ON FORMAL DEGREES OF UNIPOTENT REPRESENTATIONS

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Abstract. Let G be a reductive p-adic group which splits over an unramified extension of the ground field. Hiraga, Ichino and Ikeda conjectured that the formal degree of a square-integrable G-representation π can be expressed in terms of the adjoint γ-factor of the enhanced L-parameter of π. A similar conjecture was posed for the Plancherel densities of tempered irreducible G-representations.

We prove these conjectures for unipotent G-representations. We also derive explicit formulas for the involved adjoint γ-factors.

Contents

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Introduction

Let $\mathcal{G}$ be a connected reductive group defined over a non-archimedean local field K, and write $G = \mathcal{G}(K)$. We are interested in irreducible G-representations, always tacitly assumed to be smooth and over the complex numbers. The most basic example of such representations are the unramified or spherical representations [Mac, Sat] of G, which play a fundamental role in the Langlands correspondence by virtue of the Satake isomorphism.

By a famous result of Borel [Bor1, Cas], the smallest block of the category of of smooth representations of G which contains the spherical representations is the abelian subcategory generated by the unramified minimal principal series representations. The objects in this block are smooth representations which are generated by the vectors which are fixed by an Iwahori subgroup I of G. The study of such Iwahori-spherical representations is a classical topic, about which a lot is known.

The local Langlands correspondence for Iwahori-spherical representations was established by Kazhdan and Lusztig [KaLu], for $\mathcal{G}$ split simple of adjoint type. It parameterizes the irreducible Iwahori-spherical representations with enhanced unramified Deligne–Langlands parameters for G, where a certain condition is imposed on the enhancements. The category of representations of G which naturally completes this picture (by lifting the restriction on the enhancements) is the category of so-called unipotent representations, as envisaged by Lusztig. An irreducible smooth representation of G is called unipotent if its restriction to some parahoric subgroup $P_{\mathfrak{f}}$ of G contains a unipotent representation of $P_{\mathfrak{f}}$ (by which we mean a unipotent representation of the finite reductive quotient of $P_{\mathfrak{f}}$). In the special case that $P_{\mathfrak{f}}$ is an Iwahori subgroup of G, we recover the Iwahori-spherical representations.

Unipotent representations of simple adjoint groups over K were classified by Lusztig [Lus2, Lus3]. The classification has also been worked out when G splits over an unramified extension of K, in several papers. The authors exhibited a local Langlands correspondence for supercuspidal unipotent representations of reductive groups over K in [FeOp, FOS]. Next the second author generalized this to a Langlands parametrization of all tempered unipotent representations in [Opd4]. Finally, with different methods the third author constructed a local Langlands correspondence for all unipotent representations of reductive groups over K [Sol3]. In Theorem 2.1 we show that the approaches from [Opd4] and [Sol3] agree, and we derive some extra properties of these instances of a local Langlands correspondence. (Meanwhile, all this has been generalized to ramified groups [Sol5].)

Hiraga, Ichino and Ikeda [HII] suggested that, for any irreducible tempered representation π of a reductive p-adic group, there is a relation between the Plancherel density of π and the adjoint γ-factor of its L-parameter. In fact, they conjectured an explicit formula, to be sketched below in terms of a (tentative) enhanced L-parameter of π.

Let ${}^L G$ be the Langlands dual group of G, with identity component $G^\vee$. Let $\pi \in \mathrm{Irr}(G)$ be square-integrable modulo centre and suppose that (ϕ_π, ρ_π) is its enhanced L-parameter (so we need to assume that a local Langlands correspondence has been worked out for π). To measure the size of the L-packet we use the group

$$S^\sharp_{\phi_\pi} := \pi_0\big(Z_{(G/Z(G)_s)^\vee}(\phi_\pi)\big), \tag{1}$$

where $Z(G)_s$ denotes the maximal K-split central torus in G. Let $\mathbf{W}_K$ be the Weil group of K and let $\mathrm{Ad}_{G^\vee}$ denote the adjoint representation of ${}^L G$ on

$$\mathrm{Lie}(G^\vee)/\mathrm{Lie}\big(Z(G^\vee)^{\mathbf{W}_K}\big) \cong \mathrm{Lie}\big((G/Z(G)_s)^\vee\big).$$

Let $\psi : K \to \mathbb{C}^\times$ be a character of order 0, that is, trivial on the ring of integers $\mathfrak{o}_K$ but nontrivial on any larger fractional ideal. We endow K with the Haar measure that gives $\mathfrak{o}_K$ volume 1. We refer to (72) for the definition of the adjoint γ-factor $\gamma(s, \mathrm{Ad}_{G^\vee} \circ \phi, \psi)$.

We normalize the Haar measure on G as in [GaGr, HII]. (For ramified groups the normalizations in [HII, (1.1)] and [HII, Correction] are not entirely satisfactory, see [FOS, (A.23)] for an improvement.) It was conjectured in [HII, §1.4] that

$$\mathrm{fdeg}(\pi) = \dim(\rho_\pi) |S^\sharp_{\phi_\pi}|^{-1} |\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\pi, \psi)|. \tag{2}$$

More generally, let $\mathcal{P} = \mathcal{M}\mathcal{U}$ be a parabolic K-subgroup of $\mathcal{G}$, with Levi factor $\mathcal{M}$ and unipotent radical $\mathcal{U}$. Let $\pi \in \mathrm{Irr}(M)$ be square-integrable modulo centre and let $X_{\mathrm{unr}}(M)$ be the group of unitary unramified characters of M. Let $\mathcal{O} = X_{\mathrm{unr}}(M)\pi \subset \mathrm{Irr}(M)$ be the orbit in $\mathrm{Irr}(M)$ of π, under twists by $X_{\mathrm{unr}}(M)$. We define a Haar measure of $\mathrm{d}\mathcal{O}$ on $\mathcal{O}$ as in [Wal, p. 239 and 302]. This also provides a Haar measure on the family of (finite length) G-representations $I_P^G(\pi')$ with $\pi' \in \mathcal{O}$.

Denote the adjoint representation of ${}^L M$ on $\mathrm{Lie}\big(G^\vee)/\mathrm{Lie}(Z(M^\vee)^{\mathbf{W}_K}\big)$ by $\mathrm{Ad}_{G^\vee, M^\vee}$.

Conjecture 1. [HII, §1.5] *Suppose that the enhanced L-parameter of $\pi \in \mathrm{Irr}(M)$ is (ϕ_π, ρ_π). Then the Plancherel density at $I_P^G(\pi) \in \mathrm{Rep}(G)$ is*

$$c_M \dim(\rho_\pi) |S^\sharp_{\phi_\pi}|^{-1} |\gamma(0, \mathrm{Ad}_{G^\vee, M^\vee} \circ \phi_\pi, \psi)| \, \mathrm{d}\mathcal{O}(\pi),$$

for some constant $c_M \in \mathbb{R}_{>0}$ independent of K and $\mathcal{O}$.

We point out that the validity of (2) and of Conjecture 1 does not depend on the choice of the additive character $\psi : K \to \mathbb{C}^\times$. For another choice of ψ the adjoint γ-factors will be different [HII, Lemma 1.3]. But also the normalization of the Haar measure on G has to be modified, which changes the formal degrees [HII, Lemma 1.1]. These two effects precisely compensate each other.

We note that representations of the form $I_P^G(\pi)$ are tempered [Wal, Lemme III.2.3] and that almost all of them are irreducible [Wal, Proposition IV.2.2]. Every irreducible tempered G-representation appears as a direct summand of $I_P^G(\pi_M)$, for suitable choices of the involved objects [Wal, Proposition III.4.1]. Moreover, if $I_P^G(\pi_M)$ is reducible, its decomposition can be analysed quite explicitely in terms of R-groups [Sil1]. In this sense Conjecture 1 provides an expression for the Plancherel densities of all tempered irreducible G-representations.

In the remainder of the introduction we assume that G splits over an unramified field extension. The HII-conjectures were proven for supercuspidal unipotent representations in [Ree1, FeOp, Feng, FOS], for unipotent representations of simple adjoint groups in [Opd3] and for tempered unipotent representations in [Opd4]. However, in the last case the method only sufficed to establish the desired formulas up to a constant. Of course the formal degree of a square-integrable representation is just a number, so a priori one gains nothing from knowing it up to a constant. Fortunately, the formal degree of a unipotent square-integrable representation can be considered as a rational function of the cardinality q of the residue field of K

[Opd3]. Then "up to a constant" actually captures a substantial part of the information. The main result of this paper is a complete proof of the HII-conjectures for unipotent representations:

Theorem 2. *Let $\mathcal{G}$ be a connected reductive K-group which splits over an unramified extension and write $G = \mathcal{G}(K)$. Use the local Langlands correspondence for unipotent G-representations from Theorem 2.1.*

(*a*) *The HII-conjecture* (2) *holds for all unipotent, square-integrable modulo centre G-representations.*

(*b*) *Conjecture 1 holds for tempered unipotent G-representations, in the following slightly stronger form:*

$$\mathrm{d}\mu_{Pl}(I_P^G(\pi)) = \pm \dim(\rho_\pi)|S^\sharp_{\phi_\pi}|^{-1}\gamma(0, \mathrm{Ad}_{G^\vee,M^\vee} \circ \phi_\pi, \psi)\, \mathrm{d}\mathcal{O}(\pi).$$

In the appendix we work out explicit formulas for the above adjoint γ-factors, in terms of a maximal torus $T^\vee \subset G^\vee$ and the root system of $(G^\vee, T^\vee)$ (Lemma A.2 and Theorem A.4). These expressions can also be interpreted with μ-functions for a suitable affine Hecke algebra [Opd1]. The calculations entail in particular that all involved adjoint γ-factors are real numbers (Lemma A.5).

Our proof of Theorem 2 proceeds stepwise, in increasing generality. The most difficult case is unipotent square-integrable representations of semisimple groups. The argument for that case again consists of several largely independent parts. First we recall (§5.1) that (2) has already been proven for square-integrable representations of adjoint groups [Opd3, FOS].

Our main strategy is pullback of representations along the adjoint quotient map $\eta : \mathcal{G} \to \mathcal{G}_{\mathrm{ad}}$. The homomorphism of K-rational points $\eta : G \to G_{\mathrm{ad}}$ need not be surjective, so this pullback operation need not preserve irreducibility of representations. For $\pi_{\mathrm{ad}} \in \mathrm{Irr}(G_{\mathrm{ad}})$ the computation of the length of $\eta^*(\pi_{\mathrm{ad}})$ has two aspects. On the one hand we determine in §4 how many Bernstein components for G are involved. On the other hand, we study the decomposition within one Bernstein component in §3. The latter is done in terms of affine Hecke algebras, via the types and Hecke algebras from [Mor1, Mor2, Lus2]. Considerations with affine Hecke algebras also allow us to find the exact ratio between $\mathrm{fdeg}(\pi_{\mathrm{ad}})$ and the formal degree of any irreducible constituent of $\eta^*(\pi_{\mathrm{ad}})$, see Theorem 3.4.

On the Galois side of the local Langlands correspondence, the comparison between G and G_{ad} is completely accounted for by results from [Sol4]. In Lemma 2.3 we put those in the form that we actually need. With all these partial results at hand, we finish the computation of the formal degrees of unipotent square-integrable representations of semisimple groups in Theorem 5.4.

After a first version of this paper appeared, we learned that Gan and Ichino [GaIc] had already devised a different method to reduce the proof of (2) from semisimple groups to adjoint groups. Their argument is much shorter, but it applies only when K is a p-adic field and $\mathcal{G}$ is an inner form of a K-split group. We work this out in Appendix B.

The generalization from semisimple groups to square-integrable modulo centre representations of reductive groups (§5.3) is not difficult, because the unipotent representations of a p-adic torus are just the characters trivial on the unique parahoric subgroup. That proves part (a) of Theorem 2.

To get part (b) for square-integrable modulo centre representations (so with $M = G$), we need to carefully normalize the involved Plancherel measures (§6.1). In §6.2 we establish part (b) for any Levi subgroup $M \subset G$. This involves a translation to Plancherel densities for affine Hecke algebras, via the aforementioned types. In the final stage we use that Theorem 2 was already known up to constants [Opd4].

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1. Background on unipotent representations

Let K be a non-archimedean local field with ring of integers $\mathfrak{o}_K$ and uniformizer ϖ_K. Let $k = \mathfrak{o}_K / \varpi_K \mathfrak{o}_K$ be its residue field, of cardinality $q = q_K$.

Let K_s be a separable closure of K. Let $\mathbf{W}_K \subset \mathrm{Gal}(K_s/K)$ be the Weil group of K and let Frob be an arithmetic Frobenius element. Let $\mathbf{I}_K$ be the inertia subgroup of $\mathrm{Gal}(K_s/K)$, so that $\mathbf{W}_K/\mathbf{I}_K \cong \mathbb{Z}$ is generated by Frob.

Let $\mathcal{G}$ be a connected reductive K-group. Let $\mathcal{T}$ be a maximal torus of $\mathcal{G}$, and let $\Phi(\mathcal{G},\mathcal{T})$ be the associated root system. We also fix a Borel subgroup $\mathcal{B}$ of $\mathcal{G}$ containing $\mathcal{T}$, which determines a basis Δ of $\Phi(\mathcal{G},\mathcal{T})$.

Let $\Phi(\mathcal{G},\mathcal{T})^\vee$ be the dual root system of $\Phi(\mathcal{G},\mathcal{T})$, contained in the cocharacter lattice $X_*(\mathcal{T})$. The based root datum of $\mathcal{G}$ is

$$\big(X^*(\mathcal{T}), \Phi(\mathcal{G},\mathcal{T}), X_*(\mathcal{T}), \Phi(\mathcal{G},\mathcal{T})^\vee, \Delta\big).$$

Let $\mathcal{S}$ be a maximal K-split torus in $\mathcal{G}$. By [Spr, Theorem 13.3.6.(i)] applied to $Z_{\mathcal{G}}(\mathcal{S})$, we may assume that $\mathcal{T}$ is defined over K and contains $\mathcal{S}$. Then $Z_{\mathcal{G}}(\mathcal{S})$ is a minimal K-Levi subgroup of $\mathcal{G}$. Let

$$\Delta_0 := \{\alpha \in \Delta : \mathcal{S} \subset \ker \alpha\}$$

be the set of simple roots of $(Z_{\mathcal{G}}(\mathcal{S}),\mathcal{T})$. Recall from [Spr, Lemma 15.3.1] that the root system $\Phi(\mathcal{G},\mathcal{S})$ is the image of $\Phi(\mathcal{G},\mathcal{T})$ in $X^*(\mathcal{S})$, without 0. The set of simple roots of $(\mathcal{G},\mathcal{S})$ can be identified with $(\Delta \setminus \Delta_0)/\mu_{\mathcal{G}}(\mathbf{W}_K)$, where $\mu_{\mathcal{G}}$ denotes the action of $\mathrm{Gal}(K_s/K)$ on Δ determined by $(\mathcal{B},\mathcal{T})$.

We write $G = \mathcal{G}(K)$ and similarly for other K-groups. Let $\mathcal{G}^\vee$ be the split reductive group with based root datum

$$\big(X_*(\mathcal{T}), \Phi(\mathcal{G},\mathcal{T})^\vee, X^*(\mathcal{T}), \Phi(\mathcal{G},\mathcal{T}), \Delta^\vee\big).$$

Then $G^\vee = \mathcal{G}^\vee(\mathbb{C})$ is the complex dual group of $\mathcal{G}$. Via the choice of a pinning, the action $\mu_{\mathcal{G}}$ of $\mathbf{W}_K$ on the root datum of $\mathcal{G}$ determines an action of $\mathbf{W}_K$ of $G^\vee$. That action stabilizes the torus $T^\vee = X^*(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{C}^\times$ and the Borel subgroup $B^\vee$ determined by $T^\vee$ and $\Delta^\vee$. The Langlands dual group (in the version based on $\mathbf{W}_K$) of $\mathcal{G}(K)$ is ${}^L G := G^\vee \rtimes \mathbf{W}_K$.

Define the abelian group

$$\Omega = X_*(\mathcal{T})_{\mathbf{I}_K} / (\mathbb{Z}\Phi(\mathcal{G},\mathcal{T})^\vee)_{\mathbf{I}_K}.$$

Then $Z(G^\vee)$ can be identified with $\mathrm{Irr}(\Omega) = \Omega^*$, and Ω is naturally isomorphic to the group $X^*(Z(G^\vee))$ of algebraic characters of $Z(G^\vee)$. In particular

$$\Omega^{\mathbf{W}_K} \cong X^*\big(Z(G^\vee)\big)^{\mathbf{W}_K} = X^*\big(Z(G^\vee)_{\mathbf{W}_K}\big). \tag{3}$$

In [FOS] this group is called Ω^θ, while in [Sol3] the notation Ω is used for a group naturally isomorphic to (3). To indicate the underlying p-adic group and to reconcile the notations from [FOS] and [Sol3] we write

$$\Omega_G = \Omega^{\mathbf{W}_K}.$$

Kottwitz defined a natural, surjective group homomorphism $\kappa_G : G \to \Omega_G$. (The definition of Ω in [Sol3] is equivalent to $G/\ker(\kappa_G)$.) The action of $\ker(\kappa_G)$ on the Bruhat–Tits building preserves the types of facets, i.e. preserves a coloring of the vertices. Further, the kernel of κ_G contains the image (in G) of the simply connected cover of the derived group of G, see [PaRa, Appendix]. We say that a character of G is weakly unramified if it is trivial on $\ker(\kappa_G)$. Thus the group $X_{\mathrm{wr}}(G)$ of weakly unramified characters of G can be identified with the Pontryagin dual of Ω_G.

Let $Z(\mathcal{G})_s$ be the maximal K-split torus in $Z(\mathcal{G})$. As $H^1(K, Z(\mathcal{G})_s) = 1$, there is a short exact sequence

$$1 \to Z(\mathcal{G})_s(K) \to \mathcal{G}(K) \to (\mathcal{G}/Z(\mathcal{G})_s)(K) \to 1. \tag{4}$$

In view of the naturality of the Kottwitz homomorphism κ_G, this induces a short exact sequence

$$1 \to \Omega_{Z(G)_s} \to \Omega_G \to \Omega_{G/Z(G)_s} \to 1. \tag{5}$$

Recall [Lus1, Part 3] that an irreducible representation of a reductive group over a finite field is called unipotent if it appears in the Deligne–Lusztig series associated to the trivial character of a maximal torus in that group. An irreducible representation of a linear algebraic group over $\mathfrak{o}_K$ is called unipotent if its arises, by inflation, from a unipotent representation of the maximal finite reductive quotient of the group.

We call an irreducible smooth G-representation π unipotent if there exists a parahoric subgroup $P_{\mathfrak{f}} \subset G$ such that $\pi|_{P_{\mathfrak{f}}}$ contains an irreducible unipotent representation of $P_{\mathfrak{f}}$. Then the restriction of π to some smaller parahoric subgroup $P_{\mathfrak{f}'} \subset G$ contains a cuspidal unipotent representation of $P_{\mathfrak{f}'}$, as required in [Lus2]. An arbitrary smooth G-representation is unipotent if it lies in a product of Bernstein components, all whose cuspidal supports are unipotent.

The category of unipotent G-representations can be described in terms of types and affine Hecke algebras. For a facet $\mathfrak{f}$ of the Bruhat–Tits building $\mathcal{B}(\mathcal{G}, K)$ of G, let $\mathcal{G}_{\mathfrak{f}}$ be the smooth affine $\mathfrak{o}_K$-group scheme from [BrTi], such that $\mathcal{G}_{\mathfrak{f}}^\circ$ is a $\mathfrak{o}_K$-model of $\mathcal{G}$ and $\mathcal{G}_f^\circ(\mathfrak{o}_K)$ equals the parahoric subgroup $P_{\mathfrak{f}}$ of G. Then $\hat{P}_{\mathfrak{f}} := \mathcal{G}_{\mathfrak{f}}(\mathfrak{o}_K)$ is the pointwise stabilizer of $\mathfrak{f}$ in G. Let $\overline{\mathcal{G}_{\mathfrak{f}}}$ be the maximal reductive quotient of the k-group scheme obtained from $\mathcal{G}_{\mathfrak{f}}$ by reduction modulo ϖ_K. Thus

$$\overline{\mathcal{G}_{\mathfrak{f}}}(k) = \hat{P}_{\mathfrak{f}}/U_{\mathfrak{f}} \quad \text{and} \quad \overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k) = P_{\mathfrak{f}}/U_{\mathfrak{f}},$$

where $U_{\mathfrak{f}}$ is the pro-unipotent radical of $P_{\mathfrak{f}}$. We normalize the Haar measure on G as in [GaGr, HII]. When $\mathcal{G}$ splits over an unramified extension of K, the computation of the volume of the Iwahori subgroup of G [Gro, (4.11)] says that

$$\mathrm{vol}(P_{\mathfrak{f}}) = |\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k)| \, q^{-(\dim \overline{\mathcal{G}_{\mathfrak{f}}^\circ} + \dim \mathcal{G})/2} \tag{6}$$

By [DeRe, §5.1] this actually holds for every facet $\mathfrak{f}$. We note that with the counting formulas for reductive groups over finite fields [Car1, Theorem 9.4.10], $|\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k)|$ can be considered as a polynomial in $q = |k|$.

Replacing the involved objects by a suitable G-conjugate, we can achieve that $\mathfrak{f}$ lies in the closure of a fixed "standard" chamber C_0 of the apartment of $\mathcal{B}(\mathcal{G}, K)$ associated to $\mathcal{S}$. Since $\mathcal{G}$ splits over an unramified extension, the group $\Omega_G = \Omega^{\mathbf{W}_K}$

from (3) equals Ω^{Frob}. It acts naturally on $\overline{C_0}$, and we denote the setwise stablizer of $\mathfrak{f}$ by $\Omega_{G,\mathfrak{f}}$ and the pointwise stabilizer of $\mathfrak{f}$ by $\Omega_{G,\mathfrak{f},\mathrm{tor}}$. It was noted in [Sol3, (32)] that

$$\hat{P}_{\mathfrak{f}}/P_{\mathfrak{f}} \cong \Omega_{G,\mathfrak{f},\mathrm{tor}}. \tag{7}$$

Suppose that (σ, V_σ) is a cuspidal unipotent representation of $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k)$ (in particular this includes that it is irreducible). We inflate it to a representation of $P_{\mathfrak{f}}$, still denoted σ. It was shown in [MoPr, §6] and [Mor2, Theorem 4.8] that $(P_{\mathfrak{f}}, \sigma)$ is a type for G. Let $\mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)}$ be the corresponding direct factor of $\mathrm{Rep}(G)$. By [Lus2, 1.6.b]

$$\begin{array}{ll} \mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)} = \mathrm{Rep}(G)_{(P_{\mathfrak{f}'},\sigma')} & \text{if } g\mathfrak{f}' = \mathfrak{f}, \mathrm{Ad}(g)^* \sigma = \sigma' \text{ for some } g \in G \\ \mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)} \cap \mathrm{Rep}(G)_{(P_{\mathfrak{f}'},\sigma')} = \{0\} & \text{otherwise.} \end{array} \tag{8}$$

By [Lus2, §1.16] and [FOS, Lemma 15.7] σ can be extended (not uniquely) to a representation of $\overline{\mathcal{G}_{\mathfrak{f}}}(k)$, which we inflate to an irreducible representation of $\hat{P}_{\mathfrak{f}}$ that we denote by $(\hat{\sigma}, V_\sigma)$. It is known from [Mor2, Theorem 4.7] that $(\hat{P}_{\mathfrak{f}}, \hat{\sigma})$ is a type for a single Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$. Conversely, every Bernstein block consisting of unipotent G-representations is of this form. We note that $\mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)}$ is the direct sum of the $\mathrm{Rep}(G)^{\mathfrak{s}}$ associated to the different extensions of σ to $\hat{P}_{\mathfrak{f}}$.

To $(\hat{P}_{\mathfrak{f}}, \hat{\sigma})$ Bushnell and Kutzko associated the algebra

$$\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma}) = \mathrm{End}_G\big(\mathrm{ind}_{\hat{P}_{\mathfrak{f}}}^G(\hat{\sigma})\big)^{\mathrm{opp}}, \tag{9}$$

where the superscript means "opposite algebra". In [BuKu] it is shown that

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \to & \mathrm{Mod}(\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})) \\ \pi & \mapsto & \mathrm{Hom}_{\hat{P}_{\mathfrak{f}}}(\hat{\sigma}, \pi) \end{array} \tag{10}$$

is an equivalence of categories. It turns out that $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$ is an (extended) affine Hecke algebra, see [Lus2, §1] and [Sol3, §3]. Moreover a finite length representation in $\mathrm{Rep}(G)^{\mathfrak{s}}$ is tempered (resp. essentially square-integrable) if and only if the associated $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$-module is tempered (resp. essentially discrete series) [BHK, Theorem 3.3.(1)].

The (extended) affine Hecke algebra $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$ comes with the following data:

- a lattice $X_{\mathfrak{f}}$ and a complex torus $T_{\mathfrak{f}} = \mathrm{Irr}(X_{\mathfrak{f}})$;
- a root system $R_{\mathfrak{f}}$ in $X_{\mathfrak{f}}$, with a basis $\Delta_{\mathfrak{f}}$;
- a Coxeter group $W_{\mathrm{aff}} = W(R_{\mathfrak{f}}) \ltimes \mathbb{Z}R_{\mathfrak{f}}$ in $W(R_{\mathfrak{f}}) \ltimes X_{\mathfrak{f}}$;
- a set $S_{\mathfrak{f},\mathrm{aff}}$ of affine reflections, which are Coxeter generators of W_{aff};
- a parameter function $q^{\mathcal{N}} : W_{\mathrm{aff}} \to \mathbb{R}_{>0}$.

Furthermore it has a distinguished basis $\{N_w : w \in W(R_{\mathfrak{f}}) \ltimes X_{\mathfrak{f}}\}$, an involution $*$ and a trace τ. Thus $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$ has the structure of a Hilbert algebra, and one can define a Plancherel measure and formal degrees for its representations. The unit element N_e of $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$ is the central idempotent $e_{\hat{\sigma}}$ (in the group algebra of $\hat{P}_{\mathfrak{f}}$) associated to $\hat{\sigma}$. The trace τ is normalized so that

$$\tau(N_w) = \left\{ \begin{array}{ll} e_{\hat{\sigma}}(1) = \dim(\hat{\sigma}) \mathrm{vol}(\hat{P}_{\mathfrak{f}})^{-1} & w = e \\ 0 & w \neq e \end{array} \right. . \tag{11}$$

It follows from [BHK, Theorem 3.3.(2)] that, with this normalization, the equivalence of categories (10) preserves Plancherel measures and formal degrees. For affine Hecke algebras, these were analysed in depth in [Opd1, OpSo, CiOp].

Consider a discrete series representation δ of $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$, with central character $W(R_{\mathfrak{f}})r \in T_{\mathfrak{f}}/W(R_{\mathfrak{f}})$. By [Opd1] its formal degree can be expressed as

$$\text{fdeg}(\delta) = \pm \dim(\hat{\sigma})\text{vol}(\hat{P}_{\mathfrak{f}})^{-1} d_{\mathcal{H},\delta} m(q^{\mathcal{N}})^{(r)}, \tag{12}$$

where $d_{\mathcal{H},\delta} \in \mathbb{Q}_{>0}$ is computed in [CiOp] (often it is just 1). The factor $m(q^{\mathcal{N}})$ is a rational function in $r \in T_{\mathfrak{f}}$ and the parameters $q^{\mathcal{N}}(s_\alpha)^{1/2}$ with $s_\alpha \in S_{\mathfrak{f},\text{aff}}$, while the superscript (r) indicates that we take its residue at r. We refer to (71) and (94) for the explicit definition of $m(q^{\mathcal{N}})$.

2. Langlands parameters

Recall that a Langlands parameter for G is a homomorphism

$$\phi : \mathbf{W}_K \times SL_2(\mathbb{C}) \to {}^L G = G^\vee \rtimes \mathbf{W}_K,$$

with some extra requirements. In particular $\phi|_{SL_2(\mathbb{C})}$ has to be algebraic, $\phi(\mathbf{W}_K)$ must consist of semisimple elements and ϕ must respect the projections to $\mathbf{W}_K$.

We say that a L-parameter ϕ for G is

- discrete if it does not factor through the L-group of any proper Levi subgroup of G;
- bounded if $\phi(\text{Frob}) = (s, \text{Frob})$ with s in a bounded subgroup of $G^\vee$;
- unramified if $\phi(w) = (1, w)$ for all $w \in \mathbf{I}_K$.

Let ${G^\vee}_{\text{ad}}$ be the adjoint group of $G^\vee$, and let ${G^\vee}_{\text{sc}}$ be its simply connected cover. Let $\mathcal{G}^*$ be the unique K-quasi-split inner form of $\mathcal{G}$. We consider $\mathcal{G}$ as an inner twist of $\mathcal{G}^*$, so endowed with a K_s-isomorphism $\mathcal{G} \to \mathcal{G}^*$. Via the Kottwitz isomorphism $\mathcal{G}$ is labelled by a character $\zeta_{\mathcal{G}}$ of $Z({G^\vee}_{\text{sc}})^{\mathbf{W}_K}$ (defined with respect to $\mathcal{G}^*$). We choose an extension ζ of $\zeta_{\mathcal{G}}$ to $Z({G^\vee}_{\text{sc}})$. As explained in [FOS, §1], this is related to the explicit realization of $\mathcal{G}$ as an inner twist of $\mathcal{G}^*$.

Both ${G^\vee}_{\text{ad}}$ and ${G^\vee}_{\text{sc}}$ act on $G^\vee$ by conjugation. As

$$Z_{G^\vee}(\text{im } \phi) \cap Z(G^\vee) = Z(G^\vee)^{\mathbf{W}_K},$$

we can regard $Z_{G^\vee}(\text{im } \phi)/Z(G^\vee)^{\mathbf{W}_K}$ as a subgroup of ${G^\vee}_{\text{ad}}$. Let $Z^1_{{G^\vee}_{\text{sc}}}(\text{im } \phi)$ be its inverse image in ${G^\vee}_{\text{sc}}$ (it contains $Z_{{G^\vee}_{\text{sc}}}(\text{im } \phi)$ with finite index). A subtle version of the component group of ϕ is

$$\mathcal{A}_\phi := \pi_0\big(Z^1_{{G^\vee}_{\text{sc}}}(\text{im } \phi)\big).$$

It is related to the component group $S^\sharp_\phi$ from (1) by natural maps

$$\mathcal{A}_\phi \longleftarrow \pi_0\big(Z_{{G^\vee}_{\text{sc}}}(\text{im } \phi)\big) \longrightarrow \pi_0\big(Z_{(G/Z(G)_s)^\vee}(\phi)\big) = S^\sharp_\phi,$$

of which the first is injective and, when $G/Z(G)_s$ is semisimple, the second is surjective. An enhancement of ϕ is an irreducible representation ρ of $\mathcal{A}_\phi$.

Via the canonical map $Z({G^\vee}_{\text{sc}}) \to \mathcal{A}_\phi$, ρ determines a character ζ_ρ of $Z({G^\vee}_{\text{sc}})$. We say that an enhanced L-parameter (ϕ, ρ) is relevant for G if $\zeta_\rho = \zeta$. This can be reformulated with G-relevance of ϕ in terms of Levi subgroups [HiSa, Lemma 9.1]. To be precise, in view of [Bor2, §3] there exists an enhancement ρ such that (ϕ, ρ) is G-relevant if and only if every L-Levi subgroup of ${}^L G$ containing the image of ϕ is

G-relevant. The group $G^\vee$ acts naturally on the collection of G-relevant enhanced L-parameters, by

$$g \cdot (\phi, \rho) = (g \phi g^{-1}, \rho \circ \mathrm{Ad}(g)^{-1}).$$

We denote the set of $G^\vee$-equivalence classes of G-relevant (resp. enhanced) L-parameters by $\Phi(G)$, resp. $\Phi_e(G)$. A local Langlands correspondence for G (in its modern interpretation) should be a bijection between $\Phi_e(G)$ and the set $\mathrm{Irr}(G)$ of (isomorphism classes of) irreducible smooth G-representations, with several nice properties.

We denote the set of irreducible unipotent (resp. cuspidal) G-representations by $\mathrm{Irr}_{\mathrm{unip}}(G)$ (resp. $\mathrm{Irr}_{\mathrm{cusp}}(G)$). Let $\Phi_{\mathrm{nr}}(G)$ (resp. $\Phi_{\mathrm{nr},e}(G)$) be the subset of $\Phi(G)$ (resp. $\Phi_e(G)$) formed by the unramified L-parameters. Recall from [AMS1] that there is a notion of cuspidality for enhanced L-parameters and that the cuspidal support map Sc associates to each enhanced L-parameter for G a cuspidal L-parameter for a Levi subgroup of G (unique up $G^\vee$-conjugacy).

The next theorem is a combination of the main results of [FOS, Sol3, Opd4].

Theorem 2.1. *Let $\mathcal{G}$ be a connected reductive K-group which splits over an unramified extension. There exists a bijection*

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{unip}}(G) & \longrightarrow & \Phi_{\mathrm{nr},e}(G) \\ \pi & \mapsto & (\phi_\pi, \rho_\pi) \\ \pi(\phi,\rho) & \leftarrow\!\shortmid & (\phi, \rho) \end{array}.$$

We can construct such a bijection for every group G of this kind, in a compatible way. The resulting family of bijections satisfies the following properties.

(a) Compatibility with direct products of reductive K-groups.
(b) Equivariance with respect to the canonical actions of the group $X_{\mathrm{wr}}(G)$ of weakly unramified characters of G.
(c) The central character of π equals the character of $Z(G)$ determined by ϕ_π.
(d) π is tempered if and only if ϕ_π is bounded.
(e) π is essentially square-integrable if and only if ϕ_π is discrete.
(f) π is supercuspidal if and only if (ϕ_π, ρ_π) is cuspidal.
(g) The analogous bijections for the Levi subgroups of G and the cuspidal support maps Sc form a commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{unip}}(G) & \longrightarrow & \Phi_{\mathrm{nr},e}(G) \\ \downarrow Sc & & \downarrow Sc \\ \bigsqcup_M \mathrm{Irr}_{\mathrm{cusp,unip}}(M)/N_G(M) & \longrightarrow & \bigsqcup_M \Phi_{\mathrm{nr,cusp}}(M)/N_{G^\vee}(M^\vee \rtimes \mathbf{W}_K) \end{array}.$$

Here M runs over a collection of representatives for the conjugacy classes of Levi subgroups of G.
(h) Suppose that $P = MU$ is a parabolic subgroup of G and that $(\phi, \rho^M) \in \Phi_{\mathrm{nr},e}(M)$ is bounded. Then the normalized parabolically induced representation $I_P^G \pi(\phi, \rho^M)$ is a direct sum of representations $\pi(\phi, \rho)$, with multiplicities $[\rho^M : \rho]_{\mathcal{A}_\phi^M}$.
(i) Compatibility with the Langlands classification for representations of reductive groups and the Langlands classification for enhanced L-parameters.
(j) Compatibility with restriction of scalars of reductive groups over non-archimedean local fields.

(k) Let $\tilde{\mathcal{G}}$ be a group of the same kind as $\mathcal{G}$, and let $\eta : \tilde{\mathcal{G}} \to \mathcal{G}$ be a homomorphism of K-groups such that the kernel of $\mathrm{d}\eta : \mathrm{Lie}(\tilde{\mathcal{G}}) \to \mathrm{Lie}(\mathcal{G})$ is central and the cokernel of η is a commutative K-group. Let ${}^L\eta : {}^L\tilde{G} \to {}^LG$ be the dual homomorphism and let $\phi \in \Phi_{\mathrm{nr}}(G)$.

Then the L-packet $\Pi_{{}^L\eta\circ\phi}(\tilde{G}) = \{\tilde{\pi} \in \mathrm{Irr}(\tilde{G}) : \phi_{\tilde{\pi}} = \phi\}$ consists precisely of the constituents of the completely reducible $\tilde{G}$-representations $\eta^*(\pi)$ with $\pi \in \Pi_\phi(G)$.

(l) Conjecture 1 holds for tempered unipotent G-representations, up to some rational constants that do not change if we replace $I^G_{MU}(\pi)$ by $I^G_{MU}(\chi\pi)$ with $\chi \in X_{\mathrm{unr}}(M)$.

Moreover the above properties uniquely determine the surjection

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{unip}}(G) & \to & \Phi_{\mathrm{nr}}(G)/X_{\mathrm{wr}}(G, Z(G)) \\ \pi & \mapsto & X_{\mathrm{wr}}(G, Z(G))\phi_\pi \end{array},$$

where $X_{\mathrm{wr}}(G, Z(G))$ denotes the group of weakly unramified characters of G that are trivial on $Z(G)$.

Remark. We regard this as a local Langlands correspondence for unipotent representations. We point out that for simple adjoint groups Theorem 2.1 differs somewhat from the main results of [Lus2, Lus3] – which do not satisfy (d) and (e). In [AMS2, §3.5] this is fixed by composing a parametrization of irreducible representations with the Iwahori-Matsumoto involution of a Hecke algebra, and that propagates to a difference between Theorem 2.1 and Lusztig's parametrization.

Proof. A bijection satisfying the properties (a)–(i) was exhibited in [Sol3, §5]. The construction involves some arbitrary choices, we will fix some of those here.

For property (j) see [FOS, Lemma A.3] and [Sol3, Lemma 2.4]. For property (k) we refer to [Sol4, Corollary 5.8 and §7].

Denote the set of (isomorphism classes of) tempered irreducible smooth G-representations by $\mathrm{Irr}_{\mathrm{temp}}(G)$ and let $\Phi_{\mathrm{bdd}}(G)$ be the collection of bounded L-parameters for G. It was shown in [Opd4, Theorem 4.5.1] that there exists a "Langlands parametrization"

$$\phi_{HII} : \mathrm{Irr}_{\mathrm{unip,temp}}(G) \to \Phi_{\mathrm{nr,bdd}}(G) \tag{13}$$

which satisfies the above property (l) and is unique up to twists by certain weakly unramified characters. Notice that the image of ϕ_{HII} consists of L-parameters, not enhanced as before. For supercuspidal representations both ϕ_{HII} and [Sol3] boil down to the same source, namely [FeOp, FOS]. There it is shown that, on the cuspidal level for a Levi subgroup M of G, in the bijection

$$\mathrm{Irr}_{\mathrm{unip,cusp}}(M) \to \Phi_{\mathrm{nr,cusp}}(M) : \pi \mapsto (\phi_\pi, \rho_\pi) \tag{14}$$

the L-parameter ϕ_π is canonical up to twisting by $X_{\mathrm{wr}}(M/Z(M)_s)$. For use in [Sol3] we may pick any instance of (14) from [FOS, Theorem 2]. For use in [Opd4, 4.5.1] there are some extra conditions, related to the existence of suitable spectral transfer morphisms. We fix a set $\mathfrak{Lev}(G)$ of representatives for the conjugacy classes of Levi subgroups of G. For every $M \in \mathfrak{Lev}(G)$ we choose a bijection (14) which satisfies all the requirements from [Opd4]. In this way we achieve that

$$\phi_\pi = \phi_{HII}(\pi) \in \Phi_{\mathrm{nr,bdd}}(M) \quad \text{for every tempered } \pi \in \mathrm{Irr}_{\mathrm{unip,cusp}}(M). \tag{15}$$

To prove property (l), we will show that

$$\phi_\pi = \phi_{HII}(\pi) \in \Phi_{\mathrm{nr,bdd}}(G) \quad \text{for all } \pi \in \mathrm{Irr}_{\mathrm{unip,temp}}(G). \tag{16}$$

The infinitesimal (central) character of an L-parameter ϕ is defined as

$$\text{inf.ch.}(\phi) = G^\vee\text{-conjugacy class of } \phi\Big(\mathrm{Frob}, \left(\begin{smallmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{smallmatrix}\right)\Big) \in G^\vee \mathrm{Frob}.$$

By the definition of L-parameters this is a semisimple adjoint orbit, and by [Bor2, Lemma 6.4] it corresponds to a unique $W(\mathcal{G}^\vee, \mathcal{T}^\vee)^{\mathrm{Frob}}$-orbit in $T^\vee_{\mathrm{Frob}}$ (the coinvariants of $T^\vee$ with respect to the action of $\langle \mathrm{Frob} \rangle$). That in turn can be interpreted as a central character of the Iwahori–Hecke algebra $\mathcal{H}(G^*, I^*)$ of the quasi-split inner form G^* of G.

By [Opd4, Theorems 3.8.1 and 4.5.1] the Langlands parametrization ϕ_{HII} is completely characterized by the map

$$\text{inf.ch.} \circ \phi_{HII} : \mathrm{Irr}_{\mathrm{unip,temp}}(G) \to G^\vee \mathrm{Frob} / G^\vee\text{-conjugacy}. \tag{17}$$

Hence (16) is equivalent to:

$$\text{inf.ch.}(\phi_\pi) = \text{inf.ch.}(\phi_{HII}(\pi)) \qquad \text{for all } \pi \in \mathrm{Irr}_{\mathrm{unip,temp}}(G). \tag{18}$$

By construction the cuspidal support map Sc for enhanced L-parameters preserves infinitesimal characters, see [AMS1, Definition 7.7 and (108)]. Then property (g) says that inf.ch.(ϕ_π) does not change if we replace π by its supercuspidal support.

The map (17) is constructed in [Opd4] in three steps:

- Let $\mathcal{H}_\mathfrak{s}$ be the Hecke algebra associated to a Bushnell-Kutzko type for the Bernstein block $\mathrm{Rep}(G)^\mathfrak{s}$ that contains π, as in (9). Consider the image $\pi_\mathcal{H}$ of π in $\mathrm{Irr}(\mathcal{H}_\mathfrak{s})$ under (10).
- Compute the central character of $\pi_\mathcal{H}$, an orbit for the finite Weyl group $W_\mathfrak{s}$ acting on the complex torus $T_\mathfrak{s}$ – both attached to $\mathcal{H}_\mathfrak{s}$ as described after (10) (but there in terms of $\mathfrak{f}$).
- Apply a spectral transfer morphism $\mathcal{H}_\mathfrak{s} \rightsquigarrow \mathcal{H}(G^*, I^*)$ and the associated map $T_\mathfrak{s} \to T^\vee_{\mathrm{Frob}} / K_L^n$ – see the definitions in [Opd2, §5.1]. This map sends the central character of $\pi_\mathcal{H}$ to a unique $W(\mathcal{G}^\vee, \mathcal{T}^\vee)^{\mathrm{Frob}}$-orbit in $T^\vee_{\mathrm{Frob}}$, which we interpret as a semisimple $G^\vee$-orbit in $G^\vee\mathrm{Frob}$.

For irreducible $\mathcal{H}_\mathfrak{s}$-modules, the central character map corresponds to restriction to the maximal commutative subalgebra $\mathcal{O}(T_\mathfrak{s})$ of $\mathcal{H}_\mathfrak{s}$. There is a Levi subgroup M of G with a type, covered by the type for $\mathrm{Rep}(G)^\mathfrak{s}$, whose Hecke algebra is $\mathcal{O}(T_\mathfrak{s})$. The equivalence of categories (10) is compatible with normalized parabolic induction and Jacquet restriction [Sol2, Lemma 4.1], so the central character map for $\mathcal{H}_\mathfrak{s}$ corresponds to the supercuspidal support map for $\mathrm{Rep}(G)^\mathfrak{s}$.

As in [Opd3, §3.1.1], $\mathcal{H}_\mathfrak{s} \rightsquigarrow \mathcal{H}(G^*, I^*)$ can be restricted to a spectral transfer morphism $\mathcal{O}(T_\mathfrak{s}) \rightsquigarrow \mathcal{H}(M^*, I^*)$, where the Levi subgroup M^* of G^* is the quasi-split inner form of M. Up to adjusting by an element of $W(\mathcal{G}^\vee, \mathcal{T}^\vee)^{\mathrm{Frob}}$, these two spectral transfer morphisms are represented by the same map $T_\mathfrak{s} \to T^\vee_{\mathrm{Frob}} / K_L^n$. Consequently (17) does not change if the input π is replaced by its supercuspidal support. These considerations reduce (18) and (16) to (15).

Now we have the bijection of the theorem and all its properties, except for the asserted uniqueness. The L-parameters for $\mathrm{Irr}_{\mathrm{unip,temp}}(G)$ completely determine the L-parameters for all (not necessarily tempered) irreducible unipotent G-representations, that follows from the compatibility with the Langlands classification

[Sol3, Lemma 5.10]. Hence it suffices to address the essential uniqueness for tempered representations and bounded L-parameters. For adjoint groups it was shown in [Opd4, Theorems 4.4.1.c and 4.5.1.b].

The case where $Z(\mathcal{G})$ is K-anisotropic is reduced to the adjoint case in the proof of [Opd4, Theorem 4.5.1]. This proceeds by imposing compatibility of the Langlands parametrization ϕ_{HII} with the isogeny $\mathcal{G} \to \mathcal{G}_{\text{ad}} \times \mathcal{G}/\mathcal{G}_{\text{der}}$, in the sense that:

- every irreducible tempered unipotent representation of G should be "liftable" in an essentially unique way to one of $G_{\text{ad}} \times (\mathcal{G}/\mathcal{G}_{\text{der}})(K)$,
- that should determine the L-parameters.

In this way one concludes essential uniqueness in [Opd4, Theorem 4.5.1.b], but in a weaker sense than we want. However, the compatibility of $\mathcal{G} \to \mathcal{G}_{\text{ad}} \times \mathcal{G}/\mathcal{G}_{\text{der}}$ with L-parameters actually is a requirement, it is an instance of property (k). If we invoke that, the argument for [Opd4, Theorem 4.5.1] shows that the non-uniqueness (when $Z(\mathcal{G})$ is K-anisotropic) is the essentially the same as in the adjoint case. That is, the parametrization is unique up to twists by the image of $X_{\text{wr}}(G_{\text{ad}}) \cong Z(G_{\text{ad}}{}^\vee)^{\text{Frob}}$ in ${}^L G$, which is just $X_{\text{wr}}(G)$.

Finally we consider the case where $\mathcal{G}$ is reductive and the maximal K-split central torus $Z(\mathcal{G})_s$ is nontrivial. Then $G/Z(G)_s = (\mathcal{G}/Z(\mathcal{G})_s)(K)$ does have K-anisotropic centre. The Langlands correspondence for $\text{Irr}_{\text{unip}}(G)$ is deduced from that for $\text{Irr}_{\text{unip}}(G/Z(G)_s)$, see [FOS, §15] and [Opd4, p.35]. What happens for $Z(G)$ is determined by property (c) and the natural LLC for tori. This renders a LLC for $\text{Irr}_{\text{unip}}(G)$ precisely as canonical as for $\text{Irr}_{\text{unip}}(G_{\text{ad}})$. In view of the cases considered above, the only non-uniqueness comes from twisting by $X_{\text{wr}}(G_{\text{ad}})$. This twisting goes via the image of $X_{\text{wr}}(G_{\text{ad}})$ in $X_{\text{wr}}(G)$, which consists of the weakly unramified characters of G that are trivial on $Z(G)$. $\square$

Next we recall some results from [Sol4] about the behaviour of unipotent representations and enhanced L-parameters under isogenies of reductive groups. We will formulate them for quotient maps, because we will only need them for such isogenies.

Let $\mathcal{Z}$ be a central K-subgroup of $\mathcal{G}$ and consider the quotient map

$$\eta : \mathcal{G} \to \mathcal{G}' := \mathcal{G}/\mathcal{Z}.$$

The dual homomorphism $\eta^\vee : G'^\vee \to G^\vee$ gives rise to maps

$${}^L\eta : {}^L G' \to {}^L G \quad \text{and} \quad \Phi(\eta) : \Phi(G') \to \Phi(G).$$

For $\phi' \in \Phi(G')$ and $\phi = \Phi(\eta)\phi' \in \Phi(G)$, $\mathcal{A}_{\phi'}$ is a normal subgroup of $\mathcal{A}_\phi$ and $\mathcal{A}_\phi/\mathcal{A}_{\phi'}$ is abelian [Sol4, Lemma 4.1].

The map between groups of K-rational points $\eta : G \to G'$ need not be surjective, but in any case its cokernel is compact and commutative. This implies that the pullback functor

$$\eta^* : \text{Rep}(G') \to \text{Rep}(G)$$

preserves finite length and complete reducibility [Sil2]. It is easily seen, for instance from [Sol4, Proposition 7.2], that η^* maps one Bernstein block $\text{Rep}(G')^{\mathfrak{s}'}$ into a direct sum of finitely many Bernstein blocks $\text{Rep}(G)^{\mathfrak{s}}$.

Theorem 2.2. [Sol4, Theorem 3 and Lemma 7.3]
Let $\mathcal{G}$ be a connected reductive K-group which splits over an unramified extension.

Let $(\phi',\rho') \in \Phi_{\mathrm{nr},e}(G')$ and let $\pi(\phi',\rho') \in \mathrm{Irr}(G')$ be associated to it in Theorem 2.1. Then, with $\phi = \Phi(\eta)\phi'$:

$$\eta^* \pi(\phi',\rho') = \bigoplus_{\rho \in \mathrm{Irr}(\mathcal{A}_\phi)} \mathrm{Hom}_{\mathcal{A}_\phi}\Big(\mathrm{ind}_{\mathcal{A}_{\phi'}}^{\mathcal{A}_\phi} \rho', \rho\Big) \otimes \pi(\phi,\rho) = \bigoplus_{\rho \in \mathrm{Irr}(\mathcal{A}_\phi)} \mathrm{Hom}_{\mathcal{A}_{\phi'}}(\rho',\rho) \otimes \pi(\phi,\rho).$$

Let us work out a few more features of this result.

Lemma 2.3. (*a*) *All irreducible constituents of the G-representation $\eta^*\pi(\phi',\rho')$ have the same Plancherel density and appear with the same multiplicity. This multiplicity is one if $\pi(\phi',\rho')$ is supercuspidal.*

(*b*) *All $\rho \in \mathrm{Irr}(\mathcal{A}_\phi)$ with $\mathrm{Hom}_{\mathcal{A}_{\phi'}}(\rho',\rho) \neq 0$ have the same dimension.*

(*c*) *For any such ρ, the length of the G-representation $\eta^*\pi(\phi',\rho')$ is*

$$\dim(\rho')[\mathcal{A}_\phi : \mathcal{A}_{\phi'}] \dim(\rho)^{-1}.$$

Proof. (a) We abbreviate $\pi' = \pi(\phi',\rho')$. Since this G'-representation is irreducible, all irreducible subrepresentations of $\eta^*(\pi')$ are equivalent under the action of G' on $\mathrm{Irr}(G)$. Conjugation with $g' \in G'$ defines a unimodular automorphism of G, so $\mathrm{Ad}(g')^*$ preserves the Plancherel density on $\mathrm{Irr}(G)$.

Similarly, all isotypic components of $\eta^*(\pi')$ are G'-associate. As already shown in [GeKn, Lemma 2.1], this implies that every irreducible constituent of $\eta^*(\pi')$ appears with the same multiplicity. By [Sol4, Lemma 7.1] this multiplicity is one if π' is supercuspidal.

(b) We briefly recall how to construct irreducible representations of $\mathcal{A}_\phi$ that contain ρ'. Let $(\mathcal{A}_\phi)_{\rho'}$ be the stabilizer of ρ' in $\mathcal{A}_\phi$ (with respect to the action of $\mathcal{A}_\phi$ on $\mathrm{Irr}(\mathcal{A}_{\phi'})$ coming from conjugation). The projective action of $(\mathcal{A}_\phi)_{\rho'}$ on $V_{\rho'}$ gives rise to a 2-cocycle $\kappa_{\rho'}$ and a twisted group algebra $\mathbb{C}[(\mathcal{A}_\phi)_{\rho'}, \kappa_{\rho'}]$. Clifford theory (in the version [AMS1, Proposition 1.1]) says that:

- for every $(\tau, V_\tau) \in \mathrm{Irr}\big(\mathbb{C}[(\mathcal{A}_\phi)_{\rho'}, \kappa_{\rho'}]\big)$, $\tau \ltimes \rho := \mathrm{ind}_{(\mathcal{A}_\phi)_{\rho'}}^{\mathcal{A}_\phi}(V_\tau \otimes V_{\rho'})$ is an irreducible $\mathcal{A}_\phi$-representation containing ρ';
- every irreducible $\mathcal{A}_\phi$-representation containing ρ' is of the form $\tau \ltimes \rho'$.

For $\rho = \tau \ltimes \rho'$ we see that

$$\mathrm{Hom}_{\mathcal{A}_{\phi'}}(\rho',\rho) = \mathrm{Hom}_{\mathcal{A}_{\phi'}}\big(V_{\rho'}, \mathrm{ind}_{(\mathcal{A}_\phi)_{\rho'}}^{\mathcal{A}_\phi}(V_\tau \otimes V_{\rho'})\big) \cong \mathrm{Hom}_{\mathcal{A}_{\phi'}}(V_{\rho'}, V_\tau \otimes V_{\rho'}) \cong V_\tau.$$

We can compute the dimension of $\rho = \tau \ltimes \rho'$ in these terms:

(19)
$$\dim(\rho) = [\mathcal{A}_\phi : (\mathcal{A}_\phi)_{\rho'}] \dim(V_\tau) \dim(V_{\rho'}) = [\mathcal{A}_\phi : (\mathcal{A}_\phi)_{\rho'}] \dim(\rho') \dim \mathrm{Hom}_{\mathcal{A}_{\phi'}}(\rho',\rho).$$

By Theorem 2.2

$$\mathrm{Hom}_G(\pi(\phi,\rho), \eta^*(\pi')) \cong \mathrm{Hom}_{\mathcal{A}_{\phi'}}(\rho',\rho).$$

By part (a) this space is independent of ρ (as long as it is nonzero). With (19) we conclude that $\dim(\rho)$ is the same for all such ρ.

(c) By Frobenius reciprocity

$$\mathrm{Hom}_{\mathcal{A}_\phi}\Big(\mathrm{ind}_{\mathcal{A}_{\phi'}}^{\mathcal{A}_\phi} \rho', \rho\Big) \cong \mathrm{Hom}_{\mathcal{A}_{\phi'}}(\rho',\rho).$$

Hence $\mathrm{ind}_{\mathcal{A}_{\phi'}}^{\mathcal{A}_\phi} \rho'$ is a direct sum of irreducible subrepresentations of common dimension $\dim(\rho)$. Then its length is

$$\dim\big(\mathrm{ind}_{\mathcal{A}_{\phi'}}^{\mathcal{A}_\phi} \rho'\big) \dim(\rho)^{-1} = \dim(\rho')[\mathcal{A}_\phi : \mathcal{A}_{\phi'}] \dim(\rho)^{-1}.$$

By Theorem 2.2 that is also the length of $\eta^*(\pi')$. $\square$

3. Affine Hecke algebras

From now on $\mathcal{G}$ denotes a connected reductive K-group which splits over the maximal unramified extension K_{nr} of K. In this section we assume moreover that it has anisotropic centre. Let $\mathcal{G}_{\mathrm{ad}} = \mathcal{G}/Z(\mathcal{G})$ be its adjoint group. We intend to investigate the behaviour of the formal degrees with respect to the quotient map $\eta : \mathcal{G} \to \mathcal{G}_{\mathrm{ad}}$. As preparation, we consider the analogous question for the affine Hecke algebras from Section 1.

This means that we focus on one Bernstein component $\mathrm{Rep}(G)_{(\hat{P}_{\mathfrak{f}},\hat{\sigma})}$ for G and one Bernstein component $\mathrm{Rep}(G_{\mathrm{ad}})_{(\hat{P}_{\mathfrak{f}_{\mathrm{ad}}},\hat{\sigma}_{\mathrm{ad}})}$ for G_{ad}, such that the pullback of the latter has nonzero components in the former. As already noted in [FOS, §13] and [Sol3, §3.3], we may assume that $\mathfrak{f}_{\mathrm{ad}} = \mathfrak{f}$ and that underlying cuspidal unipotent representations σ and σ_{ad} are essentially the same. That is, they are defined on the same vector space V_σ and σ is the pullback of σ_{ad} via the natural map $\overline{\mathcal{G}_f^\circ}(k) \to \overline{\mathcal{G}_{\mathrm{ad},f}^\circ}(k)$. More precisely, we may even assume that $\hat{\sigma}$ is the pullback of $\hat{\sigma}_{\mathrm{ad}}$ along $\eta : \hat{P}_{\mathfrak{f}} \to \hat{P}_{\mathfrak{f},\mathrm{ad}}$.

In this setting η induces an inclusion

$$\eta_{\mathcal{H}} : \mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma}) \to \mathcal{H}(G_{\mathrm{ad}}, \hat{P}_{\mathrm{ad},\mathfrak{f}}, \hat{\sigma}_{\mathrm{ad}}), \tag{20}$$

which we need to analyse in more detail. Let $X_{\mathfrak{f},\mathrm{ad}}$ denote the lattice $X_{\mathfrak{f}}$ for G_{ad}. From [Sol3, Proposition 3.1 and Theorem 3.3.b] we see that $X_{\mathfrak{f}}$ can be regarded as a sublattice of $X_{\mathfrak{f},\mathrm{ad}}$, and that

$$X_{\mathfrak{f},\mathrm{ad}}/X_{\mathfrak{f}} \cong \big(\Omega_{G_{\mathrm{ad}},\mathfrak{f}}/\Omega_{G_{\mathrm{ad}},\mathfrak{f},\mathrm{tor}}\big)/\big(\Omega_{G,\mathfrak{f}}/\Omega_{G,\mathfrak{f},\mathrm{tor}}\big). \tag{21}$$

To make sense of the right hand side, we remark that the natural map $\Omega_{G,\mathfrak{f}} \to \Omega_{G_{\mathrm{ad}},\mathfrak{f}}$ is injective, because $\mathcal{G}$ is K_{nr}-split. The group (21) is finite because $Z(\mathcal{G})$ is K-anisotropic. We recall from [Lus2, §1.20] and [Sol3, (42)] that $\Omega_{G_{\mathrm{ad}},\mathfrak{f}}/\Omega_{G_{\mathrm{ad}},\mathfrak{f},\mathrm{tor}}$ acts on $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$ by algebra automorphisms, and that

$$\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma}) \cong \mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma) \rtimes \Omega_{G,\mathfrak{f}}/\Omega_{G,\mathfrak{f},\mathrm{tor}}. \tag{22}$$

By [Sol3, Lemma 3.5] $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma)$ and all the data for that algebra are the same for G and for G_{ad}. So the difference between (22) and its analogue for G_{ad} lies only in the finite group $\Omega_{G,\mathfrak{f}}/\Omega_{G,\mathfrak{f},\mathrm{tor}}$. The inclusion (20) is the identity on $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma)$.

Let τ and τ_{ad} denote the normalized traces of the affine Hecke algebras $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$ and $\mathcal{H}(G_{\mathrm{ad}}, \hat{P}_{\mathrm{ad},\mathfrak{f}}, \hat{\sigma}_{\mathrm{ad}})$. Let $Z(G)_1^\circ$ be the unique parahoric subgroup of $Z(\mathcal{G})^\circ(K)$. By (6) and [GeMa, Proposition 1.4.12.c]

$$\mathrm{vol}(P_{\mathfrak{f}}) = \mathrm{vol}(P_{\mathfrak{f},\mathrm{ad}})\,\mathrm{vol}(Z(G)_1^\circ). \tag{23}$$

By (11), (7) and (23)

$$\frac{\tau(N_e)}{\tau_{\mathrm{ad}}(N_e)} = \frac{\dim(\hat{\sigma})}{\mathrm{vol}(\hat{P}_{\mathfrak{f}})}\frac{\mathrm{vol}(\hat{P}_{\mathfrak{f},\mathrm{ad}})}{\dim(\hat{\sigma}_{\mathrm{ad}})} = \frac{|\Omega_{G_{\mathrm{ad}},\mathfrak{f},\mathrm{tor}}|}{|\Omega_{G,\mathfrak{f},\mathrm{tor}}|\mathrm{vol}(Z(G)_1^\circ)}. \tag{24}$$

Both (21) and (24) contribute to the difference between the Plancherel measures for $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$ and for $\mathcal{H}(G_{\mathrm{ad}}, \hat{P}_{\mathrm{ad},\mathfrak{f}}, \hat{\sigma}_{\mathrm{ad}})$. For the latter that is clear, for the former we compute the effect below.

We abbreviate $A = \Omega_{G_{\mathrm{ad}},\mathfrak{f}} / \Omega_{G_{\mathrm{ad}},\mathfrak{f},\mathrm{tor}}$, $\mathcal{H}_{\mathrm{ad}} = \mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma) \rtimes A$ and

$$\mathcal{H} = \mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma) \rtimes \Omega_{G,\mathfrak{f}} / \Omega_{G,\mathfrak{f},\mathrm{tor}}.$$

Since the abelian group A acts on $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma)$ and (trivially) on $\Omega_{G,\mathfrak{f}} / \Omega_{G,\mathfrak{f},\mathrm{tor}}$, (22) shows that it acts on $\mathcal{H}$ by algebra automorphisms.

Lemma 3.1. *Let V be any irreducible $\mathcal{H}_{\mathrm{ad}}$-module. All the constituents of $\eta^*_{\mathcal{H}}(V)$ have the same dimension and the same Plancherel density, and they appear with the same multiplicity.*

Proof. If $V_{\mathcal{H}}$ is any irreducible submodule of $\eta^*_{\mathcal{H}}(V)$, (22) shows that

$$V = \sum\nolimits_{\omega \in A} N_\omega \cdot V_{\mathcal{H}}. \tag{25}$$

As N_ω normalizes the subalgebra $\mathcal{H}$ of $\mathcal{H}_{\mathrm{ad}}$, each $N_\omega \cdot V_{\mathcal{H}}$ is an irreducible $\mathcal{H}$-submodule of V. Consequently

(26) every constituent of $\eta^*_{\mathcal{H}}(V)$ is isomorphic to $\mathrm{Ad}(N_\omega)^* V_{\mathcal{H}}$ for some $\omega \in A$.

Taking into account that conjugation by N_ω is a trace-preserving automorphism of $\mathcal{H}$, (26) shows that all the constituents of $\eta^*_{\mathcal{H}}(V)$ have the same dimension and the Plancherel density. Further, we see from (25) that any two $\mathcal{H}$-isotypic submodules of V are in bijection, via multiplication with a suitable N_ω. Hence all constituents of $\eta^*_{\mathcal{H}}(V)$ appear with the same multiplicity in that $\mathcal{H}$-module. $\square$

This rough analysis of $\eta^*_{\mathcal{H}}$ does not yet suffice, we need more precise results from Clifford theory. Write

$$C = \mathrm{Irr}(X_{\mathfrak{f},\mathrm{ad}} / X_{\mathfrak{f}}).$$

By (21), C can also be regarded as the character group of

$$\big(\Omega_{G_{\mathrm{ad}},\mathfrak{f}} / \Omega_{G_{\mathrm{ad}},\mathfrak{f},\mathrm{tor}}\big) / \big(\Omega_{G,\mathfrak{f}} / \Omega_{G,\mathfrak{f},\mathrm{tor}}\big).$$

Using (22), every $c \in C$ determines an automorphism of $\mathcal{H}_{\mathrm{ad}}$, namely

$$c \cdot (h \otimes N_\omega) = h \otimes c(\omega) N_\omega \qquad h \in \mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma), \omega \in A.$$

We note that $\mathcal{H}_{\mathrm{ad}}^C = \mathcal{H}$.

The restriction of modules from $\mathcal{H}_{\mathrm{ad}}$ to $\mathcal{H}_{\mathrm{ad}}^C$ was investigated in [RaRa, Appendix]. Let C_V be the stabilizer (in C) of the isomorphism class of $V \in \mathrm{Irr}(\mathcal{H}_{\mathrm{ad}})$. For every $c \in C$ there exists an isomorphism of $\mathcal{H}$-modules

$$i_c : V \to c^* V.$$

By Schur's lemma i_c is unique up to scalars, and thus the i_c furnish a projective action of C on V. Our particular situation is favourable because the action of C on $\mathcal{H}_{\mathrm{ad}}$ is free, in the sense that it acts freely on a vector space basis). This can be exploited to analyse the intertwining operators i_c.

Lemma 3.2. *The group C_V acts linearly on V, by $\mathcal{H}$-module automorphisms.*

Proof. We normalize i_c by requiring that it restricts to the identity on $V_{\mathcal{H}}$. For any $\omega \in A, c \in C_V$ and $v \in V_{\mathcal{H}}$ we have

$$i_c(N_\omega \cdot v) = c(N_\omega) \cdot i_c(v) = c(\omega) N_\omega \cdot v. \tag{27}$$

In view of (25), this formula determines i_c completely. In particular $i_c \circ i_{c'} = i_{cc'}$ for all $c, c' \in C_V$. $\square$

In the remainder of this section we assume that $\mathcal{G}$ is semisimple, so that (21) and C are finite. By [RaRa, Theorem A.13] the action from Lemma 3.2 gives rise to an isomorphism of $\mathcal{H} \times \mathbb{C}[C_V]$-modules

$$V \cong \bigoplus\nolimits_{E \in \mathrm{Irr}(C_V)} V_E \otimes E. \tag{28}$$

Lemma 3.3. *For every $E \in \mathrm{Irr}(C_V)$ the $\mathcal{H}$-module $V_E = \mathrm{Hom}_{C_V}(E, V)$ is irreducible and appears with multiplicity one in $\eta^*_{\mathcal{H}}(V)$.*

Proof. By [RaRa, Theorem A.13] the $\mathcal{H}$-module V_E is either zero or irreducible. Let $A' \subset A$ be a set of representatives of $A / \cap_{c \in C} \ker(c|_A)$, so that $\mathrm{Irr}(C_V)$ is naturally in bijection with A'. From (25) and (27) we see that there is a linear bijection

$$\mathbb{C}A' \otimes \sum_{\omega \in \cap_{c \in C} \ker(c|_A)} N_\omega \cdot V_{\mathcal{H}} \to V : a \otimes v \to N_a \cdot v.$$

Hence every $E \in \mathrm{Irr}(C_V) \cong A'$ appears nontrivially in the decomposition (28). The multiplicity of V_E in V is $\dim(E)$, which is one because C_V is abelian. $\square$

For another irreducible $\mathcal{H}_{\mathrm{ad}}$-module V', [RaRa, Theorem A.13] shows how the restrictions to $\mathcal{H}$ compare:

$$\eta^*_{\mathcal{H}}(V') \begin{cases} \cong \eta^*_{\mathcal{H}}(V) & \text{if } V' \cong c^* V \text{ for some } c \in C \\ \text{has no constituents in common with } \eta^*_{\mathcal{H}}(V) & \text{otherwise.} \end{cases} \tag{29}$$

From here on we assume that V is discrete series. Casselman's criterion for discrete series representations [Opd1, Lemma 2.22] entails that $\eta^*_{\mathcal{H}} \delta'$ is direct sum of finitely many irreducible discrete series representations of $\mathcal{H}$.

Endow $\mathcal{H}_{\mathrm{aff}}$ and $\mathcal{H}_{\mathrm{ad}}$ with the trace τ' so that $\tau'(N_e) = 1$. We indicate the formal degree with respect to this renormalized trace by fdeg$'$.

Theorem 3.4. *Let $\mathcal{G}$ be a semisimple K-group which splits over an unramified extension. Let V be an irreducible discrete series representation of $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma) \rtimes \Omega_{G_{\mathrm{ad}},\mathfrak{f}} / \Omega_{G_{\mathrm{ad}},\mathfrak{f},\mathrm{tor}}$ and let $\eta^*_{\mathcal{H}} V$ be its pullback to $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma) \rtimes \Omega_{G,\mathfrak{f}} / \Omega_{G,\mathfrak{f},\mathrm{tor}}$ via (20) and (22). Then*

$$\frac{\mathrm{fdeg}'(\eta^*_{\mathcal{H}} V)}{\mathrm{fdeg}'(V)} = |C| = \Big[\frac{\Omega_{G_{\mathrm{ad}},\mathfrak{f}}}{\Omega_{G_{\mathrm{ad}},\mathfrak{f},\mathrm{tor}}} : \frac{\Omega_{G,\mathfrak{f}}}{\Omega_{G,\mathfrak{f},\mathrm{tor}}} \Big].$$

For any irreducible constituent V_E of $\eta^*_{\mathcal{H}}(V)$:

$$\mathrm{fdeg}'(V_E) = [C : C_V] \, \mathrm{fdeg}'(V).$$

Here $|C_V|$ equals the length of $\eta^*_{\mathcal{H}}(V)$.

Proof. Let $C^*_r(\mathcal{H})$ be the C^*-completion of $\mathcal{H}$, as in [Opd1, Definition 2.4]. As V is discrete series, we know from [Opd1, §6.4] that $C^*_r(\mathcal{H}_{\mathrm{ad}})$ contains a central idempotent e_V such that

$$e_V C^*_r(\mathcal{H}_{\mathrm{ad}}) \cong \mathrm{End}_{\mathbb{C}}(V).$$

Then by definition

$$\tau'(e_V) = \dim(V) \mathrm{fdeg}'(V). \tag{30}$$

The C-orbit of V in $\mathrm{Irr}(\mathcal{H}_{\mathrm{ad}})$ has precisely $[C : C_V]$ elements, and these are all discrete series. The central idempotent

$$e_{C,V} := \sum\nolimits_{c \in C / C_V} c \cdot e_V$$

lies in $C_r^*(\mathcal{H}_{\mathrm{ad}})^C = C_r^*(\mathcal{H})$ and

$$e_{C,V} C_r^*(\mathcal{H}_{\mathrm{ad}}) \cong \bigoplus\nolimits_{c \in C/C_V} \mathrm{End}_{\mathbb{C}}(c^* V).$$

Since the action of C preserves τ', we obtain

$$\tau'(e_{C,V}) = \sum\nolimits_{c \in C/C_V} \dim(c^*V)\mathrm{fdeg}'(c^*V) = [C : C_V] \dim(V)\mathrm{fdeg}'(V).$$

With (28) and Lemma 3.3 this can be expanded as

$$\begin{aligned} \tau'(e_{C,V}) &= [C : C_V]\, \mathrm{fdeg}'(V) \sum\nolimits_{E \in \mathrm{Irr}(C_V)} \dim(V_E) \dim(E) \\ &= [C : C_V]\, \mathrm{fdeg}'(V) |C_V| \dim(V_E) = |C|\, \mathrm{fdeg}'(V) \dim(V_E). \end{aligned} \tag{31}$$

From (28) and (29) we see that

$$e_{C,V} C_r^*(\mathcal{H}) \cong \bigoplus\nolimits_{E \in \mathrm{Irr}(C_V)} \mathrm{End}_{\mathbb{C}}(V_E).$$

Considering τ' as trace for $\mathcal{H}$, using Lemma 3.1 and the commutativity of C_V, we find

$$\tau'(e_{C,V}) = \sum\nolimits_{E \in \mathrm{Irr}(C_V)} \dim(V_E)\mathrm{fdeg}'(V_E) = |C_V| \dim(V_E)\mathrm{fdeg}'(V_E). \tag{32}$$

Now we compare (31) and (32), for any constituent V_E of $\eta^*_{\mathcal{H}}(V)$, and we find the desired formula for $\mathrm{fdeg}'(V_E)$.

From that formula and (28) we deduce:

$$\mathrm{fdeg}'(\eta^*_{\mathcal{H}} V) = \sum_{E \in \mathrm{Irr}(C_V)} \dim(E)\mathrm{fdeg}(V_E) = |C_V| [C : C_V]\, \mathrm{fdeg}(V) = |C|\, \mathrm{fdeg}(V).$$

For the interpretation of $|C_V|$ we refer to Lemma 3.3. □

We note two direct consequences of Theorem 3.4 and (24):

$$\frac{\mathrm{fdeg}(\eta^*_{\mathcal{H}} V)}{\mathrm{fdeg}(V)} = \frac{|C|\, |\Omega_{G_{\mathrm{ad}},\mathfrak{f},\mathrm{tor}}|}{\mathrm{vol}(Z(G)^\circ_1)\, |\Omega_{G,\mathfrak{f},\mathrm{tor}}|} = \frac{[\Omega_{G_{\mathrm{ad}},\mathfrak{f}} : \Omega_{G,\mathfrak{f}}]}{\mathrm{vol}(Z(G)^\circ_1)}, \tag{33}$$

$$\text{length of } \eta^*_{\mathcal{H}}(V) \cdot \mathrm{fdeg}(V_E) = [\Omega_{G_{\mathrm{ad}},\mathfrak{f}} : \Omega_{G,\mathfrak{f}}]\, \mathrm{fdeg}(V)\, \mathrm{vol}(Z(G)^\circ_1)^{-1}. \tag{34}$$

4. Pullback of representations

In this section $\mathcal{G}$ still denotes a K_{nr}-split connected reductive K-group with anisotropic centre. We would like to apply Lemma 2.3 to (34), but to do so we first have to find the relation between the length of $\eta^*_{\mathcal{H}}(V)$ and the length of the pullback of the associated G_{ad}-representation. This involves the number of G-orbits of facets and the number of Bernstein components obtained from $(\hat{P}_{\mathfrak{f},\mathrm{ad}}, \hat{\sigma}_{\mathrm{ad}})$ under pullback along η.

For a facet $\mathfrak{f}$ of $\mathcal{B}(\mathcal{G}, K)$, let $\mathrm{Rep}(G)_{\mathfrak{f}}$ be the sum of the subcategories $\mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)}$, where σ runs over the irreducible representations of $P_{\mathfrak{f}}$ inflated from cuspidal representations of $\overline{\mathcal{G}^\circ_{\mathfrak{f}}}(k)$. In Section 1 we saw that this is a direct sum of finitely many Bernstein blocks, which by [Mor2, Corollary 3.10] all come from supercuspidal Bernstein components of the same Levi subgroup of $\mathcal{G}$. By (8)

$$\begin{array}{ll} \mathrm{Rep}(G)_{\mathfrak{f}} = \mathrm{Rep}(G)_{\mathfrak{f}'} & \text{if } g\mathfrak{f}' = \mathfrak{f} \text{ for some } g \in G \\ \mathrm{Rep}(G)_{\mathfrak{f}} \cap \mathrm{Rep}(G)_{\mathfrak{f}'} = \{0\} & \text{otherwise.} \end{array} \tag{35}$$

Let $\eta^*(\mathrm{Rep}(G_{\mathrm{ad}})_{\mathfrak{f}})$ the pullback of $\mathrm{Rep}(G_{\mathrm{ad}})_{\mathfrak{f}}$ along $\eta : G \to G_{\mathrm{ad}}$.

Lemma 4.1. *The number of different subcategories* $\mathrm{Rep}(G)_{\mathfrak{f}'}$ *involved nontrivially in* $\eta^*(\mathrm{Rep}(G_{\mathrm{ad}})_{\mathfrak{f}})$ *is* $|\Omega_{G_{\mathrm{ad}}} \cdot \mathfrak{f}| \, |\Omega_G \cdot \mathfrak{f}|^{-1}$.

Proof. Since $\overline{C_0}$ contains a fundamental domain for the G-action on $\mathcal{B}(\mathcal{G}, K)$, it suffices by (35) to consider facets in $\overline{C_0}$. Since the kernel of $\kappa_G : G \to \Omega_G$ acts type-preservingly on $\mathcal{B}(\mathcal{G}, K)$, the G-association classes of facets in $\overline{C_0}$ are precisely the Ω_G-orbits of facets in $\overline{C_0}$.

Since Ω_G embeds in the abelian group $\Omega_{G_{\mathrm{ad}}}$, all Ω_G-orbits in $\Omega_{G_{\mathrm{ad}}} \cdot \mathfrak{f}$ have the same length. The number of such Ω_G-orbits is $|\Omega_{G_{\mathrm{ad}}} \cdot \mathfrak{f}| \, |\Omega_G \cdot \mathfrak{f}|^{-1}$.

It is clear from the definitions that $\eta^*(\mathrm{Rep}(G_{\mathrm{ad}})_{\mathfrak{f}})$ has nonzero parts in the $\mathrm{Rep}(G)_{\mathfrak{f}'}$ with $\mathfrak{f}' \in \Omega_{G_{\mathrm{ad}}} \cdot \mathfrak{f}$, and maps to zero in all other subcategories $\mathrm{Rep}(G)_{\mathfrak{f}''}$. In view of (35), the number of different $\mathrm{Rep}(G)_{\mathfrak{f}'}$ involved here equals the number of Ω_G-orbits in $\Omega_{G_{\mathrm{ad}}} \cdot \mathfrak{f}$. □

There exists a Levi subgroup $M = \mathcal{M}(K)$ such that $(\hat{P}_{\mathfrak{f}} \cap M, \hat{\sigma})$ is a type for a supercuspidal Bernstein block $\mathfrak{s}_M$ of $\mathrm{Rep}(M)$, covered by $(\hat{P}_{\mathfrak{f}}, \hat{\sigma})$ [Mor2, Corollary 3.10]. We will often denote objects associated to M with an additional subscript, e.g. $P_{M,\mathfrak{f}} = P_{\mathfrak{f}} \cap M$. We note that, by [Mor2, Theorem 2.1], $\mathfrak{f}$ is contained in a minimal facet $\mathfrak{f}_M$ of $\mathcal{B}(\mathcal{M}_{\mathrm{ad}}, K)$ and $P_{M,\mathfrak{f}} = P_{\mathfrak{f}_M}$ is a maximal parahoric subgroup of M.

Recall from [Tits, §1.2] that the apartment $\mathbb{A}$ of $\mathcal{B}(\mathcal{G}, K)$ associated to $\mathcal{S}$ admits a canonical decomposition

$$\mathbb{A} = \mathbb{A}_{M_{\mathrm{ad}}} \times X_*(Z(\mathcal{M})_s) \otimes_{\mathbb{Z}} \mathbb{R}, \tag{36}$$

where $\mathbb{A}_{M_{\mathrm{ad}}}$ is the apartment of $\mathcal{B}(\mathcal{M}_{\mathrm{ad}}, K) = \mathcal{B}(\mathcal{M}/Z(\mathcal{M})_s, K)$ associated to $\mathcal{S}/Z(\mathcal{M})_s$. With (36) we can express $\mathfrak{f}$ as $\mathfrak{f}_M \times \mathfrak{f}^M$, where $\mathfrak{f}_M$ is a vertex of $\mathcal{B}(\mathcal{M}_{\mathrm{ad}}, K)$ and $\mathfrak{f}^M$ is an open subset of $X_*(Z(\mathcal{M})_s) \otimes_{\mathbb{Z}} \mathbb{R}$. Now

$$\bar{\mathfrak{f}} = \big(\overline{\mathfrak{f}_M} \times X_*(Z(\mathcal{M})_s) \otimes_{\mathbb{Z}} \mathbb{R} \big) \cap \overline{C_0}, \tag{37}$$

so that $\mathfrak{f}$ and $\mathfrak{f}_M$ determine each other.

Lemma 4.2. *Let* $\mathfrak{f}' \in N_{G_{\mathrm{ad}}}(M) \cdot \mathfrak{f}$ *such that* $\mathrm{Rep}(M)_{\mathfrak{f}_M} \neq \mathrm{Rep}(M)_{\mathfrak{f}'_M}$. *Then* $\mathrm{Rep}(G)_{\mathfrak{f}} \neq \mathrm{Rep}(G)_{\mathfrak{f}'}$.

Proof. Suppose that $\mathrm{Rep}(G)_{\mathfrak{f}} = \mathrm{Rep}(G)_{\mathfrak{f}'}$. Then any inertial equivalence class $\mathfrak{s} = [M, \pi_M]_G$ with $\mathrm{Rep}(G)_{\mathfrak{s}} \subset \mathrm{Rep}(G)_{\mathfrak{f}}$ equals an inertial equivalence class $\mathfrak{s}'$ with $\mathrm{Rep}(G)_{\mathfrak{s}'} \subset \mathrm{Rep}(G)_{\mathfrak{f}'}$. By assumption $\mathfrak{f}'$ also admits a decomposition (37), with the same M. Hence we may assume that $\mathfrak{s}' = [M, \pi'_M]_G$.

This means that there exist $g \in N_G(M)$ and $\chi \in X_{\mathrm{nr}}(M)$ such that $\pi'_M = \mathrm{Ad}(g)^*(\pi_M \otimes \chi)$. Since $N_G(M)/M$ only depends on G up to isogenies, we may assume that g lies in the image of $G_{\mathrm{sc}} \to G$. In particular g lies in the kernel of κ_G and acts type-preservingly on $\mathcal{B}(\mathcal{G}, K)$.

By [Lus2, §1] or [FOS, (1.18)] we can write $\pi_M \otimes \chi = \mathrm{ind}_{N_M(P_{\mathfrak{f}_M})}^M (\tilde{\sigma})$ for some extension $\tilde{\sigma}$ of σ to $N_M(P_{\mathfrak{f}_M})$. Then

$$\mathrm{Ad}(g)^*(\pi_M \otimes \chi) \cong \mathrm{ind}_{g N_M(P_{\mathfrak{f}_M}) g^{-1}}^M (\mathrm{Ad}(g)^* \tilde{\sigma}) = \mathrm{ind}_{N_M(P_{g \cdot \mathfrak{f}_M})}^M (\mathrm{Ad}(g)^* \tilde{\sigma}).$$

With (35) this implies $mg \cdot \mathfrak{f}_M = \mathfrak{f}'_M$ for some $m \in M$. Then (37) shows that also $mg \cdot \mathfrak{f} = \mathfrak{f}$. Since $\mathfrak{f} \cup \mathfrak{f}' \subset \overline{C_0}$ and $g \in \ker(\kappa_G)$, it follows that $\kappa_G(m)\mathfrak{f} = \mathfrak{f}'$. From (37) and the naturality of the Kottwitz homomorphism we deduce that $\kappa_M(m)\mathfrak{f}_M = \mathfrak{f}'_M$. By (35) this contradicts the assumption of the lemma. □

We write $\mathcal{M}_{AD} = \mathcal{M}/Z(\mathcal{G})$ and we restrict η to $\eta_M : \mathcal{M} \to \mathcal{M}_{AD}$. Let $\mathcal{P}$ be a parabolic K-subgroup of $\mathcal{G}$ with Levi factor $\mathcal{M}$ and put $\mathcal{P}_{AD} = \mathcal{P}/Z(\mathcal{G})$. The normalized parabolic induction functors form a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}(G_{\mathrm{ad}}) & \xrightarrow{\eta^*} & \mathrm{Rep}(G) \\ \uparrow I_{P_{AD}}^{G_{\mathrm{ad}}} & & \uparrow I_P^G \\ \mathrm{Rep}(M_{AD}) & \xrightarrow{\eta_M^*} & \mathrm{Rep}(M) \end{array} . \tag{38}$$

Lemma 4.3. *Let $\pi_{\mathrm{ad}} \in \mathrm{Rep}(G_{\mathrm{ad}})_{(\hat{P}_{\mathrm{ad},\mathfrak{f}},\hat{\sigma}_{\mathrm{ad}})}$ and let $\pi_{\mathrm{ad},\mathcal{H}}$ be the associated module of $\mathcal{H}(G_{\mathrm{ad}}, \hat{P}_{\mathrm{ad},\mathfrak{f}}, \hat{\sigma}_{\mathrm{ad}})$. Then the length of $\eta^*(\pi_{\mathrm{ad}}) \in \mathrm{Rep}(G)$ equals $|\Omega_{G_{\mathrm{ad}}} \cdot \mathfrak{f}|\, |\Omega_G \cdot \mathfrak{f}|^{-1}$ times the length of $\eta^*_{\mathcal{H}}(\pi_{\mathrm{ad},\mathcal{H}}) \in \mathrm{Rep}(\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma}))$.*

Proof. In every subcategory $\mathrm{Rep}(G)_{g\mathfrak{f}}$ with $g \in G_{\mathrm{ad}}$, $\eta^*(\pi_{\mathrm{ad}})$ has a nonzero component. These components are associated by the automorphisms $\mathrm{Ad}(g)$ of G, so they all have the same length. Lemma 4.1 tells us that the number of such components is $|\Omega_{G_{\mathrm{ad}}} \cdot \mathfrak{f}|\, |\Omega_G \cdot \mathfrak{f}|^{-1}$.

Hence it suffices to consider the projection $\pi_{\mathfrak{f}}$ of $\eta^*(\pi_{\mathrm{ad}})$ to $\mathrm{Rep}(G)_{\mathfrak{f}}$, and we have to show that its length equals that of $\eta^*_{\mathcal{H}}(\pi_{\mathrm{ad},\mathcal{H}})$. From the commutative diagram (38) we see that

$$\begin{aligned} &\mathrm{Rep}(G)_{\mathfrak{f}} \cap \overline{\eta^*\big(\mathrm{Rep}(G_{\mathrm{ad}})_{(\hat{P}_{\mathrm{ad},\mathfrak{f}},\hat{\sigma}_{\mathrm{ad}})}\big)} \\ &\qquad = \overline{I_P^G(\mathrm{Rep}(M)_{\mathfrak{f}_M})} \cap \overline{I_P^G\big(\eta_M^*(\mathrm{Rep}(M_{AD})_{(\hat{P}_{\mathrm{ad},\mathfrak{f}_M},\hat{\sigma}_{\mathrm{ad}})})\big)}, \end{aligned} \tag{39}$$

where the $\overline{X}$ indicates that we take the sum of all Bernstein components appearing in X. It is known from [Sol4, (7.8)] that $\eta_M^*\big(\mathrm{Rep}(M_{AD})_{(\hat{P}_{\mathrm{ad},\mathfrak{f}_M},\hat{\sigma}_{\mathrm{ad}})}\big)$ involves just one Bernstein component of $\mathrm{Rep}(M)_{\mathfrak{f}'_M}$ for every facet $\mathfrak{f}'_M \in M_{AD} \cdot \mathfrak{f}$, and no others. By Lemma 4.2 these different Bernstein components remain different upon parabolic induction to G. Hence we can identify the right hand side of (39) as

$$\begin{aligned} &\overline{I_P^G\big(\text{projection of } \eta_M^*(\mathrm{Rep}(M_{AD})_{(\hat{P}_{\mathrm{ad},\mathfrak{f}_M},\hat{\sigma}_{\mathrm{ad}})}) \text{ to } \mathrm{Rep}(M)_{\mathfrak{f}_M})\big)} \\ &\qquad = \overline{I_P^G\big(\mathrm{Rep}(M)_{(\hat{P}_{\mathfrak{f}_M},\hat{\sigma})}\big)} = \mathrm{Rep}(G)_{(\hat{P}_{\mathfrak{f}},\hat{\sigma})} \end{aligned} \tag{40}$$

From (39) and (40) we see that the projection $\pi_{\mathfrak{f}}$ of $\eta^*(\pi_{\mathrm{ad}})$ to $\mathrm{Rep}(G)_{\mathfrak{f}}$ equals its projection to $\mathrm{Rep}(G)_{(\hat{P}_{\mathfrak{f}},\hat{\sigma})}$. The latter category is equivalent with $\mathrm{Rep}(\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma}))$, via (10). Hence $\pi_{\mathfrak{f}}$ maps to $\eta^*_{\mathcal{H}}(\pi_{\mathrm{ad},\mathcal{H}})$ by (10), and in particular these two representations have the same length. $\square$

5. Computation of formal degrees

Now $\mathcal{G}$ denotes a connected reductive K-group which splits over an unramified extension. We will compute the formal degrees of square-integrable unipotent G-representations in increasing generality.

By the choice of a Haar measure on G, we make $C_c^\infty(G)$ into a convolution algebra, denoted $\mathcal{H}(G)$. The Plancherel theorem asserts that there exists a unique Borel measure μ_{Pl} on $\mathrm{Irr}(G)$ such that

$$f(1) = \int_{\mathrm{Irr}(G)} \mathrm{tr}\, \pi(f)\, \mathrm{d}\mu_{Pl}(\pi) \qquad \forall f \in \mathcal{H}(G).$$

The support of μ_{Pl} is precisely the collection $\mathrm{Irr}_{\mathrm{temp}}(G)$ of tempered irreducible G-representations. For a selfadjoint idempotent $e \in \mathcal{H}(G)$ we write

$$\mathrm{Irr}(G)^e = \{(\pi, V_\pi) \in \mathrm{Irr}(G) : eV_\pi \neq 0\}.$$

If it is nonzero, eV_π is an irreducible representation of the Hilbert algebra $e\mathcal{H}(G)e$. Suppose in addition that $\dim(eV_\pi) = d \in \mathbb{N}$ for all $\pi \in \mathrm{Irr}(G)^e$. Then [BHK, Theorem 2.3 and Proposition 2.1] tell us that

$$\mu_{Pl}(\mathrm{Irr}(G)^e) = e(1)d^{-1}. \tag{41}$$

For an important special case, suppose that (σ, V_σ) is an irreducible representation of a compact open subgroup J of G and that $e_\sigma \in \mathcal{H}(J)$ is the associated central idempotent. When $\mathrm{ind}_J^G(\sigma)$ is irreducible, the centre $Z(G)$ is compact, $\mathrm{ind}_J^G(\sigma)$ is supercuspidal and $e_\sigma \mathcal{H}(G) e_\sigma \cong \mathrm{End}_{\mathbb{C}}(V_\sigma)$. Applying (41) to e_σ, we find

$$\mu_{Pl}\big(\mathrm{ind}_J^G(\sigma)\big) = \frac{e_\sigma(1)}{\dim(\sigma)} = \frac{\dim(\sigma)}{\mathrm{vol}(J)}. \tag{42}$$

Recall that a G-representation (π, V_π) is square-integrable modulo centre if $Z(G)$ acts on V_π by a unitary character and V_π is square-integrable as representation of the derived group of G. Such a representation has a G-invariant inner product and is completely reducible. The formal degree of an irreducible square-integrable modulo centre G-representation is defined as the unique number $\mathrm{fdeg}(\pi) \in \mathbb{R}_{>0}$ such that

$$\int_{G/Z(G)_s} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, \mathrm{fdeg}(\pi) \mathrm{d}g = \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle} \quad \text{for all } v_i \in V_\pi. \tag{43}$$

When π is actually square-integrable (which can happen only if $Z(G)$ is compact), (43) entails that the formal degree of π is its mass with respect to the Plancherel measure μ_{Pl} on $\mathrm{Irr}(G)$ [Dix, Proposition 18.8.5]. The formal degree can be extended to finite length square-integrable modulo centre representations by additivity.

The above depends on the choice of a Haar measure, we need to make that explicit. Fix an additive character $\psi : K \to \mathbb{C}^\times$ which is trivial on $\mathfrak{o}_K$ but nontrivial on $\varpi_K^{-1}\mathfrak{o}_K$. We endow G (and all other reductive p-adic groups) with the Haar measure as in [HII]. As ψ has order 0, this agrees with the Haar measure in [GaGr]. Since $\mathcal{G}$ splits over K_{nr} it also agrees with [Gro, §4], which we used to get (6).

5.1. Adjoint groups.

It is known from [Lus1, Theorems 3.22 and 3.29] that the cuspidal unipotent representations of $\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k)$ depend functorially on the finite field k. This means that σ is part of a family of representations $\sigma_{k'}$ of $\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k')$, one for every finite field k' containing k. Moreover, the dimension of $\sigma_{k'}$ is a particular polynomial in $q' = |k'|$, a product of a rational number without factors p, a power of q' and terms $(q'^n - 1)^{\pm 1}$ with $n \in \mathbb{N}$.

When K' is an unramified extension of K, $\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k')$ is the finite reductive quotient of $\mathcal{G}_{\mathfrak{f}}^\circ(\mathfrak{o}_{K'})$, a parahoric subgroup of $\mathcal{G}(K')$. As already remarked after (6), the volume of $\mathcal{G}_{\mathfrak{f}}^\circ(K')$ is a rational function in q', with the same kind of (rational) factors as $\dim(\sigma_{k'})$. This enables one to vary q, while keeping $\mathfrak{f}, \mathcal{G}_{\mathfrak{f}}^\circ$ and σ essentially constant.

A similar variation is possible for the affine Hecke algebra

$$\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma}) = \mathcal{H}(X_{\mathfrak{f}}, R_{\mathfrak{f}}, q^{\mathcal{N}}). \tag{44}$$

There it means that the parameter function $q^{\mathcal{N}}$ can be replaced by $q'^{\mathcal{N}} = (q^{\mathcal{N}})^{[k':k]}$, obtaining a new Hecke algebra $\mathcal{H}(X_{\mathfrak{f}}, R_{\mathfrak{f}}, q'^{\mathcal{N}})$. Any discrete series representation δ of (44) naturally gives rise to a discrete series representation δ' of $\mathcal{H}(X_{\mathfrak{f}}, R_{\mathfrak{f}}, q'^{\mathcal{N}})$ and conversely, see [Opd1, §5.2] and [Sol1, Corollary 4.2.2]. In this way one can consider $\mathrm{fdeg}(\delta)$ as a function of q.

Further, unramified L-parameters ϕ can be made into functions of q. Replacing ϕ by a $G^\vee$-conjugate, we may assume that $\phi(\mathrm{Frob}) = t\mathrm{Frob}$ with $t \in (T^\vee)^{\mathbf{W}_F,\circ}$. For $q' = |k'|$ one takes ϕ' with $\phi'(\mathrm{Frob}) = t^{[k':k]}\mathrm{Frob}$ and $\phi' = \phi$ on $\mathbf{I}_K \times SL_2(\mathbb{C})$. It is easily seen from the explicit formulas in [GrRe, §4] that this makes the L-functions, ϵ-factors and γ-factors of ϕ into meromorphic functions of q.

Theorem 5.1. *Suppose that $\mathcal{G}$ is simple and splits over an unramified extension. Let $\pi \in \mathrm{Irr}(G)$ be square-integrable and unipotent.*

(*a*) *The HII-conjecture* (2) *holds in this setting, and more precisely*

$$\mathrm{fdeg}(\pi) = \pm \frac{\dim(\sigma)}{\mathrm{vol}(P_{\mathfrak{f}})\,[\hat{P}_{\mathfrak{f}} : P_{\mathfrak{f}}]} d_{\mathcal{H},\pi} m(q^{\mathcal{N}})^{(r)} = \pm \frac{\dim(\rho_\pi)}{|S^\sharp_{\phi_\pi}|} \gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\pi, \psi),$$

where $m(q^{\mathcal{N}})^{(r)}$ is as an (12).

(*b*) *The expressions $\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\pi, \psi)$, $\dim(\sigma)$, $\mathrm{vol}(P_{\mathfrak{f}})$ and $m(q^{\mathcal{N}})^{(r)}$ are nonzero rational functions of q. Each of them is a product of a rational constant and factors of the form $q^{m/2}$ with $m \in \mathbb{Z}$ and $(q^n - 1)^{\pm 1}$ with $n \in \mathbb{N}$.*

Proof. By [FOS, Theorem A.1 and Lemma A.3], the objects in Theorem 2.1 do not change under Weil restriction for reductive groups, with respect to finite unramified extensions. Hence we may assume that $\mathcal{G}$ is, in addition, absolutely simple. A first expression for $\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\pi, \psi)$ was given in [Opd3, (38)], we provide the proof of that in the appendix (Theorem A.4). Then we can use the results from [Opd3], which were obtained by classification.

For (a) see [Opd3, Theorem 4.11]. For (b) see [Opd3, Proposition 2.5 and (38)] and [GrRe, §4.2]. □

Exactly the same argument as for [FOS, Proposition 12.2] extends Theorem 5.1 to all adjoint groups.

Corollary 5.2. *Suppose that $\mathcal{G}_{\mathrm{ad}}$ is adjoint and splits over an unramified extension of K. Then* (2) *holds for all irreducible square-integrable unipotent G-representations.*

The functions $\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\pi, \psi)$, $\dim(\sigma)$, $\mathrm{vol}(P_{\mathfrak{f}})$ and $m(q^{\mathcal{N}})^{(r)}$ for G_{ad} are the products of the corresponding functions for the K-simple factors of $\mathcal{G}_{\mathrm{ad}}$.

5.2. **Semisimple groups.**

In the subsection $\mathcal{G}$ is a semisimple K-group which splits over an unramified extension, and $G = \mathcal{G}(K)$ as usual.

Proposition 5.3. *Let $\pi \in \mathrm{Irr}_{\mathrm{unip}}(G)$.*

(*a*) *The central character of π is trivial.*

(*b*) *There exists a $\pi_{\mathrm{ad}} \in \mathrm{Irr}_{\mathrm{unip}}(G_{\mathrm{ad}})$ such that π is a constituent of the pullback $\eta^*(\pi_{\mathrm{ad}})$.*

(*c*) *π is unitarizable, tempered, square-integrable or cuspidal if and only if π_{ad} is unitarizable, tempered, square-integrable or cuspidal respectively.*

Proof. (a) By definition (see Section 1) there exists a parahoric subgroup $P_{\mathfrak{f}} \subset G$ and an irreducible unipotent $P_{\mathfrak{f}}$-representation σ, inflated from the finite reductive quotient $P_{\mathfrak{f}}/U_{\mathfrak{f}} = \overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k)$, such that π is a constituent of $\mathrm{ind}_{P_{\mathfrak{f}}}^{G}(\sigma)$. It is known from [Lus1, Proposition 3.15] that the adjoint quotient map

$$\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k) \to \overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}_{\mathrm{ad}}(k) \quad \text{induces a bijection} \quad \mathrm{Irr}_{\mathrm{unip}}\big(\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}_{\mathrm{ad}}(k)\big) \to \mathrm{Irr}_{\mathrm{unip}}\big(\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k)\big). \tag{45}$$

In particular σ is the pullback of a unique $\sigma_{\mathrm{ad}} \in \mathrm{Irr}_{\mathrm{unip}}\big(\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}_{\mathrm{ad}}(k)\big)$, and the $Z(\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}})(k)$-character of σ is trivial.

We claim that $P_{\mathfrak{f}}$ contains $Z(G)$. The semisimplicity of $\mathcal{G}$ implies that $Z(G)$ is finite and fixes the entire Bruhat–Tits building $\mathcal{B}(\mathcal{G}, K)$. Since $\mathcal{G}$ splits over an unramified extension K'/K, there exists a K'-split maximal K-torus $\mathcal{T}$ of $\mathcal{G}$. By the maximality $Z(\mathcal{G}) \subset \mathcal{T}$. Since $\mathcal{T}$ splits over an unramified extension, the Kottwitz homomorphism

$$\kappa_T : T \to X^*(\widehat{T}^{\mathbf{I}_K})^{\theta_K} = X^*(\widehat{T})^{\theta_K}$$

is determined by the canonical homomorphism

$$\nu : T \to X_*(\mathcal{S}) \otimes \mathbb{R}.$$

(with $\mathcal{S} \subset \mathcal{T}$ maximal K-split). Then $\nu(Z(G))$ is a finite subgroup of $X_*(\mathcal{S}) \otimes \mathbb{R}$, so $\nu(Z(G)) = 0$. Consequently $\kappa_T(Z(G)) = 0$, which by the functoriality of the Kottwitz homomorphism implies that $\kappa_G(Z(G)) = 0$. Thus $Z(G)$ stabilizes fixes the facet $\mathfrak{f}$ and lies in the kernel of the Kottwitz homomorphism κ_G, which proves the claim.

From the claim we see that $Z(G)U_{\mathfrak{f}}/U_{\mathfrak{f}}$ is a central subgroup of $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k)$. Hence the $Z(G)$-character of σ is trivial. It follows that $Z(G)$ acts trivially on $\mathrm{ind}_{P_{\mathfrak{f}}}^{G}(\sigma)$ and in particular on π.
(b) By part (a) we can regard π as a representation of the normal subgroup $G_1 := G/Z(G)$ of G_{ad}. From the long exact sequence in Galois cohomology we see that G_{ad}/G_1 is isomorphic to a closed subgroup of $H^1(F, Z(\mathcal{G}))$. In particular G_{ad}/G_1 is compact and abelian.

These properties suffice to apply [Tad, §2] to $G_1 \subset G$ and the G_1-representation π. Then [Tad, Proposition 2.2] shows that $\mathrm{ind}_{G_1}^{G_{\mathrm{ad}}}(\pi)$ has an irreducible subrepresentation π_{ad} of G_{ad} such that π is a constituent of the pullback of π_{ad} to G_1.

In the proof of part (a) we saw that $\pi|_{P_{\mathfrak{f}}}$ contains a subrepresentation isomorphic to σ. Since $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}$ is an $\mathfrak{o}_K$-model of $\mathcal{G}$ [BrTi] and $Z(G)$ is finite, the homomorphisms

$$\mathcal{G}(F) \to \mathcal{G}_{\mathrm{ad}}(F) \quad \text{and} \quad \overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(\mathfrak{o}_K) \to \overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}_{\mathrm{ad}}(\mathfrak{o}_K)$$

have the same kernel and the same cokernel. Hence $G/P_{\mathfrak{f}} \to G_{\mathrm{ad}}/P_{\mathfrak{f},\mathrm{ad}}$ is bijective, and the pullback of $\mathrm{ind}_{P_{\mathfrak{f},\mathrm{ad}}}^{G_{\mathrm{ad}}}(\sigma_{\mathrm{ad}})$ to G is isomorphic with $\mathrm{ind}_{P_{\mathfrak{f}}}^{G}(\sigma)$. By Frobenius reciprocity

$$\begin{aligned} \mathrm{Hom}_{G_{\mathrm{ad}}}\big(\mathrm{ind}_{P_{\mathfrak{f},\mathrm{ad}}}^{G_{\mathrm{ad}}}(\sigma_{\mathrm{ad}}), \mathrm{ind}_{G_1}^{G_{\mathrm{ad}}}(\pi)\big) &\cong \mathrm{Hom}_{G_1}\big(\mathrm{ind}_{P_{\mathfrak{f},\mathrm{ad}}}^{G_{\mathrm{ad}}}(\sigma_{\mathrm{ad}}), \pi\big) \\ &= \mathrm{Hom}_{G}\big(\mathrm{ind}_{P_{\mathfrak{f}}}^{G}(\sigma), \pi\big) \cong \mathrm{Hom}_{P_{\mathfrak{f}}}(\sigma, \pi) \neq 0. \end{aligned} \tag{46}$$

Let π_2 be the subrepresentation of $\mathrm{ind}_{G_1}^{G_{\mathrm{ad}}}(\pi)$ generated by the images of all possible G_{ad}-homomorphisms from $\mathrm{ind}_{P_{\mathfrak{f},\mathrm{ad}}}^{G_{\mathrm{ad}}}(\sigma_{\mathrm{ad}})$, that is, the component of $\mathrm{ind}_{G_1}^{G_{\mathrm{ad}}}(\pi)$ in $\mathrm{Rep}(G_{\mathrm{ad}})_{(P_{\mathfrak{f},\mathrm{ad}}, \sigma_{\mathrm{ad}})}$. The arguments in the proof of [Tad, Proposition 2.2] also work

with π_2 instead of $\mathrm{ind}_{G_1}^{G_{\mathrm{ad}}}(\pi)$, and show that we can find a π_{ad} as above already in π_2. Then π_{ad} is unipotent, irreducible and its pullback to G contains π.
(c) This a small variation on [Tad, Proposition 2.7], applied to the inclusion $G_1 \to G$. Here we regard a representation of $G_1 = G/Z(G)$ as tempered or cuspidal if its inflation to G is tempered or cuspidal. □

Proposition 5.3 guarantees that the next result applies to all square-integrable unipotent G-representations.

Theorem 5.4. *Let $\delta \in \mathrm{Irr}_{\mathrm{unip}}(G)$ and $\delta_{\mathrm{ad}} \in \mathrm{Irr}_{\mathrm{unip}}(G_{\mathrm{ad}})$ be square-integrable, such that δ is a constituent of $\eta^*(\delta_{\mathrm{ad}})$.*
(a) Their formal degrees, normalized as in [HII]*, are related as*

$$\frac{\mathrm{fdeg}(\eta^*(\delta_{\mathrm{ad}}))}{\mathrm{fdeg}(\delta_{\mathrm{ad}})} = \frac{\mathrm{fdeg}(\delta) \cdot \text{length of } \eta(\delta_{\mathrm{ad}})}{\mathrm{fdeg}(\delta_{\mathrm{ad}})} = \frac{|\Omega_{G_{\mathrm{ad}}}|}{|\Omega_G|}.$$

(b) $\mathrm{fdeg}(\delta) = \pm \dim(\rho_\delta) \left| S^\sharp_{\phi_\delta} \right|^{-1} \gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\delta, \psi)$.

Proof. (a) The first equality sign is a consequence of Lemma 2.3.a.

Write $V = \mathrm{Hom}_{\hat{P}_{\mathfrak{f},\mathrm{ad}}}(\hat\sigma_{\mathrm{ad}}, \delta_{\mathrm{ad}})$ and $V_E = \mathrm{Hom}_{\hat{P}_{\mathfrak{f},\mathrm{ad}}}(\hat\sigma_{\mathrm{ad}}, \delta)$. Recall from [BHK] that δ and V_E have the same formal degree, and similarly for δ_{ad} and V. With (34) and Lemma 4.3 we compute

$$\frac{\mathrm{fdeg}(\delta)}{\mathrm{fdeg}(\delta_{\mathrm{ad}})} = \frac{\mathrm{fdeg}(V_E)}{\mathrm{fdeg}(V)} = \frac{[\Omega_{G_{\mathrm{ad}},\mathfrak{f}} : \Omega_{G,\mathfrak{f}}]}{\text{length of } \eta^*_{\mathcal{H}}(V)} = \frac{|\Omega_{G_{\mathrm{ad}},\mathfrak{f}}||\Omega_{G_{\mathrm{ad}}} \cdot \mathfrak{f}|}{|\Omega_{G,\mathfrak{f}}||\Omega_G \cdot \mathfrak{f}| \text{ length of } \eta^*(\delta_{\mathrm{ad}})}. \tag{47}$$

By the orbit counting lemma and (3) this equals

$$\frac{|\Omega_{G_{\mathrm{ad}}}|}{|\Omega_G| \text{ length of } \eta^*(\delta_{\mathrm{ad}})} = \frac{|Z({G^\vee}_{\mathrm{sc}})^{\mathbf{W}_K}|}{|Z(G^\vee)^{\mathbf{W}_K}| \text{ length of } \eta^*(\delta_{\mathrm{ad}})}. \tag{48}$$

Rearranging (47) and (48) yields the desired equality.
(b) By Theorem 2.2 ϕ_δ is the composition of $\phi_{\delta_{\mathrm{ad}}}$ with the quotient map

$${}^L G_{\mathrm{ad}} = {G^\vee}_{\mathrm{sc}} \rtimes \mathbf{W}_K \to G^\vee \rtimes \mathbf{W}_K = {}^L G.$$

From Lemma 2.3.c we know that $\eta^*(\delta_{\mathrm{ad}})$ is the direct sum of exactly

$$\dim(\rho_{\delta_{\mathrm{ad}}})[\mathcal{A}_{\phi_\delta} : \mathcal{A}_{\phi_{\delta_{\mathrm{ad}}}}] \dim(\rho_\delta)^{-1} \tag{49}$$

irreducible G-representations. It is known from [FOS, proof of Lemma 13.2] that

$$\left[\mathcal{A}_{\phi_\delta} : \mathcal{A}_{\phi_{\delta_{\mathrm{ad}}}}\right] = \left[Z_{{G^\vee}_{\mathrm{sc}}}(\phi_\delta) : Z_{{G^\vee}_{\mathrm{sc}}}(\phi_{\delta_{\mathrm{ad}}})\right] = \left[S^\sharp_{\phi_\delta} : S^\sharp_{\phi_{\delta_{\mathrm{ad}}}}\right] \frac{|Z({G^\vee}_{\mathrm{sc}})^{\mathbf{W}_K}|}{|Z(G^\vee)^{\mathbf{W}_K}|}. \tag{50}$$

From (47), (48), (49) and (50) we deduce

$$\frac{\mathrm{fdeg}(\delta)}{\mathrm{fdeg}(\delta_{\mathrm{ad}})} = \frac{|Z({G^\vee}_{\mathrm{sc}})^{\mathbf{W}_K}|}{|Z(G^\vee)^{\mathbf{W}_K}|} \frac{\left|\mathcal{A}_{\phi_{\delta_{\mathrm{ad}}}}\right| \dim(\rho_\delta)}{|\mathcal{A}_{\phi_\delta}| \dim(\rho_{\delta_{\mathrm{ad}}})} = \frac{\left|S^\sharp_{\phi_{\delta_{\mathrm{ad}}}}\right| \dim(\rho_\delta)}{|S^\sharp_{\phi_\delta}| \dim(\rho_{\delta_{\mathrm{ad}}})}. \tag{51}$$

As $\mathrm{Lie}({G^\vee}_{\mathrm{sc}}) = \mathrm{Lie}(G^\vee)$,

$$\gamma(s, \mathrm{Ad}_{G^\vee} \circ \phi_\delta, \psi) = \gamma(s, \mathrm{Ad}_{{G^\vee}_{\mathrm{sc}}} \circ \phi_{\delta_{\mathrm{ad}}}, \psi) \qquad \text{for all } s \in \mathbb{C}.$$

Then (51) says

$$\frac{\mathrm{fdeg}(\delta)}{\mathrm{fdeg}(\delta_{\mathrm{ad}})} = \frac{\dim(\rho_\delta)\, |S^\sharp_{\phi_{\delta_{\mathrm{ad}}}}|}{\dim(\rho_{\delta_{\mathrm{ad}}})\, |S^\sharp_{\phi_\delta}|} \frac{\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\delta, \psi)}{\gamma(0, \mathrm{Ad}_{{G^\vee}_{\mathrm{sc}}} \circ \phi_{\delta_{\mathrm{ad}}}, \psi)}.$$

Combining that with Theorem 5.1.a, we obtain the desired formula for $\mathrm{fdeg}(\delta)$. $\square$

5.3. **Reductive groups.**

To prove the HII conjecture for unipotent representations of a reductive group G, we want to compare its representations with those of $G/Z(G)_s$ and with those of the derived group $G_{\mathrm{der}} := \mathcal{G}_{\mathrm{der}}(K)$. (Notice that G_{der} may be larger than the derived group of G as an abstract group.)

We start with some preparations for the case that $Z(\mathcal{G})^\circ$ is K-anisotropic. Let $\overline{Z(\mathcal{G})^\circ}$ be the connected reductive k-group associated by Bruhat–Tits to the unique vertex of $\mathcal{B}(Z(\mathcal{G})^\circ, K)$.

For any Levi K-subgroup $\mathcal{M}$ of $\mathcal{G}$, $\mathcal{M}_d := \mathcal{M} \cap \mathcal{G}_{\mathrm{der}}$ is a Levi K-subgroup of $\mathcal{G}_{\mathrm{der}}$. Furthermore $Z(\mathcal{M})_s \subset \mathcal{M}_d$, for example because $\mathrm{Lie}(Z(\mathcal{M})_s) \subset \mathrm{Lie}(\mathcal{G}_{\mathrm{der}})$. We note also that $\mathcal{M}_d/Z(\mathcal{M})_s$ is the derived group of $\mathcal{M}/Z(\mathcal{M})_s$.

Lemma 5.5. *Suppose that $Z(\mathcal{G})^\circ$ is K-anisotropic. The inclusion $M_d \to M$ induces:*

(*a*) *a bijection* $\mathrm{Irr}_{\mathrm{unip,cusp}}(M) \to \mathrm{Irr}_{\mathrm{unip,cusp}}(M_d)$*;*

(*b*) *for every minimal facet $\mathfrak{f}$ of $\mathcal{B}(\mathcal{M}_{\mathrm{ad}}, K)$, a bijection between the types $(\hat{P}_{\mathfrak{f}}, \hat{\sigma})$ for M and for M_d.*

Proof. (a) Let $X_{\mathrm{wr}}(M)$ be the group of weakly unramified characters, i.e. those characters $M \to \mathbb{C}^\times$ that are trivial on the kernel of the Kottwitz homomorphism κ_M. From the short exact sequence (4) (for $\mathcal{M}$ and for $\mathcal{M}_d$) we deduce that there are natural isomorphisms

$$X_{\mathrm{wr}}(M)/X_{\mathrm{wr}}(M/Z(M)_s) \cong X_{\mathrm{wr}}(Z(M)_s) \cong X_{\mathrm{wr}}(M_d)/X_{\mathrm{wr}}(M_d/Z(M)_s). \tag{52}$$

By [FOS, (15.6)] every irreducible cuspidal unipotent M-representation is of the form $\pi_{M/Z(M)_s} \otimes \chi_M$ with $\pi_{M/Z(M)_s} \in \mathrm{Irr}_{\mathrm{unip,cusp}}(M/Z(M)_s)$ and $\chi_M \in X_{\mathrm{nr}}(M)$. Using weakly unramified characters, we can formulate this more precisely as a bijection

$$\mathrm{Irr}_{\mathrm{unip,cusp}}(M/Z(M)_s) \underset{X_{\mathrm{wr}}(M/Z(M)_s)}{\times} X_{\mathrm{wr}}(M) \to \mathrm{Irr}_{\mathrm{unip,cusp}}(M). \tag{53}$$

Similarly there is a bijection

$$\mathrm{Irr}_{\mathrm{unip,cusp}}(M_d/Z(M)_s) \underset{X_{\mathrm{wr}}(M_d/Z(M)_s)}{\times} X_{\mathrm{wr}}(M_d) \to \mathrm{Irr}_{\mathrm{unip,cusp}}(M_d). \tag{54}$$

We note that $Z(\mathcal{M}/Z(\mathcal{M})_s)^\circ$ is isogenous to $Z(\mathcal{G})^\circ$, and in particular it is K-anisotropic. Hence we may apply [FOS, Lemma 15.3], which tells us that the inclusion $M_d/Z(M)_s \to M/Z(M)_s$ induces a bijection

$$\mathrm{Irr}_{\mathrm{unip,cusp}}(M/Z(M)_s) \to \mathrm{Irr}_{\mathrm{unip,cusp}}(M_d/Z(M)_s). \tag{55}$$

Combining (55) and (52) with (53) and (54), we obtain the required bijection.
(b) The (semisimple) Bruhat–Tits buildings of $M, M_d, M/Z(M)_s$ and $M_d/Z(M)_s$ can be identified [Tits, §2]. In particular these buildings have the same collections of facets $\mathfrak{f}$. The group $\overline{\mathcal{M}_{\mathfrak{f}}^\circ}$ is isogenous to the direct product of $\overline{\mathcal{M}_{d,\mathfrak{f}}^\circ}$ and the k-torus $\overline{Z(\mathcal{G})^\circ}$. The only cuspidal unipotent representation of $\overline{Z(\mathcal{G})^\circ}(k)$ is the trivial representation. The collection of cuspidal unipotent representations of $\big(\overline{\mathcal{M}_{d,\mathfrak{f}}^\circ} \times \overline{Z(\mathcal{G})^\circ}\big)(k)$ does not change under isogenies of k-groups [Lus1, §3], so it is the same as for $\overline{\mathcal{M}_{\mathfrak{f}}^\circ}(k)$. As the semisimple group $\mathcal{M}_d/Z(\mathcal{M})_s$ is the derived group of $\mathcal{M}/Z(\mathcal{M})_s$,

[FOS, Lemma 15.2] says that $\Omega_{M/Z(M)_s} = \Omega_{M_d/Z(M)_s}$. Combining that with (5), we find that

$$\Omega_M = \Omega_{M_d}. \tag{56}$$

With (7) we deduce that $M_d \to M$ induces a bijection between the indicated collections of types. □

The behaviour of formal degrees of supercuspidal unipotent representations under pullback from M to M_d was analysed in [FOS, (16.13)]. That and Lemma 5.5 can be generalized to all (square-integrable) unipotent representations:

Lemma 5.6. *Suppose that $Z(\mathcal{G})^\circ$ is K-anisotropic. Let $(\hat{P}_{\mathfrak{f},M}, \hat\sigma)$ and $(\hat{P}_{\mathfrak{f},M_d}, \hat\sigma)$ be as in Lemma 5.5.b.*

(*a*) *The inclusion $G_{\mathrm{der}} \to G$ induces an algebra isomorphism*

$$\mathcal{H}(G_{\mathrm{der}}, \hat{P}_{\mathfrak{f},G_{\mathrm{der}}}, \hat\sigma) \to \mathcal{H}(G, \hat{P}_{\mathfrak{f},G}, \hat\sigma).$$

(*b*) *Suppose that $\delta \in \mathrm{Irr}(G)_{(\hat{P}_{\mathfrak{f},G},\hat\sigma)}$ is square-integrable, and let δ_{der} be its pullback to G_{der}. Then δ_{der} is irreducible and*

$$\frac{\mathrm{fdeg}(\delta)}{\mathrm{fdeg}(\delta_{\mathrm{der}})} = \frac{q^{(\dim \overline{Z(\mathcal{G})^\circ} + \dim Z(\mathcal{G})^\circ)/2}}{|\overline{Z(\mathcal{G})^\circ}(k)|}.$$

Proof. (a) By [Sol3, Lemma 3.5 and (42)] these two affine Hecke algebras differ only in the involved lattices $X_{\mathfrak{f}}$. From (56) and the proof of [Sol3, Theorem 3.3.b] we see that

$$X_{\mathfrak{f},G} = \Omega_{M,\mathfrak{f}_M} / \Omega_{M,\mathfrak{f}_M,\mathrm{tor}} = \Omega_{M_d,\mathfrak{f}_M} / \Omega_{M_d,\mathfrak{f}_M,\mathrm{tor}} = X_{\mathfrak{f},G_{\mathrm{der}}}.$$

Hence $\mathcal{H}(G_{\mathrm{der}}, \hat{P}_{\mathfrak{f},G_{\mathrm{der}}}, \hat\sigma)$ can be identified with $\mathcal{H}(G, \hat{P}_{\mathfrak{f},G}, \hat\sigma)$, and the canonical map between them is an isomorphism. We note that nevertheless the traces of these algebras may be normalized differently.
(b) Let $\delta_{\mathcal{H}}$ be the $\mathcal{H}(G, \hat{P}_{\mathfrak{f},G}, \hat\sigma)$-module associated to δ via (10). By Lemma 5.5 $\delta_{\mathrm{der}} \in \mathrm{Rep}(G_{\mathrm{der}})_{(\hat{P}_{\mathfrak{f},G_{\mathrm{der}}},\hat\sigma)}$. By part (a) the $\mathcal{H}(G_{\mathrm{der}}, \hat{P}_{\mathfrak{f},G_{\mathrm{der}}}, \hat\sigma)$-module $\delta_{\mathrm{der},\mathcal{H}}$ can be identified with $\delta_{\mathcal{H}}$, and in particular it is irreducible. From [BHK] and (12) we see that

$$\frac{\mathrm{fdeg}(\delta)}{\mathrm{fdeg}(\delta_{\mathrm{der}})} = \frac{\mathrm{fdeg}(\delta_{\mathcal{H}})}{\mathrm{fdeg}(\delta_{\mathrm{der},\mathcal{H}})} = \frac{\mathrm{vol}(\hat{P}_{\mathfrak{f},G_{\mathrm{der}}})}{\mathrm{vol}(\hat{P}_{\mathfrak{f},G})}.$$

By (7) and (56) this equals

$$\frac{\mathrm{vol}(P_{\mathfrak{f},G_{\mathrm{der}}}) |\Omega_{G_{\mathrm{der}},\mathfrak{f},\mathrm{tor}}|}{\mathrm{vol}(P_{\mathfrak{f},G}) |\Omega_{G,\mathfrak{f},\mathrm{tor}}|} = \frac{\mathrm{vol}(P_{\mathfrak{f},G_{\mathrm{der}}}) |\Omega_{M_{\mathrm{der}},\mathfrak{f},\mathrm{tor}}|}{\mathrm{vol}(P_{\mathfrak{f},G}) |\Omega_{M,\mathfrak{f},\mathrm{tor}}|} = \frac{\mathrm{vol}(P_{\mathfrak{f},G_{\mathrm{der}}})}{\mathrm{vol}(P_{\mathfrak{f},G})}. \tag{57}$$

These volumes, with respect to our normalized Haar measures, are expressed in terms of k-groups in (6). Since $\overline{\mathcal{G}_{\mathfrak{f}}^\circ}$ is isogenous to $\overline{\mathcal{G}_{\mathrm{der},\mathfrak{f}}^\circ} \times \overline{Z(\mathcal{G})^\circ}$, we have [GeMa, Proposition 1.4.12.c]

$$|\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k)| = |\overline{\mathcal{G}_{\mathrm{der},\mathfrak{f}}^\circ}(k)| \, |\overline{Z(\mathcal{G})^\circ}(k)|.$$

With that and (6), (57) becomes

$$\frac{\mathrm{vol}(P_{\mathfrak{f},G_{\mathrm{der}}})}{\mathrm{vol}(P_{\mathfrak{f},G})} = \frac{|\overline{\mathcal{G}_{\mathrm{der},\mathfrak{f}}^\circ}(k)| \, q^{-(\dim \overline{\mathcal{G}_{\mathrm{der},\mathfrak{f}}^\circ} + \dim \mathcal{G}_{\mathrm{der}})/2}}{|\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k)| \, q^{-(\dim \overline{\mathcal{G}_{\mathfrak{f}}^\circ} + \dim \mathcal{G})/2}} = \frac{q^{(\dim \overline{Z(\mathcal{G})^\circ} + \dim Z(\mathcal{G})^\circ)/2}}{|\overline{Z(\mathcal{G})^\circ}(k)|}. \tag{58}$$

□

With all preparations complete, we can prove our main result, the HII conjecture (2) for unipotent representations.

Theorem 5.7. *Let $\mathcal{G}$ be a connected reductive K-group which splits over an unramified extension. Let $\delta \in \mathrm{Irr}_{\mathrm{unip}}(G)$ be square-integrable modulo centre and let $(\phi_\delta, \rho_\delta)$ be its enhanced L-parameter via Theorem 2.1. Let $\psi : K \to \mathbb{C}^\times$ have order 0 and normalize the Haar measure on G as in* [HII] *and* [GaGr]. *Then*

$$\mathrm{fdeg}(\delta) = \pm \dim(\rho_\delta) |S^\sharp_{\phi_\delta}|^{-1} \gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\delta, \psi).$$

Proof. For the moment we assume that $Z(\mathcal{G})^\circ$ is K-anisotropic. Then Lemma 5.6 tells us that the pullback δ_{der} of δ along $G_{\mathrm{der}} \to G$ is irreducible, so that Theorem 5.4.b applies to $\delta_{\mathrm{der}} \in \mathrm{Irr}_{\mathrm{unip}}(G_{\mathrm{der}})$.

In the proof of [FOS, Lemma 16.3] it was shown that

$$\frac{\dim(\rho_\delta)\, |S^\sharp_{\phi_{\delta_{\mathrm{der}}}}|\, \gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\delta, \psi)}{\dim(\rho_{\delta_{\mathrm{der}}})\, |S^\sharp_{\phi_\delta}|\, \gamma(0, \mathrm{Ad}_{G^\vee_{\mathrm{der}}} \circ \phi_{\delta_{\mathrm{der}}}, \psi)} = \frac{q^{(\dim \overline{Z(\mathcal{G})^\circ} + \dim Z(\mathcal{G})^\circ)/2}}{|\overline{Z(\mathcal{G})^\circ}(k)|}.$$

By Lemma 5.6.b the right hand side equals $\mathrm{fdeg}(\delta)\mathrm{fdeg}(\delta_{\mathrm{der}})^{-1}$. Combining that with the formula for $\mathrm{fdeg}(\delta_{\mathrm{der}})$ from Theorem 5.4.b, we find the desired expression for $\mathrm{fdeg}(\delta)$.

Now we consider any $\mathcal{G}$ as in the statement of the theorem. The connected reductive K-group $\mathcal{G}/Z(\mathcal{G})_s$ has K-anisotropic connected centre. It was shown in [FOS, proof of Theorem 3 on page 43] how the theorem for G can be derived from the theorem for $G/Z(G)_s$. Although [FOS] is formulated only for supercuspidal representations, this proof also works for square-integrable modulo centre representations when we use the local Langlands correspondence from Theorem 2.1 (especially part (b) on the compatibility with weakly unramified characters). □

6. Extension to tempered representations

6.1. Normalization of densities.

In this paragraph we study the Plancherel densities for essentially square-integrable representations of a reductive group G with non-compact centre.

We fix an essentially square-integrable unipotent $\pi \in \mathrm{Irr}(G)$, trivial on the maximal central split torus $Z(G)_s$. Recall that we have canonical Haar measures and hence canonical Plancherel measures for G and for $G/Z(G)_s$. Further, Conjecture 1 and [Wal] impose a measure on $\mathcal{O} = X_{\mathrm{unr}}(G)\pi \subset \mathrm{Irr}(G)$. Our conventions force us to slightly modify the latter measure. We propose a new normalization and we check that it results in a nice formula for the Plancherel mass of $\mathcal{O}$.

Let G^1 be the subgroup of G generated by all compact subgroups and let $Z(G)^1_s$ be the unique maximal compact subgroup of $Z(G)_s$. We endow $X_{\mathrm{unr}}(Z(G)_s)$ with the Haar measure of total mass $\mathrm{vol}(Z(G)^1_s)^{-1}$. Following [Wal, p. 302], we decree that the covering maps

$$\begin{array}{ccccc} X_{\mathrm{unr}}(Z(G)_s) & \leftarrow & X_{\mathrm{unr}}(G) & \rightarrow & \mathcal{O} \\ \chi|_{Z(G)_s} & \leftarrow\!\shortmid & \chi & \mapsto & \chi \otimes \pi \end{array}$$

are locally measure preserving. We denote the associated density on $\mathcal{O}$ by $\mathrm{d}\mathcal{O}$. Notice that the degree of $X_{\mathrm{unr}}(G) \to X_{\mathrm{unr}}(Z(G)_s)$ equals $[G : G^1 Z(G)_s]$. Write

$$\mathcal{O} \cap \mathrm{Irr}(G/Z(G)_s) = \{\pi \otimes \chi \in \mathcal{O} : Z(G)_s \subset \ker \chi\}.$$

Tensoring π with χ gives a covering map

$$\ker\big(X_{\mathrm{unr}}(G) \to X_{\mathrm{unr}}(Z(G)_s)\big) \to \mathcal{O} \cap \mathrm{Irr}(G/Z(G)_s),$$

whose degree equals the degree of $X_{\mathrm{unr}}(G) \to \mathcal{O}$. Hence the number of elements of any fiber of $X_{\mathrm{unr}}(G) \to \mathcal{O}$ is

$$[G : G^1 Z(G)_s]\, |\mathcal{O} \cap \mathrm{Irr}(G/Z(G)_s)|^{-1}.$$

It follows that

$$\mathrm{vol}(X_{\mathrm{unr}}(G)) = [G : G^1 Z(G)_s]\, \mathrm{vol}(Z(G)^1_s)^{-1}, \tag{59}$$

$$\mathrm{vol}(\mathcal{O}) = |\mathcal{O} \cap \mathrm{Irr}(G/Z(G)_s)|\, \mathrm{vol}(Z(G)^1_s)^{-1}. \tag{60}$$

Lemma 6.1. *The Plancherel density on $\mathcal{O}$ is $\mathrm{fdeg}(\pi)\, \mathrm{d}\mathcal{O}$ and*

$$\mu_{Pl}(\mathcal{O}) = \mathrm{fdeg}(\pi)\mathrm{vol}(\mathcal{O}).$$

Proof. Choose a test function $f \in C_r^*(G)$ such that f is supported on G^1, $\mathrm{tr}\,\pi(f) = 1$ and f acts as 0 on all irreducible G-representations outside $\mathcal{O}$. Then f is $Z(G)^1_s$-invariant and $\mathrm{tr}(\pi \otimes \chi)(f) = 1$ for all $\chi \in X_{\mathrm{unr}}(G)$. By definition

$$f(1) = \int_{\mathcal{O}} \mathrm{tr}\,\pi(f)\, \mathrm{d}\mu_{Pl}(\pi) = \mu_{Pl}(\mathcal{O}). \tag{61}$$

Since f is $Z(G)^1_s$-invariant, it defines a function f_1 on

$$G^1/Z(G)^1_s \cong G^1 Z(G)_s / Z(G)_s,$$

which we extend by zero to whole of $G/Z(G)_s$. Due to the difference in the Haar measures, f and f_1 act differently on representations of $G/Z(G)_s$. Instead, the function $f_2 := \mathrm{vol}(Z(G)^1_s) f_1$ has the same action as f on any smooth $G/Z(G)_s$-representation. This can be seen by expressing f on a small subset of the form $X \cong X/Z(G)^1_s \times Z(G)^1_s$ as

$$f\big|_X = f_2\big|_{X/Z(G)^1_s} \cdot \frac{1_{Z(G)^1_s}}{\mathrm{vol}(Z(G)^1_s)}.$$

In view of the construction of f, the function f_2 detects only the G-representations $\chi \otimes \pi$ with $\chi \in X_{\mathrm{unr}}(G)$ and $Z(G)_s \subset \ker \chi$. All these representations have the same Plancherel density (both for G and for $G/Z(G)_s$). The Plancherel formula for $G/Z(G)_s$ gives

$$\begin{aligned} f(1)\mathrm{vol}(Z(G)^1_s) = f_2(1) &= \int_{\mathcal{O} \cap \mathrm{Irr}(G/Z(G)_s)} \mathrm{tr}\,\pi(f_2)\, \mathrm{d}\mu_{Pl,G/Z(G)_s}(\pi) \\ &= \mu_{Pl,G/Z(G)_s}(\mathcal{O} \cap \mathrm{Irr}(G/Z(G)_s)) = |\mathcal{O} \cap \mathrm{Irr}(G/Z(G)_s)| \mu_{Pl,G/Z(G)_s}(\pi). \end{aligned} \tag{62}$$

Comparing with (60) and (61), we find

$$\mu_{Pl}(\mathcal{O}) = \mathrm{vol}(Z(G)^1_s)^{-1} |\mathcal{O} \cap \mathrm{Irr}(G/Z(G)_s)| \mu_{Pl,G/Z(G)_s}(\pi) = \mathrm{vol}(\mathcal{O})\mathrm{fdeg}(\pi).$$

As tensoring with unramified unitary characters preserves the Plancherel density, this means that $\mathrm{fdeg}(\pi)\, \mathrm{d}\mathcal{O}$ is the Plancherel density on $\mathcal{O}$. □

Let $(\hat{P}_{\mathfrak{f}}, \hat{\sigma})$ be the unipotent type such that $\pi \in \mathrm{Irr}(G)_{(\hat{P}_{\mathfrak{f}}, \hat{\sigma})}$. We abbreviate $\mathcal{H} = \mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$. The representation $\hat{\sigma}$ is trivial on $Z(G)^1_s$, so $(\hat{P}_{\mathfrak{f}}, \hat{\sigma})$ descends to a type $(\hat{P}_{\mathfrak{f}}/Z(M)^1_s, \hat{\sigma})$ for the group $G/Z(G)_s$. This type can detect more than one

Bernstein component, because $\hat{P}_{G/Z(G)_s,\mathfrak{f}}$ can properly contain $\hat{P}_{\mathfrak{f}}/Z(M)^1_s$. Let σ' be the (unique) extension of $\hat{\sigma}$ to $\hat{P}_{G/Z(G)_s,\mathfrak{f}}$ which is contained in π. Then

$$\mathcal{H}_{ss} := \mathcal{H}\big(G/Z(G)_s, \hat{P}_{G/Z(G)_s,\mathfrak{f}}, \sigma'\big)$$

is naturally a quotient of $\mathcal{H}$, obtained by mapping the generators $N_w \in \mathcal{H}$ with $w \in Z(G)/Z(G)^1_s$ to suitable scalars. The traces τ and τ_{ss} of $\mathcal{H}$ and $\mathcal{H}_{ss}$, normalized as in (11), differ at the unit element:

Lemma 6.2. $\displaystyle \frac{\tau_{ss}(N_e)}{\tau(N_e)} = \frac{|\Omega_{G,\mathfrak{f},\mathrm{tor}}| \operatorname{vol}(Z(G)^1_s)}{|\Omega_{G/Z(G)_s,\mathfrak{f},\mathrm{tor}}|} = \frac{|\Omega_{G,\mathfrak{f},\mathrm{tor}}|}{|\Omega_{G/Z(G)_s,\mathfrak{f},\mathrm{tor}}|}\Big(\frac{q-1}{q}\Big)^{\dim(Z(\mathcal{G})_s)}.$

Proof. By (11) and (7)

$$\frac{\tau_{ss}(N_e)}{\tau(N_e)} = \frac{\dim(\sigma')\mathrm{vol}(\hat{P}_{G/Z(G)_s,\mathfrak{f}})^{-1}}{\dim(\hat{\sigma})\operatorname{vol}(\hat{P}_{G,\mathfrak{f}})^{-1}} = \frac{|\Omega_{G,\mathfrak{f},\mathrm{tor}}| \operatorname{vol}(P_{G,\mathfrak{f}})}{|\Omega_{G/Z(G)_s,\mathfrak{f},\mathrm{tor}}|\mathrm{vol}(P_{G/Z(G)_s,\mathfrak{f}})}.$$

Let $\overline{Z(\mathcal{G})_s} \cong GL_1^{\dim Z(\mathcal{G})_s}$ be the connected reductive k-group associated to the unique vertex of $\mathcal{B}(Z(\mathcal{G})_s, K)$. Since $\overline{\mathcal{G}^\circ_{\mathfrak{f}}}$ is isogenous to $\overline{(\mathcal{G}/Z(\mathcal{G})_s)^\circ_{\mathfrak{f}}} \times \overline{Z(\mathcal{G})_s}$, a calculation analogous to (58) shows that

$$\frac{\mathrm{vol}(P_{\mathfrak{f}})}{\mathrm{vol}(P_{G/Z(G)_s,\mathfrak{f}})} = \frac{|\overline{Z(\mathcal{G})_s}(k)|}{q^{\dim(\overline{Z(\mathcal{G})_s})}} = \mathrm{vol}(Z(G)^1_s).$$

Finally, we note that $|\overline{Z(\mathcal{G})_s}(k)| = |GL_1(k)|^{\dim Z(\mathcal{G})_s} = (q-1)^{\dim Z(\mathcal{G})_s}$. □

As only the trivial element of $\Omega_{Z(G)_s}$ fixes any point of the standard apartment $\mathbb{A}$ of $\mathcal{B}(\mathcal{G}, K)$, (5) entails that the natural map

$$\Omega_{G,\mathfrak{f},\mathrm{tor}} \to \Omega_{G/Z(G)_s,\mathfrak{f},\mathrm{tor}}$$

is injective. However, in general it need not be surjective.

We write $T_{\mathfrak{f},ss} = \mathrm{Hom}(X_{G/Z(G)_s,\mathfrak{f}}, \mathbb{C}^\times)$, a subtorus of $T_{\mathfrak{f}} = \mathrm{Hom}(X_{\mathfrak{f}}, \mathbb{C}^\times)$. The image of $X_{\mathrm{nr}}(G)$ in $T_{\mathfrak{f}}$ is another algebraic subtorus $T_{\mathfrak{f},Z}$, which is complementary in the sense that

$$T_{\mathfrak{f},ss}T_{\mathfrak{f},Z} = T_{\mathfrak{f}} \quad \text{and} \quad |T_{\mathfrak{f},ss} \cap T_{\mathfrak{f},Z}| < \infty.$$

Let $T_{\mathfrak{f},un} = \mathrm{Hom}(X_{\mathfrak{f}}, S^1)$, the maximal compact real subtorus of $T_{\mathfrak{f}}$. We define $T_{\mathfrak{f},ss,un}$ and $T_{\mathfrak{f},Z,un}$ similarly. Write $\pi_{\mathcal{H}} = \mathrm{Hom}_{\hat{P}_{\mathfrak{f}}}(\hat{\sigma}, \pi) \in \mathrm{Mod}(\mathcal{H})$. By (10) the map

$$X_{\mathrm{unr}}(G) \to \mathcal{O} : \chi \mapsto \chi \otimes \pi$$

induces a surjection

$$T_{\mathfrak{f},Z,un} \to T_{\mathfrak{f},Z,un}\pi_{\mathcal{H}} = \{\mathrm{Hom}_{\hat{P}_{\mathfrak{f}}}(\hat{\sigma}, \chi \otimes \pi) : \chi \in X_{\mathrm{unr}}(G)\}.$$

Furthermore $T_{\mathfrak{f},Z,un}\pi_{\mathcal{H}}$ is in bijection with $\mathcal{O}$ via (10).

Let $\mathrm{d}t_Z$ be the Haar measure on $T_{\mathfrak{f},Z,un}\pi_{\mathcal{H}}$ with total volume 1. Since (10) preserves Plancherel measures [BHK], $\mu_{Pl,G}\big|_{\mathcal{O}}$ and $\mu_{Pl,\mathcal{H}}\big|_{T_{\mathfrak{f},Z,un}\pi_{\mathcal{H}}}$ agree. With Lemma 6.1 we find that

$$\mathrm{d}\mu_{Pl,\mathcal{H}}(\pi_{\mathcal{H}}) = \mathrm{fdeg}_{\mathcal{H}_{ss}}(\pi_{\mathcal{H}})\, |\mathcal{O} \cap \mathrm{Irr}(G/Z(G)_s)| \operatorname{vol}(Z(G)^1_s)^{-1} \mathrm{d}t_Z. \tag{63}$$

This can also be formulated entirely in terms of affine Hecke algebras:

Lemma 6.3. $\mathrm{d}\mu_{Pl,\mathcal{H}}(\pi_{\mathcal{H}}) = \mathrm{fdeg}_{\mathcal{H}_{ss}}(\pi_{\mathcal{H}})\tau(N_e)\tau_{ss}(N_e)^{-1}|(T_{\mathfrak{f},ss} \cap T_{\mathfrak{f},Z})\pi_{\mathcal{H}}|\, \mathrm{d}t_Z.$

Proof. Consider an arbitrary extension σ'' of $\hat\sigma$ to $\hat P_{G/Z(G)_s,\mathfrak f}$. From [Lus2, §1.20] or [Sol3, (40)] we see that the number of elements of $\mathcal O$ that contain σ'' equals the number of elements that contain σ'. The number of possible extensions σ'' is $|\Omega_{G/Z(G)_s,\mathfrak f,\mathrm{tor}}|\,|\Omega_{G,\mathfrak f,\mathrm{tor}}|^{-1}$, and hence

$$\big|\mathcal O \cap \mathrm{Irr}(G/Z(G)_s)\big| = |\Omega_{G/Z(G)_s,\mathfrak f,\mathrm{tor}}|\,|\Omega_{G,\mathfrak f,\mathrm{tor}}|^{-1}\big|T_{\mathfrak f,Z}\pi_{\mathcal H} \cap \mathrm{Irr}(\mathcal H_{ss})\big|. \tag{64}$$

An $\mathcal H$-representation $t\otimes\pi_{\mathcal H} \in T_{\mathfrak f}\pi_{\mathcal H}$ descends to $\mathcal H_{ss}$ if and only if $t \in T_{\mathfrak f,ss}$. Therefore

$$|T_{\mathfrak f,Z}\pi_{\mathcal H} \cap \mathrm{Irr}(\mathcal H_{ss})| = |(T_{\mathfrak f,ss} \cap T_{\mathfrak f,Z})\pi_{\mathcal H}|.$$

Combine that with (63), (64) and Lemma 6.1. □

We remark that Lemma 6.3 is in accordance with a comparison formula for Plancherel measures of affine Hecke algebras [Opd1, (4.96)].

6.2. **Parabolic induction and Plancherel densities.**

Let $\mathcal M$ be a Levi K-subgroup of $\mathcal G$ and let $\pi_M \in \mathrm{Irr}_{\mathrm{unip}}(M)$ be essentially square-integrable. As before we write $\mathcal O = X_{\mathrm{unr}}(M)\pi_M$. Let $\mathcal P$ be a parabolic K-subgroup of $\mathcal G$ with Levi factor $\mathcal M$ and denote the normalized parabolic induction functor by I_P^G. We want to express the Plancherel density on the family of finite length tempered unitary G-representations

$$I_P^G(\mathcal O) = \{ I_P^G(\chi\otimes\pi_M) : \chi \in X_{\mathrm{unr}}(M)\}. \tag{65}$$

Recall that the infinitesimal character of ϕ_M is

$$\mathrm{inf.ch.}(\phi_M) = \phi_M\big(\mathrm{Frob}, \big(\begin{smallmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2}\end{smallmatrix}\big)\big).$$

We abbreviate $\mathcal H = \mathcal H(G,\hat P_{\mathfrak f},\hat\sigma)$ and $\mathcal H^M = \mathcal H(M,\hat P_{M,\mathfrak f},\hat\sigma)$, where $\pi_M \in \mathrm{Irr}(M)_{(\hat P_{M,\mathfrak f},\hat\sigma)}$. From the proof of Theorem 2.1, which can be retraced to [AMS3, Theorem 3.18.b], one sees that the central character of

$$\pi_{M,\mathcal H} = \mathrm{Hom}_{\hat P_{M,\mathfrak f}}(\hat\sigma,\pi_M) \in \mathrm{Mod}(\mathcal H^M)$$

is completely determined by inf.ch.(ϕ_M). More precisely, choose a basepoint for the appropriate Bernstein component of enhanced L-parameters as in [Sol3, Lemma 3.4] and pick $t_M \in T^\vee$ such that inf.ch.(ϕ_M) equals t_M times the basepoint. Then the central character of $\pi_{M,\mathcal H}$ is $W(R_{M,\mathfrak f})t_M \in T_{M,\mathfrak f}/W(R_{M,\mathfrak f})$.

For $t \in Z(M^\vee)^{\theta,\circ} \cong X_{\mathrm{nr}}(M)$ the M-representation $t\otimes\pi_M$ corresponds to $t\otimes\pi_{M,\mathcal H} \in \mathrm{Irr}(\mathcal H^M)$ and its enhanced L-parameter is $(t\phi_M,\rho_M)$, where $t\phi_M$ is defined in (95).

Lemma 6.4. *For $t \in X_{\mathrm{unr}}(M)$, the Plancherel density $d\mu_{Pl}\big(I_P^G(t\otimes\pi_M)\big)$ equals*

$$\pm q^{(\dim\mathcal G-\dim\mathcal M)/2} m_{\mathfrak s}^M(tt_M)\gamma(0,\mathrm{Ad}_{M^\vee}\circ t\phi_M,\psi)\dim(\rho_M)|S^\sharp_{\phi_M}|^{-1}\mathrm{d}\mathcal O(t\otimes\pi_M).$$

The factor $m_{\mathfrak s}^M(tt_M)$, which depends on the Bernstein component $\mathrm{Rep}(G)^{\mathfrak s}$ *containing $I_P^G(\pi_M)$, is defined in* [Opd1, (3.57)] *and* [Opd4, (2.17)].

Proof. Notice that $t\otimes\pi_M$ is still essentially square integrable, for that property is stable under tensoring by unitary characters. By the expression for Plancherel densities in affine Hecke algebras [Opd1, (4.96)]:

$$\mu_{Pl,\mathcal H}\big(\mathrm{ind}_{\mathcal H^M}^{\mathcal H}(t\otimes\pi_{M,\mathcal H})\big) = m_{\mathfrak s}^M(tt_M)\tau(N_e)\tau_{\mathcal H^M}(N_e)^{-1}\mu_{Pl,\mathcal H^M}(t\otimes\pi_{M,\mathcal H}) \tag{66}$$

The factor $\tau(N_e)\tau_{\mathcal{H}^M}(N_e)^{-1}$ appears because in [Opd1] the traces of the Hecke algebras are normalized by $\tau'(N_e) = 1$. From (11), (6) and (7) we see that

$$\tau(N_e) / \tau_{\mathcal{H}^M}(N_e) = \frac{\mathrm{vol}(\hat{P}_{\mathfrak{f},M})}{\mathrm{vol}(\hat{P}_{\mathfrak{f}})} = \frac{|\overline{\mathcal{M}_{\mathfrak{f}}}(k_F)|\, q^{(\dim \overline{\mathcal{G}_{\mathfrak{f}}^\circ} + \dim \mathcal{G})/2}}{|\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)|\, q^{(\dim \overline{\mathcal{M}_{\mathfrak{f}}^\circ} + \dim \mathcal{M})/2}}. \tag{67}$$

By [Mor2, Theorem 2.1] $\overline{\mathcal{M}_{\mathfrak{f}}} \cong \overline{\mathcal{G}_{\mathfrak{f}}}$, so that the right hand side of (67) reduces to $q^{(\dim \mathcal{G} - \dim \mathcal{M})/2}$. By Lemma 6.3, (63) and (67), (66) equals

$$m_{\mathfrak{s}}^M(tt_M) q^{(\dim \mathcal{G} - \dim \mathcal{M})/2} \mathrm{fdeg}_{\mathcal{H}_{ss}}(\pi_{M,\mathcal{H}})\, |\mathcal{O} \cap \mathrm{Irr}(G/Z(G)_s)|\, \mathrm{vol}(Z(G)_s^1)^{-1} \mathrm{d}t_Z.$$

It is known from [Sol2, Lemma 4.1] that normalized parabolic induction commutes with the functor (10). As (10) preserves Plancherel densities, (66) and (60) yield

$$\mu_{Pl}\big(I_P^G(t \otimes \pi_M)\big) = q^{(\dim \mathcal{G} - \dim \mathcal{M})/2} m_{\mathfrak{s}}^M(tt_M)\, \mathrm{fdeg}(t \otimes \pi_M)\, \mathrm{d}\mathcal{O}(t \otimes \pi_M).$$

Applying Theorem 5.7 to $t \otimes \pi_M \in \mathrm{Irr}_{\mathrm{unip}}(M)$, we obtain the required formula. $\square$

On the other hand, we already know from Theorem 2.1.l that Conjecture 1 holds up to some constant $C_{\mathcal{O}} \in \mathbb{Q}_{>0}$, that is:

$$\mu_{Pl}\big(I_P^G(t \otimes \pi_M)\big) = \pm C_{\mathcal{O}}\, \gamma(0, \mathrm{Ad}_{G^\vee, M^\vee} \circ t\phi_M, \psi) \dim(\rho_M) |S_{\phi_M}^\sharp|^{-1} \mathrm{d}\mathcal{O}(t \otimes \pi_M). \tag{68}$$

Theorem 6.5. *Let $\mathcal{G}$ be a connected reductive K-group which spits over an unramified extension. Let $\mathcal{M}$ be a Levi K-subgroup of $\mathcal{G}$ and let $\pi_M \in \mathrm{Irr}_{\mathrm{unip}}(M)$ be square-integrable modulo centre. Let $\mathcal{O} = X_{\mathrm{unr}}(M)\pi_M$ be the associated orbit in $\mathrm{Irr}_{\mathrm{unip}}(M)$ and define $I_P^G(\mathcal{O})$ as in (65).*

Let $(\phi_M, \rho_M) \in \Phi_{\mathrm{nr},e}(M)$ be the enhanced L-parameter of π_M, as in Theorem 2.1. With the normalization from (60), the Plancherel density on $I_P^G(\mathcal{O})$ is

$$\pm \dim(\rho_M) |S_{\phi_M}^\sharp|^{-1} \gamma(0, \mathrm{Ad}_{G^\vee, M^\vee} \circ \phi_M, \psi)\, \mathrm{d}\mathcal{O}(\pi_M).$$

That is, Conjecture 1 holds for $\mathrm{Irr}_{\mathrm{unip}}(G)$, with $c_M = 1$.

Proof. We may assume that $\mathcal{M}$ is a standard Levi subgroup, that is, $\mathcal{M}$ contains the standard maximal K-split torus $\mathcal{S}$ and the standard maximal K-torus $\mathcal{T}$. Let $\mathcal{G}^*$ be the quasi-split inner form of $\mathcal{G}$. We may identify $\mathcal{T}$ with a maximal K-torus of $\mathcal{G}^*$. Let $\mathcal{M}^* \subset \mathcal{G}^*$ be the Levi subgroup such that $\Phi(\mathcal{M}^*, \mathcal{T}) = \Phi(\mathcal{M}, \mathcal{T})$.

Write inf.ch.$(\phi_M) = r_M \theta$ with $r_M \in \hat{T}^{\circ,\theta}$ (which can be achieved by replacing ϕ_M with an equivalent L-parameter). By Lemma A.3 and (96)

$$\gamma(0, \mathrm{Ad}_{G^\vee, M^\vee} \circ t\phi_M, \psi) = \pm \gamma(0, \mathrm{Ad}_{M^\vee} \circ t\phi_M, \psi) q^{(\dim \mathcal{G} - \dim \mathcal{M})/2} m^{M^*}(tr_M). \tag{69}$$

Comparing Lemma 6.4, (68) and (69), we see that

$$m_{\mathfrak{s}}^M(tt_M) = \pm C_{\mathcal{O}} m^{M^*}(tr_M) \qquad \forall t \in X_{\mathrm{unr}}(M). \tag{70}$$

Let $R_0^{\mathfrak{s}}$ denote the root system associated with the Hecke algebra $\mathcal{H}_{\mathfrak{s}}$ from (9), and let $q^{\mathcal{N}}$ denote the parameter function of $\mathcal{H}_{\mathfrak{s}}$, as in [Opd3, Section 2] and as described after (10). Let $m_\pm^{\mathfrak{s}}$ be the corresponding parameter functions on $R_0^{\mathfrak{s}}$. Let $w_{\mathfrak{s}}^M \in W_{\mathfrak{s}}$ denote the shortest length representative of the coset $w_{\mathfrak{s},0} W_{\mathfrak{s},M}$ in $W_{\mathfrak{s}}/W_{\mathfrak{s},M}$ of the longest element $w_{\mathfrak{s},0}$. Like in Appendix A.2, these parameters can be used to define μ-functions. From [Opd1, Proposition 3.27(ii)] we see that for $t \in X_{\mathrm{unr}}(M)$:

$$m_{\mathfrak{s}}^M(tt_M) = q^{\mathcal{N}}(w_{\mathfrak{s}}^M)^{-1} \prod_{a \in R_0^{\mathfrak{s}} \setminus R_{M,0}^{\mathfrak{s}}} \frac{(1 - \gamma_a^{-2}(tt_M))}{(1 + q^{-m_-^{\mathfrak{s}}(\gamma_a)} \gamma_a^{-1}(tt_M))(1 - q^{-m_+^{\mathfrak{s}}(\gamma_a)} \gamma_a^{-1}(tt_M))}. \tag{71}$$

This is analogous to the formula (94) for $m^{M^*}(tr_M)$. The differences are that for M^* the product runs over more roots and that the parameters $m_\pm^{\mathfrak{s}}(\gamma_a)$ need not equal $m_\pm(\gamma_a)$.

Since π_M is essentially square-integrable, $\pi_{M,\mathcal{H}}$ is essentially discrete series. Together with [Opd1, Lemma 3.31 and Proposition A.4] this implies that $X_{\mathrm{nr}}(M)t_M$ is a residual coset (of minimal dimension) for $\mathcal{H}^M$. Moreover $m_\pm^{\mathfrak{s}}(\gamma_a) \in \mathbb{Z}$, so the value $\gamma_a(t_M) \in \mathbb{C}^\times$ is a root of unity times an integral power of q [Opd1, Theorem A.7]. In particular $\lim_{q\to 1} \gamma_a(t_M)$ is well-defined, and a root of unity in $\mathbb{C}$.

The discrete unramified L-parameter for M^* determines an L-packet of essentially square-integrable M^*-representations. The Iwahori-spherical members of that packet correspond to a finite set of essentially discrete series representations of the parabolic subalgebra $\mathcal{H}(M^*, I^* \cap M^*)$ of $\mathcal{H}(G^*, I^*)$, with central character

$$W(M^\vee, T^\vee)^\theta r_M \in T_\theta^\vee / W(M^\vee, T^\vee)^\theta .$$

As above for t_M, $X_{\mathrm{nr}}(M^*)r_M$ is a residual coset (of minimal dimension) for $\mathcal{H}(M^*, I^* \cap M^*)$. It follows as above that the values $\gamma_a(r_M)$, with $a \in (\Phi \setminus \Phi_{M^*})/\theta$ as in (94), are products of roots of unity and integral powers of q.

Taking this dependence of t_M and r_M on q into account, we regard both sides of (70) as rational functions in t and in q. Fix $t \in X_{\mathrm{unr}}(M)$ such that both tt_M and tr_M are in generic position with respect to all the involved roots. Then (71) and (94) entail

$$\lim_{q\to 1} m_{\mathfrak{s}}^M(tt_M) = 1 \qquad \text{and} \qquad \lim_{q \to 1} m^{M^*}(tr_M) = 1.$$

Combining that with (70), we find $C_{\mathcal{O}} = 1$ and $m_{\mathfrak{s}}^M(tt_M) = \pm m^{M^*}(tr_M)$ for all $t \in X_{\mathrm{unr}}(M)$. Then (69) becomes

$$\gamma(0, \mathrm{Ad}_{G^\vee, M^\vee} \circ t\phi_M, \psi) = \pm m^M(tt_M) \gamma(0, \mathrm{Ad}_{M^\vee} \circ t\phi_M, \psi).$$

Hence the expression in Lemma 6.4 equals (68) with $C_{\mathcal{O}} = 1$, as required. □

Appendix A. Adjoint γ-factors

Let (ρ, V) be a finite dimensional Weil–Deligne representation over $\mathbb{C}$, that is, a semisimple representation of $\mathbf{W}_K$ on V together with a nilpotent operator $N \in \mathrm{End}_{\mathbb{C}}(V)$, such that

$$\rho(w) N \rho(w)^{-1} = \|w\| N \quad \text{for all } w \in \mathbf{W}_K .$$

The contragredient of (ρ, V) is the contragredient $(\rho^\vee, V^\vee)$ as $\mathbf{W}_K$-representation, together with the nilpotent operator $N^\vee \in \mathrm{End}_{\mathbb{C}}(V^\vee)$ which sends λ to $-\lambda \circ N$. We write $V_N = \ker(N)$ and we fix an additive character $\psi : K \to \mathbb{C}^\times$.

We define a new Weil–Deligne representation (ρ_0, V), by decreeing that as $\mathbf{W}_K$-representation it is the same as (ρ, V), but with nilpotent operator $N_0 = 0$. Recall from [Tate, §4.1.6] that the local factors of (ρ, V) are defined, as meromorphic functions of $s \in \mathbb{C}$, by:

$$\begin{aligned} L(s,\rho) &= \det\big(1 - q^{-s}\rho(\mathrm{Frob}) | V_N^{\mathbf{I}_K}\big)^{-1}, \\ \epsilon(s,\rho,\psi) &= \epsilon(s,\rho_0,\psi) \det\big(- q^{-s}\rho(\mathrm{Frob}) | V^{\mathbf{I}_K} / V_N^{\mathbf{I}_K}\big), \\ \gamma(s,\rho,\psi) &= \epsilon(s,\rho,\psi) L(1-s,\rho^\vee) L(s,\rho)^{-1}. \end{aligned} \tag{72}$$

The ϵ-factor can be described further with [Tate, §3.4] and the Artin conductor $\mathbf{a}(V)$:

$$\epsilon(s, \rho_0, \psi) = \epsilon(\rho_0 \otimes \| \cdot \|^{1/2}, \psi) q^{\mathbf{a}(V)(1/2-s)}.$$

We note that in [Tate] ϵ-factors also depend on a Haar measure dx on K. In view of [Tate, (3.4.3)] it is harmless to fix any normalization of dx, and we do so by giving $\mathfrak{o}_K$ volume 1.

It is well-known, for instance from [GrRe, Proposition 2.2], that ρ gives rise to a semisimple representation

$$\tilde\rho : \mathbf{W}_K \times SL_2(\mathbb{C}) \to \mathrm{Aut}_{\mathbb{C}}(V), \text{ such that } N = \mathrm{d}\tilde\rho|_{SL_2(\mathbb{C})} \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right). \tag{73}$$

Such a $\tilde\rho$ is unique up to conjugacy in $\mathrm{Aut}_{\mathbb{C}}(V)$, and it determines ρ. We say that (ρ, V) is self-dual if it is isomorphic to its contragredient. This is equivalent to self-duality of $\tilde\rho$.

A.1. **Independence of the nilpotent operator.**

In view of the known properties of γ-factors for representations of $GL_n(K)$ [Jac, (2.7.3)], one can expect a relation between the γ-factors of ρ and of ρ_0.

Proposition A.1. *Let (ρ, V) be a finite dimensional self-dual Weil–Deligne representation over $\mathbb{C}$. Then*

$$\gamma(0, \rho, \psi) = \pm\gamma(0, \rho_0, \psi).$$

That is, up to a sign the γ-factor of ρ at $s = 0$ does not depend on the nilpotent operator N.

Proof. Let (ρ', V') be the sum of the irreducible nontrivial $\mathbf{I}_K$-subrepresentations of (ρ, V). We denote the irreducible $SL_2(\mathbb{C})$-representation of dimension $n + 1$ by $(\sigma_n, \mathrm{Sym}^n)$ and we write

$$V_n := \mathrm{Hom}_{\mathbf{I}_K \times SL_2(\mathbb{C})}(\mathrm{triv} \otimes \sigma_n, \tilde\rho).$$

We can decompose the $\mathbf{W}_K \times SL_2(\mathbb{C})$-representation $\tilde\rho$ as

$$V = V' \oplus \bigoplus\nolimits_{n=0}^{\infty} V_n \otimes \mathrm{Sym}^n. \tag{74}$$

In view of the additivity of the local factors (72), it suffices to prove the proposition for each of the direct summands in (74) separately. It follows directly from the definitions that

$$\gamma(s, \rho', \psi) = \epsilon(s, \rho'_0, \psi) = \gamma(s, \rho'_0, \psi),$$

so we only have to consider $V_n \otimes \mathrm{Sym}^n$ for a fixed (but arbitrary) $n \in \mathbb{Z}_{\geq 0}$. Since $\mathbf{I}_K$ is normal in $\mathbf{W}_K$, $\tilde\rho$ induces an action of $\mathbf{W}_K/\mathbf{I}_K \cong \mathbb{Z}$ on V_n. We decompose it as

$$V_n = \bigoplus\nolimits_{\chi \in \mathrm{Irr}(\mathbf{W}_K/\mathbf{I}_K)} \mathbb{C}_\chi^{m_\chi}, \tag{75}$$

where m_χ denotes the multiplicity of χ in V_n.

Since $(\tilde\rho, V)$ is self-dual and Sym^n is self-dual (as $SL_2(\mathbb{C})$-representation), the $\mathbf{W}_K$-representation V_n is also self-dual. Hence

$$m_{\chi^{-1}} = m_\chi \quad \text{for all } \chi \in \mathrm{Irr}(\mathbf{W}_K/\mathbf{I}_K). \tag{76}$$

To simplify the notation, we assume from now on that $V = V_n \otimes \mathrm{Sym}^n$, and in particular that $V^{\mathbf{I}_K} = V$. The relation between ρ and $\tilde\rho$ entails that

$$\rho(\mathrm{Frob}) = \tilde\rho(\mathrm{Frob}) \otimes \tilde\rho\big(1, \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}\big) = \tilde\rho(\mathrm{Frob}) \otimes \sigma_n \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}. \tag{77}$$

From (76) we see that

$$\det(\tilde{\rho}(\mathrm{Frob})|V_n) = \prod\nolimits_{\chi \in \mathrm{Irr}(\mathbf{W}_K/\mathbf{I}_K)} \chi(\mathrm{Frob})^{m_\chi} = (-1)^{m_{\chi_-}}, \tag{78}$$

where χ_- denotes the unique quadratic character of $\mathbf{W}_K/\mathbf{I}_K$. As $\sigma_n(SL_2(\mathbb{C})) \subset SL_{n+1}(\mathbb{C})$, (77) and (78) yield

$$\det(\rho(\mathrm{Frob})|V) = \det(\tilde{\rho}(\mathrm{Frob})|V_n)^{n+1} = (-1)^{(n+1)m_{\chi_-}}. \tag{79}$$

Since Sym^n_N is one-dimensional with $\left(\begin{smallmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{smallmatrix}\right)$ acting as $q^{-n/2}$,

$$\det\left(\sigma_n\left(\begin{smallmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{smallmatrix}\right)\big|\mathrm{Sym}^n/\mathrm{Sym}^n_N\right) = q^{n/2}. \tag{80}$$

With (77)–(80), we can express the ϵ-factor as

$$\begin{aligned} \epsilon(s,\rho,V) &= \epsilon(s,\rho_0,\psi)\det\big(-q^{-s}\rho(\mathrm{Frob})|V_n \otimes \mathrm{Sym}^n/\mathrm{Sym}^n_N\big) \\ &= \epsilon(s,\rho_0,\psi)(-q^{-s})^{\dim(V_n)n}(-1)^{nm_{\chi_-}} q^{\dim(V_n)n/2}. \end{aligned} \tag{81}$$

Using self-duality, the definitions (72) and (81), we compute

$$\begin{aligned} \frac{\gamma(s,\rho,\psi)}{\gamma(s,\rho_0,\psi)} &= \frac{(-1)^{nm_{\chi_-}} q^{\dim(V_n)n/2}}{(-q^s)^{\dim(V_n)n}} \frac{\det\big(1-q^{s-1}\rho(\mathrm{Frob})|V_n \otimes \mathrm{Sym}^n/\mathrm{Sym}^n_N\big)}{\det\big(1-q^{-s}\rho(\mathrm{Frob})|V_n \otimes \mathrm{Sym}^n/\mathrm{Sym}^n_N\big)} \\ &= \frac{(-1)^{nm_{\chi_-}} q^{\dim(V_n)n/2}}{(-q^s)^{\dim(V_n)n}} \prod_\chi \prod_{k=1}^n \left(\frac{1-q^{s-1}\chi(\mathrm{Frob})q^{k-n/2}}{1-q^{-s}\chi(\mathrm{Frob})q^{k-n/2}}\right)^{m_\chi} \\ &= \frac{(-1)^{nm_{\chi_-}} q^{\dim(V_n)n/2}}{(-1)^{\dim(V_n)n}} \prod_\chi \prod_{k=1}^n \left(\frac{1-q^{s}\chi(\mathrm{Frob})q^{k-1-n/2}}{q^s-\chi(\mathrm{Frob})q^{k-n/2}}\right)^{m_\chi}. \end{aligned} \tag{82}$$

When s goes to 0, the products over k in (82) attain telescopic behaviour, and all terms in the numerator (except $k=1$) cancel against all terms in the denominator (except $k=n$). This is obvious when $\chi(\mathrm{Frob})q^{k-n/2} \neq 1$, while we pick up an extra factor -1 if $\chi(\mathrm{Frob})q^{k-n/2} = 1$. Collecting all factors -1 in one symbol $\pm$, (82) yields

$$\begin{aligned} \frac{\gamma(0,\rho,\psi)}{\gamma(0,\rho_0,\psi)} &= \pm q^{\dim(V_n)n/2} \lim_{s\to 0} \prod_\chi \left(\frac{1-q^s\chi(\mathrm{Frob})q^{-n/2}}{q^s-\chi(\mathrm{Frob})q^{n/2}}\right)^{m_\chi} \\ &= \pm \lim_{s\to 0} \prod_\chi \chi(\mathrm{Frob})^{m_\chi} \left(\frac{q^{n/2}\chi(\mathrm{Frob})^{-1}-q^s}{q^s-\chi(\mathrm{Frob})q^{n/2}}\right)^{m_\chi} \end{aligned} \tag{83}$$

In view of (78), all the terms $\chi(\mathrm{Frob})^{m_\chi}$ together just contribute a sign, so we may omit them (or rather, put them into $\pm$). When $\chi(\mathrm{Frob}) \neq q^{\pm n/2}$, (76) shows that the terms in (83) associated to χ will cancel against the terms associated to χ^{-1}, up to a sign. Thus only the characters $\chi^{\pm 1}$ with $\chi(\mathrm{Frob}) = q^{n/2}$ remain in (82) upon taking the limit $s \to 0$, and for those we compute:

$$\frac{\gamma(0,\rho,\psi)}{\gamma(0,\rho_0,\psi)} = \pm \lim_{s\to 0}\left(\frac{1-q^s}{q^s-q^n}\right)^{m_\chi}\left(\frac{q^n-q^s}{q^s-1}\right)^{m_\chi} = \pm(-1)^{2m_\chi} = \pm 1. \tag{84}$$

This concludes the proof, and we note that by retracing the various steps one can find an explicit (but involved) formula for the sign. $\square$

The adjoint γ-factor of a L-parameter ϕ for $G = \mathcal{G}(K)$ comes from a Weil–Deligne representation on $\mathrm{Lie}(G^\vee)/\mathrm{Lie}(Z(G^\vee)^{\mathbf{W}_K})$, which is self-dual with respect to the Killing form [GrRe, §3.2]. Proposition A.1 says that the γ-factor of $\tilde{\rho} = \mathrm{Ad}_{G^\vee} \circ \phi$ equals the γ-factor of ρ_0 (both at $s = 0$ and up to a sign). We note that

$$\rho_0(\mathrm{Frob}) = \mathrm{Ad}_{G^\vee}\Big(\phi\big(\mathrm{Frob}, \big(\begin{smallmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{smallmatrix}\big)\big)\Big), \tag{85}$$

where we recognize the right hand side as the adjoint representation $\mathrm{Ad}_{G^\vee}$ applied to the infinitesimal character of ϕ. In these terms, Proposition A.1 says that $\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi, \psi)$ depends only on $\phi|_{\mathbf{I}_K}$ and the infinitesimal character of ϕ.

A.2. **Relation with μ-functions.**

The goal of this paragraph to relate adjoint γ-factors of unramified L-parameters to μ-functions for Iwahori–Hecke algebras. The desired equality was already claimed in [Opd3, (38)], we take this opportunity to work out the proof.

From now on the additive character $\psi : K \to \mathbb{C}^\times$ has order zero, like in the body of the paper. We assume that $\mathcal{G}$ is unramified over K, that is, $\mathcal{G}$ is quasi-split and splits over an unramified extension of K. Fix a pinning of the Lie algebra $\mathrm{Lie}(G^\vee)$ and let θ denote the pinned automorphism of $\mathrm{Lie}(G^\vee)$ induced by Frob. The quotient $\mathcal{G}/Z(\mathcal{G})_s$ defines a θ-stable reductive subgroup $\hat{G} \subset G^\vee$ with Lie algebra

$$\hat{\mathfrak{g}} := \mathrm{Lie}(\hat{G}) \cong \mathrm{Lie}(G^\vee)/\mathrm{Lie}(Z(G^\vee)^\theta).$$

Let $\mathrm{Ad}_{G^\vee}$ denote the adjoint action of ${}^L\mathcal{G}$ on $\hat{\mathfrak{g}}$. Let us denote the distinguished Cartan subgroup of $\hat{G}$ by $\hat{T}$, with corresponding Cartan subalgebra $\hat{\mathfrak{t}} := \mathrm{Lie}(\hat{T})$ of $\hat{\mathfrak{g}}$. Clearly

$$T^\vee = Z(G^\vee)^\theta \hat{T} = Z(G^\vee)^{\theta,\circ} \hat{T}, \tag{86}$$

where $Z(G^\vee)^{\theta,\circ}$ is the identity component of $Z(G^\vee)^\theta$. By [Ree2, Section 3.3], the Lie algebra $\hat{\mathfrak{g}}^\theta$ is semisimple.

Lemma A.2. *Let ϕ_T be an unramified L-parameter for T and write $\phi_T(\mathrm{Frob}) = r\theta$. Then*

$$\gamma(s, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{t}}} \circ \phi_T, \psi) = \frac{\det(1 - q^{-s}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{t}}})}{\det(1 - q^{s-1}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{t}}})}.$$

For s near 0 this can be expressed as

$$\gamma(s, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{t}}} \circ \phi_T, \psi) = s^{|\Delta/\theta|} \frac{n_1 \log(q)^{|\Delta/\theta|} \prod_{a \in \Delta/\theta} |a \cap \Delta|}{\det(1 - q^{-1}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{t}}})} + \mathcal{O}(s^{|\Delta/\theta|+1}).$$

Here n_1 is a positive integer which reduces to 1 if $Z(\mathcal{G})^\circ$ is K-split.

Proof. For any unramified representation ρ of $\mathbf{W}_K$ and an additive character ψ of order 0, [Tate, (3.2.6) and (3.4.2)] say that

$$\epsilon(s, \rho, \psi) = 1 \qquad \text{for all } s \in \mathbb{C}. \tag{87}$$

This applies to $\rho = \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{t}}} \circ \phi$, and moreover ρ is self-dual with respect to the Killing form. Knowing that, the definitions (72) yield the asserted formula for $\gamma(s, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{t}}} \circ \phi_T, \psi)$.

The finite order map $\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{t}}}$ cannot have an eigenvalue $q \in \mathbb{R}_{>1}$. Hence the denominator $\det(1 - q^{s-1}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{t}}})$ is regular at $s = 0$, and behaves as expected.

The numerator $\det(1 - q^{-s}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{t}}})$ can be analysed by splitting

$$\hat{\mathfrak{t}} = (1 - \theta)Z(\hat{\mathfrak{g}}) \oplus (\hat{\mathfrak{t}} \cap \hat{\mathfrak{g}}_{\mathrm{der}}). \tag{88}$$

On the first summand of (88) we get

$$\lim_{s\to 0} \det(1 - q^{-s}\mathrm{Ad}_{G^\vee}(r\theta)|_{(1-\theta)Z(\hat{\mathfrak{g}})}) = \det(1-\theta|_{(1-\theta)Z(\hat{\mathfrak{g}})}). \tag{89}$$

Identifying $(1-\theta)Z(\hat{\mathfrak{g}})$ with the Lie algebra of the complex dual group of $Z(\mathcal{G})^\circ/Z(\mathcal{G})_s$, we see that (89) can be computed as the determinant of a linear transformation of a (co)character lattice, so in particular it is an integer. More precisely, as θ has finite order but no eigenvalues 1 on the involved lattice, (89) equals the natural number

$$n_1 := \det\big(1-\theta|_{X^*(Z(\mathcal{G})^\circ/Z(\mathcal{G})_s)}\big) \in \mathbb{N}.$$

If $Z(\mathcal{G})^\circ$ is K-split, then $Z(\mathcal{G})^\circ/Z(\mathcal{G})_s = 1$ and $n_1 = 1$.

The basis of $\hat{\mathfrak{t}} \cap \hat{\mathfrak{g}}_{\mathrm{der}}$ consisting of the simple coroots is permuted by θ, with orbits of length $|a \cap \Delta|$. For the second summand in (88) we find a contribution of

$$\lim_{s\to 0} \det\big(1 - q^{-s}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{t}}\cap\hat{\mathfrak{g}}_{\mathrm{der}}}\big) = \lim_{s\to 0} \prod\nolimits_{a\in\Delta/\theta} (1 - q^{-s|a\cap\Delta|}). \tag{90}$$

The leading order term of (90) for s near 0 is

$$\prod\nolimits_{a\in\Delta/\theta} \big(s\,|a\cap\Delta|\,\log(q)\big) = s^{|\Delta/\theta|}\log(q)^{|\Delta/\theta|} \prod\nolimits_{a\in\Delta/\theta} |a\cap\Delta|. \tag{91}$$ □

We start the definition of the μ-functions for the relevant Hecke algebras. Let Φ/θ be the set of equivalence classes in the root system Φ of $(\hat{\mathfrak{g}}, \hat{\mathfrak{t}})$ as defined in [Ree2, Section 3.3], and Δ/θ the set of equivalence classes of the basis Δ of Φ. For each $a \in \Phi/\theta$ put

$$\gamma_a := \sum\nolimits_{\alpha\in a} \alpha|_{\hat{\mathfrak{t}}^\theta}.$$

Then $\Phi_\theta = \{\gamma_a\}_{a\in\Phi/\theta}$ is a reduced root system on $\hat{\mathfrak{t}}^\theta$. With Φ_θ we also consider its (untwisted) affine extension $\Phi_\theta^{(1)} = \Phi_\theta \times \mathbb{Z}$, naturally indexed by $\Phi/\theta \times \mathbb{Z}$, that is, we will denote the affine root (γ_a, n) with $a \in \Phi/\theta$ and $n \in \mathbb{Z}$ by $\gamma_{(a,n)}$.

Recall the Kac root system $\hat{\Phi}_\theta = \{\beta_a \mid a \in \Phi/\theta\}$, where $\beta_a = \alpha|_{\hat{\mathfrak{t}}^\theta}$ for an $\alpha \in a$ such that $\beta_a/2$ is not of this form. This root system has a twisted affine extension with "Kac diagram" $\mathcal{D}(\hat{\mathfrak{g}}, \theta)$ [Ree2, Section 3.4]. By [Ree2, Section 3], $\hat{\Phi}_\theta$ is the root system of $\hat{\mathfrak{g}}^\theta$. For each $a \in \Phi/\theta \times \mathbb{Z}$ there exists a positive integer f_a such that $\gamma_a = f_a\beta_a$. If $\gamma_a \in \Phi_\theta$ is a minimal root, then by construction $f_{(a,1)}$ is the order of θ on the union of the components of Φ which intersect a.

We say that $a \in \Phi/\theta$ (or $\alpha \in a$) has:

- type I if the θ-orbit of α consists of mutually orthogonal roots;
- type II if a contains a triple $\{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$ with $\alpha_2 \in \langle\theta\rangle\alpha_1$.

Type II only occurs if some irreducible component of Φ has type A_{2n} and a power of θ acts on it by the nontrivial diagram automorphism.

From [Ree2, Table 2] we see that for every root of type I and every $e \in \mathbb{Z}$:

$$f_{(a,e)} = f_a = |a|.$$

On the other hand, for $a \in \Phi/\theta$ of type II :

$$f_{(a,e)} = \begin{cases} f_a = 4|a|/3 & \text{if } e \text{ is even,} \\ f_a/2 = 2|a|/3 & \text{if } e \text{ is odd.} \end{cases}$$

Recall that $\mathcal{S} \subset \mathcal{G}$ denotes a maximal K-split torus of $\mathcal{G}$ contained in $\mathcal{T}$. Let $\Phi(\mathcal{G},\mathcal{S})_0$ (resp. $\Phi(\mathcal{G},\mathcal{S})_1$) be the set of indivisible (resp. non-multipliable) roots of $\Phi(\mathcal{G},\mathcal{S})$. From [Ree2, (26)] we conclude that $\Phi_\theta^\vee = \Phi(\mathcal{G},\mathcal{S})_0$ and $\Phi_\theta = \Phi(\mathcal{G},\mathcal{S})_0^\vee$.

Let $I \subset G$ be an Iwahori subgroup and let $\mathcal{H}(G, I)$ be the Iwahori–Hecke algebra of G. We write the underlying root datum as $(R_0, X_*(\mathcal{S}), R_0^\vee, X^*(\mathcal{S}))$ and the parameter functions on R_0 as $m_\pm$. That means that the q-parameter for any simple reflection $s_\alpha \in W(R_0)$ is $q^{\mathcal{N}}(s_\alpha) = q^{m^+(\alpha)}$, while the simple affine reflection s'_α with linear part conjugate to s_α (if it exists) has q-parameter $q^{\mathcal{N}}(s'_\alpha) = q^{m^-(\alpha)}$.

Then R_m (in the sense of [Opd3, Subsection 2.3.3]) is equal to Φ_θ (cf. [FeOp, Section 4.2]) or equivalently, $R_0^\vee = \Phi(\mathcal{G}, \mathcal{S})_1$. We identify the roots of R_0 with $\{\gamma_a\}_{a \in \Phi/\theta}$. If $a \in \Phi/\theta$ is of type I, then $\rho_0(\mathrm{Frob}) = \mathrm{Ad}_{G^\vee}(r\theta)$ and the parameters of $\mathcal{H}(G, I)$ on Φ_θ are given by:

$$m_+(\gamma_a) = f_a \qquad \text{and} \qquad m_-(\gamma_a) = 0. \tag{92}$$

If $a \in \Phi/\theta$ is of type II, then these parameters are given by:

$$\begin{aligned} m_+(\gamma_a) &= f_{(a,0)}/2 = 2|a|/3, \\ m_-(\gamma_a) &= f_{(a,1)}/2 = |a|/3. \end{aligned} \tag{93}$$

Another way of expressing this is that the linearly extended parameter function $m_R^\vee$ on the affine Kac roots $\mathcal{D}(\hat{\mathfrak{g}}, \theta)$ is constant and equal to 1, see [FeOp, Proposition 4.2.1].

For use in Paragraph 6.2 we need a μ-function of $\mathcal{H}(G, I)$ relative to a Levi subgroup (or equivalently, relative to a parabolic subalgebra). Let $\mathcal{P}$ be a standard parabolic K-subgroup of $\mathcal{G}$, with standard Levi factor $\mathcal{M}$. Let $\Phi_M \subset \Phi$ be the corresponding parabolic root subsystem. We recall from [Opd1, (3.57) and (4.96)] that

$$\begin{aligned} m^M(t) &= q^{(\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{m}})/2} \prod_{a \in (\Phi \setminus \Phi_M)/\theta} \frac{(\gamma_a^2(t) - 1)}{(q^{m_-(\gamma_a)}\gamma_a(t) + 1)(q^{m_+(\gamma_a)}\gamma_a(t) - 1)} \\ &= q^{(\dim \hat{\mathfrak{m}} - \dim \hat{\mathfrak{g}})/2} \prod_{a \in (\Phi \setminus \Phi_M)/\theta} \frac{(1 - \gamma_a^{-2}(t))}{(1 + q^{-m_-(\gamma_a)}\gamma_a^{-1}(t))(1 - q^{-m_+(\gamma_a)}\gamma_a^{-1}(t))}, \end{aligned} \tag{94}$$

a rational function of $t \in T^\vee/(1-\theta)T^\vee$. We note that for M equal to the maximal torus T, $m^T(t)$ involves all roots from Φ/θ.

We denote the adjoint representation of ${}^L M$ on $\mathrm{Lie}(G^\vee)/\mathrm{Lie}(M^\vee) = \hat{\mathfrak{g}}/\hat{\mathfrak{m}}$ by $\mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{g}}/\hat{\mathfrak{m}}}$. Let ϕ_M be an unramified L-parameter for $M = \mathcal{M}(K)$ and write

$$r_M\theta = \phi_M\big(\mathrm{Frob}, \big(\begin{smallmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{smallmatrix}\big)\big).$$

Upon replacing ϕ_M by an equivalent L-parameter we may assume that $r_M \in \hat{T}^{\theta,0}$ [Ree2, Lemma 3.2]. For $z \in Z(M^\vee)^\circ_\theta \cong X_{\mathrm{nr}}(M)$ we define another unramified L-parameter $z\phi_M \in \Phi(M)$ by

$$(z\phi_M) = \phi_M \text{ on } \mathbf{I}_K \times SL_2(\mathbb{C}), \quad (z\phi_M)(\mathrm{Frob}) = z(\phi_M(\mathrm{Frob})). \tag{95}$$

By the additivity of γ-factors

$$\gamma(s, \mathrm{Ad}_{G^\vee, M^\vee} \circ z\phi_M, \psi) = \gamma(s, \mathrm{Ad}_{M^\vee} \circ z\phi_M, \psi)\, \gamma(s, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{g}}/\hat{\mathfrak{m}}} \circ z\phi_M, \psi). \tag{96}$$

Lemma A.3. *There is an equality of rational functions of $z \in Z(M^\vee)^\circ_\theta$:*

$$\gamma(0, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{g}}/\hat{\mathfrak{m}}} \circ z\phi_M, \psi) = \pm q^{(\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{m}})/2} m^M(zr_M).$$

Proof. Let ρ, ρ_0 be the associated self-dual Weil–Deligne representations as in (73). By Proposition A.1

$$\gamma(0, \rho, \psi) = \pm\gamma(0, \rho_0, \psi). \tag{97}$$

Using the definitions (72) we plug (87) into (97), and we obtain

$$\gamma(0, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{g}}/\hat{\mathfrak{m}}} \circ \phi, \psi) = \pm \lim_{s\to 0} L(1-s, \rho_0) L(s, \rho_0)^{-1}. \tag{98}$$

For $a \in \Phi/\theta$ let $\hat{\mathfrak{g}}_a \subset \hat{\mathfrak{g}}$ be the subspace $\sum_{\alpha \in a} \hat{\mathfrak{g}}_\alpha$, so that we have a $\mathrm{Ad}_{G^\vee}(zr_M\theta)$-stable decomposition

$$\hat{\mathfrak{g}} = \hat{\mathfrak{m}} \oplus \bigoplus\nolimits_{a \in (\Phi \setminus \Phi_M)/\theta} \hat{\mathfrak{g}}_a. \tag{99}$$

Using (99) and (98) we see that

$$\begin{aligned} \gamma(0, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{g}}/\hat{\mathfrak{m}}} \circ z\phi_M, \psi) &= \pm \lim_{s \to 0} \frac{\det\left(1 - q^{-s}\mathrm{Ad}_{G^\vee}(zr_M\theta)|_{\hat{\mathfrak{g}}/\hat{\mathfrak{m}}}\right)}{\det\left(1 - q^{s-1}\mathrm{Ad}_{G^\vee}(zr_M\theta)|_{\hat{\mathfrak{g}}/\hat{\mathfrak{m}}}\right)} \\ &= \pm \lim_{s\to 0} \prod_{a \in (\Phi\setminus\Phi_M)/\theta} \frac{\det(1 - q^{-s}\mathrm{Ad}_{G^\vee}(zr_M\theta)|_{\hat{\mathfrak{g}_a}})}{\det(1 - q^{s-1}\mathrm{Ad}_{G^\vee}(zr_M\theta)|_{\hat{\mathfrak{g}_a}})}. \end{aligned} \tag{100}$$

In [Ree2, Section 3.4] the characteristic polynomial of $\mathrm{Ad}_{G^\vee}(r\theta)$ on $\hat{\mathfrak{g}}_a$ was determined. (Strictly speaking Reeder only treats the case where Φ is irreducible, but his calculations generalize readily.) For $a \in \Phi/\theta$ of type I this gives

$$\det(1 - q^{-s}\mathrm{Ad}_{G^\vee}(zr_M\theta)|_{\hat{\mathfrak{g}_a}}) = 1 - q^{-sm_+(\gamma_a)}\gamma_a(zr_M), \tag{101}$$

while for $a \in \Phi/\theta$ of type II:

$$\det(1 - q^{-s}\mathrm{Ad}_{G^\vee}(zr_M\theta)|_{\hat{\mathfrak{g}_a}}) = (1 + q^{-sm_-(\gamma_a)}\gamma_a(zr_M))(1 - q^{-sm_+(\gamma_a)}\gamma_a(zr_M)). \tag{102}$$

With these expressions for the characteristic polynomials, (100) becomes precisely

$$\pm q^{(\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{m}})/2} m^M(zr_M).$$

Finally we note that (101) and (102) are regular for z in a dense Zariski-open subset of $Z(M^\vee)^\circ_\theta$, so that (100) defines a rational function on $Z(M^\vee)^\circ_\theta$. □

Consider an Iwahori subgroup $I_{ss} \subset G/Z(G)_s$. The Iwahori–Hecke algebra $\mathcal{H}_{ss} := \mathcal{H}(G/Z(G)_s, I_{ss})$ of $G/Z(G)_s$ is a quotient of $\mathcal{H}(G, I)$. It has the same root system and the same parameter functions $m_\pm$, and hence (essentially) the same relative μ-function $m^M = m^{M/Z(G)_s}$.

Let $\overline{I_{ss}}$ be the maximal finite reductive quotient of the Iwahori subgroup $I_{ss} \subset G/Z(G)_s$. By [Car2, Proposition 3.3.5]

$$|\overline{I_{ss}}| = \det(q - \mathrm{Ad}(\theta)|_{\hat{\mathfrak{t}}}) = q^{\dim \hat{\mathfrak{t}}} \det(1 - q^{-1}\mathrm{Ad}(\theta)|_{\hat{\mathfrak{t}}}). \tag{103}$$

Recall from (6) that the normalized Haar measure on $G/Z(G)_s$ satisfies

$$\mathrm{vol}(I_{ss}) = q^{-(\dim(\hat{\mathfrak{t}}) + \dim(\hat{\mathfrak{g}})/2} \det(q - \mathrm{Ad}(\theta)|_{\hat{\mathfrak{t}}}). \tag{104}$$

The μ-function of the Iwahori–Hecke algebra $\mathcal{H}_{ss}$ is denoted m_T in [Opd1, Theorem 3.25]. In our setting, we replace the subscript T (the torus associated to an affine Hecke algebra) by the relevant group. With the above Haar measure and the normalization convention [Opd3, §2.4.1 and Proposition 2.5], the μ-function for $G/Z(G)_s$ becomes:

$$
\begin{aligned}
(105)\quad m_{G/Z(G)_s}(t) &= \mathrm{vol}(I_{ss})^{-1} m^T(t) \\
&= \frac{q^{\dim(\hat{\mathfrak{t}})}}{\det(q - \mathrm{Ad}(\theta)|_{\hat{\mathfrak{t}}})} \prod_{a \in \Phi/\theta} \frac{(1-\gamma_a^{-2}(t))}{(1+q^{-m_-(\gamma_a)}\gamma_a^{-1}(t))(1-q^{-m_+(\gamma_a)}\gamma_a^{-1}(t))}.
\end{aligned}
$$

Here t lies in $\hat{T}/(1-\theta)\hat{T}$, the torus associated to $\mathcal{H}_{ss}$. However, as the roots γ_a are trivial on $Z(G^\vee)^\theta$, (86) entails that we may just as well consider $m_{G/Z(G)_s}$ as a $Z(G^\vee)^\theta$-invariant function on $T^\vee/(1-\theta)T^\vee = \mathrm{Hom}(X_*(\mathcal{S}), \mathbb{C}^\times)$.

Recall from [GrRe, Proposition 3.2] that $\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi, \psi)$ is nonzero if and only if ϕ is discrete. Observe that it is a priori clear that (85), and hence the γ-value (72) for the adjoint representation $\tilde{\rho}$, is invariant under $X_{\mathrm{nr}}(G) \cong Z(G^\vee)^{\theta,\circ}$. Therefore it suffices to consider a discrete unramified L-parameter for $G/Z(G)_s$ in the next theorem.

Theorem A.4. *Let $\mathcal{G}$ be an unramified reductive K-group and fix an additive character ψ of order 0. Let ϕ be an unramified discrete L-parameter for $G/Z(G)_s$, write $\tilde{\rho} = \mathrm{Ad}_{G^\vee} \circ \phi$ and $\rho_0(\mathrm{Frob}) = \mathrm{Ad}_{G^\vee}(r\theta)$ as in (85). By [Ree2, Lemma 3.2] we may assume that $r \in \hat{T}^{\theta,0}$. There exists $d \in \mathbb{Q}^\times$ such that, as rational functions in q:*

$$
\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) = d\, m_{G/Z(G)_s}^{(\{r\})} =
$$
$$
\frac{d}{\det(1 - q^{-1}\mathrm{Ad}(\theta)|_{\hat{\mathfrak{t}}})} \frac{\prod'_{a\in\Phi/\theta}(1+\gamma_a^{-1}(r))}{\prod'_{a\in\Phi/\theta}(1+q^{-m_-(\gamma_a)}\gamma_a^{-1}(r))} \frac{\prod'_{a\in\Phi/\theta}(1-\gamma_a^{-1}(r))}{\prod'_{a\in\Phi/\theta}(1-q^{-m_+(\gamma_a)}\gamma_a^{-1}(r))},
$$

where $\prod'_{a\in\Phi/\theta}$ denotes the product in which zero factors are omitted.

The constant d equals ± 1 if $\mathcal{G}$ is semisimple and K-split, while in general d is of the form $\pm n_1 2^{n_2} 3^{n_3}$ with $n_1, n_2, n_3 \in \mathbb{Z}$.

Proof. By the additivity of γ-factors

$$
(106)\qquad \gamma(s, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) = \gamma(s, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{t}}} \circ \phi, \psi)\gamma(s, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{g}}/\hat{\mathfrak{t}}} \circ \phi, \psi).
$$

Define $\phi_T : \mathbf{W}_K \to {}^L T$ by $\phi_T(\mathrm{Frob}) = r\theta$. By Proposition A.1, applied to the factor for $\hat{\mathfrak{t}}$, (106) equals

$$
\pm\gamma(s, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{t}}} \circ \phi_T, \psi_0)\gamma(s, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{g}}/\hat{\mathfrak{t}}} \circ \phi, \psi).
$$

With Lemma A.2 and (100), we find that (106) equals

$$
(107)\qquad \pm \frac{\det(1 - q^{-s}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{t}}})}{\det(1 - q^{s-1}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{t}}})} \prod_{a\in\Phi/\theta} \frac{\det(1 - q^{-s}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{g}}_a})}{\det(1 - q^{s-1}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{g}}_a})}.
$$

The behavior for $s \to 0$ was already analysed in Lemmas A.2 and A.3. In the current situation we can do better, by comparing the poles and the zeros.

Let $r = sc \in \hat{T}^{\theta,\circ}$ be the polar decomposition of r, with s a torsion element and c in the positive part of a real split subtorus. Since ϕ is unramified and discrete, $H := Z_{\hat{G}}(s\theta)$ is a semisimple group and

$$
\phi' := \phi|_{SL_2(\mathbb{C})} : SL_2(\mathbb{C}) \to H
$$

has finite centraliser in H [Ree2, §3.3]. This means that $s\theta \in \hat{G}\theta$ is an isolated torsion element [Ree2, Section 3.8], and the root system of H is a maximal proper subdiagram of $\mathcal{D}(\hat{\mathfrak{g}}, \theta)$. Moreover, ϕ' corresponds to a distinguished unipotent orbit of H, and

$$c = \phi' \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \in \hat{T}^{\theta,\circ} \subset H. \tag{108}$$

It follows [Opd1, Appendix A] that the image of $r \in \hat{T}^{\theta,\circ}$ in $\hat{T}/(1-\theta)\hat{T}$ is a residual point for $\mathcal{H}$ (or equivalently for $m_{G/Z(G)_s}$).

Now we analyse the product obtained from (107) by applying (101) and (102):

$$\prod_{a \in \Phi/\theta} \frac{\det(1 - q^{-s}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{g}_a}})}{\det(1 - q^{s-1}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{g}_a}})} = \prod_{a \in \Phi/\theta} \frac{1 - q^{-sm_+(\gamma_a)}\gamma_a(r)}{1 - q^{(s-1)m_+(\gamma_a)}\gamma_a(r)} \frac{1 + q^{-sm_-(\gamma_a)}\gamma_a(r)}{1 + q^{(s-1)m_-(\gamma_a)}\gamma_a(r)}.$$

The residuality of r means that the pole order of this expression at $s = 0$ is precisely $\dim(\hat{\mathfrak{t}}^\theta) = |\Delta/\theta|$. Notice that the terms with $m_-(\gamma_a) = 0$ in the numerator cancel out against the same kind of terms in the denominator.

Consider a linear factor $1 \pm q^m \gamma_a(r)$, of the numerator or the denominator, which has a zero at $s = 0$. Its leading order term near $s = 0$ is linear, namely $s \log(q) m_\pm(\gamma_a)$ for the numerator and $-s \log(q) m_\pm(\gamma_a)$ for the denominator.

Let N be the subset of $(a, \epsilon) \in \Phi/\theta \times \{\pm 1\}$ for which the corresponding term in the numerator has a pole at $s = 0$, but with $m_\epsilon(\gamma_a) \neq 0$. Similarly we define P for the denominator. Then

$$\frac{\prod_{(a,\epsilon)\in N} 1 - \epsilon q^{-sm_\epsilon(\gamma_a)}\gamma_a(r)}{\prod_{(a,\epsilon)\in P} 1 - \epsilon q^{(s-1)m_\epsilon(\gamma_a)}\gamma_a(r)} = \frac{s^{-|\Delta/\theta|}}{\log(q)^{|\Delta/\theta|}} \frac{\prod_{(a,\epsilon)\in N} m_\epsilon(\gamma_a)}{\prod_{(a,\epsilon)\in P} -m_\epsilon(\gamma_a)} + \mathcal{O}(s^{1-|\Delta/\theta|}). \tag{109}$$

It follows that

$$\begin{aligned} &\prod_{a \in \Phi/\theta} \frac{\det(1 - q^{-s}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{g}_a}})}{\det(1 - q^{s-1}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{g}_a}})} = \\ &\frac{\prod'_{a\in\Phi/\theta}(1 + \gamma_a^{-1}(r))}{\prod'_{a\in\Phi/\theta}(1 + q^{-m_-(\gamma_a)}\gamma_a^{-1}(r))} \frac{\prod'_{a\in\Phi/\theta}(1 - \gamma_a^{-1}(r))}{\prod'_{a\in\Phi/\theta}(1 - q^{-m_+(\gamma_a)}\gamma_a^{-1}(r))} \times (109). \end{aligned} \tag{110}$$

From (107), Lemma A.2 and (110) we conclude

$$\begin{aligned} \gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) = &\frac{\pm(-1)^{|P|} n_1 \prod_{a\in\Delta/\theta} |a \cap \Delta|}{\det(1 - q^{-1}\mathrm{Ad}_{G^\vee}(r\theta)|_{\hat{\mathfrak{t}}})} \frac{\prod_{(a,\epsilon)\in N} m_\epsilon(\gamma_a)}{\prod_{(a,\epsilon)\in P} m_\epsilon(\gamma_a)} \times \\ &\frac{\prod'_{a\in\Phi/\theta}(1 + \gamma_a^{-1}(r))}{\prod'_{a\in\Phi/\theta}(1 + q^{-m_-(\gamma_a)}\gamma_a^{-1}(r))} \frac{\prod'_{a\in\Phi/\theta}(1 - \gamma_a^{-1}(r))}{\prod'_{a\in\Phi/\theta}(1 - q^{-m_+(\gamma_a)}\gamma_a^{-1}(r))}. \end{aligned} \tag{111}$$

It remains to analyse the expression

$$\frac{\prod_{a\in\Delta/\theta} |a \cap \Delta| \prod_{(a,\epsilon)\in N} m_\epsilon(\gamma_a)}{\prod_{(a,\epsilon)\in P} m_\epsilon(\gamma_a)}. \tag{112}$$

Since we omitted the terms with $m_\pm(\gamma_a) = 0$, (112) is a nonzero rational number. It factors as a product, over the irreducible components R_i of Φ/θ, of the terms with $a \in R_i$. The restriction of r to any of the R_i is still a residual point, so there as many terms with $a \in R_i$ in numerator as in the denominator. Let $|\theta_i|$ be the number of irreducible components of Φ that go into R_i, and pick one such component Φ_i. Then

$|a| = |\theta_i|\,|a \cap \Phi_i|$ for $a \in R_i$. The factor $|\theta_i|$ appears equally often in the numerator and in the denominator of (112), so it cancels. Writing $m_{\pm,i}(\gamma_a) := m_\pm(\gamma_a)|\theta_i|^{-1}$, we find that (112) equals

$$\prod_i \frac{\prod_{a \in R_i \cap (\Delta/\theta)} |a \cap \Delta \cap \Phi_i| \prod_{(a,\epsilon) \in N : a \in R_i} m_{\epsilon,i}(\gamma_a)}{\prod_{(a,\epsilon) \in P : a \in R_i} m_{\epsilon,i}(\gamma_a)}. \tag{113}$$

The formulas (92) and (93) entail that each of the factors in (113) is the length of an orbit of an automorphism of a connected Dynkin diagram of finite type. That is: they are 1, 2 or 3, where 3 can only occur for an exceptional automorphism of D_4. Hence (113) is of the form $2^{n_2} 3^{n_3}$ with $n_2, n_3 \in \mathbb{Z}$. We insert this into (111), and we obtain the claimed formula for the adjoint γ-factor.

When $\mathcal{G}$ is an almost direct product of restrictions of scalars of split groups, all the factors in (113) are one. In the special case where $\mathcal{G}$ is K-split, also $n_1 = 1$ so that (111) becomes the desired expression with $d = \pm 1$. $\square$

We conclude this appendix by showing that adjoint γ-factors of bounded unramified L-parameters have real values. Notice that every such L-parameter arises from a discrete unramified L-parameter for a Levi subgroup $M \subset G$, via an inclusion ${}^L M \to {}^L G$.

Lemma A.5. (*a*) *In the notations from Theorem A.4,* $\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) \in \mathbb{R}^\times$.

(*b*) *Suppose that* $\phi_M \in \Phi(M)$ *is a discrete bounded unramified L-parameter and that* $z \in X_{\mathrm{unr}}(M)$. *Then* $\gamma(0, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{g}}/\hat{\mathfrak{m}}} \circ z\phi_M, \psi) \in \mathbb{R}$.

Proof. (a) For $t \in T^\vee$ we define $\overline{t} \in T^\vee$ by $x(\overline{t}) = \overline{x(t)}$ for all $x \in X^*(T^\vee)$. From (108) we see that $\overline{r}^{-1} = \overline{sc}^{-1} = sc^{-1}$ is conjugate to sc by the element $w_c := \phi'\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We note that w_c commutes with s and with θ, and that it normalizes $T^\vee$. Hence it defines an element of $W(G^\vee, T^\vee)^\theta$.

Since $\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) = \gamma(0, \mathrm{Ad}_{G^\vee} \circ w_c \phi w_c^{-1}, \psi)$, the expression (107) does not change if we replace $r\theta$ by $w_c r \theta w_c^{-1} = \overline{r}^{-1}\theta$. As the product in (107) runs over all roots (both positive and negative), we may further replace $\overline{r}^{-1}\theta$ by $\overline{r}\theta$ without changing the value. Continuing the calculation from the proof of Theorem A.4 with $\overline{r}\theta$ we end up with $\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) = d\, m_{G/Z(G)_s}^{(\{\overline{r}\})}$, which is exactly the complex conjugate of $d\, m_{G/Z(G)_s}^{(\{r\})} = \gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi, \psi)$.

(b) As observed before, we may assume that $r_M \in \hat{T}^{\theta,\circ}$. Replacing ϕ_M by $t\phi_M$ (and z by zt^{-1}) for a suitable $t \in X_{\mathrm{unr}}(M)$, we can further achieve that $r_M \in \hat{T}^{\theta,\circ} \cap M^\vee_{\mathrm{der}}$. In the proof of part (a) we showed that r_M is conjugate to $\overline{r_M}^{-1}$ by an element $w_c \in N_{M^\vee}(T^\vee)^\theta$. As $z = \overline{z}^{-1} \in Z(M^\vee)^{\theta,\circ}$ is fixed by w_c, we have $w_c z r_M \theta w_c^{-1} = \overline{z r_M}^{-1}\theta$.

Since $W(M^\vee, T^\vee)^\theta$ acts on $\hat{\mathfrak{g}}/\hat{\mathfrak{m}}$, it is clear that (100) does not change if we conjugate $z\phi_M$ and $zr\theta$ by w_c. Further, from (101) and (102) we see that (100) is invariant under replacing zr_M by $(zr_M)^{-1}$. From (100) with $\overline{zr_M}$ instead of zr_M we obtain

$$\gamma(0, \mathrm{Ad}_{G^\vee}|_{\hat{\mathfrak{g}}/\hat{\mathfrak{m}}} \circ z\phi_M, \psi) = \pm q^{(\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{m}})/2} m^M(\overline{zr_M}).$$

From (94) we see that this the complex conjugate of $\pm q^{(\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{m}})/2} m^M(zr_M)$. In combination with Lemma A.2 that means that it is a real number, or ∞ if zr_M happens to be a pole.

However, the latter can not happen, and that can be seen with the residual cosets from [Opd1, Appendix A]. Namely, if zr_M were a pole of m^M, the tempered residual coset $X_{\text{unr}}(M)r_M$ for $m_{G/Z(G)_s}$ would contain a tempered residual coset (with the point zr_M) of smaller dimension. But that is excluded by [Opd1, Theorem A.17]. $\square$

Appendix B. The case $\text{char}(K) = 0$

Throughout this section we assume that the field K underlying G is p-adic. Then the arguments from [GaIc] with Galois cohomology are available. For some reductive groups this allows us to reduce the proof of Theorem 2 to the case of adjoint groups, much quicker than we do in the body of the paper.

Let $\mathcal{G}$ and $\mathcal{G}'$ be connected reductive K-groups which split over an unramified extension. We assume that

$$\mathcal{G}'_{\text{der}} \subset \mathcal{G} \subset \mathcal{G}' \text{ and the canonical map } \mathcal{G} \to \mathcal{G}'/Z(\mathcal{G}')_s \text{ is a central isogeny.} \tag{114}$$

In particular $Z(\mathcal{G})_s = \{1\}$ and $\mathcal{G}'$ is generated by $\mathcal{G}$ and $Z(\mathcal{G}')_s$. It was shown by Tadić [Tad, §2] that:

Proposition B.1. (a) *Every irreducible representation of G appears in an irreducible representation of G'.*

(b) *For $\pi, \pi' \in \text{Irr}(G')$ the following are equivalent:*
  - (i) $\text{Res}^{G'}_G(\pi)$ *and* $\text{Res}^{G'}_G(\pi')$ *have a common irreducible subquotient;*
  - (ii) $\text{Res}^{G'}_G(\pi) \cong \text{Res}^{G'}_G(\pi')$*;*
  - (iii) *there is a $\gamma \in \text{Irr}(G'/G)$ such that $\pi' \cong \pi \otimes \gamma$.*

(c) *The restriction of $(\pi', V') \in \text{Irr}(G')$ to G is a finite direct sum of irreducible G-representations, each one appearing with the same multiplicity.*

(d) *Let (π, V) be an irreducible G-subrepresentation of (π', V'). Then the stabilizer in G' of V is an open, finite index normal subgroup which contains G and the centre of G'.*

For $(\pi', V') \in \text{Irr}(G')$ we write

$$X^G(\pi') = \{\gamma \in \text{Irr}(G'/G) : \pi' \otimes \gamma \cong \pi'\}.$$

By Proposition B.1.d this is a finite group. For every $\gamma \in X^G(\pi)$ there exists a nonzero intertwining operator

$$I(\gamma, \pi') \in \text{Hom}_{G'}(\pi' \otimes \gamma, \pi') \subset \text{End}_G(\pi').$$

By Schur's lemma it is unique up to a scalar. These operators determine a 2-cocycle $\kappa_{\pi'}$ of $X^G(\pi')$ by

$$I(\gamma, \pi') I(\gamma', \pi') = \kappa_{\pi'}(\gamma, \gamma') I(\gamma\gamma', \pi').$$

According to [HiSa, Lemma 2.4], the $I(\gamma, \pi')$ form a basis of the G-intertwining algebra of (π', V'):

$$\text{End}_G(\text{Res}^{G'}_G(\pi')) \cong \mathbb{C}[X^G(\pi'), \kappa_{\pi'}], \tag{115}$$

where the right hand side denotes the twisted group algebra of $X^G(\pi')$. By [HiSa, Corollary 2.10], the decomposition of (π', V') into irreducible representations of $\mathbb{C}G \otimes \text{End}_G(\text{Res}^{G'}_G(\pi'))$ is

$$\pi' = \bigoplus_{\eta \in \text{Irr}(\mathbb{C}[X^G(\pi'), \kappa_{\pi'}])} \eta \otimes \text{Hom}_{\mathbb{C}[X^G(\pi'), \kappa_{\pi'}]}(\eta, \pi') = \bigoplus_{\eta \in \text{Irr}(\mathbb{C}[X^G(\pi'), \kappa_{\pi'}])} \eta \otimes \pi'_\eta. \tag{116}$$

Assume now that π' is in addition square-integrable modulo centre. The crucial contribution from [GaIc, Lemma 13.2] says

$$\frac{\mathrm{fdeg}(\pi'_\eta)}{\mathrm{fdeg}(\pi')} = \frac{\big|Z\big((G'/Z(G')_s)^\vee\big)^{\mathbf{W}_K}\big| \dim \eta}{|Z(G^\vee)^{\mathbf{W}_K}|\, |X^G(\pi')|}. \tag{117}$$

We note that here it is essential that K is a p-adic field. For local fields of positive characteristic p, the proof of [GaIc, Lemma 13.2] breaks down if p divides the order of the kernel of $\mathcal{G} \to \mathcal{G}'/Z(\mathcal{G}')_s$.

For unipotent representations we can reformulate (117) in terms of enhanced L-parameters:

Lemma B.2. *Let π' be an irreducible unipotent square-integrable modulo centre representation of G' and let π be an irreducible constituent of $\mathrm{Res}^{G'}_G(\pi')$. Then*

$$\frac{\mathrm{fdeg}(\pi)}{\mathrm{fdeg}(\pi')} = \frac{|S^\sharp_{\phi_{\pi'}}| \dim(\rho_\pi)}{|S^\sharp_{\phi_\pi}| \dim(\rho_{\pi'})}.$$

Proof. From (116) we see that

$$\mathrm{End}_G(\pi', V') \cong \bigoplus_{\eta \in \mathrm{Irr}(\mathbb{C}[X^G(\pi'),\kappa_{\pi'}])} \mathrm{End}_{\mathbb{C}}(\eta) \otimes \mathbb{C}\mathrm{id}_{V'_\eta}.$$

Together with (115) and Proposition B.1.c that yields

$$|X^G(\pi')| = \dim(\eta)^2 \, |\mathrm{Irr}(\mathbb{C}[X^G(\pi'), \kappa_{\pi'}])|. \tag{118}$$

Theorem 2.2 also holds for the inclusion $i : \mathcal{G} \to \mathcal{G}'$, see [Sol4, Theorem 3]. Combining that with (116) we obtain $\phi_\pi = \Phi(i)\phi_{\pi'}$ and

$$\bigoplus_{\eta \in \mathrm{Irr}(\mathbb{C}[X^G(\pi'),\kappa_{\pi'}])} \eta \otimes \pi'_\eta \cong \bigoplus_{\rho \in \mathrm{Irr}(\mathcal{A}_{\phi_\pi})} \mathrm{Hom}_{\mathcal{A}_{\phi_\pi}}\Big(\mathrm{ind}_{\mathcal{A}_{\phi_{\pi'}}}^{\mathcal{A}_{\phi_\pi}} \rho_{\pi'}, \rho\Big) \otimes \pi(\phi_\pi, \rho). \tag{119}$$

Then Proposition B.1.c shows that

$$\dim(\eta) = \dim \mathrm{Hom}_{\mathcal{A}_{\phi_\pi}}\Big(\mathrm{ind}_{\mathcal{A}_{\phi_{\pi'}}}^{\mathcal{A}_{\phi_\pi}} \rho_{\pi'}, \rho\Big) \tag{120}$$

for any $\eta \in \mathrm{Irr}(\mathbb{C}[X^G(\pi'), \kappa_{\pi'}])$ and any ρ appearing in $\mathrm{ind}_{\mathcal{A}_{\phi_{\pi'}}}^{\mathcal{A}_{\phi_\pi}} \rho_{\pi'}$. Another consequence of (119) is

$$|\mathrm{Irr}(\mathbb{C}[X^G(\pi'), \kappa_{\pi'}])| = |\{\rho \in \mathrm{Irr}(\mathcal{A}_{\phi_\pi}) : \rho \text{ appears in } \mathrm{ind}_{\mathcal{A}_{\phi_{\pi'}}}^{\mathcal{A}_{\phi_\pi}} \rho_{\pi'}\}| \tag{121}$$

With (118), (120) and (121) we compute

$$\begin{aligned} \frac{|X^G(\pi')|}{\dim(\eta)} &= \dim(\eta)\, |\mathrm{Irr}(\mathbb{C}[X^G(\pi'), \kappa_{\pi'}])| \\ &= \dim \mathrm{Hom}_{\mathcal{A}_{\phi_\pi}}\Big(\mathrm{ind}_{\mathcal{A}_{\phi_{\pi'}}}^{\mathcal{A}_{\phi_\pi}} \rho_{\pi'}, \rho\Big) |\{\rho \in \mathrm{Irr}(\mathcal{A}_{\phi_\pi}) : \rho \text{ appears in } \mathrm{ind}_{\mathcal{A}_{\phi_{\pi'}}}^{\mathcal{A}_{\phi_\pi}} \rho_{\pi'}\}| \qquad (122) \\ &= \text{length of } \mathrm{ind}_{\mathcal{A}_{\phi_{\pi'}}}^{\mathcal{A}_{\phi_\pi}} \rho_{\pi'} \text{ in } \mathrm{Rep}(\mathcal{A}_{\phi_\pi}). \end{aligned}$$

By (120) all irreducible constituents of $\mathrm{ind}_{\mathcal{A}_{\phi_{\pi'}}}^{\mathcal{A}_{\phi_\pi}} \rho_{\pi'}$ have the same dimension. We continue (122):

$$= \frac{\dim \left(\mathrm{ind}_{\mathcal{A}_{\phi_{\pi'}}}^{\mathcal{A}_{\phi_\pi}} \rho_{\pi'} \right)}{\dim \rho_\pi} = \frac{|\mathcal{A}_{\phi_\pi}| \dim(\rho_{\pi'})}{|\mathcal{A}_{\phi_{\pi'}}| \dim(\rho_\pi)}. \tag{123}$$

From the proof of [FOS, Lemma 13.2] and the discreteness of $\phi_{\pi'}$ we see that

$$[\mathcal{A}_{\phi_{\pi'}} : Z_{G^\vee{}_{\mathrm{sc}}}(\phi_{\pi'})] = [Z(G^\vee{}_{\mathrm{sc}}) : Z(G^\vee{}_{\mathrm{sc}})^{\mathbf{W}_K}] = [\mathcal{A}_{\phi_\pi} : Z_{G^\vee{}_{\mathrm{sc}}}(\phi_\pi)] \tag{124}$$

Further, from the definition (1) of $S^\sharp_{\phi_{\pi'}}$ we deduce

$$\frac{|Z_{G'^\vee{}_{\mathrm{sc}}}(\phi_{\pi'})|}{|S^\sharp_{\phi_{\pi'}}|} = \frac{\left| \ker \left(Z_{G'^\vee{}_{\mathrm{sc}}}(\phi_{\pi'}) \to Z_{G'^\vee{}_{\mathrm{der}}}(\phi_{\pi'}) \right) \right|}{[S^\sharp_{\phi_{\pi'}} : Z_{G'^\vee{}_{\mathrm{der}}}(\phi_{\pi'})]} = \tag{125}$$

$$\frac{\left| Z(G'^\vee{}_{\mathrm{sc}})^{\mathbf{W}_K} \right| \left| Z(G'^\vee{}_{\mathrm{der}})^{\mathbf{W}_K} \right|^{-1}}{[Z((G'/Z(G')_s)^\vee)^{\mathbf{W}_K} : Z(G'^\vee{}_{\mathrm{der}})^{\mathbf{W}_K}]} = \frac{\left| Z(G'^\vee{}_{\mathrm{sc}})^{\mathbf{W}_K} \right|}{\left| Z((G'/Z(G')_s)^\vee)^{\mathbf{W}_K} \right|}.$$

Now we continue (123). With (124) we replace $\mathcal{A}_{\phi_\pi}$ by $Z_{G^\vee{}_{\mathrm{sc}}}(\phi_\pi)$, and simultaneously for π'. Then we use (125) to replace $Z_{G'^\vee{}_{\mathrm{sc}}}(\phi_{\pi'})$ by $S^\sharp_{\phi_{\pi'}}$, with a correction term involving central elements in complex dual groups. We do the same with $Z_{G^\vee{}_{\mathrm{sc}}}(\phi_\pi)$, and using $G^\vee{}_{\mathrm{sc}} = G'^\vee{}_{\mathrm{sc}}$ we find that (122) equals

$$\frac{|S^\sharp_{\phi_\pi}|}{|S^\sharp_{\phi_{\pi'}}|} \frac{|Z((G'/Z(G')_s)^\vee)^{\mathbf{W}_K}|}{|Z(G^\vee)^{\mathbf{W}_K}|} \frac{\dim(\rho_{\pi'})}{\dim(\rho_\pi)}.$$

That equality can be rearranged to

$$\frac{|Z((G'/Z(G')_s)^\vee)^{\mathbf{W}_K}|}{|Z(G^\vee)^{\mathbf{W}_K}|} \frac{\dim(\eta)}{|X^G(\pi')|} = \frac{|S^\sharp_{\phi_{\pi'}}|}{|S^\sharp_{\phi_\pi}|} \frac{\dim(\rho_\pi)}{\dim(\rho_{\pi'})}. \tag{126}$$

Finally we combine (126) with (117). □

Corollary B.3. *Suppose that the HII conjecture (as in* (2) *and* [HII, §1.4]*) holds for a unipotent square-integrable modulo centre* $\pi' \in \mathrm{Irr}(G')$*. Then the HII conjecture holds for every irreducible constituent* π *of* $\mathrm{Res}^{G'}_G(\pi')$*.*

Proof. Lemma B.2 and the assumption entail

$$\mathrm{fdeg}(\pi) = \mathrm{fdeg}(\pi') \frac{|S^\sharp_{\phi_{\pi'}}| \dim(\rho_\pi)}{|S^\sharp_{\phi_\pi}| \dim(\rho_{\pi'})} = |\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_{\pi'}, \psi)| \frac{\dim(\rho_\pi)}{|S^\sharp_{\phi_\pi}|}.$$

Since $\phi_\pi = \Phi(i)\phi_{\pi'}$ and $\mathcal{G} \to \mathcal{G}'/Z(\mathcal{G}')_s$ is a central isogeny, $\mathrm{Ad}_{G^\vee} \circ \phi_\pi = \mathrm{Ad}_{G'^\vee} \circ \phi_{\pi'}$ as representations of $\mathbf{W}_K \times SL_2(\mathbb{C})$. In particular $\phi_{\pi'}$ and ϕ_π have the same adjoint γ-factors. □

While Corollary B.3 applies in large generality, it is not clear whether it can be used to reduce the HII conjecture for G to that for G_{ad}. The problem lies in the existence of a group $\mathcal{G}'$ satisfying condition (114), such that at the same time $\mathcal{G}'/Z(\mathcal{G}')_s$ equals $\mathcal{G}_{\mathrm{ad}}$. (Instead of $\mathcal{G}_{\mathrm{ad}}$, the direct product of an adjoint group with a K-anisotropic torus would also be acceptable, because the HII conjecture for anisotropic tori is known [HII, Correction].)

When $\mathcal{G}$ is an inner form of a K-split semisimple group, such a $\mathcal{G}'$ always exists. Indeed, then $\mathrm{Gal}(\overline{K}/K)$ acts trivially on $\mathrm{Hom}(Z(\mathcal{G}), GL_1)$ and the construction of Langlands [Lan, p. 120–121] suffices. It yields a reductive K-group $\mathcal{G}'$ with connected K-split centre, such that $\mathcal{G}'/Z(\mathcal{G}')_s = \mathcal{G}_{\mathrm{ad}}$ and (114) holds.

For outer forms of semisimple K-groups, we do not know whether a suitable $\mathcal{G}'$ exists. Langlands' construction produces a group with a connected centre, that however need not be split. For instance if $\mathcal{G}$ is a special unitary group, Langlands' method yields a full unitary group, which in this setting is not easier.

Theorem B.4. *Let K be a p-adic field. Let $\mathcal{G}$ be a semisimple K-group, which is an inner form of a K-split group and splits over an unramified extension. Then the HII conjecture* [HII, §1.4] *holds for all square-integrable unipotent irreducible G-representations.*

Proof. Let $\mathcal{G}'$ be as indicated above, so as constructed in [Lan]. Then $\mathcal{G}'/Z(\mathcal{G}') = \mathcal{G}_{\mathrm{ad}}$, for which the HII conjecture (2) was shown in [Opd3]. (See §5.1 for the details.) With the method from [FOS, proof of Theorem 3 on page 43] one can derive the HII conjecture for $\mathcal{G}'$ from that for $\mathcal{G}'/Z(\mathcal{G}')$.

Consider an irreducible unipotent square-integrable G-representation π, with enhanced L-parameter (ϕ_π, ρ_π) from Theorem 2.1. We note that ϕ_π is discrete. By condition (114), $G'^\vee \to G^\vee$ is a surjection with commutative kernel. As ϕ_π is unramified and $\mathcal{G}$ splits over an unramified extension, ϕ_π factors via $\mathbf{W}_K/\mathbf{I}_K \times SL_2(\mathbb{C})$. That makes it easy to lift ϕ_π to an unramified L-parameter $\phi' \in \Phi(G')$, necessarily discrete. Then $\mathcal{A}_{\phi_\pi} \subset \mathcal{A}_{\phi'}$ [Sol4, Proposition 5.4.a]. Let ρ' be a representation of $\mathcal{A}_{\phi'}$ that contains ρ_π and define $\pi' = \pi(\phi', \rho')$. By Theorem 2.1.e, π' is unipotent and essentially square-integrable. By tensoring π' with a suitable unramified character (automatically trivial on G) we can achieve that π' is in fact square-integrable modulo centre.

By Theorem 2.2, π appears in $\mathrm{Res}_G^{G'}(\pi')$. Now apply Corollary B.3. □

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