{ \mathrm{ind}_{\mathcal{O}(V)}^A \mathbb{C}_v : v \in V \}$.
- Define $\delta_{\mathbf{i}} \in \mathrm{Irr}(\mathcal{O}(V_{\mathbf{i}}) \rtimes \mathbb{C}[G_{\mathbf{i}}, \natural])$ by $\delta_{\mathbf{i}}(T_{\pm \mathbf{i}}) = i$ and $\delta_i(f) = f(0)$ for $f \in \mathcal{O}(V_{\mathbf{i}})$. Take
$$\mathfrak{F}(\mathbf{i}, \delta_{\mathbf{i}}) = \big\{ \mathrm{ind}_{\mathcal{O}(V) \rtimes \mathbb{C}[G_{\mathbf{i}}, \natural]}^A (\mathbb{C}_v \otimes \delta_{\mathbf{i}}) : v \in V^{\mathbf{i}} = \{0\} \times \mathbb{C} \big\}$$
- Define $\delta_{\mathbf{j}} \in \mathrm{Irr}(\mathcal{O}(V_{\mathbf{j}}) \rtimes \mathbb{C}[G_{\mathbf{j}}, \natural])$ by $\delta_{\mathbf{j}}(T_{\pm \mathbf{i}}) = -i$ and $\delta_j(f) = f(0)$ for $f \in \mathcal{O}(V_{\mathbf{j}})$. Take
$$\mathfrak{F}(\mathbf{j}, \delta_{\mathbf{j}}) = \big\{ \mathrm{ind}_{\mathcal{O}(V) \rtimes \mathbb{C}[G_{\mathbf{j}}, \natural]}^A (\mathbb{C}_v \otimes \delta_{\mathbf{j}}) : v \in V^{\mathbf{j}} = \{0\} \times \mathbb{C} \big\}$$

Composing $\mathbb{C}_v \otimes \delta_{\mathbf{i}}$ with conjugation by $T_{\pm \mathbf{j}}$ gives

$$\mathbb{C}_{-v} \otimes \delta_{-\mathbf{i}} : f \otimes T_{\pm \mathbf{i}} \mapsto -i f(-v).$$

Hence $\mathfrak{F}(\mathbf{i}, \delta_{\mathbf{i}})$ is not stable under elements of $G \setminus G_{\mathbf{i}}$, and similarly for $\mathfrak{F}(\mathbf{j}, \delta_{\mathbf{j}})$. The only relations in $\mathbb{Q} \otimes_{\mathbb{Z}} R(A)$ between the members of these families are:

$$\begin{aligned}
&\pi(\emptyset, \mathrm{triv}, v_{\mathbf{i}}) = \pi(\mathbf{i}, \delta_{\mathbf{i}}, v_{\mathbf{i}}) + \pi(\mathbf{i}, \delta_{-\mathbf{i}}, v_{\mathbf{i}}) = \pi(\mathbf{i}, \delta_{\mathbf{i}}, v_{\mathbf{i}}) + \pi(\mathbf{i}, \delta_{\mathbf{i}}, -v_{\mathbf{i}}) \qquad v_{\mathbf{i}} \in V^{\mathbf{i}}, \\
&\pi(\emptyset, \mathrm{triv}, v_{\mathbf{j}}) = \pi(\mathbf{j}, \delta_{\mathbf{j}}, v_{\mathbf{j}}) + \pi(\mathbf{j}, \delta_{-\mathbf{j}}, v_{\mathbf{j}}) = \pi(\mathbf{j}, \delta_{\mathbf{j}}, v_{\mathbf{j}}) + \pi(\mathbf{j}, \delta_{\mathbf{j}}, -v_{\mathbf{j}}) \qquad v_{\mathbf{j}} \in V^{\mathbf{j}}, \\
&\pi(\emptyset, \mathrm{triv}, 0) = 2\pi(\mathbf{i}, \delta_{\mathbf{i}}, 0) = 2\pi(\mathbf{j}, \delta_{\mathbf{j}}, 0).
\end{aligned}$$

Comparing traces of representations we find

$$\begin{aligned}
4\nu_{\pm \mathbf{1}, v} &= \pi(\emptyset, \mathrm{triv}, v) \qquad v \in V, \\
4\nu_{\pm \mathbf{i}, v_{\mathbf{i}}} &= -2i\pi(\mathbf{i}, \delta_{\mathbf{i}}, v_{\mathbf{i}}) + i\pi(\emptyset, \mathrm{triv}, v_{\mathbf{i}}) \qquad v_{\mathbf{i}} \in V^{\mathbf{i}}, \\
4\nu_{\pm \mathbf{j}, v_{\mathbf{j}}} &= -2i\pi(\mathbf{j}, \delta_{\mathbf{j}}, v_{\mathbf{j}}) + i\pi(\emptyset, \mathrm{triv}, v_{\mathbf{j}}) \qquad v_{\mathbf{j}} \in V^{\mathbf{j}}.
\end{aligned}$$

Notice that $\nu_{\pm \mathbf{i}, 0} = \nu_{\pm \mathbf{j}, 0} = 0 \neq \nu_{\pm \mathbf{1}, 0}$. In case $V^g \subset V^{\mathbf{j}}$, $\phi_{g, \mathbf{j}}$ equals

$$\mathrm{Res}_{V^g}^{V^{\mathbf{j}}} : \mathcal{O}(V^{\mathbf{j}}) \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}[G, \natural] \underset{\mathbb{C}[G_{\mathbf{j}}, \natural]}{\otimes} \mathbb{C}) \to \mathcal{O}(V^g) \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}[G, \natural] \underset{\mathbb{C}[G_{\mathbf{j}}, \natural]}{\otimes} \mathbb{C}).$$

The maps $\phi_{g,\emptyset}$ and $\phi_{g,\mathbf{i}}$ admit similar descriptions (for the latter provided that $V^g \subset V^{\mathbf{i}}$). Thus $HH_n(\phi^*)$ equals

$$\sum\nolimits_{g\in G} \frac{1}{4} HH_n(\phi_{g,\emptyset}) + \frac{i}{4} HH_n(\phi_{\pm\mathbf{1},\mathbf{i}}) - \frac{i}{2} HH_n(\phi_{\pm\mathbf{i},\mathbf{i}}) - \frac{i}{4} HH_n(\phi_{\pm\mathbf{1},\mathbf{j}}) + \frac{i}{2} HH_n(\phi_{\pm\mathbf{j},\mathbf{j}}).$$

Theorem 1.13 provides an injection

$$HH_n(\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]) \to \Omega^n(V) \oplus \Omega^n(V^{\mathbf{i}}) \oplus \Omega^n(V^{\mathbf{j}})$$

whose image is precisely

$$HH_n(\phi^*)^{-1} \Big(\Omega^n(V)^G \oplus (\Omega^n(V^{\mathbf{i}}) \otimes \natural^{\pm\mathbf{i}})^G \oplus (\Omega^n(V^{\mathbf{j}}) \otimes \natural^{\pm\mathbf{j}})^G \Big).$$

We note that in degree $n = 1$ the specialization of $HH_1(A)$ at $Gv = (0,0)$ is $0 \oplus \mathbb{C}dz_2 \oplus \mathbb{C}dz_1$. Remarkably, the dimension of this space is larger than the number of irreducible A-representations with $\mathcal{O}(V)^G$-character $(0,0)$.

Most results in Section 1 are also valid in a smooth setting. Let V be a smooth manifold on which G acts by diffeomorphisms, so that G also acts on $C^\infty(V)$. We compute the Hochschild homology of $C^\infty(V) \rtimes \mathbb{C}[G, \natural]$, with respect to the complete bornological tensor product or equivalently the complete projective tensor product. Recall that the smooth version of the HKR theorem was proven by Connes:

$$HH_n(C^\infty(V)) \cong \Omega^n_{sm}(V),$$

where Ω^n_{sm} stands for smooth differential forms of degree n. The results of Paragraph 1.1 hold in that setting, for Paragraphs 1.2–1.3 our results remain valid in a smooth setting with V a real vector space.

2. Twisted graded Hecke algebras

We will adapt the computations from Paragraph 1.1 to graded Hecke algebras extended with a twisted group algebra. Let $(X, \Phi, Y, \Phi^\vee, \Delta)$ be a based root datum with Weyl group $W = W(\Phi)$. We write

$$\mathfrak{t}_\mathbb{R} = \mathbb{R} \otimes_Z Y, \qquad \mathfrak{t} = \mathbb{C} \otimes_\mathbb{Z} Y, \qquad \mathfrak{t}^* = \mathbb{C} \otimes_\mathbb{Z} X.$$

Let Γ be a finite group acting on the root datum and let $\natural : \Gamma \times \Gamma \to \mathbb{C}^\times$ be a 2-cocycle. We regard it also as a 2-cocycle of the group $W\Gamma = W \rtimes \Gamma$. Let $k : \Delta \to \mathbb{C}$ be a $W\Gamma$-invariant parameter function, with values denoted k_α. The twisted graded Hecke algebra $\mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural)$ associated to these data is the vector space $\mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[W\Gamma, \natural]$ with multiplication defined by

- $\mathcal{O}(\mathfrak{t})$ and $\mathbb{C}[W\Gamma, \natural]$ are embedded as unital subalgebras,
- for $\alpha \in \Delta$ and $f \in \mathcal{O}(\mathfrak{t})$:
$$f T_{s_\alpha} - T_{s_\alpha} s_\alpha(f) = k_\alpha (f - s_\alpha(f)) \alpha^{-1},$$
- for $\gamma \in \Gamma$ and $f \in \mathcal{O}(\mathfrak{t})$:
$$T_\gamma f T_\gamma^{-1} = \gamma(f) = [\lambda \mapsto f(\gamma^{-1}\lambda)] \qquad \lambda \in \mathfrak{t}.$$

When $\natural$ is trivial, we omit it from the notation and we speak of a graded Hecke algebra (or an extended grade Hecke algebra if Γ is nontrivial). This relates to the notation in the introduction by

$$\mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural) = \mathbb{H}(\mathfrak{t}, W, k) \rtimes \mathbb{C}[\Gamma, \natural].$$

Notice that for $k = 0$ we recover the twisted crossed product $\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \natural]$. Multiplication with $\epsilon \in \mathbb{C}^\times$ defines a bijection $m_\epsilon : \mathfrak{t}^* \to \mathfrak{t}^*$, which extends to an algebra automorphism of $\mathcal{O}(\mathfrak{t})$. From the above multiplication rules we see that it extends even further, to an algebra isomorphism

$$m_\epsilon : \mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural) \to \mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural) \tag{2.1}$$

which is the identity on $\mathbb{C}[W\Gamma, \natural]$. For $\epsilon = 0$ the homomorphism m_0 is well-defined, but not bijective. Like in (1.1), let

$$1 \to Z \to \tilde{\Gamma} \to \Gamma \to 1$$

be a finite central extension such that $\natural$ becomes trivial in $H^2(\tilde{\Gamma}, \mathbb{C}^\times)$, and let $p_\natural \in \mathbb{C}[Z]$ be the associated minimal central idempotent. Then

$$\mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural) \cong p_\natural \mathbb{H}(\mathfrak{t}, W\tilde{\Gamma}, k),$$

a direct summand of the extended graded Hecke algebra

$$\mathbb{H}(\mathfrak{t}, W\tilde{\Gamma}, k) = \mathbb{H}(\mathfrak{t}, W, k) \rtimes \tilde{\Gamma}. \tag{2.2}$$

The Hochschild homology of (2.2) was computed in [Sol1, Theorem 3.4]. It is isomorphic to $HH_*(\mathcal{O}(\mathfrak{t}) \rtimes W\tilde{\Gamma})$, which we already know from (1.8). The arguments in [Sol1] make use of the subcomplexes

$$wC_*(\mathcal{O}(\mathfrak{t}/(w-1)\mathfrak{t})) \quad \text{of} \quad C_*(\mathbb{H}(\mathfrak{t}, W\tilde{\Gamma}, k)). \tag{2.3}$$

For each $w \in \langle W\tilde{\Gamma} \rangle$, this subcomplex contributes precisely $\Omega^n(\mathfrak{t}^w)^{Z_{W\tilde{\Gamma}}(w)}$ to $HH_n(\mathbb{H}(\mathfrak{t}, W\tilde{\Gamma}, k))$, and $HH_n(\mathbb{H}(\mathfrak{t}, W\tilde{\Gamma}, k))$ is the direct sum of these contributions. This works for every parameter function k, and in particular yields a canonical $\mathbb{C}$-linear bijection

$$HH_n(\mathbb{H}(\mathfrak{t}, W\tilde{\Gamma}, k)) \to HH_n(\mathcal{O}(\mathfrak{t}) \rtimes W\tilde{\Gamma}). \tag{2.4}$$

The constructions involved in (2.4) affect neither $\mathbb{C}[Z]$ nor the central idempotent $p_\natural$. Hence

$$\begin{aligned} HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural)) \cong HH_n(p_\natural \mathbb{H}(\mathfrak{t}, W\tilde{\Gamma}, k)) = p_\natural HH_n(\mathbb{H}(\mathfrak{t}, W\tilde{\Gamma}, k)) \cong \\ p_\natural HH_n(\mathcal{O}(\mathfrak{t}) \rtimes W\tilde{\Gamma}) = HH_n(p_\natural \mathcal{O}(\mathfrak{t}) \rtimes W\tilde{\Gamma}) \cong HH_n(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\tilde{\Gamma}, \natural]). \end{aligned} \tag{2.5}$$

The second line of (2.5) is an instance of Theorem 1.2. We conclude that there is an isomorphism of vector spaces

$$HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural)) \cong \bigoplus\nolimits_{w \in \langle W\Gamma \rangle} \big(\Omega^n(\mathfrak{t}^w) \otimes \natural^w \big)^{Z_{W\Gamma}(w)}, \tag{2.6}$$

where the summand indexed by w arises from the differential complex $T_w C_*(\mathfrak{t}/(w-1)\mathfrak{t})$, which does not depend on k.

Although the isomorphism (2.6) is in general not canonical, we see from the proofs of [Sol1, Theorem 3.4] and Theorem 1.2 that it depends only on some choices in $\mathbb{C}[W\tilde{\Gamma}, \natural]$. These choices can be made independently of k, so (2.6) provides an identification with $HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, k', \natural))$ for any parameter function $k' : \Delta \to \mathbb{C}$. Unfortunately the isomorphism (2.6) is (to all appearances) not induced by an algebra homomorphism from $\mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural)$ to $\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \natural]$, which makes it difficult to handle.

We warn that usually (2.6) is not an isomorphism of $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-modules, for the $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-module structure on $HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural))$ is a bit more complicated than suggested by (2.6).

2.1. **Representation theory.**

Like in Theorem 1.13 we want to obtain an expression for $HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural))$ in terms of algebraic families of representations. For $\mathbb{H}(\mathfrak{t}, W, \natural) \rtimes \tilde{\Gamma}$ that was achieved in [Sol4, Theorem 3.1]. However, the families of (virtual) representations used in [Sol4] do not seem to be available in our more general setting with a nontrivial 2-cocycle. To overcome that we will modify some arguments from [Sol4], so that they become available in larger generality.

Firstly, we need to specify our parabolic subalgebras. For every $P \subset \Delta$ there is a standard parabolic subalgebra $\mathbb{H}(\mathfrak{t}, W_P, k)$ of $\mathbb{H}(\mathfrak{t}, W, k)$. But that does not yet mimic the situation for reductive groups well enough. To that end we need to allow several parabolic subalgebras with underlying root datum $(X, \Phi_P, Y, \Phi_P^\vee, P)$, namely of the form $\mathbb{H}(\mathfrak{t}, W_P \rtimes \Gamma')$ where $\Gamma' \subset \Gamma$ stabilizes P. More precisely, we assume that we are given a finite set Δ' with a surjection to Δ (written as $Q \mapsto \Delta_Q$) and for each $Q \subset \Delta'$ a subgroup $\Gamma_Q \subset \Gamma$ stabilizing Δ_Q. We abbreviate the group $W_{\Delta_Q} \rtimes \Gamma_Q$ to $(W\Gamma)_Q$. Furthermore, we assume that the collection of parabolic subalgebras

$$\mathbb{H}^Q = \mathbb{H}(\mathfrak{t}, (W\Gamma)_Q, k, \natural) \quad \text{of} \quad \mathbb{H} = \mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural)$$

satisfies the conditions listed just before Definition 1.8. Here the role of V is played by the vector space $\mathfrak{t}$ and

$$\mathfrak{t}^Q \subset \mathfrak{t}^{(W\Gamma)_Q}, \qquad \mathbb{C}\Delta_Q \subset \mathfrak{t}_Q, \qquad \mathfrak{t}^Q \oplus \mathfrak{t}_Q = \mathfrak{t}.$$

Let us abbreviate $\mathbb{H}_Q = \mathbb{H}(\mathfrak{t}_Q, (W\Gamma)_Q, k, \natural)$, so that

$$\mathbb{H}^Q = \mathcal{O}(\mathfrak{t}^Q) \otimes_{\mathbb{C}} \mathbb{H}_Q \quad \text{as algebras.}$$

With the following result we will reduce several issues for $\mathbb{H}$ to the simpler algebra

$$\mathbb{H}(\mathfrak{t}, W\Gamma, 0, \natural) = \mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \natural].$$

Theorem 2.1. *Assume that $k_\alpha \in \mathbb{R}$ for every $\alpha \in \Delta$, and let $\epsilon \in \mathbb{R}_{\geq 0}$. There exists a natural bijection*

$$\zeta_\epsilon : R(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural)) \to R(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)),$$

and similarly for all its parabolic subalgebras, with the following properties:

(i) $\zeta_\epsilon(\pi)$ is a tempered virtual $\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)$-representation if and only if π is a tempered virtual $\mathbb{H}$-representation;

(ii) ζ_ϵ commutes with parabolic induction and character twists, in the sense that

$$\zeta_\epsilon\big(\mathrm{ind}_{\mathbb{H}^Q}^{\mathbb{H}}(\mathbb{C}_\lambda \otimes \sigma)\big) = \mathrm{ind}_{\mathbb{H}(\mathfrak{t},(W\Gamma)_Q,\epsilon k,\natural)}^{\mathbb{H}(\mathfrak{t},W\Gamma,\epsilon k,\natural)}\big(\mathbb{C}_\lambda \otimes \zeta_\epsilon(\sigma)\big)$$

for a tempered $\sigma \in R(\mathbb{H}_Q)$ and $\lambda \in \mathfrak{t}^Q$;

(iii) if $\lambda \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}$ and π is a virtual representation with $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-character in $W\Gamma\lambda + \mathfrak{t}_{\mathbb{R}}$, then so is $\zeta_\epsilon(\pi)$;

(iv) if π is tempered and admits a $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-character in $\mathfrak{t}_{\mathbb{R}}$, then $\zeta_\epsilon(\pi) = \pi \circ m_\epsilon$, with m_ϵ as in (2.1);

(v) ζ_ϵ preserves the underlying $\mathbb{C}[W\Gamma, \natural]$-representations.

Proof. Almost all claims were proven, for $\mathbb{H}(\mathfrak{t}, W\Gamma, k)$ with $\epsilon = 0$, in [Sol3, §2.3] and [Sol4, Theorem 2.4]. The bijectivity follows from [Sol5, Theorem 1.9]. Item (v) is not mentioned explicitly in these references, but it is a direct consequence of the properties (ii) and (iv). With that at hand, we can restrict

$$\zeta_0 : R(\mathbb{H}(\mathfrak{t}, W\Gamma, k)) \to R(\mathbb{H}(\mathfrak{t}, W\Gamma, 0))$$

to the image of $p_\natural$ on both sides. That yields the desired map ζ_0.

Now we consider $\epsilon \in \mathbb{R}_{>0}$, which is actually easier, because the two involved algebras are isomorphic via m_ϵ. This case is not mentioned in [Sol3] or [Sol4], but it can be derived from related results for affine Hecke algebras [Sol3, §4] similarly to the case $\epsilon = 0$. Alternatively one can obtain ζ_ϵ as $(\zeta_0')^{-1}\zeta_0$, where ζ_0' means ζ_0 for the algebra $\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)$. □

To make full use of Theorem 2.1, we assume from now on that $k_\alpha \in \mathbb{R}$ for all $\alpha \in \Delta$. We note that the maps ζ_ϵ in Theorem 2.1 are well-defined and bijective for any $\epsilon \in \mathbb{C}$. Only for $\epsilon \notin \mathbb{R}_{\geq 0}$ they have fewer nice properties with respect to temperedness, see [Sol4, §2.2].

Let $\mathrm{Rep}_{f,t}(\mathbb{H})$ be the category of finite dimensional tempered $\mathbb{H}$-modules. For a discrete series representation δ (see [Sol1, §5]) of a parabolic subalgebra $\mathbb{H}_Q$, let $\mathrm{Rep}_{f,t}^{[Q,\delta]}(\mathbb{H})$ be the full subcategory of $\mathrm{Rep}_f(\mathbb{H})$ generated by the subquotients of the representations

$$\pi(Q, \delta, \lambda) = \mathrm{ind}_{\mathbb{H}^Q}^{\mathbb{H}}(\mathbb{C}_\lambda \otimes \delta) \text{ with } \lambda \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^Q.$$

Choose a set $\Delta_{\mathbb{H}}$ of representatives $\mathfrak{d} = [Q, \delta]$ for such pairs up to $W\Gamma$-equivalence. Define $(W\Gamma)_{\mathfrak{d}}$ as the stabilizer of (Q, δ).

Theorem 2.2. *The Grothendieck group $R_t(\mathbb{H})$ of $\mathrm{Rep}_{f,t}(\mathbb{H})$ decomposes as a direct sum $\bigoplus_{\mathfrak{d} \in \Delta_{\mathbb{H}}} R_t(\mathbb{H})^{\mathfrak{d}}$.*

Proof. By the main result of [DeOp], an analogue of the Plancherel isomorphism [Wal] for affine Hecke algebras, our theorem holds for affine Hecke algebras with positive parameters. That extends to twisted affine Hecke algebras $\mathcal{H}$ with positive parameters [Sol3, Theorem 3.2.2]. Choose such an algebra $\mathcal{H}$, whose associated graded Hecke algebra, via the localization process from [Sol3, §2.1], is $\mathbb{H}$. This is possible because $\mathbb{H}$ has real parameters. Then [Sol3, Theorem 2.1.3] provides an equivalence between the subcategory of $\mathrm{Rep}_f(\mathbb{H})$ formed by the representations with all their $\mathcal{O}(\mathfrak{t})$-weights in a certain analytically open neighborhood U of $\mathfrak{t}_{\mathbb{R}}$ in $\mathfrak{t}$, and an analogous category of $\mathcal{H}$-modules. This equivalence preserves temperedness [Slo, (2.11)], so the subcategory $\mathrm{Rep}_{f,U}(\mathbb{H})$ of $\mathrm{Rep}_f(\mathbb{H})$ determined by $U \subset \mathfrak{t}$ decomposes in the required way. In particular the Grothendieck group of $\mathrm{Rep}_{f,U}(\mathbb{H})$ has the desired property.

Let $\eta \in \mathbb{R}_{>0}$ and $\lambda_1, \lambda_2 \in \mathfrak{t}_{\mathbb{R}}$ so that $\lambda_1 + \sqrt{-1}\lambda_2 \in U$. $\mathrm{Rep}_{f,t}^{[Q,\delta]}(\mathbb{H})$ only has nonzero objects with $\mathcal{O}(\mathfrak{t})$-weights of this form if the central character of δ is $(W\Gamma)_Q\lambda_1$.

By [Sol2, Proposition 10.1] and the complete decomposability of $\pi(Q, \delta, \lambda)$ for $\lambda \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^Q$ [Sol2, Proposition 7.2], there is a natural bijection between the set of irreducible objects of $\mathrm{Rep}_{f,t}^{[Q,\delta]}(\mathbb{H})$ with central character $\lambda_1 + \sqrt{-1}\lambda_2$ and the analogous set for $\lambda_1 + \sqrt{-1}\eta\lambda_2$. Hence the proven property of the part of $R_t(\mathbb{H})$ coming from $\mathrm{Rep}_{f,U}(\mathbb{H})$ extends to the whole of $R_t(\mathbb{H})$. □

The decomposition from Theorem 2.2 is available for $\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)$ for any $\epsilon \in \mathbb{R}$. It is compatible with ζ_ϵ for $\epsilon > 0$, but not for $\epsilon < 0$ (because ζ_{-1} does not preserve temperedness). Theorem 2.2 is hardly helpful in the case $\epsilon = 0$, because the parabolic subalgebras $\mathbb{H}(\mathfrak{t}, (W\Gamma)_Q, 0, \natural)$ with $Q \neq \emptyset$ do not have any discrete series representations. (The map ζ_0 does not preserve the discrete series property.)

Let us consider a finite set of algebraic families of $\mathbb{H}$-representations $\mathfrak{F}(Q_i, \sigma_i)$ as in Definition 1.8. We assume that σ_i is irreducible elliptic and that the representations

$$\pi(Q_i, \sigma_i, \lambda_i) = \mathrm{ind}_{\mathbb{H}^Q}^{\mathbb{H}}(\mathbb{C}_{\lambda_i} \otimes \sigma_i) \quad \lambda_i \in \mathfrak{t}^{Q_i}, i = 1, \ldots, n_{\mathfrak{F}} \tag{2.7}$$

span $\mathbb{Q} \otimes_{\mathbb{Z}} R(\mathbb{H})$. It follows from the Langlands classification for graded Hecke algebras [Eve] (which can be generalized to $\mathbb{H}$ with the method from [Sol3, §2.2]) that every irreducible elliptic $\mathbb{H}$-representation is tempered and admits an $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-character in

$$\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}\Phi \ \oplus \ \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}\Phi^\vee)^\perp .$$

In particular every $\sigma_i \in \mathrm{Irr}(\mathbb{H}_{Q_i})$ is tempered and admits an $\mathcal{O}(\mathfrak{t}_{Q_i})^{(W\Gamma)_{Q_i}}$-character in $\mathbb{R}\Delta_Q$. Then $\pi(Q_i, \sigma, \lambda_i)$ is tempered if and only if $\lambda_i \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^{Q_i}$, see [Sol2, Lemma 2.2]. Every tempered family $\mathfrak{F}^t(Q_i, \sigma_i)$ belongs to $R_t(\mathbb{H})^{\mathfrak{d}}$ for a unique $\mathfrak{d} = [Q, \delta] \in \Delta_{\mathbb{H}}$. We denote that as $i \prec \mathfrak{d}$. In this situation we may and will assume that $Q_i \supset Q$ and that σ_i is a subquotient of $\mathrm{ind}_{\mathbb{H}^Q}^{\mathbb{H}^{Q_i}} \delta$.

By Theorem 2.1, $\zeta_\epsilon(\sigma_i) = \sigma_i \circ m_\epsilon$ is a finite dimensional elliptic representation of $\mathbb{H}(\mathfrak{t}_{Q_i}, (W\Gamma)_{Q_i}, \epsilon k, \natural)$. Further,

$$\zeta_\epsilon(\pi(Q_i, \sigma_i, \lambda_i)) = \pi(Q_i, \zeta_\epsilon(\sigma_i), \lambda_i) \qquad \text{for } \lambda_i \in \mathfrak{t}^{Q_i},$$

so ζ_ϵ sends $\mathfrak{F}(Q_i, \sigma_i)$ to the algebraic family $\mathfrak{F}(Q_i, \zeta_\epsilon(\sigma_i))$. The bijectivity of ζ_ϵ implies that the members of the algebraic families $\{\mathfrak{F}(Q_i, \sigma_\epsilon(\sigma_i))\}_{i=1}^{n_{\mathfrak{F}}}$ span $\mathbb{Q} \otimes_{\mathbb{Z}} R(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural))$. Thus Lemma 1.9 and Theorem 1.13 apply to the families $\mathfrak{F}(Q_i, \zeta_0(\sigma_i))$.

Recall from (2.6) that we can identify the $HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural))$ for all $\epsilon \in \mathbb{C}$ with one fixed vector space. For each $\epsilon \in \mathbb{C}$ we have the algebra homomorphisms

$$\mathcal{F}_{Q_i, \zeta_\epsilon(\sigma_i)} : \mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural) \to \mathcal{O}(\mathfrak{t}^{Q_i}) \otimes \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}[W\Gamma, \natural] \underset{\mathbb{C}[(W\Gamma)_Q, \natural]}{\otimes} V_{\sigma_i}\big). \tag{2.8}$$

Lemma 2.3. *The map $\mathcal{F}_{Q_i, \zeta_\epsilon(\sigma_i)}$ is a homomorphism of $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-algebras, for a module structure that depends on ϵ. In particular the induced map on Hochschild homology is $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-linear.*

Proof. As $\mathcal{O}(\mathfrak{t})^{W\Gamma} \subset Z(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural))$, it acts naturally on $HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural))$. By the irreducibility of σ_i, every $\zeta_\epsilon(\sigma_i)$ admits a $\mathcal{O}(t_{Q_i})^{(W\Gamma)_{Q_i})}$-character. It depends linearly on ϵ. By [BaMo, Theorem 6.4], every $\pi(Q_i, \zeta_\epsilon(\sigma_i), \lambda_i)$ admits a $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-character, which implies that the image of $\mathcal{O}(\mathfrak{t})^{W\Gamma}$ under $\mathcal{F}_{Q_i, \zeta_\epsilon(\sigma_i)}$ is central. That turns

$$\mathcal{O}(\mathfrak{t}^{Q_i}) \otimes \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}[W\Gamma, \natural] \underset{\mathbb{C}[(W\Gamma)_Q, \natural]}{\otimes} V_{\sigma_i}\big)$$

into a $\mathcal{O}(t)^{W\Gamma}$-algebra and makes $\mathcal{F}_{Q_i, \zeta_\epsilon(\sigma_i)}$ $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-linear. The final claim is one of the functorial properties of Hochschild homology [Lod, §1.1]. $\square$

The algebra homomorphisms (2.8) and (2.6) give rise to a family of maps

$$HH_n(\mathcal{F}_\epsilon) \ : \ \bigoplus\nolimits_{w \in \langle W\Gamma \rangle} \big(\Omega^n(\mathfrak{t}^w) \otimes \natural^w\big)^{Z_{W\Gamma}(w)} \to \bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \Omega^n(\mathfrak{t}^{Q_i}) \qquad \epsilon \in \mathbb{C}.$$

The discussion after (2.6) and the construction of ζ_ϵ entail that these maps depend algebraically on ϵ.

Lemma 2.4. *The map $HH_n(\mathcal{F}_\epsilon)$ is injective, for every $\epsilon \in \mathbb{C}$.*

Proof. Consider a nonzero element x of the domain of $HH_n(\mathcal{F}_\epsilon)$. From Theorem 1.13 we know that $HH_n(\mathcal{F}_0)$ is injective, so $HH_n(\mathcal{F}_0)x \neq 0$. As $HH_n(\mathcal{F}_\epsilon)x$ depends algebraically on $\epsilon \in \mathbb{C}$, it is nonzero when $|\epsilon|$ is sufficiently small.

For any $\eta \in \mathbb{C}^\times$ we have the algebra isomorphism

$$m_\eta : \mathbb{H}(t, W\Gamma, \epsilon k, \natural) \to \mathbb{H}(t, W\Gamma, \eta^{-1}\epsilon k, \natural),$$

which sends a family $\mathfrak{F}(Q_i, \zeta_\epsilon(\sigma_i))$ to $\mathfrak{F}(Q_i, \zeta_{\eta^{-1}\epsilon}(\sigma_i))$, with an additional scaling by η on $\mathfrak{t}^{Q_i}$. The latter respects the entire structure, so we can conclude that $HH_n(\mathcal{F}_{\eta^{-1}\epsilon})x \neq 0$ for all $\eta \in \mathbb{C}^\times$. $\square$

We want to analyse the maps $HH_n(\mathcal{F}_\epsilon)$ like in Proposition 1.11. For $g \in \langle W\Gamma \rangle$ we choose $\lambda_{g,i} \in \mathbb{C}$ and $\phi_{g,i} : \mathfrak{t}^g \to \mathfrak{t}^{Q_i}$ as in Lemma 1.10, so that

$$\nu_{g,v} = \sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} \operatorname{tr} \pi(Q_i, \zeta_0(\sigma_i), \phi_{g,i}(v)) \quad \text{in } \mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H}(\mathfrak{t}, W\Gamma, 0, \natural)). \tag{2.9}$$

Although $\nu_{g,v}$ has only been defined when $(W\Gamma)_v \cap Z_G(g) \subset \ker(\natural^g)$, the right hand side of (2.9) makes sense for any $v \in \mathfrak{t}^g$. For $\epsilon \in \mathbb{C}$, and $\mathfrak{d} \in \Delta_{\mathbb{H}}$ we define the algebraic families of virtual $\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)$-representations

$$\begin{aligned} \nu_{g,v}^{\epsilon,\mathfrak{d}} &= \sum\nolimits_{i=1, i \prec \mathfrak{d}}^{n_{\mathfrak{F}}} \lambda_{g,i} \operatorname{tr} \pi(Q_i, \zeta_\epsilon(\sigma_i), \phi_{g,i}(v)) \qquad v \in \mathfrak{t}^g, \\ \nu_{g,v}^{\epsilon} &= \sum\nolimits_{\mathfrak{d} \in \Delta_{\mathbb{H}}} \nu_{g,v}^{\epsilon,\mathfrak{d}}. \end{aligned} \tag{2.10}$$

For all $v \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^g, v_i \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^{Q_i}$ we have

$$\nu_{g,v}^{0,\mathfrak{d}}, \pi(Q_i, \zeta_0(\sigma_i), v_i) \in \zeta_0(R_t(\mathbb{H})^{\mathfrak{d}}).$$

Hence the proof of Lemma 1.10 can played entirely in $\mathbb{Q} \otimes_{\mathbb{Z}} \zeta_0(R_t(\mathbb{H})^{\mathfrak{d}})$, which means that we do not need all elements of $W\Gamma$ for the $\phi_{g,i}$, only those of $(W\Gamma)_{\mathfrak{d}}$.

Like in (1.45), for $g \in \langle W\Gamma \rangle, h \in W\Gamma$ we define

$$\lambda_{hgh^{-1},i} = \natural^g(h)\lambda_{g,i} \quad \text{and} \quad \phi_{hgh^{-1},i} = \phi_{g,i} \circ h^{-1}.$$

This is consistent with the equality $\nu_{hgh^{-1},hv} = \natural^g(h)\nu_{g,v}$ from (1.20). Then we define $\nu_{hgh^{-1},v'}$ as in (2.9) and $\nu^\epsilon_{hgh^{-1},v'}$ as in (2.10). The bijectivity in Theorem 2.1 implies that the $\nu^\epsilon_{hgh^{-1},v'}$ satisfy the same consistency relations.

Lemma 2.5. *Let $g, h \in W\Gamma$ and $v \in \mathfrak{t}^g$.*

(a) There is an equality

$$\nu^\epsilon_{hgh^{-1},hv} = \natural^g(h)\nu^\epsilon_{g,v} \quad \textit{in} \quad \mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)).$$

(b) For $\mathfrak{d} \in \Delta_{\mathbb{H}}$ we have

$$\nu^{\epsilon,\mathfrak{d}}_{hgh^{-1},hv} = \natural^g(h)\nu^{\epsilon,\mathfrak{d}}_{g,v}.$$

Proof. (a) By (1.20) and the above conventions, this holds when $g \in \langle W\Gamma \rangle$ and $(W\Gamma)_v \cap Z_{W\Gamma}(g) \subset \ker(\natural^g)$. That extends to all $v \in V^g$ by continuity.

For an arbitrary element $g' = wgw^{-1}$ of $W\Gamma$, we get

$$\nu^\epsilon_{hg'h^{-1},hv'} = \nu^\epsilon_{hwgw^{-1}h^{-1},hv'} = \natural^g(hw)\nu^\epsilon_{g,w^{-1}v'} = \natural^g(hw)\natural^g(w)^{-1}\nu^\epsilon_{wgw^{-1},v'}.$$

By Lemma 1.3.b the right hand side equals $\natural^{g'}(h)\nu^{\epsilon}_{g',v'}$.

(b) Since every $\phi_{g,i}$ stabilizes $\sqrt{-1}\mathfrak{t}_{\mathbb{R}}$, it makes sense to restrict our attention to the $\nu^{\epsilon,\mathfrak{d}}_{g,v}$ with $v \in \sqrt{-1}\mathfrak{t}^{g}_{\mathbb{R}}$. These are precisely the v for which the associated representations $\pi(Q_i, \sigma_i, \phi_{g,i}(v))$ are tempered.

Theorem 2.2 says that for various $\mathfrak{d}$ the components $\nu^{\epsilon,\mathfrak{d}}_{g,v}$ of $\nu^{1}_{g,v}$ live in linearly independent parts of $\mathbb{Q} \otimes_{\mathbb{Z}} R(\mathbb{H})$. Then the property of $\nu^{\epsilon}_{g,v}$ from Lemma 2.5.a must also hold for all of its $\mathfrak{d}$-components, so

$$\nu^{\epsilon,\mathfrak{d}}_{hgh^{-1},hv} = \natural^{g}(h)\nu^{\epsilon,\mathfrak{d}}_{g,v} \qquad v \in \mathfrak{t}^{g}_{\mathbb{R}}.$$

Both sides of this equality extend algebraically to $v \in \mathfrak{t}^{g}$, so the equality as well. $\square$

By Lemma 1.9 and Theorem 2.1, the $\nu^{\epsilon}_{g,v}$ with $g \in \langle W\Gamma\rangle$ and $(W\Gamma)_v \cap Z_{W\Gamma}(g) \subset \ker(\natural^{g})$ span $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural))$, and a basis is obtained by dividing out the $W\Gamma$-equivariance relations from Lemma 2.5.a.

In general the virtual representations $\nu^{\epsilon}_{g,v}$ do not admit a central character, but their summands $\nu^{\epsilon,\mathfrak{d}}_{g,v}$ do:

Lemma 2.6. *Let $cc(\delta) \in \mathfrak{t}_{\mathbb{R},Q}$ be an $\mathcal{O}(\mathfrak{t}_Q)$-weight of δ, in other words, a representative of the central character of δ. Let $g \in W\Gamma, v \in \mathfrak{t}^{g}$. Then the virtual $\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)$-representation $\nu^{\epsilon,\mathfrak{d}}_{g,v}$ admits the central character*

$$W\Gamma(\epsilon\, cc(\delta) + v) = W\Gamma((W\Gamma)_Q\, \epsilon\, cc(\delta) + v).$$

Proof. For $\epsilon = 0$, by construction $\nu^{0}_{g,v} = \nu_{g,v}$ has central character $W\Gamma v$. Hence all the $\pi(Q_i, \zeta_0(\sigma_i), \phi_{g,i}(v))$ with $\lambda_{g,i} \neq 0$ have central character $W\Gamma v$.

Consider $i \prec \mathfrak{d}$. Since σ_i is a subquotient of $\mathrm{ind}_{\mathbb{H}^{Q}}^{\mathbb{H}^{Q_i}}(\delta)$, $(W\Gamma)_{Q_i} cc(\delta)$ is the central character of σ_i. By the invertibility of intertwining operators for tempered parabolically induced representations and the theory of R-groups for graded Hecke algebras [Sol6, §3.5 and 4.1], $\mathrm{ind}_{\mathbb{H}^{Q}}^{\mathbb{H}^{Q_i}}(\delta)$ is a direct sum of irreducible representations with exactly the same $\mathcal{O}(\mathfrak{t})$-weights. Thus every $\mathcal{O}(\mathfrak{t})$-weight of δ, and in particular $cc(\delta)$, is also a $\mathcal{O}(\mathfrak{t})$-weight of σ_i.

By Theorem 2.1.iv, $\epsilon cc(\delta) + \phi_{g,i}(v)$ is an $\mathcal{O}(\mathfrak{t})$-weight of $\zeta_\epsilon(\sigma_i) \otimes \phi_{g,i}(v)$. Then the central character of $\pi(Q_i, \zeta_\epsilon(\sigma_i), \phi_{g,i}(v))$ is

$$W\Gamma(\epsilon cc(\delta) + \phi_{g,i}(v)) = W\Gamma(\epsilon \phi_{g,i}^{-1} cc(\delta) + v). \tag{2.11}$$

Here $\phi_{g,i} \in (W\Gamma)_{\mathfrak{d}}$, so $\phi_{g,i}^{-1} cc(\delta)$ is still a $\mathcal{O}(\mathfrak{t})$-weight of δ. Thus all $\pi(Q_i, \zeta_\epsilon(\sigma_i), \phi_{g,i}(v))$ with $\lambda_{g,i} \neq 0$ and $i \prec \mathfrak{d}$ have the same central character (2.11), and so does their linear combination $\nu^{\epsilon,\mathfrak{d}}_{g,v}$. $\square$

2.2. **Hochschild homology.**

As $\phi_{g,i}$ is given by an element of $W\Gamma$, it extends naturally to a linear bijection $\phi_{g,i} : \mathfrak{t} \to \mathfrak{t}$. Thus $\phi_{g,i} : \mathfrak{t}^{g} \to \mathfrak{t}^{Q_i}$ admits a one-sided inverse

$$\mathfrak{t}^{Q_i} \xrightarrow{(\mathrm{id},0)} \mathfrak{t}^{Q_i} \oplus \mathfrak{t}_{Q_i} = \mathfrak{t} \xrightarrow{\phi_{g,i}^{-1}} \mathfrak{t} \to \mathfrak{t}/(g-1)\mathfrak{t} \cong \mathfrak{t}^{g}. \tag{2.12}$$

The algebra homomorphism $\phi^{*}_{g,i} : \mathcal{O}(\mathfrak{t}^{Q_i}) \to \mathcal{O}(\mathfrak{t}^{g})$ also has a one-sided inverse, namely composing functions with (2.12). Like in (1.33), for each pair (g, i) the map $\phi^{*}_{g_i}$ induces an algebra homomorphism

(2.13)

$$\phi^{*}_{g,i} : \mathcal{O}(\mathfrak{t}^{Q_i}) \otimes \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}[W\Gamma, \natural] \underset{\mathbb{C}[(W\Gamma)_Q, \natural}{\otimes} V_{\sigma_i}\big) \to \mathcal{O}(V^{g}) \otimes \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}[W\Gamma, \natural] \underset{\mathbb{C}[(W\Gamma)_Q, \natural}{\otimes} V_{\sigma_i}\big).$$

The map $\phi^*_{g,i}$ is $\mathcal{O}(\mathfrak{t}^g)$-linear if we let $\mathcal{O}(\mathfrak{t}^g)$ act on its domain via composition with (2.12). On the other hand, $\phi^*_{g,i}$ is $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-linear if we endow both sides with the $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-module structure coming from the central characters of the involved $\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)$-representations. That works for any $\epsilon \in \mathbb{C}$, but the resulting module structures depend on ϵ.

The maps on Hochschild homology induced by the $\phi^*_{g,i}$ can be summed with coefficients $\lambda_{g,i}$, and that yields a map

$$HH_n(\phi^*_{\mathfrak{d}}) = \bigoplus_{g \in \langle W\Gamma \rangle} \sum_{i=1, i \prec \mathfrak{d}}^{n_{\mathfrak{F}}} \lambda_{g,i} HH_n(\phi^*_{g,i}) \; : \; \bigoplus_{i=1, i \prec \mathfrak{d}}^{n_{\mathfrak{F}}} \Omega^n(\mathfrak{t}^{Q_i}) \to \bigoplus_{g \in \langle W\Gamma \rangle} \Omega^n(\mathfrak{t}^g). \tag{2.14}$$

By Lemma 2.6 it is $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-linear if we let $\mathcal{O}(\mathfrak{t})^{W\Gamma}$ act on $\Omega^n(\mathfrak{t}^g)$ via the central character of the underlying virtual representation $\nu^{\epsilon,\mathfrak{d}}_{g,v}$. On the other hand, the map

$$HH_n(\phi^*) = \bigoplus_{\mathfrak{d} \in \Delta_{\mathbb{H}}} HH_n(\phi^*_{\mathfrak{d}}) \; : \; \bigoplus_{i=1}^{n_{\mathfrak{F}}} \Omega^n(\mathfrak{t}^{Q_i}) \to \bigoplus_{g \in \langle W\Gamma \rangle} \Omega^n(\mathfrak{t}^g)$$

is in general not $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-linear for these module structures.

Lemma 2.7. *The map $HH_n(\phi^*)$ is injective. For each $\epsilon \in \mathbb{C}$, the image of $HH_n(\phi^*_{\mathfrak{d}}) \circ HH_n(\mathcal{F}_\epsilon)$ is contained in $\bigoplus_{g \in \langle W\Gamma \rangle} (\Omega^n(\mathfrak{t}^g) \otimes \natural^g)^{Z_{W\Gamma}(g)}$.*

Proof. The injectivity can be shown in the same way as in Lemma 1.12. Note that

$$HH_n(\phi^*_{\mathfrak{d}}) \circ HH_n(\mathcal{F}_\epsilon) = \bigoplus\nolimits_{g \in \langle W\Gamma \rangle} \sum\nolimits_{i=1, i \prec \mathfrak{d}}^{n_{\mathfrak{F}}} \lambda_{g,i} HH_n(\phi^*_{g,i} \circ \mathcal{F}_{Q_i, \zeta_\epsilon(\sigma_i)}).$$

The specialization of this expression at (g, v) comes from the virtual representation $\nu^{\epsilon,\mathfrak{d}}_{g,v}$ of $\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)$. By Lemma 1.7 the map

$$\mathrm{ev}_v \circ \mathrm{gtr} \circ \sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} C_* \big(\phi^*_{g,i} \circ \mathcal{F}_{Q_i, \zeta_\epsilon(\sigma_i)} \big)$$

cannot distinguish equivalent virtual representations. In combination with Lemma 2.5.b we find that the image of $\sum_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} HH_n(\phi^*_{g,i} \circ \mathcal{F}_{Q_i, \zeta_\epsilon(\sigma_i)})$ consists of differential forms that transform as $(\natural^g)^{-1}$ under the action of $Z_{W\Gamma}(g)$. $\square$

With the procedure described before (1.32) we can achieve that our set of algebraic families of $\mathbb{H}$-representations $\mathfrak{F}(Q_i, \sigma_i)$ minimally spans $\mathbb{Q} \otimes_{\mathbb{Z}} R(\mathbb{H})$. Thus our new, reduced collection of algebraic families satisfies the three properties listed directly after (1.32).

We are ready to prove the description of the Hochschild homology of $\mathbb{H}$ in the style of the trace Paley–Wiener theorem for reductive p-adic groups [BDK]. Recall that still all parameters k_α are real.

Theorem 2.8. *In the above setting we fix $\epsilon \in \mathbb{C}$.*

(*a*) *The map*

$$HH_n(\mathcal{F}_\epsilon) : HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)) \to \bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \Omega^n(\mathfrak{t}^{Q_i})$$

is a $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-linear injection, for the module structure from Lemma 2.3.

(b) *$HH_n(\phi^*)$ is a bijection*

$$HH_n(\mathcal{F}_\epsilon)HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)) \to \bigoplus\nolimits_{g\in\langle W\Gamma\rangle} (\Omega^n(\mathfrak{t}^g) \otimes \natural^g)^{Z_{W\Gamma}(g)},$$

and it is $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-linear with respect to the natural module structures.

(c) *In degree $n = 0$ the condition on an element ω of $\bigoplus_{i=1}^{n_{\mathfrak{F}}} \mathcal{O}(\mathfrak{t}^{Q_i})$ to belong to the image of $HH_0(\mathcal{F}_\epsilon)$ is:*

$$\text{if } \mu_j \in \mathbb{C}, 1 \leq i_j \leq n_{\mathfrak{F}}, \lambda_{i_j} \in \mathfrak{t}^{Q_{i_j}} \text{ and } \sum\nolimits_j \mu_j \pi(Q_{i_j}, \zeta_\epsilon(\sigma_{i_j}), \lambda_{i_j}) = 0$$
$$\text{in } \mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)), \text{ then } \sum\nolimits_j \mu_j \omega(\lambda_{i_j}) = 0.$$

Equivalently, ω must determine a linear function on $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural))$, via the natural pairing (1.23). This yields an isomorphism of $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-modules

$$HH_0(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)) \cong \big(\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural))\big)^*_{\mathrm{reg}}.$$

Proof. (a) is just a restatement of Lemmas 2.3 and 2.4.
(b) By Lemma 1.12 and the third property (as listed after (1.32)) of our set of algebraic families $\mathfrak{F}(Q_i, \sigma_i)$, $HH_n(\phi^*)$ is injective.

Consider a finite subset S of $\bigsqcup_{i=1}^{n_{\mathfrak{F}}} \mathfrak{t}^{Q_i}$ and let $I_S \subset \bigoplus_{i=1}^{n_{\mathfrak{F}}} \mathcal{O}(\mathfrak{t}^{Q_i})$ be the ideal of functions that vanish on S. For $m \in \mathbb{N}$ let J_n^m (resp. $\tilde{J}^n_{m,\epsilon}$) be the image of

$$HH_n(\phi^*)^{-1}\Big(\bigoplus\nolimits_{g\in\langle W\Gamma\rangle} (\Omega^n(\mathfrak{t}^g) \otimes \natural^g)^{Z_{W\Gamma}(g)}\Big) \tag{2.15}$$

(resp. $HH_n(\mathcal{F}_\epsilon)$) in

$$\bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \Omega^n(\mathfrak{t}^{Q_i})/I_S^m \Omega^n(\mathfrak{t}^{Q_i}). \tag{2.16}$$

By Theorem 1.13 $\tilde{J}^n_{m,0} = J^n_m$. As $HH_n(\mathcal{F}_\epsilon)$ depends algebraically on $\epsilon \in \mathbb{C}$ and J^n_m has finite dimension, $\tilde{J}^n_{m,\epsilon} = J^n_m$ when $|\epsilon|$ is sufficiently small. The argument with m_η in the proof of Lemma 2.4 then shows that $\tilde{J}^n_{m,\epsilon} = J^n_m$ for all $\epsilon \in \mathbb{C}$.

Fix an $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-character $W\Gamma\lambda$. Choose S so that it contains all $\lambda_i \in \mathfrak{t}^{Q_i}$ for which $W\Gamma\lambda$ is the central character of $\pi(Q_i, \zeta_\epsilon(\sigma_i), \lambda_i)$. With Lemma 2.3 it follows from the above that the map $HH_n(\mathcal{F}_\epsilon)$ induces a surjection between the formal completions at $W\Gamma\lambda$ of the $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-modules $HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural))$ and (2.15). Thus the quotient of (2.15) by the image of $HH_n(\mathcal{F}_\epsilon)$ is an $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-module M all whose formal completions are 0. It is finitely generated because $\Omega^n(\mathfrak{t}^{Q_i})$ is finitely generated as $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-module. A general argument, which we formulate as Lemma 2.9 below, says that $M = 0$. Hence the image of $HH_n(\mathcal{F}_\epsilon)$ is as claimed.
(c) The description of the image of $HH_0(\mathcal{F}_\epsilon)$ was proven in the case $\epsilon = 0$ in Theorem 1.13. In combination with Theorem 2.1, the same argument applies when $\epsilon \in \mathbb{C}^\times$. $\square$

Let $\mathcal{O}(V)$ be the ring of regular functions on a complex affine variety V. For each $v \in V$, let $I_v \subset \mathcal{O}(V)$ be the maximal ideal of functions vanishing at v. For any $\mathcal{O}(V)$-module M, we can form the completion

$$\hat{M}_v = \varprojlim\nolimits_n M/I_v^n M.$$

Lemma 2.9. *Let M be a finitely generated $\mathcal{O}(V)$-module, such that $\hat{M}_v = 0$ for all $v \in V$. Then $M = 0$.*

Proof. For any $m \in M$, the image of m in $\hat{M}_v$ is zero, so $m \in I_v^n M$ for all $n \in \mathbb{N}$. Hence $M = I_v^n M$ for all $v \in V$ and all $n \in \mathbb{N}$.

As M is finitely generated, we can write $M = \mathcal{O}(V)^r / N$ for some $\mathcal{O}(V)$-submodule N of $\mathcal{O}(V)^r$. In combination with the above we find

$$\mathcal{O}(V)^r / N = I_v^n (\mathcal{O}(V)^r / N) = (I_v^n \mathcal{O}(V)^r + N)/N$$

Therefore $\mathcal{O}(V)^r = I_v^n \mathcal{O}(V)^r + N$ for all $v \in V$ and all $n \in \mathbb{N}$. This is only possible when $N = \mathcal{O}(V)^r$, so $M = 0$. $\square$

Like in (1.46), we can vary on (2.14) and define the $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-linear map

$$HH_n(\tilde{\phi}^*) = \bigoplus_{g \in W\Gamma} \sum_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,i} HH_n(\phi^*_{g,i}) \; : \; \bigoplus_{i=1}^{n_{\mathfrak{F}}} \Omega^n(\mathfrak{t}^{Q_i}) \to \bigoplus_{g \in W\Gamma} \Omega^n(\mathfrak{t}^g). \tag{2.17}$$

The same arguments as for Corollary 1.14 show that:

Corollary 2.10. *There is a $\mathbb{C}$-linear bijection*

$$HH_n(\tilde{\phi}^*) \circ HH_n(\mathcal{F}_\epsilon) \; : \; HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)) \to \Big(\bigoplus\nolimits_{g \in W\Gamma} \Omega^n(\mathfrak{t}^g) \otimes \natural^g \Big)^{W\Gamma}.$$

We note that in Corollary 2.10 the target does not depend on ϵ. In fact, in Theorem 2.8 the map $HH_n(\phi^*)$ does not depend on ϵ either, and the same goes for the subspace

$$HH_n(\mathcal{F}_\epsilon) HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, \epsilon k, \natural)) \; \subset \; \bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \Omega^n(\mathfrak{t}^{Q_i}),$$

Hence we can define a $\mathbb{C}$-linear bijection

$$HH_n(\zeta_0) := HH_n(\mathcal{F}_1)^{-1} HH_n(\mathcal{F}_0) \; : \; HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, 0, \natural)) \to HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural)).$$

Proposition 2.11. $HH_n(\zeta_0)$ *is the unique $\mathbb{C}$-linear bijection*

$$HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, 0, \natural)) \to HH_n(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \natural))$$

such that

$$HH_n(\mathcal{F}_{Q,\sigma}) \circ HH_n(\zeta_0) = HH_n(\mathcal{F}_{Q,\zeta_0(\sigma)})$$

for all algebraic families of $\mathbb{H}$-representations $\mathfrak{F}(Q, \sigma)$.

Proof. By construction

$$HH_n(\mathcal{F}_{Q_i,\sigma_i}) \circ HH_n(\zeta_0) = HH_n(\mathcal{F}_{Q_i,\zeta_0(\sigma_i)}) \qquad i = 1, \ldots, n_{\mathfrak{F}}. \tag{2.18}$$

As $\mathcal{F}_1$ is built from the $\mathcal{F}_{Q_i,\sigma_i}$ and $\mathcal{F}_0$ from the $\mathcal{F}_{Q_i,\zeta_0(\sigma_i)}$, the property (2.18) already determines $HH_n(\zeta_0)$ uniquely.

It remains to check the condition for an arbitrary algebraic family $\mathfrak{F}(Q, \sigma)$. Recall that in Lemma 1.9 we exhibited a basis of $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \natural])$, consisting of some virtual representations $\nu_{g,v}$. The virtual representations $\nu^1_{g,v} = \zeta_0^{-1}(\nu_{g,v})$ form a basis of $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H})$. In (2.9) we expressed $\nu_{g,v}$ as linear combination of members of the families $\mathfrak{F}(Q_i, \zeta_0(\sigma_i))$, and by definition $\nu^1_{g,v}$ is almost the same linear combination, only with $\mathfrak{F}(Q_i, \sigma_i)$ instead. Write

$$\operatorname{tr} \pi(Q, \sigma, \lambda) = \sum\nolimits_{g,v} c(g, v, \lambda) \nu^1_{g,v},$$

then Theorem 2.1 implies

$$\operatorname{tr} \pi(Q, \zeta_0(\sigma), \lambda) = \sum\nolimits_{g,v} c(g, v, \lambda) \nu_{g,v}.$$

Hence there exist $c'(i,\lambda,v_i) \in \mathbb{C}$ such that

$$
\begin{array}{lll}
\pi(Q,\sigma,\lambda) & = & \sum_{i,v_i} c'(i,\lambda,v_i)\, \pi(Q_i,\sigma_i,v_i), \\
\pi(Q,\zeta_0(\sigma),\lambda) & = & \sum_{i,v_i} c'(i,\lambda,v_i)\, \pi(Q_i,\zeta_0(\sigma_i),v_i),
\end{array} \tag{2.19}
$$

in $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H})$ and $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \natural])$, respectively. With (2.18) we find that

$$HH_n(\pi(Q,\sigma,\lambda)) \circ HH_n(\zeta) = HH_n(\pi(Q,\zeta_0(\sigma),\lambda)).$$

The same reasoning for all $\lambda \in \mathfrak{t}^Q$ simultaneously yields the required property of $HH_n(\zeta_0)$. $\square$

It turns out that the description of $HH_n(\mathbb{H})$ in Theorem 2.8 decomposes further, such that the decomposition reveals the structure as a module over the centre. For $\mathfrak{d} \in \Delta_{\mathbb{H}}$ we define $\mathcal{F}_{\mathfrak{d}} = \bigoplus_{i \prec \mathfrak{d}} \mathcal{F}_{Q_i,\sigma_i}$.

Lemma 2.12. *(a)* $HH_n(\mathcal{F}_1)\big(HH_n(\mathbb{H})\big) = \bigoplus_{\mathfrak{d}=[Q,\delta]\in\Delta_{\mathbb{H}}} HH_n(\mathcal{F}_{\mathfrak{d}})\big(HH_n(\mathbb{H})\big)$.
(b) The subspace

$$HH_n(\mathbb{H})^{\mathfrak{d}} := HH_n(\mathcal{F}_1)^{-1}\big(HH_n(\mathcal{F}_{\mathfrak{d}})HH_n(\mathbb{H})\big)$$

of $HH_n(\mathbb{H})$ can be defined canonically, without choosing any algebraic families of representations.

Proof. (a) By definition the left hand side is contained in the right hand side. From Theorem 2.8 we know the conditions that describe the left hand side: upon applying $HH_n(\phi^*)$ one lands in $\bigoplus_{g\in\langle W\Gamma\rangle} (\Omega^n(V^g) \otimes \natural^g)^{Z_{W\Gamma}(g)}$. Recall from Lemma 2.7 that those conditions arose from the virtual $\mathbb{H}$-representations $\nu^1_{g,\lambda}$.

With Lemma 2.5.b we see, in the same way as in the proof of Lemma 2.7, that $HH_n(\phi^*_{\mathfrak{d}})$ sends the image of $HH_n(\mathcal{F}_1)$, or equivalently the image of $HH_n(\mathcal{F}_{\mathfrak{d}})$, to $\bigoplus_{g\in\langle W\Gamma\rangle} (\Omega^n(\mathfrak{t}^g) \otimes \natural^g)^{Z_W\Gamma(g)}$. Hence, for any $x \in HH_n(\mathbb{H})$:

$$HH_1(\mathcal{F}_1)x = \bigoplus\nolimits_{\mathfrak{d}\in\Delta_{\mathbb{H}}} HH_n\big(\mathcal{F}_{\mathfrak{d}}\big)x,$$

$$HH_n\big(\mathcal{F}_{\mathfrak{d}}\big)x \in HH_n(\mathcal{F}_1)HH_n(\mathbb{H}).$$

(b) This subspace is well-defined by the injectivity of $HH_n(\mathcal{F}_1)$ (Theorem 2.8.a). Consider an algebraic family $\mathfrak{F}(Q,\sigma)$ whose tempered part

$$\mathfrak{F}^t(Q,\sigma) = \{\pi(Q,\sigma,\lambda) : \lambda \in \sqrt{-1}\mathfrak{t}^Q_{\mathbb{R}}\}$$

lies in $R_t(\mathbb{H})^{\mathfrak{d}'}$, for some $\mathfrak{d}' \in \Delta_{\mathbb{H}} \setminus \{\mathfrak{d}\}$. If we express $\pi(Q,\sigma,\lambda)$ with $\lambda \in \sqrt{-1}\mathfrak{t}^Q_{\mathbb{R}}$ as in (2.19), all the coefficients $c'(i,\lambda,v_i)$ with $i \not\prec \mathfrak{d}'$ are zero. By construction

$$HH_n(\mathcal{F}_{Q_j,\sigma_j})HH_n(\mathbb{H})^{\mathfrak{d}} = 0 \quad \text{if } j \prec \mathfrak{d}'.$$

Hence $HH_n(\mathcal{F}_{Q,\sigma})HH_n(\mathbb{H})^{\mathfrak{d}}$ consists of algebraic differential forms on $\mathfrak{t}^Q$, which vanish on $\sqrt{-1}\mathfrak{t}^Q_{\mathbb{R}}$. Since $\sqrt{-1}\mathfrak{t}^Q_{\mathbb{R}}$ is Zariski-dense in $\mathfrak{t}^Q$,

$$HH_n(\mathcal{F}_{Q,\sigma})HH_n(\mathbb{H})^{\mathfrak{d}} = 0.$$

On the other hand, by Theorem 2.8 $HH_n(\mathcal{F}_{\mathfrak{d}}) = \bigoplus_{i\prec\mathfrak{d}} HH_n(\mathcal{F}_{Q_i,\sigma_i})$ is injective on $HH_n(\mathbb{H})^{\mathfrak{d}}$. Thus $HH_n(\mathbb{H})^{\mathfrak{d}}$ can be characterized as

$$\{x \in HH_n(\mathbb{H}) : HH_n(\mathcal{F}_{Q,\sigma})x = 0 \text{ for all algebraic families } \mathfrak{F}(Q,\sigma) \text{ with } \mathfrak{F}^t(Q,\sigma) \text{ in } R_t(\mathbb{H})^{\mathfrak{d}'} \text{ for some } \mathfrak{d}' \neq \mathfrak{d}\}. \quad \square$$

From Lemma 2.12 and Theorem 2.8 we conclude:

Corollary 2.13. *There is a canonical decomposition of $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-modules*

$$HH_n(\mathbb{H}) = \bigoplus\nolimits_{\mathfrak{d}\in\Delta_{\mathbb{H}}} HH_n(\mathbb{H})^{\mathfrak{d}}.$$

The injection

$$HH_n(\phi_{\mathfrak{d}}^*) \circ HH_n(\mathcal{F}_{\mathfrak{d}}) : HH_n(\mathbb{H})^{\mathfrak{d}} \to \bigoplus\nolimits_{g\in\langle W\Gamma\rangle} (\Omega^n(\mathfrak{t}^g) \otimes \natural^g)^{Z_{W\Gamma}(g)}$$

is $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-linear if we let $\mathcal{O}(\mathfrak{t})^{W\Gamma}$ act at the stalk over $g \in \langle W\Gamma\rangle, v \in \mathfrak{t}^g$ via evaluation at the central character $W\Gamma(cc(\delta)+v)$ of $\nu_{g,v}^{1,\mathfrak{d}}$.

Example 2.14. Consider the graded Hecke algebra $\mathbb{H}$ with $\mathfrak{t} = \mathbb{C}^2$, Φ of type A_2 with basis $\Delta = \{\alpha = (1,0), \beta = (-1/2, \sqrt{3}/2)\}$ and parameters $k_\alpha = k_\beta = k \in \mathbb{R}_{>0}$. The group Γ and the 2-cocycle $\natural$ are trivial. We have $W \cong S_3$, $\mathfrak{t}_\alpha = \mathbb{C}\times\{0\}$, $\mathfrak{t}^\alpha = \{0\}\times\mathbb{C}$, $\mathfrak{t}^\emptyset = \mathfrak{t}$ and $\mathfrak{t}^\Delta = \{0\}$. For each $\epsilon \in \mathbb{C}$ we need three algebraic families of $\mathbb{H}(\mathfrak{t}, W, \epsilon k)$-representations, namely

- $\mathfrak{F}(\emptyset, \mathrm{triv}) = \{\pi(\emptyset, \mathrm{triv}, \lambda) = \mathrm{ind}_{\mathcal{O}(\mathfrak{t})}^{\mathbb{H}(\mathfrak{t},W,\epsilon k)}(\mathbb{C}_\lambda) : \lambda \in \mathfrak{t}\}$,
- $\mathfrak{F}(\{\alpha\}, \mathrm{St}_\alpha)$, where the Steinberg representation of $\mathbb{H}_\alpha$ is defined by $\mathrm{St}_\alpha|_{\mathbb{C}[W_\alpha]} = \mathrm{sgn}_{W_\alpha}$ and $\mathrm{St}_\alpha|_{\mathcal{O}(\mathfrak{t}_\alpha)} = \mathbb{C}_{-k}$,
- $\mathfrak{F}(\Delta, \mathrm{St})$, where the Steinberg representation of $\mathbb{H}$ is defined by $\mathrm{St}|_{\mathbb{C}[W]} = \mathrm{sgn}_W$ and $\mathrm{St}|_{\mathcal{O}(\mathfrak{t})} = \mathbb{C}_{(-k,-\sqrt{3}k)}$.

Let us identify the virtual representations $\nu_{g,\lambda}^\epsilon$. All maps $\phi_{g,i}$ are the identity, and the scalars $\lambda_{g,i}$ can be determined from direct calculations in the algebra $\mathcal{O}(\mathfrak{t}) \rtimes W = \mathbb{H}(\mathfrak{t}, W, 0)$. The latter reduces further to a calculation in $\mathbb{C}[W]$ because $\mathcal{O}(\mathfrak{t})$ acts as evaluation at 0 on all the relevant representations.

- $\nu_{\mathrm{id},\lambda}^\epsilon = \mathrm{tr}\,\pi(\emptyset, \mathrm{triv}, \lambda)$,
- $\nu_{s_\alpha,\lambda}^\epsilon = -\mathrm{tr}\,\pi(\{\alpha\}, \mathrm{St}_\alpha, 0) + \mathrm{tr}\,\pi(\emptyset, \mathrm{triv}, 0)/2$, because
$$\mathrm{tr}\,\mathrm{ind}_{W_\alpha}^W(\mathrm{sgn}_{W_\alpha}) + \mathrm{tr}\,\mathrm{ind}_{\{\mathrm{id}\}}^W(\mathrm{triv})/2$$
is the trace function on W associated to the conjugacy class of s_α,
- $\nu_{s_\alpha s_\beta,0}^\epsilon = \mathrm{tr}\,\mathrm{St} - \mathrm{tr}\,\pi(\{\alpha\}, \mathrm{St}_\alpha, 0) + \mathrm{tr}\,\pi(\emptyset, \mathrm{triv}, 0)/3$, because
$$\mathrm{tr}\,\mathrm{sgn}_W - \mathrm{tr}\,\mathrm{ind}_{W_\alpha}^W(\mathrm{sgn}_{W_\alpha}) + \mathrm{tr}\,\mathrm{ind}_{\{\mathrm{id}\}}^W(\mathrm{triv})/3$$
is the trace function on W associated to the conjugacy class of $s_\alpha s_\beta$.

When $\epsilon = 0$ or $g = \mathrm{id}$, $\nu_{g,\lambda}^\epsilon$ has central character $W\lambda$. In all other cases $\nu_{g,\lambda}^\epsilon$ does not admit a central character.

In this example $Z_W(s_\alpha) = \{\mathrm{id}, s_\alpha\}$ acts trivially on $\mathfrak{t}^\alpha$, and $Z_W(s_\alpha s_\beta)$ acts trivially on $\mathfrak{t}^\Delta = \{0\}$. Hence the components of $HH_n(\phi^*)$ indexed by s_α and $s_\alpha s_\beta$ do not impose any further restriction on the image of $HH_n(\mathcal{F}_\epsilon)$. The component of $HH_n(\phi^*)$ indexed by id must have image invariant under W, and by the expression for $\nu_{\mathrm{id},\lambda}^\epsilon$ that only puts a condition on the image of $HH_n(\mathcal{F}_{\emptyset,\mathrm{triv}})$. Thus Theorem 2.8 provides a bijection

$$HH_n(\mathcal{F}_\epsilon) : HH_n(\mathbb{H}(\mathfrak{t}, W, \epsilon k)) \to \Omega^n(\mathfrak{t})^W \oplus \Omega^n(\mathfrak{t}^\alpha) \oplus \Omega^n(\{0\}). \tag{2.20}$$

The $\mathcal{O}(\mathfrak{t})^W$-module structure on the right hand is standard on $\Omega^n(\mathfrak{t})^W$, via evaluations at $(-\epsilon k, 0) + \mathfrak{t}^\alpha$ on $\Omega^n(\mathfrak{t}^\alpha)$ and as evaluation at $(-\epsilon k, -\epsilon\sqrt{3}k)$ on $\Omega^n(\{0\})$.

The decomposition of $R_t(\mathbb{H})$ from Theorem 2.2 has three direct summands, indexed precisely by the above three families. Here $R_t(\mathbb{H})^{[P,\delta]}$ is spanned by

$$\{\pi(P, \delta, \lambda) : \lambda \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^P\}.$$

For $\epsilon \in \mathbb{R}_{>0}$, the right hand side of (2.20) is also the decomposition of $HH_n(\mathbb{H}(\mathfrak{t}, W, \epsilon k))$ from Lemma 2.12.a.

2.3. A Morita equivalent algebra.

With applications to p-adic groups in mind we also consider some algebras that are Morita equivalent to twisted graded Hecke algebras. Suppose that $W\Gamma$ is a subgroup of some finite group G and that $\natural$ extends to a 2-cocycle of G (still denoted $\natural$). Then $\mathbb{C}[W\Gamma, \natural]$ is a subalgebra of $\mathbb{C}[G, \natural]$ and G acts on the space

$$V := G \times_{W\Gamma} \mathfrak{t}$$

by left multiplication. We fix a set of representatives $[G/W\Gamma] \subset G$ for $G/W\Gamma$, with $1 \in [G/W\Gamma]$. Consider a $g \in [G/W\Gamma]$. In $g\mathfrak{t}$ we have the root system $g(\Phi)$ with Weyl group $gWg^{-1} \subset G$. We define twisted graded Hecke algebra

$$\mathbb{H}_g = \mathbb{H}(g\mathfrak{t}, gW\Gamma g^{-1}, k^g, \natural),$$

where $k^g_{g\alpha} = k_\alpha$. By construction there is an algebra isomorphism

$$\begin{array}{cccc} \mathrm{Ad}(T_g): & \mathbb{H} & \to & \mathbb{H}_g \\ & fT_w & \mapsto & (f \circ g^{-1}) T_g T_w T_g^{-1} \end{array} \qquad f \in \mathcal{O}(\mathfrak{t}), w \in W\Gamma.$$

Next we the define $\mathbb{H}(V, G, k, \natural)$ as the vector space $\mathcal{O}(V) \otimes_{\mathbb{C}} \mathbb{C}[G, \natural]$ with the multiplication rules:

- $\mathcal{O}(V)$ and $\mathbb{C}[G, \natural]$ are embedded as unital subalgebras,
- for each $g \in [G/W\Gamma]$, $\mathbb{H}_g$ is embedded as a subalgebra with underlying vector space $\mathcal{O}(g\mathfrak{t}) \otimes \mathbb{C}[gW\Gamma g^{-1}, \natural]$,
- if $g, \tilde{g} \in [G/W\Gamma]$ and $g \neq \tilde{g}$, then $h\tilde{h} = 0$ for all $h \in \mathbb{H}_g, \tilde{h} \in \mathbb{H}_{\tilde{g}}$,
- $T_{\tilde{g}} T_g^{-1} h T_g T_{\tilde{g}}^{-1} = \mathrm{Ad}(T_{\tilde{g}}) \mathrm{Ad}(T_g)^{-1} h$ for $g, \tilde{g} \in [G/W\Gamma], h \in \mathbb{H}_g$.

It is easily checked that this determines an associative algebra, which in the case $k = 0$ reduces to $\mathcal{O}(V) \rtimes \mathbb{C}[G, \natural]$. The algebras $\mathbb{H}(V, G, k, \natural)$ and $\mathbb{H}$ are Morita equivalent via the bimodules $1_{\mathfrak{t}} \mathbb{H}(V, G, k, \natural)$ and $\mathbb{H}(V, G, k, \natural) 1_{\mathfrak{t}}$. In particular

$$\text{the inclusion } \mathbb{H} \to \mathbb{H}(V, G, k, \natural) \text{ is a Morita equivalence} \tag{2.21}$$

and induces an isomorphism on Hochschild homology. We want to express $HH_n(\mathbb{H}(V, G, k, \natural))$ so that all the subalgebras $\mathbb{H}_g$ participate on equal terms.

The family of $\mathbb{H}$-representations $\mathfrak{F}(Q_i, \sigma_i)$ gives rise to a family of $\mathbb{H}(V, G, k, \natural)$-representations (still denoted $\mathfrak{F}(Q_i, \sigma_i)$) by applying (2.21), or equivalently by inducing from $\mathbb{H}$ to $\mathbb{H}(V, G, k, \natural)$. The natural isomorphism

$$\mathrm{ind}_{\mathbb{H}}^{\mathbb{H}(V,G,k,\natural)} \pi(Q_i, \sigma_i, \lambda_i) \cong \mathrm{ind}_{\mathbb{H}_g}^{\mathbb{H}(V,G,k,\natural)} \pi(g(Q_i), \mathrm{Ad}(T_g) \cdot \sigma_i, g(\lambda_i)) \tag{2.22}$$

shows that $\mathfrak{F}(Q_i, \sigma_i)$ can also be obtained from the family of $\mathbb{H}_g$-representations $\mathfrak{F}(g(Q_i), \mathrm{Ad}(T_g) \cdot \sigma_i)$. The $\mathcal{O}(\mathfrak{t})^{W\Gamma}$-algebra homomorphism $\mathcal{F}_{Q_i, \sigma_i}$ extends naturally to $\mathbb{H}(G, V, k, \natural)$ with the same formulas, only now for the representations (2.22). Similarly $HH_n(\mathcal{F}_1)$ extends naturally to

$$HH_n(\mathcal{F}_1) : HH_n(\mathbb{H}(V, G, k, \natural)) \to \bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \Omega^n(\mathfrak{t}^{Q_i}).$$

From the canonical isomorphism

$$\Omega^n(g^{-1}) : \bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \Omega^n(\mathfrak{t}^{Q_i}) \to \bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \Omega^n((g\mathfrak{t})^{g(Q_i)}) \tag{2.23}$$

and (2.22) we see that $HH_n(\mathcal{F}_1)$ arises by performing the analogous constructions to the family of $\mathbb{H}_g$-representations $\mathfrak{F}(g(Q_i), \mathrm{Ad}(T_g) \cdot \sigma_i)$.

For $g \in [G/W\Gamma], w \in W\Gamma, 1 \leq i \leq n_{\mathfrak{F}}, v \in g(\mathfrak{t}^w)$ we define

$$\begin{aligned}\phi_{g,w,i} &= \phi_{w,i} \circ g^{-1} : g(\mathfrak{t}^w) \to \mathfrak{t}^{Q_i},\\ \lambda_{g,w,i} &= \natural^w(g)\lambda_{w,i},\\ \nu_{g,w,v} &= \sum\nolimits_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,w,i} \mathrm{ind}_{\mathbb{H}}^{\mathbb{H}(V,G,k,\natural)} \pi(Q_i, \sigma_i, \phi_{g,w,i}(v)).\end{aligned}$$

We let G act on $[G/W\Gamma] \times W\Gamma$ by

$$\tilde{g}(g, w) = (h, \tilde{w} w \tilde{w}^{-1}) \quad \text{if} \quad \tilde{g} g = h \tilde{w} \text{ with } h \in [G/\Gamma], \tilde{w} \in W\Gamma.$$

That can be identified with left multiplication action of G of $G \times_{W\Gamma} W\Gamma$, where $W\Gamma$ acts from the right on G by multiplication and from the left on itself by conjugation.

Lemma 2.15. *For $h \in G, g \in [G/W\Gamma], w \in W\Gamma$ and $v \in g(\mathfrak{t}^w)$:*

$$\nu_{h(g,w),hv} = \natural^{gwg^{-1}}(h)\nu_{g,w,v}.$$

Proof. By definition $\nu_{g,w,v} = \natural^w(g)\nu_{1,w,g^{-1}v}$. Since

$$\nu_{1,w,v} = \mathrm{ind}_{\mathbb{H}}^{\mathbb{H}(V,G,k,\natural)}(\nu^1_{w,v})$$

and since $\mathrm{ind}_{\mathbb{H}}^{\mathbb{H}(V,G,k,\natural)}$ is a Morita equivalence, the $\nu_{1,w,v}$ satisfy the same relations as the $\nu^1_{w,v}$. In particular, by Lemma 2.5.a

$$\nu_{1,\gamma w \gamma^{-1},hv} = \natural^w(\gamma)\nu_{1,w,v} \quad \text{for } \gamma \in W\Gamma.$$

From these properties we deduce, for $h = \tilde{g}\tilde{w} \in G$:

$$\begin{aligned}\nu_{h(1,w),hv} = \nu_{\tilde{g},\tilde{w}w\tilde{w}^{-1},\tilde{g}\tilde{w}v} &= \natural^{\tilde{w}w\tilde{w}^{-1}}(\tilde{g})\nu_{1,\tilde{w}w\tilde{w}^{-1},\tilde{w}v}\\ &= \natural^{\tilde{w}w\tilde{w}^{-1}}(\tilde{g})\natural^w(\tilde{w})\nu_{1,w,v} = \natural^w(h)\nu_{1,w,v},\end{aligned}$$

where the last step relies on Lemma 1.3. It follows that

$$\nu_{h(g,w),hv} = \nu_{hg(1,w),hv} = \natural^w(hg)\nu_{1,w,g^{-1}v} = \natural^w(hg)\natural^w(g)^{-1}\nu_{g,w,v} = \natural^{gwg^{-1}}(h)\nu_{g,w,v},$$

where we used Lemma 1.3 again. □

The version of (1.46) and (2.17) for $\mathbb{H}(V,G,k,\natural)$ is the map $HH_n(\tilde{\phi}^*)$ defined as

$$\bigoplus_{g\in[G/W\Gamma]} \bigoplus_{w\in W\Gamma} \sum_{i=1}^{n_{\mathfrak{F}}} \lambda_{g,w,i} HH_n(\phi^*_{g,w,i}) \ : \ \bigoplus_{i=1}^{n_{\mathfrak{F}}} \Omega^n(\mathfrak{t}^{Q_i}) \to \bigoplus_{g\in[G/W\Gamma]} \bigoplus_{w\in W\Gamma} \Omega^n(g \cdot \mathfrak{t}^w).$$

By (2.22) and (2.23), all the subalgebras $\mathbb{H}_g$ are involved in the same way in $HH_n(\tilde{\phi}^*)$. The map $HH_n(\tilde{\phi}^*)$ is injective for the same reasons as $HH_n(\phi^*)$, see Lemmas 1.12 and 2.7.

Proposition 2.16. (a) *The map*

$$HH_n(\mathcal{F}_1) : HH_n(\mathbb{H}(V,G,k,\natural)) \to \bigoplus\nolimits_{i=1}^{n_{\mathfrak{F}}} \Omega^n(\mathfrak{t}^{Q_i})$$

is an $\mathcal{O}(V)^G$-linear injection.

(b) *The $\mathbb{C}$-linear map*

$$HH_n(\tilde{\phi}^*) : HH_n(\mathcal{F}_1) HH_n(\mathbb{H}(V,G,k,\natural)) \to \Big(\bigoplus_{g\in[G/W\Gamma]} \bigoplus_{w\in W\Gamma} \Omega^n(g \cdot \mathfrak{t}^w) \otimes \natural^{gwg^{-1}} \Big)^G$$

is bijective.

(c) *$HH_0(\mathcal{F}_1)HH_0(\mathbb{H}(V,G,k,\natural))$ equals the set of $\sum_{i=1}^{n_{\mathfrak{F}}} \omega_i \in \bigoplus_{i=1}^{n_{\mathfrak{F}}} \mathcal{O}(\mathfrak{t}^{Q_i})$ for which the map*

$$\mathrm{ind}_{\mathbb{H}}^{\mathbb{H}(V,G,k,\natural)} \pi(Q_i,\sigma_i,\phi_{g,w,i}(v)) \;\mapsto\; \omega_i(\lambda_i) \qquad i=1,\ldots,n_{\mathfrak{F}}, \lambda_i \in \mathfrak{t}^{Q_i}$$

descends to a linear function on $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H}(V,G,k,\natural))$. This provides an isomorphism of $\mathcal{O}(V)^G$-modules

$$HH_0(\mathcal{F}_1)HH_0(\mathbb{H}(V,G,k,\natural)) \cong \big(\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbb{H}(V,G,k,\natural))\big)^*_{\mathrm{reg}}.$$

Proof. (a) Lemma 2.3 says that $HH_n(\mathcal{F}_1)$ is a homomorphism of modules over $\mathcal{O}(V)^G = \mathcal{O}(\mathfrak{t})^{W\Gamma}$. By Lemma 2.4 and (2.21), it is injective.
(b) From Corollary 2.10 we know that the projection of the image of $HH_n(\tilde{\phi}^*)HH_n(\mathcal{F}_1)$ on the summands indexed by $g=1$ is precisely

$$\Big(\bigoplus\nolimits_{w \in W\Gamma} \Omega^n(\mathfrak{t}^w) \otimes \natural^w\Big)^{W\Gamma}. \tag{2.24}$$

By Lemma 2.15 this image consists of G-invariant elements. Hence every element in the image of $HH_n(\tilde{\phi}^*)HH_n(\mathcal{F}_1)$ is determined by its summands with $g=1$. From the natural isomorphism of (2.24) with the asserted image (via removing the summands with $g \neq 1$) we see that $HH_n(\tilde{\phi}^*)HH_n(\mathcal{F}_1)$ indeed has that image.
(c) This follows from Theorem 2.8 and the Morita equivalence (2.21). $\square$

We define a $\mathbb{H}(V,G,k,\natural)$-representation π to be tempered if the $\mathbb{H}$-representation $1_{\mathfrak{t}}\pi$ is tempered. The decomposition from Theorem 2.2 also holds for the category of finite dimensional tempered $\mathbb{H}(V,G,k,\natural)$-representations, by the Morita equivalence (2.21). Hence Lemmas 2.5.b and 2.12 pertain to $\mathbb{H}(V,G,k,\natural)$ as well. Consequently there is a canonical decomposition like in Corollary 2.13:

$$\begin{array}{lll} HH_n(\mathbb{H}(V,G,k,\natural))^{\mathfrak{d}} & = & HH_n(\mathcal{F}_1)^{-1} HH_n(\mathcal{F}_{\mathfrak{d}}) HH_n(\mathbb{H}(V,G,k,\natural))^{\mathfrak{d}}, \\ HH_n(\mathbb{H}(V,G,k,\natural)) & = & \bigoplus_{\mathfrak{d} \in \Delta_{\mathbb{H}}} HH_n(\mathbb{H}(V,G,k,\natural))^{\mathfrak{d}}. \end{array} \tag{2.25}$$

Furthermore $HH_n(\tilde{\phi}^*) \circ HH_n(\mathcal{F}_{\mathfrak{d}})$ is $\mathcal{O}(V)^G$-linear on $HH_n(\mathbb{H}(V,G,k,\natural))^{\mathfrak{d}}$ if we endow the target with the module structure coming from the central characters of the virtual $\mathbb{H}(V,G,k,\natural)$-representations

$$\nu^{\mathfrak{d}}_{g,w,v} = \sum\nolimits_{i=1, i \prec \mathfrak{d}}^{n_{\mathfrak{F}}} \lambda_{g,w,i} \, \mathrm{tr} \, \mathrm{ind}_{\mathbb{H}}^{\mathbb{H}(V,G,k,\natural)} \pi(Q_i,\sigma_i,\phi_{g,w,i}(v)). \tag{2.26}$$

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