AFFINE HECKE ALGEBRAS FOR CLASSICAL p-ADIC GROUPS

ANNE-MARIE AUBERT, AHMED MOUSSAOUI, AND MAARTEN SOLLEVELD

Abstract. We consider four classes of classical groups over a non-archimedean local field F: symplectic, (special) orthogonal, general (s)pin and unitary. These groups need not be quasi-split over F. The main goal of the paper is to obtain a local Langlands correspondence for any group G of this kind, via Hecke algebras.

To each Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ in the category $\mathrm{Rep}(G)$ of smooth complex G-representations, an (extended) affine Hecke algebra $\mathcal{H}(\mathfrak{s})$ can be associated with the method of Heiermann. On the other hand, to each Bernstein component $\Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee}$ of the space $\Phi_{\mathrm{e}}(G)$ of enhanced L-parameters for G, one can also associate an (extended) affine Hecke algebra, say $\mathcal{H}(\mathfrak{s}^\vee)$. For the supercuspidal representations underlying $\mathrm{Rep}(G)^{\mathfrak{s}}$, a local Langlands correspondence is available via endoscopy, due to Mœglin and Arthur. Using that we assign to each $\mathrm{Rep}(G)^{\mathfrak{s}}$ a unique $\Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee}$.

Our main new result is an algebra isomorphism $\mathcal{H}(\mathfrak{s})^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^\vee)$, canonical up to inner automorphisms. In combination with earlier work, that provides an injective local Langlands correspondence $\mathrm{Irr}(G) \to \Phi_{\mathrm{e}}(G)$ which satisfies Borel's desiderata. When F has characteristic zero, this parametrization map is in fact bijective. When F has positive characteristic it is probably bijective as well, but we could not show that in all cases.

Our framework is suitable to (re)prove many results about smooth G-representations (not necessarily irreducible), and to relate them to the geometry of a space of L-parameters. In particular our Langlands parametrization yields an independent way to classify discrete series G-representations in terms of Jordan blocks and supercuspidal representations of Levi subgroups. We show that it coincides with the classification of the discrete series obtained twenty years ago by Mœglin and Tadić.

Contents

Introduction 2
1. General spin groups 7
1.1. Properties of Langlands parameters 8
1.2. Hecke algebras for Langlands parameters 11
2. Mœglin's classification of discrete series representations 18
2.1. The method of close local fields 20
2.2. Parametrization of essentially square-integrable representations 22
3. Comparison of Hecke algebras for Bernstein components 25
3.1. Versions for G^+ 32
3.2. Langlands parameters via Hecke algebras 36
4. Comparison of Langlands parameters 39
4.1. Cuspidal supports of essentially square-integrable representations 39
4.2. Jordan blocks of discrete series representations 42

Date: June 23, 2025.
2010 *Mathematics Subject Classification.* 20G25,20C08,22E50,11S37.

4.3. Intertwining operators for discrete series representations 45
4.4. Tempered representations 51
4.5. Irreducible smooth representations 54
5. Unitary groups 56
References 60

Introduction

In the theory of linear algebraic groups, the classical groups play a special role. As the stabilizer groups of bilinear/hermitian forms, they can arise from many directions and have various applications. Within the representation theory of reductive p-adic groups, the main advantage of classical groups is their explicit structure. It enables precise, combinatorial methods to study representations, on a level which is hard to reach for other reductive groups. Such methods have been pursued by many mathematicians, see for instance [Art, GGP, Hei4, KiMa, MoTa].

In this paper we translate the (smooth, complex) representation theory of classical p-adic groups to affine Hecke algebras arising from Langlands parameters. This is part of a long-term program [AMS1, AMS2, AMS3] that applies to all reductive p-adic groups and aims to establish instances of Langlands correspondences via Hecke algebras. The method has already proven successful for principal series representations of split groups [ABPS3] and for unipotent representations [Sol3]. For classical groups, our Hecke algebra methods provide alternative proofs of many earlier results (e.g. the classification of discrete series representations) and install a framework in which one can easily establish many new results that involve categories of smooth representations.

Let F be any non-archimedean local field (p-adic or a local function field). We will consider classical F-groups in a broad sense, namely

- symplectic groups;
- (special) orthogonal groups associated to symmetric bilinear forms on a finite dimensional F-vector space V;
- general (s)pin groups associated to such bilinear forms;
- unitary groups associated to hermitian forms on vector spaces over a separable quadratic extension of F.

We stress that these groups do not have to be quasi-split, we allow pure inner forms. For $G = \mathrm{SO}(V)$ and $G = \mathrm{GSpin}(V)$ we write, respectively, $G^+ = \mathrm{O}(V)$ and $G^+ = \mathrm{GPin}(V)$, otherwise $G^+ = G$. An advantage of including general spin groups is that they provide information about all representations of spin groups, including those that do not factor through special orthogonal groups.

General linear groups could also figure in the list, they are very classical (but note that they do not come from a nondegenerate bilinear form). We excluded them because for $\mathrm{GL}_n(F)$ everything we will discuss has been known for a long time already, see [HaTa, Hen, Sch, LRS] for the local Langlands correspondence and [BuKu, AMS1] for the Hecke algebras.

The common feature of all the above groups G is that their Levi subgroups are isomorphic to $G' \times \mathrm{GL}_{n_1}(F') \times \cdots \times \mathrm{GL}_{n_k}(F')$, where G' is a group in the same family as G but of smaller rank, and $F' = F$ unless G is a unitary group, then

$[F' : F] = 2$. It is this structure which enables the aforementioned "combinatorial" approach to representations of classical groups. In a sense that approach is recursive, relating G-representations to similar groups of smaller rank and to representations of $\mathrm{GL}_n(F)$, which are understood well already. However, such a reduction strategy does not say much about supercuspidal G-representations. The crucial technique to analyse those is endoscopy, as in [Art, Mok, KMSW, MoRe]. From the work of Arthur and Mœglin, the following version of a local Langlands correspondence (for the discrete objects) can be distilled.

Theorem A. (Arthur, Mœglin, see Theorem 2.1)
Let F be a p-adic field and let G be one of the connected classical groups listed above.

(*a*) *Let π be a discrete series representation of G^+. Then the L-parameter of π can be obtained from the set of Jordan blocks of π, by taking the L-parameters of all $\mathrm{GL}_n(F)$-representations in $\mathrm{Jord}(\pi)$ and combining those via block-diagonal matrices.*
(*b*) *Pick a Whittaker datum for the quasi-split inner form of G. This determines an extension of part (a) to an injection from the discrete series of G^+ to the set of enhanced bounded discrete L-parameters for G (where the component groups of L-parameters are computed in the possibly disconnected group $G^{\vee +}$).*
(*c*) *When $G^+ \neq G$, it can be described explicitly in terms of $\mathrm{Jord}(\pi)$ whether or not $\mathrm{Res}_G^{G^+}(\pi)$ is irreducible.*

Mœglin characterized also supercuspidality in the context of Theorem A, both for G-representations and for enhanced L-parameters. We refer to Section 2 for the notations and more background. For now, we make a couple of remarks to aid the correct interpretation of Theorem A. Firstly, note that in part (b) no bijectivity is claimed, although that is known for many of these groups. Secondly, we have to warn that not all details of the proof of Theorem A have been worked out (we ourselves did not try, we only provide the relevant references). Further, Theorem A relies heavily on endoscopy, that is the reason why F needs to have characteristic zero.

Nevertheless, Theorem A should also hold for classical groups over local function fields, see [GaVa, GaLo] for some instances. In Paragraph 2.1 we attempt to derive that with the method of close local fields. We managed to prove that in Proposition 2.3, under Hypothesis 2.2 on depths of representations in Jordan blocks (the hypothesis most probably holds always). Unfortunately our arguments do not suffice to prove surjectivity in Theorem A.b in complete generality for classical groups over local function fields, even if we would know such surjectivity for the analogous groups over p-adic fields.

For the purposes of this paper, we only need to know Theorem A for supercuspidal G^+-representations. Indeed, the remainder of Theorem A follows from those cases with either [Mœ1, MoTa] or with our results discussed below and the detailed knowledge of the discrete series of the Hecke algebras from [AMS2, AMS3]. Consequently all results in paper hold for G and G^+ as soon as we know Theorem A for supercuspidal representations of G^+ and of the groups of smaller rank in the same family.

Next we discuss our new results. An important aspect is their canonicity, by which we mean that they do not depend on arbitrary choices. For results that are not entirely canonical, we indicate the freedom in the choices.

Recall that the category of smooth complex representations of any classical F-group G admits the Bernstein decomposition

$$\mathrm{Rep}(G) = \prod\nolimits_{\mathfrak{s}} \mathrm{Rep}(G)^{\mathfrak{s}}, \tag{1}$$

indexed by inertial equivalence classes $\mathfrak{s}$. Each such $\mathfrak{s}$ is a G-conjugacy class of pairs $(L, X_{\mathrm{nr}}(L) \cdot \sigma)$, where σ is an irreducible supercuspidal representation of a Levi subgroup L of G and $X_{\mathrm{nr}}(L)$ is the group of unramified characters of L. Every Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ is equivalent with the category of right modules of some finitely generated algebra $\mathcal{H}(\mathfrak{s})$, often an affine Hecke algebra. Usually these Hecke algebras arise via types (in the sense of Bushnell–Kutzko). For classical groups such types are indeed available [MiSt], but it has turned out to be difficult to analyse the Hecke algebras via those types. Instead we follow the approach of Heiermann [Hei2, Hei3, Hei4], who constructed $\mathcal{H}(\mathfrak{s})$ as the G-endomorphism algebra of a canonical progenerator $\Pi_{\mathfrak{s}}$ of $\mathrm{Rep}(G)^{\mathfrak{s}}$. (For $\mathrm{GSpin}(V)$ we use the more general results from [Sol4, Sol5].) We recall that by design $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, ?)$ provides an equivalence of categories

$$\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{Mod}(\mathcal{H}(\mathfrak{s})^{\mathrm{op}}) = \mathrm{Mod}(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{\mathrm{op}}). \tag{2}$$

Here A^{op} denotes the opposite algebra of A, so that $\mathrm{Mod}(A^{\mathrm{op}})$ is the category of right A-modules. These algebras $\mathcal{H}(\mathfrak{s})$ have been described explicitly in terms of the Jordan blocks of the underlying supercuspidal representations (of a Levi subgroup of G). They are extended affine Hecke algebras [Hei3, Sol4]. That links them to Theorem A and hence to Langlands parameters. These links were investigated in [Hei4], where it was shown that each $\mathrm{Rep}(G)^{\mathfrak{s}}$ equivalent to a Bernstein block of unipotent representations in another group. Unfortunately these equivalences are far from canonical, as most of the comparison steps in [Hei4] involve arbitrary choices. Also, the back-and-forth between various Langlands parameters and Hecke algebras entails that in [Hei4] there is no construction of representations in $\mathrm{Rep}(G)^{\mathfrak{s}}$ with an unambiguous relation to Langlands parameters for G.

Fortunately, objects of the above kinds are also available directly for L-parameters. Indeed, in [AMS1, §8] the space of enhanced L-parameters (of any connected reductive p-adic group G) is partitioned into Bernstein components:

$$\Phi_{\mathrm{e}}(G) = \bigsqcup\nolimits_{\mathfrak{s}^\vee} \Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee}, \tag{3}$$

indexed by inertial equivalence classes $\mathfrak{s}^\vee$ for $\Phi_e(G)$. By definition each $\mathfrak{s}^\vee$ is a $G^\vee$-conjugacy class of triples $({}^L M, X_{\mathrm{nr}}({}^L M) \cdot \phi, \epsilon)$, where ${}^L M$ is a L-Levi subgroup of ${}^L G$, $X_{\mathrm{nr}}({}^L M)$ denotes the analogue of unramified characters for ${}^L M$ and $(\phi, \epsilon) \in \Phi_{\mathrm{cusp}}(M)$ is a cuspidal enhanced L-parameter, see [AMS1, 7.1].

To each such Bernstein component, one can associate a twisted affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ [AMS3]. Here $\mathbf{z}$ is an invertible indeterminate, analogous to $\sqrt{\mathbf{q}}$ for Iwahori–Hecke algebras. Two important features of $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ were established in [AMS2, AMS3]: a construction of (irreducible) representations in terms of the geometry of a space of Langlands parameters and for each $z \in \mathbb{R}_{>0}$ a canonical bijection

$$\Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee} \longleftrightarrow \mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, z)), \tag{4}$$

where $\mathcal{H}(\mathfrak{s}^\vee, z)$ denotes the specialization of $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ at $\mathbf{z} = z$. Moreover, for $z > 1$ the bijection (4) sends bounded parameters and discrete parameters to the expected kind of representations (respectively tempered and essentially discrete series). Later we will specialize $\mathbf{z}$ to $q_F^{1/2}$, where q_F denotes the cardinality of the residue field of F. The algebras $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ are crucial, without them it is hardly possible to make the relations between $\Phi_{\mathrm{e}}(G)$ and $\mathrm{Rep}(G)$ canonical.

In Paragraph 1.2 we make the affine Hecke algebras $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ completely explicit, for any Bernstein component of $\Phi_{\mathrm{e}}(G)$ with G a classical F-group. This involves a description of the underlying root datum and of the labels (equivalently: the q-parameters) in terms of the relevant Jordan blocks. We refer to Table 2 for an overview.

Theorem A enables us to associate to each Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ a unique Bernstein component of $\Phi_{\mathrm{e}}(G)$ which we call $\Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee}$, see Theorem 3.1. When $G^+ \neq G$ (so for special orthogonal groups and general spin groups), $\mathfrak{s}^\vee$ is only canonical up to the action of the two-element group $\mathrm{Out}(\mathcal{G})$. Our most important result is a comparison of Hecke algebras on the two sides of the local Langlands correspondence:

Theorem B. (see Theorem 3.3, Proposition 3.5 and Proposition 4.6)
Let G be a connected classical group over F and fix a Whittaker datum for the quasi-split inner form of G. Suppose that the inertial equivalence classes $\mathfrak{s}$ and $\mathfrak{s}^\vee$ are matched as in Theorem 3.1. There exists an algebra isomorphism

$$\mathcal{H}(\mathfrak{s}) \cong \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$$

with the following properties.

- *On the standard maximal commutative subalgebras $\mathcal{O}(T_{\mathfrak{s}}) \subset \mathcal{H}(\mathfrak{s})$ and $\mathcal{O}(T_{\mathfrak{s}^\vee}) \subset \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$, the isomorphism is prescribed by the L-parameters of supercuspidal representations from Theorem A and from the local Langlands correspondence for general linear groups.*
- *There is a canonical bijection between the root system associated to $\mathfrak{s}$ and the root system associated to $\mathfrak{s}^\vee$.*
- *The isomorphism is canonical up to conjugation by elements of $\mathcal{O}(T_{\mathfrak{s}})^\times$ and (in the cases with $G^+ \neq G$) up to the action of* $\mathrm{Out}(\mathcal{G})$.

There exists an analogous isomorphism of Hecke algebras for G^+, which is canonical up to conjugation by elements of $\mathcal{O}(T_{\mathfrak{s}})^\times$.

Further, the algebra $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ is canonically isomorphic to its own opposite.

We remind the reader that in the case of classical groups over local function fields we need the mild Hypothesis 2.2 for Theorem B (and hence for most subsequent results). The Whittaker datum is needed for the canonicity of the above Hecke algebra isomorphism. That is a subtle affair, it relies on a normalization of certain intertwining operators in Paragraphsubsection 4.3, which in the end boils down to [Art].

As a direct consequence of Theorem B and (2) we find an equivalence of categories

$$\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{Mod}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big), \tag{5}$$

which is canonical up to inner automorphisms of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ (and up to the action of $\mathrm{Out}(\mathcal{G})$ when $G^+ \neq G$). The analogous equivalence of categories for G^+ is canonical up to inner automorphisms of the involved Hecke algebra.

With the Bernstein decompositions (1) and (3), (5) makes $\mathrm{Rep}(G)$ equivalent to the module category of a direct sum of Hecke algebras. In combination with (4) we obtain:

Theorem C. (see Theorem 3.7 and Corollary 3.8)
Theorem A (for supercuspidal representations) and Theorem B induce an injective local Langlands correspondence

$$\mathrm{Irr}(G) \hookrightarrow \Phi_{\mathrm{e}}(G).$$

It is canonical (up to the action of $\mathrm{Out}(\mathcal{G})$ *when* $G^+ \neq G$*) and it sends supercuspidal/ essentially square-integrable/tempered representations to cuspidal/discrete/bounded enhanced* L*-parameters.*

There exists an analogous parametrization of $\mathrm{Irr}(G^+)$*, which uses component groups of* L*-parameters computed in* $G^{\vee +}$ *and is entirely canonical.*

We note that (5) is much stronger than any results about the parametrization of $\mathrm{Irr}(G)$, in the sense that it deals with an entire category of representations. Indeed, earlier results about Hecke algebras entail that (5) has various consequences that involve reducible representations, see Paragraph 3.2. Furthermore the equivalence of categories (5) makes it possible to relate $\mathrm{Rep}(G)$ to the complex geometry of the space/stack of L-parameters, as in [Sol6].

In Theorem C we do not claim surjectivity of the parametrization map, because for that we would need surjectivity in Theorem A.b, which we do not know when F is a local function field. That is in fact the only obstruction: the image of the map in Theorem C is the union of all Bernstein components of $\Phi_{\mathrm{e}}(G)$ whose underlying cuspidal L-parameters can be reached via Theorem A. So in all the cases where the surjectivity of Theorem A.b has been proven, we also get surjectivity in Theorem C.

Theorem C yields in particular a classification of the discrete series of G^+, in terms of the bounded discrete enhanced L-parameters in the image of the parametrization map. On the other hand, Theorem A also classifies discrete series representations of G^+. For supercuspidal representations these two methods agree, that is a starting point of our setup. We obtain two independent ways to classify the discrete series in terms of supercuspidal representations of Levi subgroups: with Hecke algebras via Theorem C and with Jordan blocks as in [Mœ1, MoTa, KiMa].

Moreover both methods can be pushed further, to classify all irreducible smooth G^+-representations. Indeed, in Theorem C that comes at the same time as the discrete series (in the underlying proofs from [AMS2] the discreteness of representations is actually analysed last). Irreducible tempered G^+-representations are classified with endoscopy and Jordan blocks in [MoTa, MoRe]. The step from tempered representations to all irreducible smooth representations via the Langlands classification is well-known for G and we show how this technique can be made to work for G^+. With that extension the papers [MoTa, MoRe] also classify $\mathrm{Irr}(G^+)$.

Theorem D. (see Theorem 4.12)
Let G *be a connected classical group over* F *and fix a Whittaker datum for the quasi-split inner form of* G*. The following two ways to parametrize* $\mathrm{Irr}(G^+)$ *with enhanced* L*-parameters coincide:*

- *with Hecke algebras as in Theorem C,*
- *with endoscopy and Jordan blocks (as in Theorem A) and the Langlands classification.*

Because the two strategies are very different, it is rather cumbersome to check that they agree. We do this step by step in Section 4, in the following order: completely positive discrete series, all discrete series, irreducible tempered, all irreducible representations. The most difficult part concerns the enhancements of L-parameters for discrete series representations. To match those for the two methods in Theorem D, we have to investigate normalizations of intertwining operators. That also informs us how to make Theorem B canonical.

A few words about the setup of the paper are in order. As we mentioned at the start of the introduction, we consider four classes of classical groups. For all classes the proofs of our results are extremely similar, yet not entirely the same. For symplectic and (special) orthogonal groups, almost everything that we show about Hecke algebras was known already, from [Hei2, Hei3] for p-adic groups and from [Mou, AMS3] for Langlands parameters.

Instead we focus on general (s)pin groups in Sections1 and 3. That is a little bit more involved because the center $\mathrm{Z}(G)$ of such a group G is not compact. The proofs for symplectic and (special) orthogonal groups can be recovered by restricting from $G^\vee$ to its derived group, where $G^\vee$ denotes the complex reductive group with root datum dual to that of G. Sections 2 and 4 are written so that they apply equally well to symplectic, (special) orthogonal and general (s)pin groups.

For unitary groups, the necessary changes affect the notations so much that we discuss them in the separate Section 5. We check carefully which modifications are needed to make Sections1–4 work for unitary groups. It turns out that for unramified unitary groups some calculations in Paragraph 1.2 have different outcomes, which we record.

1. General spin groups

Let F be a non-archimedean local field with absolute Weil group $\mathbf{W}_F$. Consider a finite dimensional F-vector space V endowed with a symmetric bilinear form. The associated general pin group, denoted $\mathrm{GPin}(V)$, contains the general spin group $\mathrm{GSpin}(V)$ with index two. Both are subgroups of the multiplicative group of the Clifford algebra of V. For the root datum of $\mathrm{GSpin}(V)$ we refer to [AsSh, §2]. Simultaneously we consider the groups $\mathrm{GSpin}(V')$, where $\dim(V') = \dim(V)$ and $\mathrm{disc}(V) = \mathrm{disc}(V')$. The equivalence classes of such groups are naturally in bijection with:

- equivalence classes of symmetric bilinear forms of the same dimension and the same discriminant as V,
- pure inner twists of $\mathrm{SO}(V)$,
- rigid inner twists of $\mathrm{GSpin}(V)$ [Kal1], with respect to the central subgroup

$$\{\pm 1\} = \ker\big(\mathrm{Spin}(V) \to \mathrm{SO}(V)\big) = \ker\big(\mathrm{GSpin}(V) \to \mathrm{SO}(V) \times F^\times\big).$$

To align with the other classical groups, we refer to the above groups as the pure inner twists of $\mathrm{GSpin}(V)$. Let us list all the possibilities:

- for dim $= 2n+1$, the split group $\mathrm{GSpin}_{2n+1}(F)$ of F-rank n and one pure inner twist $\mathrm{GSpin}'_{2n+1}(F)$ of F-rank $n-1$,
- for dim $= 2n$, the split group $\mathrm{GSpin}_{2n}(F)$ of F-rank n and one pure inner twist $\mathrm{GSpin}'_{2n}(F)$ of F-rank $n-1$,
- for dim $= 2n$, the quasi-split group $\mathrm{GSpin}^*_{2n}(F)$ of F-rank $n-1$ and one pure inner twist $\mathrm{GSpin}^{*'}_{2n+1}(F)$, which is also quasi-split.

For any of these groups G, we write

$$G^+ = \begin{cases} \mathrm{GPin}(V) & \text{if } \dim V \text{ is even} \\ \mathrm{GSpin}(V) & \text{if } \dim V \text{ is odd} \end{cases} . \tag{1.1}$$

All inner twists share the same Langlands dual group, so for that we have precisely three possibilities:

- $\mathrm{GSpin}^\vee_{2n+1} = \mathrm{GSp}_{2n}(\mathbb{C})$, and since one of the p-adic groups is split we may take ${}^L\mathrm{GSpin}_{2n+1} = \mathrm{GSp}_{2n}(\mathbb{C})$,
- $\mathrm{GSpin}^\vee_{2n} = \mathrm{GSO}_{2n}(\mathbb{C})$, and again one of the p-adic groups is split so we take ${}^L\mathrm{GSpin}_{2n} = \mathrm{GSO}_{2n}(\mathbb{C})$,
- $\mathrm{GSpin}^{*\vee}_{2n} = \mathrm{GSO}_{2n}(\mathbb{C})$, and $\mathbf{W}_F$ acts on it via passing to a quotient $\mathbf{W}_F/\mathbf{W}_E$ of order two and then conjugation by an element of $\mathrm{O}_{2n}(\mathbb{C}) \setminus \mathrm{SO}_{2n}(\mathbb{C})$. We may take ${}^L\mathrm{GSpin}^*_{2n} = \mathrm{GO}_{2n}(\mathbb{C})$, where we remember that every Langlands parameter for $\mathrm{GSpin}^*_{2n}(F)$ sends $\mathbf{W}_E$ to $\mathrm{GSO}_{2n}(\mathbb{C})$ and $\mathbf{W}_F \setminus \mathbf{W}_E$ to $\mathrm{GO}_{2n}(\mathbb{C}) \setminus \mathrm{GSO}_{2n}(\mathbb{C})$.

We write ${}^L G_n$ or ${}^L G$ for ${}^L\mathrm{GSpin}_{2n+1}$, ${}^L\mathrm{GSpin}_{2n}$ or ${}^L\mathrm{GSpin}^*_{2n}$. We also write

$$G^{\vee +} = \begin{cases} \mathrm{GO}_{2n}(\mathbb{C}) \\ \mathrm{GSp}_{2n}(\mathbb{C}) \end{cases}, \quad G^{\vee +}_{\mathrm{der}} = \begin{cases} \mathrm{O}_{2n}(\mathbb{C}) & \text{if } \dim V = 2n \\ \mathrm{Sp}_{2n}(\mathbb{C}) & \text{if } \dim V = 2n+1 \end{cases} .$$

Langlands parameters for G are group homomorphisms $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$, with the usual requirements as for instance in [AMS1, Definition 6.4]. Considered up to $G^\vee$-conjugation, these form the set $\Phi(G)$. Langlands parameters for G^+ take values in ${}^L G$ and are considered up to conjugation by $G^{\vee +}$.

1.1. Properties of Langlands parameters.

Let us investigate when a Langlands parameter ϕ for G is discrete. The image of ϕ is contained in ${}^L G$, so in $\mathrm{GSp}_{2n}(\mathbb{C})$ or in $\mathrm{GO}_{2n}(\mathbb{C})$. In the former case ϕ is an L-parameter for $\mathrm{GSpin}_{2n+1}(F)$ or $\mathrm{GSpin}'_{2n+1}(F)$, in the latter case for a general spin group of even size. We can distinguish two subcases:

- when $\mathrm{im}(\phi) \subset \mathrm{GSO}_{2n}(\mathbb{C})$, where ϕ is an L-parameter for $\mathrm{GSpin}_{2n}(F)$ or $\mathrm{GSpin}'_{2n}(F)$,
- otherwise there is an index two subgroup $\mathbf{W}_E \subset \mathbf{W}_F$ such that $\phi(\mathbf{W}_E \times \mathrm{SL}_2(\mathbb{C})) \subset \mathrm{GSO}_{2n}(\mathbb{C})$. Then ϕ is an L-parameter for a group $\mathrm{GSpin}^*_{2n}(F)$ or $\mathrm{GSpin}^{*'}_{2n}(F)$ which splits over E.

We suppose that the bilinear form B_J on $\mathbb{C}^{2n}$ from which $G^\vee$ is defined is given by a (skew-)symmetric matrix $J \in \mathrm{GL}_{2n}(\mathbb{C})$. Let $\mu_G^\vee : {}^L G \to \mathbb{C}^\times$ be the similitude character, that is

$$B_J(gv_1, gv_2) = \mu_G^\vee(g) B_J(v_1, v_2) \qquad v_1, v_2 \in \mathbb{C}^{2n}, g \in {}^L G. \tag{1.2}$$

Recall that (1.2) holds for $g = \phi(w)$ with $w \in \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$. Hence the map

$$\begin{array}{cccc} \tilde{B}_J : & \mathbb{C}^{2n} & \to & (\mathbb{C}^{2n})^\vee \\ & v & \mapsto & [v' \mapsto B_J(v', v)] \end{array}$$

provides an isomorphism of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$-representations

$$\phi \xrightarrow{\sim} \phi^\vee \otimes \mu_G^\vee \circ \phi \qquad \text{or equivalently} \qquad \phi \otimes (\mu_G^\vee \circ \phi)^{-1} \xrightarrow{\sim} \phi^\vee. \tag{1.3}$$

Here $\phi^\vee$ denotes the contragredient of ϕ. The adjoint map

$$\tilde{B_J}^\vee : \phi \xrightarrow{\sim} \big(\phi \otimes (\mu_G^\vee \circ \phi)^{-1}\big)^\vee = \phi^\vee \otimes \mu_G^\vee \circ \phi$$

is also an isomorphism of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$-representations. Suppose that V_1 is an irreducible subrepresentation of $(\phi, \mathbb{C}^{2n})$, on which B_J is nondegenerate. By Schur's lemma there exists $c_1 \in \mathbb{C}^\times$ such that $\tilde{B_J}^\vee|_{V_1} = c_1 \tilde{B_J}|_{V_1}$. Then

$$\tilde{B_J}|_{V_1} = \tilde{B_J}^{\vee\vee}|_{V_1} = c_1 \tilde{B_J}^\vee|_{V_1} = c_1^2 \tilde{B_J}|_{V_1},$$

so $c_1 \in \{1, -1\}$. This says that (V_1, B_J) has a well-defined sign c_1.

Since ${}^L G = \mathbb{C}^\times {}^L G_{\mathrm{der}}$ and ${}^L G_{\mathrm{der}} = \mathrm{Sp}_{2n}(\mathbb{C})$ or ${}^L G_{\mathrm{der}} \subset \mathrm{O}_{2n}(\mathbb{C})$, the decomposition of $(\phi, \mathbb{C}^{2n})$ in irreducible subrepresentations can be carried out just like for orthogonal or symplectic representations. For those kinds of representations we use the instructive paper [GGP]. Thus we decompose

$$(\phi, \mathbb{C}^{2n}) = \bigoplus\nolimits_{\psi \in \mathrm{Irr}(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}))} N_\psi \otimes V_\psi, \tag{1.4}$$

where V_ψ is the space of the representation ψ and N_ψ is the multiplicity space (with a trivial action). By [GGP, Theorem 8.1] the right hand side of (1.4) determines ϕ up to $G^\vee$-conjugacy, apart from some exceptional cases in which it is up to $\mathrm{GO}_{2n}(\mathbb{C})$-conjugacy. Further, by [GGP, §4] B_J induces bilinear a form on each N_ψ and

$$\begin{aligned} \mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi) &:= \mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}))) = \\ \mathrm{S}\big(\prod\nolimits_{\psi \in I^+} \mathrm{O}(N_\psi) \otimes \mathrm{Id}_{V_\psi}\big) &\times \prod\nolimits_{\psi \in I^-} \mathrm{Sp}(N_\psi) \otimes \mathrm{Id}_{V_\psi} \times \prod\nolimits_{\psi \in I^0} \mathrm{GL}(N_\psi) \otimes \mathrm{Id}_{V_\psi \oplus V_\psi^\vee}, \end{aligned} \tag{1.5}$$

where $\mathrm{S}(H)$ denotes the subgroup of elements in H with determinant equal to 1. Here we abbreviated

$$\begin{aligned} I^\pm &= \{\psi \in \mathrm{Irr}(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})) \colon \psi \cong \psi^\vee \otimes \mu_G^\vee \circ \phi, \mathrm{sgn}(\psi) = \pm \mathrm{sgn}(G^\vee{}_{\mathrm{der}})\}, \\ I^0 &= \{\psi \in \mathrm{Irr}(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})) : \psi \not\cong \psi^\vee \otimes \mu_G^\vee \circ \phi\} / (\psi \sim \psi^\vee \otimes \mu_G^\vee \circ \phi). \end{aligned}$$

If $G^\vee = \mathrm{GSO}_{2n}(\mathbb{C})$, then we may also consider (1.5) with $G^{\vee +} = \mathrm{GO}_{2n}(\mathbb{C})$ instead of $G^\vee$. That results in omitting the S from the second line of (1.5).
Recall that ϕ is discrete if and only if $\mathrm{Z}_{G^\vee}(\phi)/\mathrm{Z}(G^\vee)^{\mathbf{W}_F}$ is finite, which is equivalent to: $\mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi)$ is finite. From (1.5) we see that that is the case if and only if

$$N_\tau = 0 \text{ for } \tau \in I^- \cup I^0 \qquad \text{and} \qquad \dim(N_\tau) \leq 1 \text{ for } \tau \in I^+. \tag{1.6}$$

From now on we assume that ϕ is discrete. Thus each $\tau \otimes P_a$ has multiplicity at most one in ϕ.

Recall that $\mathrm{SL}_2(\mathbb{C})$ has a unique irreducible representation $(P_a, \mathbb{C}^a)$ of dimension $a \in \mathbb{Z}_{>0}$, and that it is self-dual with sign $(-1)^{a-1}$. Let $\mathrm{Jord}(\phi)$ be the set of pairs $(\tau, a) \in \mathrm{Irr}(\mathbf{W}_F) \times \mathbb{Z}_{>0}$ for which $\tau \otimes P_a$ occurs in $(\phi, \mathbb{C}^{2n})$. The set $\mathrm{Jord}(\phi)$ describes the Jordan decomposition of the unipotent element $u_\phi = \phi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$: for each $(\tau, a) \in \mathrm{Jord}(\phi)$, u_ϕ has $\dim \tau$ Jordan blocks of size a. We abbreviate

$$\mathrm{Jord}_\tau(\phi) = \{a \in \mathbb{Z}_{>0} : (\tau, a) \in \mathrm{Jord}(\phi)\}.$$

We define

$$\begin{aligned} \mathrm{Irr}(\mathbf{W}_F)^\pm_\phi &= \{\tau \in \mathrm{Irr}(\mathbf{W}_F) : \tau \cong \tau^\vee \otimes \mu_G^\vee \circ \phi, \mathrm{sgn}(\tau) = \pm \mathrm{sgn}(G^\vee{}_{\mathrm{der}})\}, \\ \mathrm{Irr}(\mathbf{W}_F)^0_\phi &= \{\tau \in \mathrm{Irr}(\mathbf{W}_F) : \tau \not\cong \tau^\vee \otimes \mu_G^\vee \circ \phi\} / (\tau \sim \tau^\vee \otimes \mu_G^\vee \circ \phi). \end{aligned}$$

Then we can express (1.4) more precisely as
(1.7)

$$(\phi, \mathbb{C}^{2n}) = \bigoplus_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi} \tau \otimes \Big(\bigoplus_{a \text{ odd}: (\tau,a) \in \mathrm{Jord}(\phi)} P_a \Big) \oplus \bigoplus_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^-_\phi} \tau \otimes \Big(\bigoplus_{a \text{ even}: (\tau,a) \in \mathrm{Jord}(\phi)} P_a \Big) .$$

The setup with rigid inner forms from [Kal1] entails that we should consider centralizers of L-parameters in

$$(G/\{\pm 1\})^\vee = (\mathrm{SO}(V) \times \mathrm{GL}_1(F))^\vee = \mathrm{SO}(V)^\vee \times \mathrm{GL}_1(\mathbb{C}).$$

The factor $\mathrm{GL}_1(\mathbb{C})$ is central, connected and fixed by $\mathbf{W}_F$, so does not influence the component groups and may be omitted. Therefore we may and will compute component groups of L-parameters for our pure inner twists of $\mathrm{GSpin}(V)$ in ${G^\vee}_{\mathrm{der}} = \mathrm{SO}(V)^\vee$, which equals $\mathrm{SO}_{2n}(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$. We put

$$\mathcal{S}_\phi = \pi_0(\mathrm{Z}_{{G^\vee}_{\mathrm{der}}}(\phi)) \quad \text{and} \quad \mathcal{S}_\phi^+ = \pi_0(\mathrm{Z}_{{G^\vee}^+_{\mathrm{der}}}(\phi)). \tag{1.8}$$

and we use the irreducible representations of $\mathcal{S}_\phi$ as enhancements of ϕ. From (1.5) we see that every $(\tau, a) \in \mathrm{Jord}(\phi)$ contributes a generator $z_{\tau,a}$ of order two to $\mathcal{S}_\phi^+$. Then $\mathcal{S}_\phi^+$ is the $\mathbb{F}_2$-vector space with these $z_{\tau,a}$ as basis. Here $z_{\tau,a}$ acts as -1 on $\tau \otimes P_a$ and as 1 on the other summands of (1.7). The group $\mathcal{S}_\phi$ has index at most two in $\mathcal{S}_\phi^+$ and consists of all products of the $z_{\tau,a}$ such that the determinant is 1. Thus every element of $\mathcal{S}_\phi$ involves an even number of $z_{\tau,a}$ with $a \dim \tau$ odd.

A character ϵ of $\mathcal{S}_\phi$ is a G-relevant enhancement of ϕ if and only if ϵ restricted to $\mathrm{Z}({G^\vee}_{\mathrm{der}})^{\mathbf{W}_F}$ encodes G via the Kottwitz isomorphism, i.e. it is quadratic if G is a "prime" form (with notation as above (1.1)) and trivial otherwise. Here the image of $\mathrm{Z}({G^\vee}_{\mathrm{der}})^{\mathbf{W}_F}$ in $\mathcal{S}_\phi$ is generated by

$$\prod\nolimits_{(\tau,a) \in \mathrm{Jord}(\phi):\, a \dim \tau \text{ odd}} z_{\tau,a}, \tag{1.9}$$

which is an element of order at most two. We refine $\Phi(G)$ and $\Phi(G^+)$ to sets of enhanced L-parameters by

$$\begin{array}{lll} \Phi_e(G) & = & \{(\phi,\rho) : \phi \in \Phi(G), \rho \in \mathrm{Irr}(\mathcal{S}_\phi) \text{ is } G\text{-relevant}\}, \\ \Phi_e(G^+) & = & \{(\phi,\rho) : \phi \in \Phi(G), \rho \in \mathrm{Irr}(\mathcal{S}_\phi^+) \text{ is } G\text{-relevant}\}. \end{array} \tag{1.10}$$

We want to make explicit which enhancements of ϕ are cuspidal. Like in (1.7)

$$\begin{aligned} \mathrm{Z}_{{G^\vee}_{\mathrm{der}}}(\phi(\mathbf{W}_F)) = \mathrm{S}\Big(& \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi} \mathrm{Id}_{V_\tau} \otimes \mathrm{O}\big(\bigoplus_{a \text{ odd}: (\tau,a) \in \mathrm{Jord}(\phi)} \mathbb{C}^a \big) \Big) \times \\ & \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^-_\phi} \mathrm{Id}_{V_\tau} \otimes \mathrm{Sp}\big(\bigoplus_{a \text{ even}: (\tau,a) \in \mathrm{Jord}(\phi)} \mathbb{C}^a \big). \end{aligned} \tag{1.11}$$

This brings us to the setting of [Mœ1, MoTa] and [AMS3, §5.3]. In the latter it is checked that (ϕ, ϵ) is cuspidal if and only if the following conditions are met:

- $\mathrm{Jord}(\phi)$ does not have holes, that is, if $(\tau, a) \in \mathrm{Jord}(\phi)$ and $a > 2$, then also $(\tau, a-2) \in \mathrm{Jord}(\phi)$,
- ϵ is alternated, in the sense that for all $(\tau, a), (\tau, a+2), (\tau', 2) \in \mathrm{Jord}(\phi)$:

$$\epsilon_\pi(z_{\tau,a} z_{\tau,a+2}) = -1 \quad \text{and} \quad \epsilon_\pi(z_{\tau',2}) = -1. \tag{1.12}$$

1.2. **Hecke algebras for Langlands parameters.**

We will work out the Hecke algebras associated in [AMS3, §3] to Bernstein components of enhanced L-parameters for G. Although in [AMS3] we used an alternative group $\mathcal{S}_\phi$ coming from the simply connected cover of $G^\vee{}_{\mathrm{der}}$, the constructions work equally well with $\mathcal{S}_\phi$ as above.

For the convenience of the reader, we start by recalling the definition of the precise kind of (extended) affine Hecke algebra that will be involved [AMS3, Proposition 2.2]. We use the following data:

- a root datum $\mathcal{R} = (R, X, R^\vee, Y)$, endowed with a notion of positive and simple roots in R,
- the Weyl group $W(R)$,
- $W(R)$-invariant parameter functions $\lambda : R \to \mathbb{Z}_{\geq 0}$ and $\lambda^* : \{\alpha \in R_{\mathrm{red}} : \alpha^\vee \in 2Y\} \to \mathbb{Z}_{\geq 0}$,
- an invertible indeterminate $\mathbf{z}$,
- (optional) a finite group Γ acting on X and R, stabilizing λ, λ^* and the set of positive roots.

Then $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{z}) \rtimes \Gamma$ is the vector space $\mathbb{C}[X] \otimes \mathbb{C}[W(R)] \otimes \mathbb{C}[\Gamma] \otimes \mathbb{C}[\mathbf{z}, \mathbf{z}^{-1}]$ with the multiplication rules:

- $\mathbb{C}[X], \mathbb{C}[\Gamma]$ and $\mathbb{C}[\mathbf{z}, \mathbf{z}^{-1}]$ are embedded as subalgebras.
- $\mathbb{C}[\mathbf{z}, \mathbf{z}^{-1}]$ is central.
- the $\mathbb{C}[\mathbf{z}, \mathbf{z}^{-1}]$-span of $W(R)$ is the Iwahori–Hecke algebra $\mathcal{H}(W(R), \mathbf{z}^{2\lambda})$ of $W(R)$ with parameters $\mathbf{z}^{2\lambda(\alpha)}$. That is, it has a $\mathbb{C}[\mathbf{z}, \mathbf{z}^{-1}]$-basis $\{N_w : w \in W(R)\}$ such that
$$\begin{array}{rcll} N_w N_v & = & N_{wv} & \text{if } \ell(w) + \ell(v) = \ell(wv), \\ (N_{s_\alpha} + \mathbf{z}_j^{-\lambda(\alpha)})(N_{s_\alpha} - \mathbf{z}_j^{\lambda(\alpha)}) & = & 0 & \text{if } \alpha \in R_{\mathrm{red}} \text{ is a simple root.} \end{array}$$
- For $w \in W(R), x \in X$ and $\gamma \in \Gamma$, corresponding to $N_\gamma \in \mathbb{C}[\Gamma]$:
$$N_\gamma N_w \theta_x N_\gamma^{-1} = N_{\gamma w \gamma^{-1}} \theta_{\gamma(x)},$$
- For a simple root $\alpha \in R_{\mathrm{red}}$ and $x \in X$ corresponding to $\theta_x \in \mathbb{C}[X]$:

$$\theta_x N_{s_\alpha} - N_{s_\alpha} \theta_{s_\alpha(x)} =$$
$$\begin{cases} \big(\mathbf{z}^{\lambda(\alpha)} - \mathbf{z}^{-\lambda(\alpha)}\big)(\theta_x - \theta_{s_\alpha(x)})/(\theta_0 - \theta_{-\alpha}) & \alpha^\vee \notin 2Y \\ \big(\mathbf{z}^{\lambda(\alpha)} - \mathbf{z}^{-\lambda(\alpha)} + \theta_{-\alpha}(\mathbf{z}^{\lambda^*(\alpha)} - \mathbf{z}^{-\lambda^*(\alpha)})\big)(\theta_x - \theta_{s_\alpha(x)})/(\theta_0 - \theta_{-2\alpha}) & \alpha^\vee \in 2Y \end{cases}.$$

We can specialize $\mathbf{z}$ to any $z \in \mathbb{C}^\times$, that gives another extended affine Hecke algebra
$$\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, z) \rtimes \Gamma = \big(\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{z}) \rtimes \Gamma\big)/(\mathbf{z} - z).$$
When Γ is trivial or absent, we have the (non-extended) affine Hecke algebras $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{z})$ and $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, z)$.

We fix a minimal F-Levi subgroup $L_{\min}$ of $G = \mathrm{GSpin}(V)$ as in [AsSh, §2]. We call a Levi subgroup L standard if it contains $L_{\min}$. It is shown in [AsSh, §2] that every such L has the form
$$L = G_{n_-} \times \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_k}(F), \tag{1.13}$$
where $n_- \in \mathbb{N}$, $G_{n_-} = \mathcal{G}_{n_-}(F) = \mathrm{GSpin}(V_-)$ with $\mathrm{disc}(V_-) = \mathrm{disc}(V)$ and
$$\dim(V) - \dim(V_-) = 2(n_1 + \cdots + n_k).$$

The embedding $L \to G$ will be described in (3.3).

The group ${}^L G_{n_-}$ has the same type as ${}^L G$ (but smaller rank) and

$$ {}^L L = {}^L G_{n_-} \times \mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_k}(\mathbb{C}). \tag{1.14}$$

Assume that the (skew-)symmetric matrix $J \in \mathrm{GL}_{2n}(\mathbb{C})$ defining the bilinear form has the following simple shape: the isotropic part is built from matrices $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ placed in rows and columns $j, 2n+1-j$. Then the embedding ${}^L L \to {}^L G$ is given by

$$ (h_-, h_1, \ldots, h_k) \mapsto \big(h_1, \ldots, h_k, h_-, \mu_G^\vee(h_-) J h_k^{-T} J^{-1}, \ldots, \mu_G^\vee(h_-) J h_1^{-T} J^{-1} \big), \tag{1.15}$$

where for an invertible matrix m we denote the inverse-transpose by $m^{-T} = (m^{-1})^T$.

Consider a Langlands parameter $\phi \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L L$. With (1.14) and (1.15) we can write

$$ \phi = \bigoplus\nolimits_j \phi_j \oplus \phi_- \oplus \bigoplus\nolimits_j \phi_j^\vee \otimes \mu_G^\vee \circ \phi \cong \phi_- \oplus \bigoplus\nolimits_j \phi_j \oplus (\phi_j^\vee \otimes \mu_G^\vee \circ \phi), \tag{1.16}$$

where $\phi_- \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G_{n-}$ and $\phi_j \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{n_j}(\mathbb{C})$. Clearly ϕ is discrete if and only if ϕ_- and all the ϕ_j are discrete. Notice that $\mathcal{S}_\phi = \mathcal{S}_{\phi_-}$ because $\mathrm{Z}_{\mathrm{GL}_{n_j}(\mathbb{C})}(\phi_j)$ is connected. An enhancement

$$ \epsilon \in \mathrm{Irr}(\mathcal{S}_\phi) = \mathrm{Irr}(\mathcal{S}_{\phi_-}) $$

is cuspidal if and only (ϕ_-, ϵ) and all the (ϕ_j, triv) are cuspidal. For (ϕ_-, ϵ) cuspidality was analysed after (1.11), while for (ϕ_j, triv) it means that ϕ_j is trivial on $\mathrm{SL}_2(\mathbb{C})$ and ϕ_j is irreducible as representation of $\mathbf{W}_F$ [AMS1, Example 6.11].

Let $\Phi_{\mathrm{cusp}}(L)$ denote the set of $L^\vee$-conjugacy classes of cuspidal enhanced L-parameters for L. The group $Z(L^\vee)^{\mathbf{I}_F}$ acts naturally on $\Phi(L), \Phi_e(L)$ and $\Phi_{\mathrm{cusp}}(L)$ [AMS1, (100)]. From now on we assume that $(\phi, \epsilon) \in \Phi_{\mathrm{cusp}}(L)$. Following [AMS1, §8] this gives a subset

$$ \mathfrak{s}_L^\vee = (Z(L^\vee)^{\mathbf{I}_F, \circ} \cdot \phi, \epsilon) \subset \Phi_{\mathrm{cusp}}(L) $$

and a Bernstein component $\Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee} \subset \Phi_{\mathrm{e}}(G)$. For $\tau \in \mathrm{Irr}(\mathbf{W}_F)$, let ℓ_τ be the multiplicity of τ in ϕ_- (regarded as $\mathbf{W}_F$-representation via the standard embedding ${}^L G_{n_-} \to \mathrm{GL}_{2n_-}$) and let e_τ be the sum of the multiplicities of τ in the $\mathrm{GL}_{n_j}(\mathbb{C})$. Then (1.16) and (1.7) become

$$ \begin{aligned} \phi &= \phi_- \oplus \bigoplus_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^\pm_\phi} 2 e_\tau \tau \oplus \bigoplus_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^0_\phi} e_\tau (\tau \oplus \tau^\vee \otimes \mu_G^\vee \circ \phi), \\ \phi|_{\mathbf{W}_F} &= \bigoplus_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^\pm_\phi} (2 e_\tau + \ell_\tau) \tau \oplus \bigoplus_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^0_\phi} e_\tau (\tau \oplus \tau^\vee \otimes \mu_G^\vee \circ \phi) \end{aligned} $$

From (1.5) we deduce

$$ \begin{aligned} \mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi(\mathbf{W}_F)) = \mathrm{S}\Big(& \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi} \mathrm{O}_{2e_\tau + \ell_\tau}(\mathbb{C}) \otimes \mathrm{Id}_{V_\tau} \Big) \times \\ & \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^-_\phi} \mathrm{Sp}_{2e_\tau + \ell_\tau}(\mathbb{C}) \otimes \mathrm{Id}_{V_\pi} \times \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^0_\phi} \mathrm{GL}_{e_\tau}(\mathbb{C}) \otimes \mathrm{Id}_{V_\tau \oplus V_\tau^\vee}. \end{aligned} \tag{1.17}$$

Relevant for the determination of Hecke algebras are furthermore

$$\begin{aligned}
&\mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi(\mathbf{W}_F)) \cap {}^L L = \mathrm{S}\Big(\prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi} \mathrm{O}_{\ell_\tau}(\mathbb{C}) \times (\mathbb{C}^\times)^{e_\tau} \otimes \mathrm{Id}_{V_\tau} \Big) \times \\
&\qquad \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^-_\phi} \mathrm{Sp}_{\ell_\tau}(\mathbb{C}) \times (\mathbb{C}^\times)^{e_\tau} \otimes \mathrm{Id}_{V_\pi} \times \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^0_\phi} (\mathbb{C}^\times)^{e_\tau} \otimes \mathrm{Id}_{V_\tau \oplus V_\tau^\vee}, \\
&G^\vee_\phi = \mathrm{Z}_{G^\vee}(\phi(\mathbf{W}_F)) = \mathbb{C}^\times \mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi(\mathbf{W}_F)), \\
&M = G^\vee_\phi \cap {}^L L = \mathbb{C}^\times \big(\mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi(\mathbf{W}_F)) \cap {}^L L \big), \\
&T = \mathrm{Z}(M)^\circ \cong \mathbb{C}^\times \Big(\prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^\pm_\phi} (\mathbb{C}^\times)^{e_\tau} \times \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^0_\phi} (\mathbb{C}^\times)^{e_\tau} \Big) = \mathrm{Z}({}^L L).
\end{aligned}$$

If $G^\vee = \mathrm{GSO}_{2n}(\mathbb{C})$, we may extend it to $G^{\vee +} := \mathrm{GO}_{2n}(\mathbb{C})$. That means omitting the S from (1.17), which makes the group (at most) a factor 2 bigger, so that it decomposes naturally as a product over the involved τ's:

$$\mathrm{Z}_{G^{\vee +}_{\mathrm{der}}}(\phi(\mathbf{W}_F)) = \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^\pm_\phi \cup \mathrm{Irr}(\mathbf{W}_F)^0_\phi} G^\vee_{\phi,\tau}. \tag{1.18}$$

Then the root system $R(G^\vee_\phi, T)$ decomposes canonically as a disjoint union of the root systems

$$R_\tau := R(G^\vee_{\phi,\tau} T, T) = R(G^\vee_{\phi,\tau}, T \cap G^\vee_{\phi,\tau}).$$

In [AMS3, §1] a graded Hecke algebra is attached to the data $(G^\vee_\phi, M, u_\phi, \epsilon)$. The maximal commutative subalgebra is $\mathcal{O}(\mathrm{Lie}(T))$, the root system is R_τ and the parameters of the roots come from [Lus2]. The root system and the parameter functions $c : R_\tau \to \mathbb{Z}_{\geq 0}$ (which are used to construct graded Hecke algebras) were worked out in [AMS3, §5.3]. To write down the parameters uniformly, we define

$$a_\tau = \begin{cases} \max \mathrm{Jord}_\tau(\phi_-) & \mathrm{Jord}_\tau(\phi_-) \neq \emptyset \\ 0 & \mathrm{Jord}_\tau(\phi_-) = \emptyset, \tau \in \mathrm{Irr}(\mathbf{W}_F)^-_\phi \\ -1 & \mathrm{Jord}_\tau(\phi_-) = \emptyset, \tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi \end{cases}. \tag{1.19}$$

Notice that now a_τ is odd for $\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi$ and even for $\tau \in \mathrm{Irr}(\mathbf{W}_F)^-_\phi$. For $e_\tau = 0$ the torus $T \cap G^\vee_{\phi,\tau}$ reduces to 1, and there are no roots. Otherwise we denote a root of length $\sqrt{2}$ by α and a root of length 1 by β. Now the root systems and the parameter can be expressed as in Table 1. When $e_\tau = 1$, we must regard D_{e_τ} and $A_{e_\tau - 1}$ as the empty root system. Although β is not a root in C_n or D_n, [AMS3, §3.2] still allows us to attach a useful parameter $c(\beta)$.

TABLE 1. Root systems and graded Hecke algebra parameters for τ

$\tau \in$	ℓ_τ	R_τ	$c(\alpha)$	$c(\beta)$
$\mathrm{Irr}(\mathbf{W}_F)^+_\phi$	$= 0$	D_{e_τ}	2	$0 = 1 + a_\tau$
$\mathrm{Irr}(\mathbf{W}_F)^+_\phi$	> 0	B_{e_τ}	2	$1 + a_\tau$
$\mathrm{Irr}(\mathbf{W}_F)^-_\phi$	$= 0$	C_{e_τ}	2	$c(2\beta) = 2, c(\beta) = 1 = 1 + a_\tau$
$\mathrm{Irr}(\mathbf{W}_F)^-_\phi$	> 0	BC_{e_τ}	2	$1 + a_\tau$
$\mathrm{Irr}(\mathbf{W}_F)^0_\phi$	$= 0$	$A_{e_\tau - 1}$	2	$--$

Recall that the cuspidal supports in the Bernstein component $\Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee}$ are precisely the twists of (ϕ, ϵ) by elements of

$$X_{\mathrm{nr}}({}^L L) := \mathrm{Z}(L^\vee \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F} \cong X_{\mathrm{nr}}(L).$$

Here $\mathbf{W}_F$ acts trivially on the type GL factors of ${}^L L$ and on $Z(G_{n_-}) \cong \mathbb{C}^\times$, so

$$X_{\mathrm{nr}}({}^L L) \cong \mathrm{Z}(G_{n_-}) \times \prod\nolimits_j \mathbb{C}^\times \mathrm{Id} = \mathrm{Z}({}^L L) = T. \tag{1.20}$$

Without changing $\Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee}$, we can bring (ϕ, ϵ) in a somewhat better position:

- if $\phi_j : \mathbf{W}_F \to \mathrm{GL}_{n_j}(\mathbb{C})$ differs from $\phi_j^\vee \otimes \mu_G^\vee \circ \phi$ by $z \in \mathrm{Z}(\mathrm{GL}_{n_j}(\mathbb{C})) \cong X_{\mathrm{nr}}(\mathrm{GL}_{n_j}(F))$, then we replace ϕ_j by $z^{1/2}\phi_j$, so that
$$z^{1/2}\phi_j \cong z^{-1/2}\phi_j^\vee \otimes \mu_G^\vee \circ \phi \cong (z^{1/2}\phi_j)^\vee \otimes \mu_G^\vee \circ \phi.$$
- if $n_i = n_j$ and ϕ_i, ϕ_j differ by an element of $X_{\mathrm{nr}}(\mathrm{GL}_{n_i}(F))$, then we adjust one of them so that actually $\phi_i = \phi_j$,
- if $\phi_i, \phi_j \in \mathrm{Irr}(\mathbf{W}_F)^0_\phi$ and $\phi_i, \phi_j^\vee \otimes \mu_G^\vee \circ \phi$ differ by an element of $\mathrm{GL}_{n_i}(\mathbb{C})$, then we replace ϕ_j by ϕ_i.

Let $\tau' \in \mathrm{Irr}(\mathbf{W}_F)^\pm_\phi$ be a twist of $\tau \in \mathrm{Irr}(\mathbf{W}_F)^\pm_\phi$ by an unramified character, such that τ' is equivalent with ${\tau'}^\vee \otimes \mu_G^\vee \circ \phi$ but not with τ. By the above assumptions on ϕ, $e_{\tau'} = 0$ if $e_\tau > 0$. Still, ℓ_τ and $\ell_{\tau'}$ can be nonzero simultaneously. If $e_\tau > 0$ and $\ell_\tau < \ell_{\tau'}$ (resp. $\ell_\tau = \ell_{\tau'} = 0$ and $a_\tau = 0 < a_{\tau'}$) then we change $\phi_j = \tau$ to $\phi_j = \tau'$, so that the roles of ℓ_τ and $\ell_{\tau'}$ (resp. of a_τ and $a_{\tau'}$) are exchanged.

Let $\mathrm{Z}(L^\vee)^\circ_\phi \subset \mathrm{Z}(L^\vee)^\circ = T$ be the subgroup of elements z such that $z\phi$ is equivalent with ϕ in $\Phi_{\mathrm{e}}(L)^{\mathfrak{s}^\vee}$. It is finite and the map

$$\mathrm{Z}(L^\vee)^\circ / \mathrm{Z}(L^\vee)^\circ_\phi \to \Phi_{\mathrm{e}}(L)^{\mathfrak{s}^\vee} : z \mapsto (z\phi, \epsilon)$$

is a bijection. The affine Hecke algebra we are constructing has the underlying complex torus

$$T_{\mathfrak{s}^\vee} := \mathrm{Z}(L^\vee)^\circ / \mathrm{Z}(L^\vee)^\circ_\phi = T \big/ \prod\nolimits_j Z(\mathrm{GL}_{n_j}(\mathbb{C}))_{\phi_j}. \tag{1.21}$$

Let t_τ be the torsion number of $\tau \in \mathrm{Irr}(\mathbf{W}_F)$, that is, the order of the group $\mathrm{Z}(\mathrm{GL}_{d_\tau}(\mathbb{C}))_\tau$. Then t_τ is also the number of irreducible constituents θ of $\mathrm{Res}^{\mathbf{W}_F}_{\mathbf{I}_F} \tau$. We need to distinguish two cases:

(i) $\theta \cong \theta^\vee \otimes \mu_G^\vee \circ \phi$. Then the same goes for all constituents of $\mathrm{Res}^{\mathbf{W}_F}_{\mathbf{I}_F} \tau$, because all those are in one orbit for $\mathbf{W}_F$. The proof of [Sol5, Proposition 4.10.a] (which concerns self-dual representations of $\mathbf{W}_F$) applies and shows that τ and τ' have the same sign.

(ii) $\theta \not\cong \theta^\vee \otimes \mu_G^\vee \circ \phi$ for all eligible θ. Again the proof of [Sol5, Proposition 4.10.a] applies, now it shows that τ and τ' have different signs.

According to these two cases, we divise a new partition of $\mathrm{Irr}(\mathbf{W}_F)^\pm_\phi$:

- $\mathrm{Irr}(\mathbf{W}_F)^{++}$ is the set of all $\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi$ in case (i) above, modulo the relation $\tau \sim \tau'$;
- $\mathrm{Irr}(\mathbf{W}_F)^{--}$ is defined in the same way, only starting from $\mathrm{Irr}(\mathbf{W}_F)^-_\phi$;
- $\mathrm{Irr}(\mathbf{W}_F)^{+-}$ is the set of all $\tau \in \mathrm{Irr}(\mathbf{W}_F)^\pm_\phi$ in case (ii) above, modulo the relation $\tau \sim \tau'$;
- $\mathrm{Irr}'(\mathbf{W}_F)^\pm_\phi = \mathrm{Irr}(\mathbf{W}_F)^{++} \cup \mathrm{Irr}(\mathbf{W}_F)^{--} \cup \mathrm{Irr}(\mathbf{W}_F)^{+-} = \mathrm{Irr}(\mathbf{W}_F)^\pm_\phi / (\tau \sim \tau')$.

A computation like for (1.17) yields

$$
\begin{aligned}
\mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi(\mathbf{I}_F)) = \mathrm{S}\Big(& \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^{++}_\phi} \mathrm{O}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau} \Big) \times \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^{--}_\phi} \mathrm{Sp}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau} \\
& \times \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^{+-}_\phi} \mathrm{GL}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau/2} \times \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^{0}_\phi} \mathrm{GL}_{e_\tau}(\mathbb{C})^{t_\tau}.
\end{aligned} \tag{1.22}
$$

Analogous to (1.18) we decompose

$$
\begin{aligned}
& \mathrm{Z}_{G^\vee{}^+_{\mathrm{der}}}(\phi(\mathbf{I}_F)) = \prod\nolimits_\tau G^\vee_{\phi(\mathbf{I}_F),\tau}, \\
& T_{\mathfrak{s}^\vee} = \mathbb{C}^\times \underset{\{1,-1\}}{\times} \prod\nolimits_\tau T_{\mathfrak{s}^\vee,\tau} = \mathbb{C}^\times \underset{\{1,-1\}}{\times} \prod\nolimits_\tau \big(\mathbb{C}^\times / \mathrm{Z}(\mathrm{GL}_{d_\tau}(\mathbb{C}))_\tau\big)^{e_\tau}, \\
& X^*(T_{\mathfrak{s}^\vee}) \subset \mathbb{Z} \oplus \bigoplus\nolimits_\tau X^*(T_{\mathfrak{s}^\vee,\tau}) \cong \mathbb{Z} \oplus \bigoplus\nolimits_\tau \mathbb{Z}^{e_\tau}.
\end{aligned} \tag{1.23}
$$

In each of the above cases, the product or sum runs over $\mathrm{Irr}'(\mathbf{W}_F)^\pm_\phi \cup \mathrm{Irr}(\mathbf{W}_F)^0_\phi$. For comparison with [AMS3] we record the group

$$
J = \mathrm{Z}_{G^\vee}(\phi(\mathbf{I}_F)) = \mathbb{C}^\times \mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi(\mathbf{I}_F)). \tag{1.24}
$$

The root system of J° with respect to the (possibly non-maximal) torus T splits naturally as a disjoint union of root systems $R(G^\vee_{\phi(\mathbf{I}_F),\tau} T, T)$, indexed as in (1.23). We note that by the above assumptions on τ and τ' we have $\ell_\tau \geq \ell_{\tau'} = 0$ and if $\ell_\tau = 0$, then $\ell_{\tau'} = 0$ and $a_\tau \geq a_{\tau'}$. With Table 1 at hand, one checks readily that

$$
R(G^\vee_{\phi(\mathbf{I}_F),\tau} T, T) = R(G^\vee_{\phi,\tau} T, T) \tag{1.25}
$$

in all cases. (Only the dimensions of the root subspaces for $G^\vee_{\phi(\mathbf{I}_F),\tau}$ are higher than for $G^\vee_{\phi,\tau}$, namely t_τ times higher in all but ones cases.) In view of [AMS3, Proposition 3.9], this means that (ϕ, ϵ) is a good basepoint of $\Phi_{\mathrm{e}}(L)^{\mathfrak{s}^\vee}$.

Now the roots for the Hecke algebra that we are after can be found with [AMS3, Definition 3.11]. The reduced roots $\alpha \in R(G^\vee_{\phi(\mathbf{I}_F),\tau} T, T)$ need to be scaled by a certain factor $m_\alpha \in \mathbb{N}$, which we compute next. Let $B_J \supset T_J$ be a $\phi(\mathrm{Frob}_F)$-stable Borel subgroup and maximal torus of J°, such that $T_J^{\phi(\mathrm{Frob}_F)} \supset T$. A natural choice for T_J comes from the standard maximal tori $T_{J,\tau}$ in $G^\vee_{\phi(\mathbf{I}_F),\tau}$:

$$
T_J = \mathbb{C}^\times \Big(\prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^{++}_\phi \cup \mathrm{Irr}(\mathbf{W}_F)^{--}_\phi} T_{J,\tau}^{t_\tau} \times \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^{+-}_\phi} T_{J,\tau}^{t_\tau/2} \times \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^{0}_\phi} T_{J,\tau}^{t_\tau} \Big). \tag{1.26}
$$

We see that $R(G^\vee_{\phi(\mathbf{I}_F),\tau}, T_J)$ has t_τ irreducible components, unless $\tau \in \mathrm{Irr}(\mathbf{W}_F)^{+-}_\phi$, then there are $t_\tau/2$. Following [AMS3, Definition 3.11], m_α equals t_τ (or $t_\tau/2$ for $\tau \in \mathrm{Irr}(\mathbf{W}_F)^{+-}_\phi$) times a number m'_α which is m_α for $\mathrm{Res}^{\mathbf{W}_F}_{\mathbf{W}_E} \phi$ where E/F is the unramified extension of degree t_τ (or $t_\tau/2$ for $\tau \in \mathrm{Irr}(\mathbf{W}_F)^{+-}_\phi$). By definition m'_α is the smallest number such that $\ker(m'_\alpha \alpha)$ contains all $t \in T$ for which $t \mathrm{Res}^{\mathbf{W}_F}_{\mathbf{W}_E} \phi$ is equivalent with $\mathrm{Res}^{\mathbf{W}_F}_{\mathbf{W}_E} \phi$.

The group of $t \in T$ with $t\phi \cong \phi$ factors as a product indexed by all possible τ, and the contribution from one τ consists of t_τ unramified characters of $\mathbf{W}_F$. But this group of unramified characters becomes trivial if we pass from $\mathbf{W}_F$ to the Weil group of the degree t_τ unramified extension of F. Hence $m'_\alpha = 1$, unless maybe when $\tau \in \mathrm{Irr}(\mathbf{W}_F)^{+-}_\phi$. In the latter situation we usually have $m'_\alpha = 2$, because

$\ker(m'_\alpha \alpha)$ has to contain an element $t \in (1 - \phi(\mathrm{Frob}_F)^{t_\tau/2}) T_{J,\tau}$ with $\alpha(t) = -1$. The only exception occurs when $\ell_\tau = 0$ and $\alpha \in C_{e_\tau}$ is long, then $m'_\alpha = 1$. We conclude that $m_\alpha = t_\tau$ in all cases, except when $\tau \in \mathrm{Irr}(\mathbf{W}_F)^{+-}_\phi, \ell_\tau = 0$ and $\alpha \in C_{e_\tau}$ is long, then $m_\alpha = t_\tau / 2$.

Finally, we are ready to define the root datum for our affine Hecke algebra:

$$\begin{aligned} &\mathcal{R}_{\mathfrak{s}^\vee} = \big(R_{\mathfrak{s}^\vee}, X^*(T_{\mathfrak{s}^\vee}), R_{\mathfrak{s}^\vee}^\vee, X_*(T_{\mathfrak{s}^\vee}) \big), \\ &\text{where } R_{\mathfrak{s}^\vee} = \{ m_\alpha \alpha : \alpha \in R(G^\vee_\phi, T)_{\mathrm{red}} \}. \end{aligned} \tag{1.27}$$

Here $R_{\mathfrak{s}^\vee}$ is the disjoint union of root subsystems

$$R_{\mathfrak{s}^\vee, \tau} = \{ m_\alpha \alpha : \alpha \in R(G^\vee_{\phi,\tau} T, T)_{\mathrm{red}} \}.$$

Notice that $X^*(T_{\mathfrak{s}^\vee,\tau})$ arises from the part of $X^*(T)$ associated to τ by multiplication with t_τ, where $t_\tau = m_\alpha$ for most $\alpha \in R(G^\vee_{\phi,\tau} T, T)$. The multiplication rules in our affine Hecke algebra are determined by parameter functions $\lambda, \lambda^* \colon R_{\mathfrak{s}^\vee} \to \mathbb{Z}_{\geq 0}$, which come from [AMS3, Proposition 3.14]. The outcome of those constructions is summarized in [AMS3, §5.3]:

- For $\alpha \in R_{\tau,\mathrm{red}}$ a short root in a type B root system, $t_\tau = m_\alpha$, $c(\alpha) = a_\tau + 1, c^*(\alpha) = a_{\tau'} + 1$ and
$$\lambda(\alpha) = t_\tau (a_\tau + a_{\tau'} + 2)/2, \qquad \lambda^*(\alpha) = t_\tau (a_\tau - a_{\tau'})/2.$$
We note that $\lambda^*(\alpha) \geq 0$ because $\ell_\tau \geq \ell_{\tau'}$.
- For $\alpha \in R_{\tau,\mathrm{red}}, \tau \in \mathrm{Irr}(\mathbf{W}_F)^{+-}_\phi$, $\ell_\tau = 0$, α a long root of a type C root system: $c(\alpha) = 2$ and
$$\lambda(\alpha) = \lambda^*(\alpha) = m_\alpha = t_\tau / 2.$$
- For all other $\alpha \in R_{\tau,\mathrm{red}}$: $c(\alpha) = 2$ and
$$\lambda(\alpha) = \lambda^*(\alpha) = m_\alpha = t_\tau.$$

We note that the operation $\alpha \mapsto m_\alpha \alpha$ preserves the type of the root systems $R_{\tau,\mathrm{red}}$ from Table 1, except that in the case $\tau \in \mathrm{Irr}(\mathbf{W}_F)^{+-}_\phi, \ell_\tau + \ell_{\tau'} = 0$ type C_{e_τ} is turned into B_{e_τ}.

We also need to determine $W_{\mathfrak{s}^\vee}$, the stabilizer of $\mathfrak{s}_L^\vee$ in

$$N_{G^\vee}(L^\vee \rtimes \mathbf{W}_F)/L^\vee = N_{G^\vee}(L^\vee)/L^\vee. \tag{1.28}$$

Recall the embedding ${}^L L \to {}^L G$ from (1.15). For each j the group $\mathrm{N}_{G^{\vee +}}(L^\vee)$ possesses an element that exchanges h_j and $\mu_G^\vee(h_-) J h_j^{-T} J^{-1}$. In terms of representations of $\mathbf{W}_F$ (via ϕ), this

$$\text{exchanges } \tau \text{ and } \tau^\vee \otimes \mu_G^\vee \circ \phi. \tag{1.29}$$

Further $\mathrm{N}_{G^{\vee +}}(L^\vee)$ contains elements that permute the factors $\mathrm{GL}_{n_j}(\mathbb{C})$ of the same size. It follows that $\mathrm{N}_{G^{\vee +}}(L^\vee)/L^\vee$ is isomorphic with a direct product of Weyl groups of type B_{e_N}, where e_N counts the number of j's with $n_j = N$. The group (1.28) has index at most two in $\mathrm{N}_{G^{\vee +}}(L^\vee)/L^\vee$, which comes from the difference between $\mathrm{GO}_{2n}(\mathbb{C})$ and $\mathrm{GSO}_{2n}(\mathbb{C})$.

The group $W_{\mathfrak{s}^\vee}$ can be represented with elements that normalize M and T and centralize $\phi(\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C}))$, so in particular elements of J. Further $W_{\mathfrak{s}^\vee}$ contains $W(R_{\mathfrak{s}^\vee}) = W(J^\circ, T)$ as a normal subgroup. Fix a standard Borel subgroup $B^\vee$ of $G^\vee$. That determines a Borel subgroup B^J of J°, and hence a system of positive roots in $R(J^\circ, T)$ and in $R_{\mathfrak{s}^\vee}$. Let $\Gamma_{\mathfrak{s}^\vee}$ be the subgroup of $W_{\mathfrak{s}^\vee}$ that stabilizes this positive system of roots. By standard results about finite root system and Weyl groups

$$W_{\mathfrak{s}^\vee} = W(R_{\mathfrak{s}^\vee}) \rtimes \Gamma_{\mathfrak{s}^\vee}. \tag{1.30}$$

Let us determine $\Gamma_{\mathfrak{s}^\vee}$ in terms of the action of $N_{G^\vee}(L^\vee)/L^\vee$ on the type GL factors of $L^\vee$ and on the tensor factors of ϕ (as described above). The $\tau \in \mathrm{Irr}(\mathbf{W}_F)$ with $e_\tau = 0$ do not contribute. If $e_\tau > 0$, then $e_{\tau\otimes\chi} = 0$ for every unramified twist $\tau\otimes\chi$ which is not isomorphic to τ by our normalization of ϕ. Hence every element of $\mathrm{N}_{G^{\vee+}}(L^\vee)$ that stabilizes $\mathfrak{s}_L^\vee$ must already stabilize ϕ. In other words, $W_{\mathfrak{s}^\vee}$ equals the stabilizer of ϕ in $\mathrm{N}_{G^\vee}(L^\vee)/L^\vee$. Thus we can represent $W_{\mathfrak{s}^\vee}$ with elements of $\mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi)$ that normalize T. Let $W^+_{\mathfrak{s}^\vee}$ and $\Gamma^+_{\mathfrak{s}^\vee}$ be the versions of $W_{\mathfrak{s}^\vee}$ and $\Gamma_{\mathfrak{s}^\vee}$ for $G^{\vee+}$. From (1.17) we see that

$$W^+_{\mathfrak{s}^\vee} = \prod_\tau W^+_{\mathfrak{s}^\vee,\tau} \cong \prod_{\tau\in\mathrm{Irr}'(\mathbf{W}_F)^\pm_\phi} W(B_{e_\tau}) \times \prod_{\tau\in\mathrm{Irr}(\mathbf{W}_F)^\circ_\phi} W(A_{e_\tau-1}). \tag{1.31}$$

Comparing with Table 1, we find that

$$\Gamma^+_{\mathfrak{s}^\vee} \cong \prod_{\tau\in\mathrm{Irr}'(\mathbf{W}_F)^+_\phi:\ell_\tau=0} W(B_{e_\tau})/W(D_{e_\tau}) \cong \prod_{\tau\in\mathrm{Irr}'(\mathbf{W}_F)^+_\phi:\ell_\tau=0} \pi_0\big(\mathrm{O}_{2e_\tau}(\mathbb{C})\otimes \mathrm{Id}_{V_\tau}\big). \tag{1.32}$$

In (1.32) every τ contributes a factor

$$\Gamma^+_{\mathfrak{s}^\vee,\tau} = \langle r_\tau\rangle \cong \mathbb{Z}/2\mathbb{Z}$$

to $\Gamma^+_{\mathfrak{s}^\vee}$. For τ not appearing in (1.32), we may put $\Gamma^+_{\mathfrak{s}^\vee,\tau} = 1$.

When $\dim\tau$ is even, $\det(r_\tau) = 1$ and when $\dim\tau$ is odd, $\det(r_\tau) = -1$. Hence the S in (1.17) does not put any condition on the r_τ with $\dim\tau$ even. If there exists a $\tau\in\mathrm{Irr}(\mathbf{W}_F)^+_\phi$ with $\ell_\tau > 0$, then we can use $\mathrm{O}_{\ell_\tau}(\mathbb{C})$ to make the determinant of a product of r_τ's equal to 1. From [Mou, §4.1] we know that this leaves just two possibilities for $\Gamma_{\mathfrak{s}^\vee}$:

- $\Gamma_{\mathfrak{s}^\vee} = \prod_{\tau\in\mathrm{Irr}'(\mathbf{W}_F)^+_\phi:\ell_\tau=0}\langle r_\tau\rangle$,
- if $\mathcal{G}$ is a form of GSpin_{2n} and $\mathcal{L}\cong\prod_j \mathrm{GL}_{n_j}$ with $n_j\in\mathbb{Z}_{>0}$, then

$$\Gamma_{\mathfrak{s}^\vee} = \prod_{\tau\in\mathrm{Irr}'(\mathbf{W}_F)^+_\phi:\ell_\tau=0,\dim\tau\text{ even}} \langle r_\tau\rangle \times \mathrm{S}\Big(\prod_{\tau\in\mathrm{Irr}'(\mathbf{W}_F)^+_\phi:\ell_\tau=0,\dim\tau\text{ odd}}\langle r_\tau\rangle\Big),$$

  where S denotes the subgroup of elements with determinant 1.

Conceivably our affine Hecke algebra could contain the span of $\Gamma_{\mathfrak{s}^\vee}$ as a twisted group algebra. But here the 2-cocycle of $W_{\mathfrak{s}^\vee}$ involved in the Hecke algebra can be computed for each τ separately, and $\langle r_\tau\rangle\cong\mathbb{Z}/2\mathbb{Z}$ only has trivial 2-cocycles.

Let us summarise our findings. From (1.23) we know that $X^*(T_{\mathfrak{s}^\vee})$ has index two in $\mathbb{Z}\oplus\bigoplus_\tau\mathbb{Z}^{e_\tau}$, where $\mathbb{Z}^{e_\tau}\cong X^*(T_{\mathfrak{s}^\vee,\tau})$. If we replace $X^*(T_{\mathfrak{s}^\vee})$ and $X_*(T_{\mathfrak{s}^\vee})$ in $\mathcal{R}_{\mathfrak{s}^\vee}$ by $\oplus_\tau\mathbb{Z}^{e_\tau}$, we get a new root datum $\mathcal{R}_{\mathfrak{s}^\vee,\mathrm{der}}$ that decomposes naturally. More precisely, the root datum $\mathcal{R}_{\mathfrak{s}^\vee,\mathrm{der}}$, extended with the finite group $\Gamma^+_{\mathfrak{s}^\vee}$ acting on it, is a direct sum of such extended root data, where the product is indexed by $\tau\in\mathrm{Irr}'(\mathbf{W}_F)^\pm_\phi\cup\mathrm{Irr}(\mathbf{W}_F)^0_\phi$. For each such τ the data are (with α a root of length $\sqrt{2}$ and β a root of another length) are collected in Table 2. Recall from (1.19) that a_τ is odd for $\tau\in\mathrm{Irr}(\mathbf{W}_F)^+_\phi$ and even for $\tau\in\mathrm{Irr}(\mathbf{W}_F)^-_\phi$.

TABLE 2. Data from $\mathcal{R}_{\mathfrak{s}^\vee}$ for each τ

a_τ	$a_{\tau'}$	$X^*(T_{\mathfrak{s}^\vee,\tau})$	$R_{\mathfrak{s}^\vee,\tau}$	$\lambda(\alpha)$	$\lambda(\beta)$	$\lambda^*(\beta)$	$\Gamma^+_{\mathfrak{s}^\vee}$
-1	-1	$\mathbb{Z}^{e_\tau}$	D_{e_τ}	t_τ	$--$	$--$	$\mathrm{Out}(D_{e_\tau})$
0	-1	$\mathbb{Z}^{e_\tau}$	B_{e_τ}	t_τ	$t_\tau/2$	$t_\tau/2$	1
0	0	$\mathbb{Z}^{e_\tau}$	C_{e_τ}	t_τ	t_τ	t_τ	1
≥ 1	-1	$\mathbb{Z}^{e_\tau}$	B_{e_τ}	t_τ	$t_\tau(a_\tau+1)/2$	$t_\tau(a_\tau+1)/2$	1
≥ 1	≥ 0	$\mathbb{Z}^{e_\tau}$	B_{e_τ}	t_τ	$t_\tau(a_\tau+a_{\tau'}+2)/2$	$t_\tau(a_\tau-a_{\tau'})/2$	1
$\mathrm{Irr}(\mathbf{W}_F)^0_\phi$		$\mathbb{Z}^{e_\tau}$	$A_{e_\tau-1}$	t_τ	$--$	$--$	1

Here the second, third and fourth lines can be regarded as special cases of the fifth line. We write them down nevertheless, because they arise from different lines in Table 1. With Table 2 and (1.23) we can finally make the affine Hecke algebra associated in [AMS3] to $\mathfrak{s}^\vee$ (and a parameter $z \in \mathbb{C}^\times$) explicit:

$$\mathcal{H}(\mathfrak{s}^\vee, z) = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, z) \rtimes \Gamma_{\mathfrak{s}^\vee}, \tag{1.33}$$

where $\Gamma_{\mathfrak{s}^\vee}$ acts on $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, z)$ via automorphisms of $\mathcal{R}_{\mathfrak{s}^\vee}$.

2. Mœglin's classification of discrete series representations

Arthur famously proved the local Langlands correspondence for symplectic and quasi-split (special) orthogonal groups over p-adic fields [Art]. An analogue for quasi-split unitary groups was announced in [Mok] and proven (for all unitary groups) in [KMSW]. As explained in [Mœ4, MoRe], Arthur's endoscopic methods can also be applied to (special) orthogonal groups and general spin groups that are not necessarily quasi-split. In principle that should yield local Langlands correspondences for all classical groups over p-adic fields. However, not all arguments have been worked out in detail. For classical groups over local function fields far less is known, the notable exception being [GaVa]. We address that in Paragraph 2.1. Here we make Mœglin 's parametrization of discrete series representations [Mœ1, MoTa, Mœ3, Mœ4] more explicit.

Let $G = G_n = \mathcal{G}(F)$ be a symplectic group, a special orthogonal group or a general spin group. When G is an even special orthogonal group or an even general spin group, we denote by G^+ the associated orthogonal or general pin group, as in (1.1). In the other cases G^+ means just G. Let $\mathrm{Z}(G)_s$ be the maximal F-split central torus in G. It is isomorphic to $F^\times$ for general spin groups and trivial in the other cases.

We say that an irreducible smooth G-representation belongs to the discrete series if it is square-integrable modulo centre. More explicitly, that means that π has a unitary central character and its restriction to the derived group of G is square-integrable.

The group $\mathrm{GL}_m(F) \times G_n$ is a Levi subgroup of a group G_{n+m} of the same kind as G_n but of rank m higher. There is a parabolic induction functor

$$\times : \mathrm{Rep}(\mathrm{GL}_m(F)) \times \mathrm{Rep}(G_n) \to \mathrm{Rep}(G_{n+m}),$$

which up to semisimplification does not depend on the choice of a parabolic subgroup of G_{n+m} with Levi factor $\mathrm{GL}_m(F) \times G_n$. Similarly there is a parabolic induction functor

$$\times : \mathrm{Rep}(\mathrm{GL}_m(F)) \times \mathrm{Rep}(G_n^+) \to \mathrm{Rep}(G_{n+m}^+).$$

Let $\rho \in \mathrm{Irr}(\mathrm{GL}_{d_\rho}(F))$ be unitary and supercuspidal, for some d_ρ. For an integer $a \geq 1$ we can form the generalized Steinberg representation $\delta(\rho, a) \in \mathrm{Irr}(\mathrm{GL}_m(F))$ with $m = d_\rho a$. Take π in the discrete series of G_n^+ and let ν_π be the character by which $\mathrm{Z}(G)_s$ acts on π. One says that (ρ, a) lies in the Jordan block of π if $\delta(\rho, a) \times \pi$ is irreducible but there exists $a' \in a + 2\mathbb{Z}$ such that $\delta(\rho, a') \times \pi$ is reducible. We denote the set of all such pairs (ρ, a) by $\mathrm{Jord}(\pi)$. That reducibility is only possible if the nontrivial element

$$s_\alpha \in N_{G^+_{n+d_\rho a}}\big(\mathrm{GL}_{d_\rho a}(F) \times G_n^+\big) / \big(\mathrm{GL}_{d_\rho a}(F) \times G_n^+\big),$$

see (3.4), stabilizes $\delta(\rho, a') \boxtimes \pi$ up to an unramified character. That in turn implies that the version of s_α with $a = 1$ stabilizes ρ, or more explicitly

$$\rho \cong \rho^\vee \otimes \nu_\pi. \tag{2.1}$$

To $\mathrm{Jord}(\pi)$ one associates a finite group S_π, the $\mathbb{F}_2$-vector space with basis $\{z_{\rho,a} : (\rho, a) \in \mathrm{Jord}(\pi)\}$. Via endoscopy, π determines a character $\epsilon_\pi : S_\pi \to \{1, -1\}$, see [Art, Theorem 2.2.1] and [MoRe, §2.1]. This requires a Whittaker datum for the quasi-split inner form of G, which we will use as input. Alternatively, ϵ_π can be defined almost entirely using parabolic induction, see [Mœ1, p. 147–148] or Paragraphs 4.1–4.3.

Theorem 2.1. [Mœglin]
Let F be a p-adic field and consider π in the discrete series of G^+.

(a) $\mathrm{Jord}(\pi)$ *has image in* ${}^L G$*, by which we mean that the Langlands parameter of*

$$\boxtimes_{(\rho,a)\in \mathrm{Jord}(\pi)} \delta(\rho, a) \in \mathrm{Irr}\big(\prod\nolimits_{(\rho,a)\in\mathrm{Jord}(\pi)} \mathrm{GL}_{d_\rho a}(F)\big)$$

factors through ${}^L G$.

(b) $\mathrm{Jord}(\pi)$ *determines precisely the L-packet containing π.*

(c) *When $G \neq G^+$, the restriction of π to G is reducible if and only if $d_\rho a$ is even for all* $(\rho, a) \in \mathrm{Jord}(\pi)$.

(d) *Fix a Whittaker datum for the quasi-split inner form of G. That and the above determine an injection from the discrete series of G^+ to the set of pairs* $(\mathrm{Jord}, \epsilon)$ *(up to $G^{\vee +}$-conjugacy) for which* Jord *has image in* ${}^L G$ *and ϵ is G-relevant, as explained around* (1.9).

(e) *π is supercuspidal if and only if* $\mathrm{Jord}(\pi)$ *does not have holes and ϵ_π is alternated in the sense of* (1.12).

In fact there should be a bijection in part (d), but for our purposes an injection suffices. Theorem 2.1.a entails in particular

$$\sum\nolimits_{(\rho,a)\in\mathrm{Jord}(\pi)} d_\rho a = \text{size of } G^\vee, \tag{2.2}$$

where the size of an $N \times N$-matrix is N. Parts (a) and (b) of Theorem 2.1 are in proven in [Mœ3, §2.2–2.5]. When in addition G is quasi-split, parts (c) and (d) are shown in [Mœ4, §7.1]. Theorem 2.1.c–d is stated for all our G^+ in [Mœ3, §2.5], attributed to Arthur [Art]. Later this was worked out for non-quasi-split groups in [MoRe]. We note that these sources do include surjectivity in part (d). Part (e) was shown in [Mœ3, Theorem 2.5.1].

For the main results in this paper it suffices to know that Theorem 2.1 holds for supercuspidal representations. We point out that unfortunately the proof of those cases of Theorem 2.1 is not yet entirely complete (except when G has very small rank). Arthur's book [Art] relies on certain papers that were announced but have not yet appeared. Meanwhile most of this has been settled in [MoWa, AGIKMS], and it is expected that the remaining gaps in [Art] will be fixed soon. The paper [MoRe] uses [Art], and also leaves some other details to be worked out.

On the other hand, the part of Theorem 2.1 that classifies discrete series representations in terms of supercuspidal representations is documented much better. Besides the above references, it is also treated in [Mœ1, MoTa, KiMa, Xu]. We describe this explicitly in Paragraph 4.1.

2.1. The method of close local fields.

To goal of this paragraph is to deduce instances of Theorem 2.1 for groups over local function fields (for which very little is in the literature) from Theorem 2.1 for groups over p-adic fields. To this end we employ the method of close fields, a general method to transfer statements from a group over one local field to the same group over an another local field, provided these fields look sufficiently similar. Let F be a local field of positive characteristic and let F' be a local field of characteristic zero. From the classification of classical groups we see that we can define any algebraic group $\mathcal{G} = \mathcal{G}_n$ as in Section 2 simultaneously over F and over F'.

Consider π in the discrete series of $\mathcal{G}(F)^+$. Let d be the maximum of the depths of π and of all the ρ that appear in $\mathrm{Jord}(\pi)$. We denote the subcategory of $\mathrm{Rep}(\mathcal{G}(F))$ generated by the representations of depth $\leq d$ by $\mathrm{Rep}(\mathcal{G}(F))_{\leq d}$. Let F' be a p-adic field which is sufficiently close to F, with respect to the depth $\mathcal{D} := \mathcal{D}(p, \mathcal{G}, d)$ and the groups $\mathcal{G}$, GL_m, $\mathcal{G}_m$ with $m \leq \mathrm{rk}(\mathcal{G})$. Here F and F' are at least $\mathcal{D}$-close, but usually a lot closer is needed.

By [Gan] the method of close fields yields canonical equivalences of categories

$$
\begin{array}{rlcl}
\zeta^{\mathcal{G},F,F'} : & \mathrm{Rep}(\mathcal{G}_n(F))_{\leq \mathcal{D}} & \xrightarrow{\sim} & \mathrm{Rep}(\mathcal{G}_n(F'))_{\leq \mathcal{D}}, \\
\zeta^{\mathcal{G},F,F'} : & \mathrm{Rep}(\mathrm{GL}_m(F))_{\leq \mathcal{D}} & \xrightarrow{\sim} & \mathrm{Rep}(\mathrm{GL}_m(F'))_{\leq \mathcal{D}}, \\
\zeta^{\mathcal{G},F,F'} : & \mathrm{Rep}(\mathcal{G}_{n+m}(F))_{\leq \mathcal{D}} & \xrightarrow{\sim} & \mathrm{Rep}(\mathcal{G}_{n+m}(F'))_{\leq \mathcal{D}},
\end{array} \tag{2.3}
$$

for all $m \leq \mathrm{rk}(\mathcal{G}) = n$. By [Sol5, Theorem 3.5] these equivalences of categories are compatible with normalized parabolic induction. Hence the equivalences (2.3) transfer the condition that (ρ, a) belongs to $\mathrm{Jord}(\pi)$ into the condition that $\zeta^{\mathrm{GL}_{d_\rho},F,F'}(\pi)$ belongs to $\mathrm{Jord}(\zeta^{\mathcal{G},F,F'}(\pi))$. In other words, (2.3) induces an injection

$$
\mathrm{Jord}(\pi) \to \mathrm{Jord}(\zeta^{\mathcal{G},F,F'}(\pi)). \tag{2.4}
$$

Now the problem arises that $\mathrm{Jord}(\pi)$ could be too small, so that (2.4) would not be surjective. Then (2.2) fails and $\mathrm{Jord}(\pi)$ would not yield a Langlands parameter for $\mathcal{G}(F)$. For groups over p-adic fields this used to be a difficult problem [MoTa, p. 727], which has only been solved with the endoscopic methods from [Art]. To carry out the method of close fields completely, we need the following additional input.

Hypothesis 2.2. *Fix $\mathcal{G}$, a prime p and a depth $d \in \mathbb{N}$. There exists a bound $\mathcal{D}(p, \mathcal{G}, d) \in \mathbb{Z}_{\geq d}$ such that*

- *for all p-adic fields F',*
- *for all unitary supercuspidal representations $\sigma \in \mathrm{Irr}(\mathcal{G}(F'))$ of depth $\leq d$,*
- *for all $\rho \in \mathrm{Irr}(\mathrm{GL}_{d_\rho}(F'))$ occurring in $\mathrm{Jord}(\sigma)$,*

the depth of ρ is $\leq \mathcal{D}(p, \mathcal{G}, d)$.

For symplectic groups and split special orthogonal groups this assumption is known (for $p > 2$) from [GaVa, Lemma 8.2.3], in the stronger form $\mathcal{D}(p, \mathcal{G}, d) = d+1$. In fact the main results of [GaVa] imply Theorem 2.1 for these split groups, including bijectivity in part (c). For possibly non-split classical groups (with $p > 2$ but not general spin groups), it seems likely that Hypothesis 2.2 follows from [KSS].

Proposition 2.3. *Fix a prime p and a group $\mathcal{G}$ as before. Suppose that Hypothesis 2.2 holds for all $d \in \mathbb{N}$. Then Theorem 2.1 holds for $\mathcal{G}(F)$, for any local function field F.*

Proof. Write $\zeta^{\mathcal{G},F,F'}(\pi)$ as a subquotient of the parabolic induction of a supercuspidal representation $\sigma \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_r$ of a Levi subgroup of $\mathcal{G}(F')^+$. Since (normalized) parabolic induction preserves depth [MoPr, Theorem 5.2], σ and all the ρ_i have depth $\leq d$. The Jordan block of $\zeta^{\mathcal{G},F,F'}(\pi)$ consists of the Jordan block of σ and some pairs (ρ, a) where ρ is an unramified twist of ρ_i. By Hypothesis 2.2 all ρ appearing in $\mathrm{Jord}(\zeta^{\mathcal{G},F,F'}(\pi))$ have depth $\leq \mathcal{D}(p, \mathcal{G}, d)$. We note that $\rho \in \mathrm{Irr}(\mathrm{GL}_m(F'))$ where $m \leq \mathrm{rk}(\mathcal{G})$ by Theorem 2.1.a. Hence every such ρ is in the image of $\zeta^{\mathrm{GL}_m,F,F'}$ for the correct m. Then $(\zeta^{\mathrm{GL}_m,F,F'})^{-1}\rho$ lies in $\mathrm{Jord}(\pi)$, and we can conclude that (2.4) is in fact a bijection.

The L-parameter ϕ_ρ of any ρ from $\mathrm{Jord}(\pi)$ has depth $\leq \mathcal{D}$, because the local Langlands correspondence for general linear groups preserves depths [ABPS2, Proposition 4.2]. Let $\mathbf{W}_F^r$ be the r-th filtration subgroup of the absolute Galois group of F. Recall from [Del, (3.5.1)] that the D-closeness of F and F' is reflected in a group isomorphism

$$\mathbf{W}_F / \mathbf{W}_F^{\mathcal{D}+} \cong \mathbf{W}_{F'} / \mathbf{W}_{F'}^{\mathcal{D}+}. \tag{2.5}$$

Composition with (2.5) transfers ϕ_ρ to a L-parameter for $\mathrm{GL}_{d_\rho}(F')$, say $\zeta(\phi_\rho)$. When F and F' are very close (for instance $2^{d_\rho}\mathcal{D}$-close), $\zeta(\phi_\rho)$ is indeed the L-parameter of $\zeta^{\mathrm{GL}_{d_\rho},F,F'}(\rho)$ [ABPS2, Theorem 6.1]. We note that we really can chose F' that close to F: by [Del] such a field exists and the above works for any choice of F' that is $\mathcal{D}$-close to F. For such an F' composition of the L-parameter of

$$\boxtimes_{(\rho',a) \in \mathrm{Jord}(\zeta^{\mathcal{G},F,F'}(\pi))} \delta(\rho', a)$$

with (2.5) yields the L-parameter of

$$\boxtimes_{(\rho,a) \in \mathrm{Jord}(\pi)} \delta(\rho, a).$$

By Theorem 2.1.a the former parameter has image in ${}^L\mathcal{G}$, hence so does the latter parameter. We define the latter to be the L-parameter of π, like in (2.9). Then parts (a) and (b) of Theorem 2.1 hold for $\mathcal{G}(F)$.

The (partially defined) character $\epsilon_{\zeta^{\mathcal{G},F,F'}(\pi)}$ is transferred, via (2.4), to a (partially defined) character ϵ_π of S_π. Moreover ϵ_π is $\mathcal{G}(F)$-relevant because $\epsilon_{\zeta^{\mathcal{G},F,F'}(\pi)}$ is $\mathcal{G}(F')$-relevant.

Suppose that two discrete series representations $\pi, \tilde{\pi}$ of $\mathcal{G}(F)^+$ have the same Jordan block and the same ϵ. Then their transfers to representations of $\mathcal{G}(F')$ also share the same Jordan block and the same ϵ. With Theorem 2.1.d we find $\zeta^{\mathcal{G},F,F'}(\pi) \cong \zeta^{\mathcal{G},F,F'}(\tilde{\pi})$. Then (2.3) says that $\pi \cong \tilde{\pi}$.
Similarly (2.3) readily shows that Theorem 2.1.d carries over from $\mathcal{G}(F')$ to $\mathcal{G}(F)$. $\square$

2.2. Parametrization of essentially square-integrable representations.

From now on F can be any non-archimedean local field, but we need Hypothesis 2.2 if F has positive characteristic.

We note that $\mathrm{Out}(\mathcal{G})$ is trivial except for special orthogonal groups and general spin groups associated to vector spaces of even dimension $2n$. Then (for $n \neq 2$)

$$\mathrm{Out}(\mathcal{G}) \cong G^{\vee +}/G^{\vee} \cong \mathrm{O}_{2n}(\mathbb{C})/\mathrm{SO}_{2n}(\mathbb{C}). \tag{2.6}$$

When $\mathcal{G}$ is a form of SO_4, we ignore its exceptional automorphisms and instead we use (2.6) as a definition of $\mathrm{Out}(\mathcal{G})$. In particular the two-element group (2.6) acts naturally on $\mathrm{Irr}(G)$ and on $\Phi_{\mathrm{e}}(G)$.

Theorem 2.4. *Fix a Whittaker datum for the quasi-split inner form of G.*

(*a*) *Suppose that* $\mathrm{Out}(\mathcal{G})$ *is trivial. There exists a canonical injection*
- *from the set of discrete series representations of G,*
- *to the set of discrete bounded parameters in $\Phi_{\mathrm{e}}(G)$.*

(*b*) *Suppose that* $\mathrm{Out}(\mathcal{G})$ *is nontrivial. There exists an injection*
- *from the set of discrete series representations of G,*
- *to the set of discrete bounded parameters in $\Phi_{\mathrm{e}}(G)$,*

which intertwines the actions of $\mathrm{Out}(\mathcal{G})$. *The induced injection between* $\mathrm{Out}(\mathcal{G})$*-orbits in these two sets is canonical.*

(*c*) *The injection in parts (a) and (b) send supercuspidal unitary G-representations to bounded cuspidal L-parameters, and non-supercuspidal representations to non-cuspidal enhanced L-parameters.*

Proof. (a) If we apply the local Langlands correspondence for $\mathrm{GL}_{d_\rho}(F)$ to a ρ occurring in $\mathrm{Jord}(\pi)$, we obtain $\phi_\rho \in \mathrm{Irr}(\mathbf{W}_F)$. The property (2.1) translates to

$$\phi_\rho \cong \phi_\rho^\vee \otimes \phi_{\nu_\pi}. \tag{2.7}$$

From (2.7), (1.3) and Theorem 2.1 we see that

$$\{(\phi_\rho, a) : (\rho, a) \in \mathrm{Jord}(\pi)\}$$

is the set of Jordan blocks of some $\phi \in \Phi(G)$ with

$$\phi^\vee \otimes \mu_G^\vee \circ \phi \cong \phi \cong \phi^\vee \otimes \phi_{\nu_\pi}. \tag{2.8}$$

Further ϕ is unique by [GGP, Theorem 8.1], and discrete because $\mathrm{Jord}(\pi)$ does not have multiplicities. As ρ (from above) was unitary supercuspidal and in particular tempered, ϕ_ρ is bounded and therefore ϕ is also bounded.

Under the correspondence $\mathrm{Jord}(\pi) \mapsto \mathrm{Jord}(\phi)$, the group $\mathcal{S}_\pi$ becomes $\mathcal{S}_\phi$. The set of G-relevant characters of $\mathcal{S}_\phi$ is naturally in bijection with the set of partially defined characters ϵ_π of $\mathcal{S}_\pi$ which figures in Theorem 2.1.d. Thus we can define the required injection by sending π to (ϕ, ϵ) such that the local Langlands correspondence for GL_m sends

$$(\mathrm{Jord}(\pi), \epsilon_\pi) \quad \text{to} \quad (\mathrm{Jord}(\phi), \epsilon). \tag{2.9}$$

(b) The proof of part (a) applies perfectly well to the disconnected reductive group G^+. It provides a canonical injection from the discrete series representations π^+ of G^+ to the pairs (ϕ, ϵ) with $\phi \in \Phi(G)/\mathrm{Out}(\mathcal{G})$ bounded and discrete and $\epsilon \in \mathcal{S}_\phi^+$, where $\mathcal{S}_\phi^+$ is like $\mathcal{S}_\phi$ but computed in $\mathrm{O}_{2n}(\mathbb{C})$. We can distinguish two cases:

Case (i): There exists $(\tau, a) \in \mathrm{Jord}(\phi)$ with $a \dim \tau$ odd.
From (1.5) we see that the group $\mathcal{S}_\phi^+$ contains an element of $\mathrm{O}_{2n}(\mathbb{C}) \setminus \mathrm{SO}_{2n}(\mathbb{C})$. Hence the preimage of ϕ in $\Phi(G)$ is just one equivalence class.

By Theorem 2.1.c, $\pi^+ \in \mathrm{Irr}(G^+)$ restricts to an irreducible representation π of G. In particular π is stable under $\mathrm{Out}(\mathcal{G})$. Clifford theory tells us that there are precisely two inequivalent irreducible representations of G^+ that restrict to π.

As $\mathcal{S}_\phi^+ \cong \mathbb{F}_2^{|\mathrm{Jord}(\phi)|}$, we find

$$\mathcal{S}_\phi^+ \cong \mathcal{S}_\phi \times \mathbb{F}_2.$$

Hence there exist precisely two characters of $\mathcal{S}_\phi^+$ that extend $\epsilon|_{\mathcal{S}_\phi}$. We decree that the bijection for the discrete series of G sends π to $(\phi, \epsilon|_{\mathcal{S}_\phi})$, that is the only natural possibility and does not disturb the injectivity we had for G^+.

Case (ii): $a \dim \tau$ is even for all $(\tau, a) \in \mathrm{Jord}(\phi)$.
Now (1.5) shows that $\mathcal{S}_\phi^+$ does not contain any elements from $\mathrm{O}_{2n}(\mathbb{C}) \setminus \mathrm{SO}_{2n}(\mathbb{C})$, so $\mathcal{S}_\phi^+ = \mathcal{S}_\phi$. By [GGP, Theorem 8.1] the preimage of ϕ in $\Phi(\mathrm{GSpin}(V))$ consists of two equivalence classes, say ϕ' and ϕ''. Then ϕ'' is equivalent with $\mathrm{Ad}(h^\vee)\phi'$ for some $h^\vee \in \mathrm{O}_{2n}(\mathbb{C}) \setminus \mathrm{SO}_{2n}(\mathbb{C})$ and $\mathcal{S}_{\phi'}$ is canonically isomorphic with $\mathcal{S}_{\phi''}$.

By Theorem 2.1.c the restriction of π^+ to G is reducible. By Clifford theory it is the direct sum of two inequivalent irreducible G-representations say $\pi' \oplus \pi''$, and any element of $G^+ \setminus G$ exchanges π' and π''.

$$\text{We choose a bijection between } \{(\phi', \epsilon), (\phi'', \epsilon)\} \text{ and } \{\pi', \pi''\}, \tag{2.10}$$

and we decree that it gives two instances of the injection for the discrete series of G. Notice that this guarantees $\mathrm{Out}(\mathcal{G})$-equivariance on these objects.

Combining all instances, we obtain the desired injection for the discrete series of G. Its only noncanonical parts are the choices (2.10), which become invisible when we pass to $\mathrm{Out}(\mathcal{G})$-orbits.
(c) This is clear from the criteria for cuspidality on page 10 and in Theorem 2.1.e. $\square$

For the moment G is a general spin group. Since the centre of G is not compact (unlike for the other groups in Section 2), we have to distinguish between discrete series representations and essentially square-integrable representations. A G-representation π is called essentially square-integrable if its restriction to G_{der} is square-integrable. If π is in addition irreducible, then there exists an unramified character $\chi \in X_{\mathrm{nr}}(G)$ such that $\chi \otimes \pi$ has unitary central character, that is, $\chi \otimes \pi$ belongs to the discrete series. We can even achieve this with χ a real power of the norm character of $F^\times \cong G/G_{\mathrm{der}}$.

Recall from [Hai] that the group $X_{\mathrm{nr}}(G)$ of unramified characters of G is naturally isomorphic with $(\mathrm{Z}(G^\vee)^{\circ, \mathbf{I}_F})_{\mathbf{W}_F}$, which for our G is just $\mathrm{Z}(G^\vee)^\circ \cong \mathbb{C}^\times$. Similarly the group $X_{\mathrm{unr}}(G)$ of unitary unramified characters is naturally isomorphic with the maximal compact subgroup $\mathrm{Z}(G^\vee)^\circ_{\mathrm{cpt}}$ of $\mathrm{Z}(G^\vee)^\circ$. The group $X_{\mathrm{nr}}(G)$ acts on $\mathrm{Irr}(G)$ by tensoring and the group $\mathrm{Z}(G^\vee)^{\mathbf{I}_F} = \mathrm{Z}(G^\vee)$ acts on $\Phi_{\mathrm{e}}(G)$ by

$$z(\phi, \rho) = (z\rho, \phi), \qquad (z\phi)|_{\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})} = \phi|_{\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})}, \qquad (z\phi)(\mathrm{Frob}_F) = z\, \phi(\mathrm{Frob}_F).$$

Theorem 2.5. *Let G be a general spin group.*

(a) *The injection in Theorem 2.4.a is equivariant for the actions of $X_{\mathrm{unr}}(G) \cong \mathrm{Z}(G^\vee)^\circ_{\mathrm{cpt}}$, and by suitable choices the bijection in Theorem 2.4.b can be made equivariant for these actions.*

(*b*) *The injection from part (a) extends canonically to an injection*
- *from the set of irreducible essentially square-integrable G-representations,*
- *to the set of discrete parameters in $\Phi_{\mathrm{e}}(G)$.*

(*c*) *The injection in part (b) is equivariant for the actions of $X_{\mathrm{nr}}(G) \cong \mathrm{Z}(G^\vee)^\circ$, and it respects cuspidality.*

Proof. (a) Let $\pi \in \mathrm{Irr}(G)$ and $(\phi, \epsilon) \in \Phi_{\mathrm{e}}(G)$ be as in the proof of Theorem 2.4. For $\chi \in X_{\mathrm{unr}}(G)$, $\chi \otimes \pi$ is of the same kind. From the natural isomorphisms

$$(\chi \otimes \mathrm{St}(\rho, a)) \times (\chi \otimes \pi) \cong \chi \otimes (\mathrm{St}(\rho, a) \times \pi)$$

we see that

$$\mathrm{Jord}(\chi \otimes \pi) \quad \text{equals} \quad \{(\chi \otimes \rho, a) : (\rho, a) \in \mathrm{Jord}(\pi)\} =: \chi \otimes \mathrm{Jord}(\pi).$$

The properties of ϵ_π in [Mœ3, §2.5] readily imply that

$$\epsilon_{\chi \otimes \pi}(z_{\chi \otimes \rho, a}) = \epsilon_\pi(z_{\rho, a}).$$

Let $\hat{\chi} \in \mathrm{Z}(G^\vee)^\circ$ correspond to χ via $X_{\mathrm{unr}}(G) \cong \mathrm{Z}(G^\vee)^\circ_{\mathrm{cpt}}$. Then $\hat{\chi}\phi$ is still discrete and bounded, while

$$\mathrm{Jord}(\hat{\chi}\phi) \quad \text{equals} \quad \{(\hat{\chi}\tau, a) : (\tau, a) \in \mathrm{Jord}(\phi)\} =: \hat{\chi}\mathrm{Jord}(\phi).$$

The action of $\hat{\chi}$ does not change ϵ as character of

$$\mathcal{S}_\phi = \mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi) = \mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\hat{\chi}\phi) = \mathcal{S}_{\hat{\chi}\phi}.$$

However, the element $z_{\tau, a} \in \mathcal{S}_\phi$ is renamed as $z_{\hat{\chi}\tau, a}$ and to account for that we rename ϵ to $\hat{\chi}\epsilon$.

Suppose now that π and (ϕ, ϵ) are matched by Theorem 2.4, so (2.9) holds. By the known equivariance properties of the local Langlands correspondence for GL_m,

$$(\chi \otimes \mathrm{Jord}(\pi), \epsilon_{\chi \otimes \pi}) \quad \text{is sent to} \quad (\hat{\chi}\mathrm{Jord}(\phi), \hat{\chi}\epsilon).$$

In the setting of Theorem 2.4.a, this shows that $\chi \otimes \pi$ is matched with $(\hat{\chi}\phi, \hat{\chi}\epsilon)$, which is the desired equivariance.

In the setting of Theorem 2.4.b, only the choices in (2.10) could disturb this equivariance for $X_{\mathrm{unr}}(G) \cong \mathrm{Z}(G^\vee)^\circ_{\mathrm{cpt}}$. To prevent that, it suffices to make the entirety of the choices (2.10) in an equivariant way. This can be done as follows. Pick a set of representatives for the (ϕ, ϵ) with all $a \dim \tau$ even, modulo the action of $\mathrm{Z}(G^\vee)^\circ_{\mathrm{cpt}}$. For each of these ϕ's we fix a choice (2.10), say $(\phi', \epsilon) \mapsto \pi'$. Then decree that, for each $\chi \in X_{\mathrm{unr}}(G)$, $(\hat{\chi}\phi', \epsilon)$ is matched via $\chi \otimes \pi'$.
(b) By design, the set of essentially square-integrable irreducible G-representations can be expressed as

$$\text{discrete series of } G \times_{X_{\mathrm{unr}}(G)} X_{\mathrm{nr}}(G). \tag{2.11}$$

Similarly, it follows from [Hei1, Lemma 5.1] that the set of discrete parameters in $\Phi_{\mathrm{e}}(G)$ can be constructed as

$$\text{bounded discrete part of } \Phi_{\mathrm{e}}(G) \times_{\mathrm{Z}(G^\vee)^\circ_{\mathrm{cpt}}} \mathrm{Z}(G^\vee)^\circ. \tag{2.12}$$

From (2.11), (2.12) and part (a) we deduce an injection from the set of essentially square-integrable irreducible G-representations to the discrete part of $\Phi_{\mathrm{e}}(G)$, which is equivariant for $X_{\mathrm{nr}}(G) \cong \mathrm{Z}(G^\vee)^\circ$.
(c) The actions of $X_{\mathrm{nr}}(G)$ on $\mathrm{Irr}(G)$ and of $\mathrm{Z}(G^\vee)^\circ$ on $\Phi_{\mathrm{e}}(G)$ preserve cuspidality. Combine that with Theorem 2.4.c and the construction of the injection in part (b). $\square$

Now G can again be any group as in Section 2. The set of supercuspidal Bernstein components of $\mathrm{Irr}(G)$ is just $\mathrm{Irr}_{\mathrm{cusp}}(G)/X_{\mathrm{nr}}(G)$. Recall the notion of a Bernstein component of enhanced L-parameters from [AMS1, §8]. By definition, the set of cuspidal Bernstein components of $\Phi_{\mathrm{e}}(G)$ is $\Phi_{\mathrm{cusp}}(G)/\mathrm{Z}(G^\vee)^\circ$. If we apply Theorems 2.4 and 2.5 to these sets, we obtain:

Corollary 2.6. *Theorems 2.4 and 2.5.b induce an injection*

- *from the set of supercuspidal Bernstein components of* $\mathrm{Irr}(G)$*,*
- *to the set of cuspidal Bernstein components of* $\Phi_{\mathrm{e}}(G)$*.*

This injection is $\mathrm{Out}(\mathcal{G})$*-equivariant and becomes canonical if we pass to* $\mathrm{Out}(\mathcal{G})$*-orbits on both sides.*

3. Comparison of Hecke algebras for Bernstein components

In this section G is a general spin group. All our results are also valid for symplectic groups and for (special) orthogonal groups, with slightly simpler proofs, see [Hei2, Hei3, Hei4] (on the p-adic side) and [Mou] and [AMS3, §5.3] (on the Galois side). In [Sol4, §10] an extended affine Hecke algebra

$$\mathcal{H}(\mathfrak{s}) = \mathrm{End}_G(\Pi_{\mathfrak{s}})$$

was attached to $\mathfrak{s}$, where $\Pi_{\mathfrak{s}}$ is a particular progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$. We have $\Pi_{\mathfrak{s}} = I_P^G \Pi_{\mathfrak{s}_L}$, where I_P^G is the (normalized) parabolic induction functor for P a parabolic subgroup of G with Levi factor L. Further $\Pi_{\mathfrak{s}_L} := \mathrm{ind}_{L^1}^L(\sigma_1)$, with L^1 the subgroup of L generated by all compact subgroups and σ_1 an irreducible constitutent of $\mathrm{Res}^L_{L^1}\sigma$. It is shown in [Ren, §VI.10.1] that $\Pi_{\mathfrak{s}}$ is canonical, in the sense that

$$\text{up to isomorphism, } \Pi_{\mathfrak{s}} \text{ depends only on } \mathfrak{s}. \tag{3.1}$$

We note that [Sol4, §10] is applicable because the restriction of σ to L^1 is multiplicity-free, which follows from the fact that $\mathcal{L}$ is a direct product of reductive groups with centre of dimension ≤ 1. In this setup there is a natural equivalence of categories

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \xrightarrow{\sim} & \mathrm{Mod}(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{\mathrm{op}}) \\ \pi & \mapsto & \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi) \\ V \otimes_{\mathrm{End}_G(\Pi_{\mathfrak{s}})} \Pi_{\mathfrak{s}} & \leftarrow\!\shortmid & V \end{array}, \tag{3.2}$$

see [Ren, Théorème VI.10.2]. We will explain the structure of $\mathcal{H}(\mathfrak{s}) := \mathrm{End}_G(\Pi_{\mathfrak{s}})$ in terms of the data and the presentation at the start of Paragraph 1.2. From there we will see that it is self-opposite, and we will compare it with $\mathcal{H}(\mathfrak{s}^\vee, z)$.

Before we come to that, we need to match Bernstein components for $\mathrm{Irr}(G)$ and for $\Phi_{\mathrm{e}}(G)$. Suppose that the bilinear form on V is given by a symmetric matrix $\hat{J}$, such that the isotropic part is made from blocks $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ placed in rows and columns $j, \dim V + 1 - j$. Let $\mu_G : G \to F^\times$ be the spinor norm, so that $\mathrm{Spin}(V) = \ker \mu_G$. The Levi subgroup $L = \mathcal{L}(F)$ is embedded in $G = \mathrm{GSpin}(V)$ via

$$\begin{aligned} & G_{n_-}/F^\times \times \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_k}(F) \to G/F^\times \cong \mathrm{SO}(V) \\ & (g_-, g_1, \ldots, g_k) \mapsto \big(g_1, \ldots, g_k, g_-, \hat{J} g_k^{-T} \hat{J}^{-1}, \ldots, \hat{J} g_k^{-T} \hat{J}^{-1}\big) \end{aligned} \tag{3.3}$$

It is difficult to write down the actual embedding $L \to G$ in such terms, to study that the root datum from [AsSh] is more useful. The group $\mathrm{N}_{\mathrm{GPin}(V)}(L)$ contains an element that exchanges g_j and $\hat{J} g_j^{-T} \hat{J}^{-1}$, and the same time multiplies g_- with $\det(g_j)$. As automorphism of L , it is given by

$$(g_-, g_1, \ldots, g_k) \mapsto (\det(g_j) g_-, g_1, \ldots, g_{j-1}, \hat{J} g_j^{-T} \hat{J}^{-1}, g_{j+1}, \ldots, g_k). \tag{3.4}$$

We record that the effect of (3.4) on irreducible representations is

$$\sigma_- \boxtimes \sigma_1 \boxtimes \cdots \boxtimes \sigma_k \mapsto \sigma_- \boxtimes \sigma_1 \boxtimes \cdots \boxtimes \sigma_{j-1} \boxtimes (\sigma_j^\vee \otimes \nu_{\sigma_-} \circ \det) \boxtimes \sigma_{j+1} \boxtimes \cdots \boxtimes \sigma_k,$$

where ν_{σ_-} is character by which the central subgroup $F^\times \subset G_{n_-}$ acts on σ_-.

Further $\mathrm{N}_G(L)$ contains elements that act on L by permuting some type GL factors of the same size. The group $\mathrm{N}_{\mathrm{GPin}(V)}(L)/L$ is generated by elements of these two kinds, and is isomorphic to a direct product of Weyl groups of type B_e. For $\mathrm{N}_G(L)/L$ the only difference is that the elements from (3.4) are subject to a determinant condition if $\dim(V)$ is even. Notice that these descriptions match those after (1.28). Thus there are canonical isomorphisms

$$\mathrm{N}_G(L)/L \cong \mathrm{N}_{G^\vee}(L^\vee)/L^\vee \quad \text{and} \quad \mathrm{N}_{\mathrm{GPin}(V)}(L)/L \cong \mathrm{N}_{G^{\vee +}}(L^\vee)/L^\vee. \tag{3.5}$$

Theorem 3.1. (*a*) *There exists a injection*

- *from the set of supercuspidal Bernstein components of* $\mathrm{Irr}(L)$*,*
- *to the set of cuspidal Bernstein components of* $\Phi_{\mathrm{e}}(L)$*.*

This bijection is equivariant for the natural actions of (3.5) *and becomes canonical if we pass to* $\mathrm{Out}(\mathcal{G}_{n_-})$*-orbits.*

(*b*) *Let L run through a set of representatives for the conjugacy classes of Levi subgroups of* G*. The corresponding instances of part (a) provide an injection*

- *from the set of Bernstein components of* $\mathrm{Irr}(G)$*,*
- *to the set of Bernstein components of* $\Phi_{\mathrm{e}}(G)$*.*

This injection becomes canonical if we pass to $\mathrm{Out}(\mathcal{G})$*-orbits.*

Proof. (a) The injection and the canonicity follow from Corollary 2.6, while the equivariance can be seen from our explicit formulas for the actions of (3.5), namely (1.15), (1.29) and (3.4).
(b) By definition Bernstein components of $\mathrm{Irr}(G)$ are parametrized by supercuspidal Bernstein components for Levi subgroups of G. Further $\mathfrak{s}_L \subset \mathrm{Irr}_{\mathrm{cusp}}(L)$ and $\mathfrak{s}_{L'} \subset \mathrm{Irr}_{\mathrm{cusp}}(L')$ give the same Bernstein component for $\mathrm{Irr}(G)$ if and only if $\mathfrak{s}_L$ and $\mathfrak{s}_{L'}$ are G-conjugate. Analogous statements hold for Bernstein components of $\Phi_{\mathrm{e}}(G)$ [AMS1, §8], which yields the desired bijection. By the equivariance in part (a), this bijection does not depend on the choice of the representative Levi subgroups. $\square$

With Theorem 2.4 we consider $\sigma = \pi(\phi_\sigma, \epsilon_\sigma) \in \mathrm{Irr}_{\mathrm{cusp}}(L)$. Then $\mathfrak{s}_L = X_{\mathrm{nr}}(L)\sigma$ is the image of $\mathfrak{s}_L^\vee$ under Theorem 3.1.b. The injectivity and $X_{\mathrm{nr}}(L)$-equivariance in Theorem 2.5 say that this extends to an injection from $\mathfrak{s}_L$ to $\mathfrak{s}_L^\vee$. Then the equivariance in Theorem 3.1.a guarentees that the groups $W_{\mathfrak{s}}$ and $W_{\mathfrak{s}^\vee}$ are canonically isomorphic.

We may assume that σ has been normalized like ϕ after (1.20). Then the group $W_{\mathfrak{s}^\vee}$ can also be described as the stabilizer of

$$\sigma \;=\; \sigma_- \,\boxtimes\, \boxtimes_\rho \,\rho^{\boxtimes e_\rho} \;=\; \sigma_- \,\boxtimes\, \boxtimes_j \,\rho_j^{\boxtimes e_j}. \tag{3.6}$$

in $\mathrm{N}_G(L)/L$. The stabilizer $W_{\mathfrak{s}}^+$ of σ in $\mathrm{N}_{\mathrm{GPin}(V)}(L)/L$ decomposes as a direct product of subgroups $W_{\mathfrak{s},\rho}$. From (3.4) we see that

- if $\rho \not\cong \rho^\vee \otimes \nu_{\sigma_-} \circ \det$, then $W_{\mathfrak{s},\rho} \cong S_{e_\rho} \cong W(A_{e_\rho - 1})$,

- if $\rho \cong \rho^\vee \otimes \nu_{\sigma_-} \circ \det$, then $W_{\mathfrak{s},\rho} \cong W(B_{e_\rho}) = W(C_{e_\rho})$, which can sometimes be interpreted better as $W(D_{e_\rho}) \rtimes \mathrm{Aut}(D_n)$.

Using Theorem 3.1, we can express the complex torus underlying $\mathcal{H}(\mathfrak{s})$ as

$$T_{\mathfrak{s}} = \mathfrak{s}_L \cong \mathfrak{s}_L^\vee = T_{\mathfrak{s}^\vee}, \tag{3.7}$$

Further, the action of $W_{\mathfrak{s}}^+$ on $T_{\mathfrak{s}}$ can be identified with the action of $W_{\mathfrak{s}^\vee}^+$ on $T_{\mathfrak{s}^\vee}$. The map $\chi \mapsto \sigma \otimes \chi$ provides a homeomorphism

$$T_{\mathfrak{s}} \cong X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\sigma),$$

where $X_{\mathrm{nr}}(L,\sigma) \subset X_{\mathrm{nr}}(L)$ is the stabilizer of $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(L)$. Hence

$$X^*(T_{\mathfrak{s}}) \cong L_\sigma / L^1 \text{ where } L_\sigma = \bigcap\nolimits_{\chi \in X_{\mathrm{nr}}(L,\sigma)} \ker \chi.$$

More explicitly, $L/L^1 \cong \mathbb{Z} \times \prod_j \mathbb{Z}^{e_j}$ and

$$L_\sigma / L^1 \text{ is the subgroup } \mathbb{Z} \times \prod\nolimits_j (t_{\rho_j}\mathbb{Z})^{e_j},$$

where the first factor $\mathbb{Z}$ comes from $\mathrm{Z}(\mathcal{G})^\circ \cong \mathrm{GL}_1$. We recall that t_ρ denotes the torsion number of ρ, that is, the number of unramified characters of $\mathrm{GL}_{d_\rho}(F)$ that stabilize $\rho \in \mathrm{Irr}(\mathrm{GL}_{d_\rho}(F))$. We write the root datum for $\mathcal{H}(\mathfrak{s})$ as

$$\mathcal{R}_{\mathfrak{s}} = \big(\Sigma_{\mathfrak{s}}, X^*(T_{\mathfrak{s}}), \Sigma_{\mathfrak{s}}^\vee, X_*(T_{\mathfrak{s}})\big).$$

As explained in [Sol4, §3], the root system $\Sigma_{\mathfrak{s}}^\vee$ comes from the roots $\alpha \in \Sigma_{\mathrm{red}}(G, \mathrm{Z}(L))$ for which the so-called Harish-Chandra μ-function μ^α has a zero on $\mathfrak{s}_L$. Then $\Sigma_{\mathfrak{s}}$ consists of multiples of some elements of $\Sigma_{\mathrm{red}}(G,\mathrm{Z}(L))^\vee \cong \Sigma_{\mathrm{red}}(G^\vee, \mathrm{Z}(L^\vee))$, just like $R_{\mathfrak{s}^\vee}$ in (1.27).

The group $W_{\mathfrak{s}}^+$ acts naturally on $\mathcal{R}_{\mathfrak{s}}$ and contains $W(\Sigma_{\mathfrak{s}})$. Our choice of a Borel subgroup $B^\vee$ of $G^\vee$ yields a system of positive roots $\Sigma_{\mathfrak{s}}^+$ in $\Sigma_{\mathfrak{s}}$. If $\Gamma_{\mathfrak{s}}^{(+)}$ denotes the stabilizer of $\Sigma_{\mathfrak{s}}^+$ in $W_{\mathfrak{s}}^{(+)}$, then

$$W_{\mathfrak{s}}^+ = \Gamma_{\mathfrak{s}}^+ \ltimes W(\Sigma_{\mathfrak{s}}) \quad \text{and} \quad W_{\mathfrak{s}} = \Gamma_{\mathfrak{s}} \ltimes W(\Sigma_{\mathfrak{s}}). \tag{3.8}$$

To match this decomposition with (1.30), we need to compare the underlying root systems. In [Sol4, §3] an element

$$h_\alpha^\vee \in (L_\sigma \cap L_\alpha^1)/L^1 \subset L_\sigma/L^1 = X^*(T_{\mathfrak{s}}) \tag{3.9}$$

was associated to each $\alpha \in \Sigma_{\mathrm{red}}(G,\mathrm{Z}(L))$. Here L_α is the Levi subgroup of G which contains L and the root subgroups $\mathrm{U}_{\alpha'}$ (for $\alpha' \in R(G,S)$ with $\alpha'|_{\mathrm{Z}(L)} \in \mathbb{Q}\alpha$) and whose semisimple rank is one higher than that of L. In fact $(L_\sigma \cap L_\alpha^1)/L^1 \cong \mathbb{Z}$, $h_\alpha^\vee$ generates this group and is pinned down by the requirement $\nu_F(\alpha(h_\alpha^\vee)) > 0$. Then

$$\Sigma_{\mathfrak{s}} = \{h_\alpha^\vee : \mu^\alpha \text{ has a zero on } \mathfrak{s}_L\}. \tag{3.10}$$

Recall that $R_{\mathfrak{s}^\vee}$ is a disjoint union of irreducible root systems

$$R_{\mathfrak{s}^\vee,\tau} = R(G_{\phi,\tau}^\vee T, T) = R(G^\vee_{\phi(\mathbf{I}_F),\tau} T, T)$$

which are given explicitly in Table 2. Similarly, by [Hei3, Proposition 1.13], $\Sigma_{\mathfrak{s}}$ is a disjoint union of irreducible root systems $R_{\mathfrak{s},\rho}$, each one coming from the factors $\mathrm{GL}_{m_j}(F)$ of L with $\sigma_j = \rho$. By [Hei3, Proposition 1.15] (generalized to our setting) the groups $W_{\mathfrak{s}}^+$ and $\Gamma_{\mathfrak{s}}^+$ decompose canonically as direct products of subgroups $W_{\mathfrak{s},\rho}^+$ and $\Gamma_{\mathfrak{s},\rho}^+$.

We fix one ρ and we let $\tau \in \mathrm{Irr}(\mathbf{W}_F)$ be its image under the local Langlands correspondence for $\mathrm{GL}_{m_j}(F)$. By design $e_\tau = e_\rho > 0$. Recall from (1.31) and (1.32) that $W^+_{\mathfrak{s}^\vee}$ and $\Gamma^+_{\mathfrak{s}^\vee}$ decompose canonically as direct products of subgroups $W^+_{\mathfrak{s}^\vee,\tau}$ and $\Gamma^+_{\mathfrak{s},\tau}$. By Theorem 3.1.a

$$W^+_{\mathfrak{s},\rho} \cong W^+_{\mathfrak{s}^\vee,\tau} \qquad \text{for all } \tau \in \mathrm{Irr}(\mathbf{W}_F) \text{ with } e_\rho > 0. \tag{3.11}$$

Since $\mathrm{Jord}(\sigma_-)$ and $\mathrm{Jord}(\phi_-)$ correspond via the local Langlands correspondence, $\ell_\tau > 0$ if and only if ρ appears in $\mathrm{Jord}(\sigma_-)$. We write

$$a_\rho = \max\{a : (\rho, a) \in \mathrm{Jord}(\sigma_-)\},$$

which equals a_τ. Let ρ' correspond to τ' via the local Langlands correspondence for $\mathrm{GL}_{d_\rho}(F)$, so ρ' is an unramified twist of ρ which is not isomorphic to ρ, but still $\rho' \cong \rho'^\vee \otimes \nu_{\sigma_-} \circ \det$.

Proposition 3.2. *There is a canonical bijection $R_{\mathfrak{s},\rho} \to R_{\tau,\mathrm{red}}$ which respects positivity of roots. In particular $W(R_{\mathfrak{s},\rho}) \cong W(R_\tau)$ and $\Gamma^+_{\mathfrak{s},\rho} \cong \Gamma^+_{\mathfrak{s}^\vee,\tau}$.*

Proof. The proof of [Hei3, Proposition 1.13] shows that for $\mathrm{GL}_{d_\rho}(F)^{e_\rho} \subset L$ the roots $\alpha : t \mapsto t_i t_j^{-1}$ with $1 \leq i, j \leq e_\rho, i \neq j$ can be treated entirely like roots for some general linear group. Hence the associated function μ^α has a zero on $\mathfrak{s}_L$ and $h^\vee_\alpha \in \Sigma_{\mathfrak{s}}$. Thus $R_{\mathfrak{s},\rho}$ always contains a root subsystem of type $A_{e_\rho - 1}$. In terms of $\mathcal{R}_{\mathfrak{s}}$: the corresponding part of $X^*(T_{\mathfrak{s}})$ can be identified with $(t_\rho \mathbb{Z})^{e_\rho}$ and $A_{e_\rho-1}$ is embedded there as the elements $h^\vee_\alpha = t_\rho \alpha^\vee$ with $\alpha^\vee \in \mathbb{Z}^{e_\rho}$ of the usual form

$$(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0).$$

In view of the description of $W^+_{\mathfrak{s}^\vee}$ following (3.6), $R_{\mathfrak{s},\rho}$ is a S_{e_ρ}-stable reduced root subsystem of BC_{e_ρ}. In other words, it has type $A_{e_\rho-1}$, B_{e_ρ}, C_{e_ρ} or D_{e_ρ}. We check all the cases in Table 1.

- $\tau \in \mathrm{Irr}(\mathbf{W}_F)^0_\phi, \ell_\tau = 0$. Then $\rho \not\cong \rho^\vee \otimes \nu_{\sigma_-} \circ \det$ and from (3.4) we see that $R_{\mathfrak{s},\rho} \cong A_{e_\rho - 1} \cong R_\tau$.
- $\tau \in \mathrm{Irr}(\mathbf{W}_F)^\pm_\phi, \ell_\tau > 0$. Here $\rho \cong \rho^\vee \otimes \nu_{\sigma_-} \circ \det$ and $a_\tau = a_\rho > 0$. Since $(\rho, a_\rho) \in \mathrm{Jord}(\sigma_-)$, $\rho \otimes |\cdot|^{(a_\rho+1)/2} \times \sigma_-$ is reducible [Mœ4, §3.2]. Hence the automorphism (3.4) comes from a root α for which μ^α has a zero on $T_{\mathfrak{s}}$. In the picture (3.9) that becomes $h^\vee_\alpha = t_\rho \alpha^\vee$ with $\alpha^\vee$ a standard basis vector of $\mathbb{Z}^{e_\rho}$. In particular $R_{\mathfrak{s},\rho}$ has type B_{e_ρ}, just like $R_{\tau,\mathrm{red}}$.
- $\tau \in \mathrm{Irr}(\mathbf{W}_F)^-_\phi, \ell_\tau = 0$. Again $\rho \cong \rho^\vee \otimes \nu_{\sigma_-} \circ \det$, but now ρ does not occur in $\mathrm{Jord}(\sigma_-)$. Still (3.4) fixes ρ, and by [Hei2, p. 1610] $\rho \otimes |\cdot|^{1/2} \rtimes \sigma_-$ is reducible. This is like the previous case, only with $a_\tau = a_\rho = 0$. Notice that $\ell_{\tau'} = a_{\tau'} = a_{\rho'} = 0$ as well. Again we find $R_{\mathfrak{s},\rho} \cong B_{e_\rho}$, while $R_\tau \cong C_{e_\rho}$.
- $\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi, \ell_\tau = 0$. Now (3.4) fixes $\rho \cong \rho^\vee \otimes \nu_{\sigma_-} \circ \det$ although ρ does not occur in $\mathrm{Jord}(\sigma_-)$. By [Hei2, p. 1610], $\rho \times \sigma_-$ is reducible. By our assumptions on σ, $\ell_{\tau'} = 0$, so $\rho' \times \sigma_-$ is also reducible. Then the shape of μ^α [Sol4, (3.7)] entails that μ^α is constant on $T_{\mathfrak{s}}$, for α associated to (3.4). Hence $R_{\mathfrak{s},\rho}$ does not contain short roots from B_{e_ρ} or long roots from C_{e_ρ}.

  Consider a root in $D_{e_\rho} \setminus A_{e_\rho - 1}$, so of the form

$$\beta = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0).$$

Via a suitable reflection s_α with α as before, β is associate to a root $\beta' \in A_{e_\rho -1}$. Since $s_\alpha \in W_{\mathfrak s}^+$, $\mu^\beta = \mu^{\beta'} \circ s_\alpha$. As $\mu^{\beta'}$ has a zero on $T_{\mathfrak s}$, so does μ^β. Therefore $R_{\mathfrak s,\tau}$ contains

$$h_\beta^\vee = t_\rho \beta^\vee = t_\rho(0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0),$$

and $R_{\mathfrak s,\rho} \cong D_{e_\rho} \cong R_\tau$.

In all cases there is indeed a natural bijection $R_{\mathfrak s,\rho} \to R_{\tau,\mathrm{red}}$: the identity on all roots except the short roots in the second case, those are multiplied by 2. The bijection preserves positivity of roots, so it induces an isomorphism from the stabilizer $\Gamma_{\mathfrak s,\rho}^+ \subset W_{\mathfrak s,\rho}^+$ of $R_{\mathfrak s,\rho}^+$ to the stabilizer $\Gamma_{\mathfrak s^\vee,\tau}^+ \subset W_{\mathfrak s^\vee,\tau}^+$ of R_τ^+. $\square$

Now we analyse the q-parameters for $\mathcal H(\mathfrak s)$. In view of the shape of μ^α [Sol4, (3.7) and (3.10)], the condition (3.10) on $\alpha \in \Sigma_{\mathrm{red}}(G, \mathrm Z(L))$ is equivalent with $q_\alpha > 1$, where q_α comes from μ^α and will also be a q-parameter for $\mathcal H(\mathfrak s)$. The parameter functions $\lambda, \lambda^* : \Sigma_{\mathfrak s} \to \mathbb R_{\geq 0}$ and the parameters

$$q_\alpha = q_F^{(\lambda(\alpha)+\lambda^*(\alpha))/2}, \quad q_\alpha^* = q_F^{(\lambda(\alpha)-\lambda^*(\alpha))/2}$$

were computed in [Hei2, Hei3]. Although these papers were written for $\mathrm{Sp}(V)$ and $\mathrm{SO}(V)$, the same arguments apply in our setting, that was checked in [Sol4, Sol5]. The q-parameters on $R_{\mathfrak s,\rho}$ are expressed in terms of t_ρ and a_ρ. More precisely, by [Hei2, Proposition 3.4] the q-parameters are:

- if $R_{\mathfrak s,\rho} \cong B_{e_\rho}$ and α is a short root, then $q_\alpha = q_F^{t_\rho(a_\rho+1)/2}$ and $q_\alpha^* = q_F^{t_\rho(a_{\rho'}+1)/2}$,
- otherwise $q_\alpha = q_F^{t_\rho}$ and $q_\alpha^* = 1$.

With q_F as q-base that gives

- $\lambda(\alpha) = t_\rho(a_\rho + a_{\rho'} + 2)/2, \lambda^*(\alpha) = t_\rho(a_\rho - a_{\rho'})/2$ if α is a short root in B_{e_ρ},
- $\lambda(\alpha) = \lambda^*(\alpha) = t_\rho$ otherwise.

In the case $\tau, \tau' \in \mathrm{Irr}(\mathbf W_F)_\phi^-, \ell_\tau + \ell_{\tau'} = 0$ we find $q_\beta = q_\beta^* = q_F^{t_\rho/2}$ for the short roots $h_\beta^\vee$ in B_{e_ρ}. As explained in [Sol5, proof of Theorem 4.9], we may replace $h_\beta^\vee$ by a long root $(h_\beta^\vee)^2 = h_{\beta/2}^\vee$ of C_{e_ρ}, and simultaneously put

$$q_{\beta/2} = q_F^{t_\rho}, \qquad q_{\beta/2}^* = 1, \qquad \lambda(\beta/2) = \lambda^*(\beta/2) = t_\rho. \tag{3.12}$$

With that improvement, the bijection $R_{\mathfrak s,\rho} \to R_{\tau,\mathrm{red}}$ in Proposition 3.2 becomes simply the restriction of the canonical bijection $X^*(T_{\mathfrak s}) \to X_*(T_{\mathfrak s^\vee})$ to reduced roots. That yields a canonical isomorphism of root data

$$\mathcal R_{\mathfrak s} \cong \mathcal R_{\mathfrak s^\vee,\mathrm{red}},$$

where the subscript red means that (for the involved non-reduced root systems BC_e) we take only the indivisible roots and the non-multipliable coroots. Comparing with page 16, we see that the parameter functions λ, λ^* for $\mathcal H(\mathfrak s)$ are the same as those for $\mathcal H(\mathfrak s^\vee, z)$ with $z = q_F^{1/2}$. Thus we find a canonical isomorphism of affine Hecke algebras

$$\mathcal H(\mathcal R_{\mathfrak s}, \lambda, \lambda^*, q_F^{1/2}) \cong \mathcal H(\mathcal R_{\mathfrak s^\vee}, \lambda, \lambda^*, q_F^{1/2}). \tag{3.13}$$

Here we have $q_F^{1/2}$ instead of q_F because in the setup of [AMS3] the indeterminate z^2 was a replacement of the usual q in affine Hecke algebras.

We recall from (1.32) that

$$\Gamma^+_{\mathfrak{s}^\vee} = \prod\nolimits_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi, \ell_\tau = 0} \langle r_\tau \rangle,$$

where r_τ is the nontrivial automorphism of $R_{\mathfrak{s}^\vee,\tau} \cong D_{e_\tau}$. With Proposition 3.2 we deduce that

$$\Gamma^+_{\mathfrak{s}} = \prod\nolimits_{\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi, \ell_\tau = 0} \langle r_\rho \rangle,$$

where ρ corresponds to τ via the local Langlands correspondence for $\mathrm{GL}_{\dim \tau}(F)$ and r_ρ is the nontrivial automorphism of $R_{\mathfrak{s},\rho} \cong D_{e_\rho}$. For each such τ we define J_{r_τ} as in [Hei3, §4.6], it is unique up a factor ± 1. Then the arguments from [Hei3] remain valid in our setting (see [Sol4, §10]) and they show that

$$\mathcal{H}(\mathfrak{s}) \cong \mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_{\mathfrak{s}}, \tag{3.14}$$

where $\Gamma_{\mathfrak{s}}$ acts on $\mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^*, q_F^{1/2})$ via automorphisms of $\mathcal{R}_{\mathfrak{s}}$. We note that the algebra on the right is canonically isomorphic to its own opposite:

$$\begin{array}{llll} \mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_{\mathfrak{s}} & \xrightarrow{\sim} & \big(\mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_{\mathfrak{s}}\big)^{\mathrm{op}} & \\ f T'_w & \mapsto & T'_{w^{-1}} f & f \in \mathcal{O}(T_{\mathfrak{s}}), w \in W_{\mathfrak{s}}. \end{array} \tag{3.15}$$

Here T'_w with $w \in W(R_{\mathfrak{s}}) \rtimes \Gamma_{\mathfrak{s}}$ denotes a product of a standard generator of $\mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^*, q_F^{1/2})$ and an element of $\Gamma_{\mathfrak{s}}$.

Theorem 3.3. (*a*) *There exists an algebra isomorphism*

$$\mathcal{H}(\mathfrak{s}) \cong \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$$

which extends the isomorphism $\mathcal{O}(T_{\mathfrak{s}}) \cong \mathcal{O}(T_{\mathfrak{s}^\vee})$ given by Theorem 2.5. This isomorphism is canonically determined up to:

(i) the action of $\mathrm{Out}(\mathcal{G})$,
(ii) conjugation by elements of $\mathcal{O}(T_{\mathfrak{s}})^\times$,
(iii) adjusting the image of $\Gamma_{\mathfrak{s}}$ in $\mathcal{H}(\mathfrak{s})$ by a character of $\Gamma_{\mathfrak{s}}$,
(iv) Let β be a short simple root in a root system B_{e_ρ}, and suppose that $R_{\mathfrak{s},\rho}$ has type D_{e_ρ} or that $R_{\mathfrak{s},\rho}$ has type B_{e_ρ} and $q^*_\beta = 1$. Then we may replace s_β by $h^\vee_\beta s_\beta \in X^*(T_{\mathfrak{s}}) \rtimes W(R_{\mathfrak{s}})$ and T'_{s_β} by $T'_{h^\vee_\beta s_\beta}$ in $\mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma^+_{\mathfrak{s}^\vee}$.

(*b*) *Composition with (3.15) yields an algebra isomorphism*

$$\mathcal{H}(\mathfrak{s})^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}),$$

which is exactly as canonical as part (a).

Remarks. The condition $q^*_\beta = 1$ in (iv) is equivalent with $\lambda(\beta) = \lambda^*(\beta)$, and also with $\ell_{\tau'} = 0, a_{\tau'} = -1$. Canonical choices for (iii) and (iv) will come from Proposition 3.5 and Paragraph 4.3. That will render the isomorphisms in Theorem 3.3 canonical up to $\mathrm{Out}(\mathcal{G})$ and inner automorphisms.

Proof. (a) From (1.33), Theorem 2.5 and Proposition 3.2 we get an algebra isomorphism

$$\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}) \cong \mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_{\mathfrak{s}}. \tag{3.16}$$

It is canonical up to the action of $\mathrm{Out}(\mathcal{G})$ on supercuspidal representations, see Theorem 2.4. We fix a bijection $\mathfrak{s}_L \to \mathfrak{s}_L^\vee$ as in Theorem 2.5, then we do not have to worry about $\mathrm{Out}(\mathcal{G})$ any more. We compose (3.16) with (3.14) to obtain the isomorphism in the statement.

Any two such isomorphisms differ by an automorphism ψ of $\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_\mathfrak{s}$. We need to investigate the possibilities for ψ. Since the isomorphism

$$\mathcal{O}(T_{\mathfrak{s}^\vee}) \cong \mathcal{O}(T_\mathfrak{s}) \subset \mathcal{H}(\mathfrak{s})$$

has been fixed, ψ is the identity on $\mathcal{O}(T_\mathfrak{s})$. Any such ψ extends naturally to an automorphism ψ_e of

$$\mathbb{C}(T_\mathfrak{s})^{W_\mathfrak{s}} \underset{\mathcal{O}(T_\mathfrak{s})^{W_\mathfrak{s}}}{\otimes} \mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_\mathfrak{s},$$

an algebra which by [Lus3, §5] is isomorphic to $\mathbb{C}(T_\mathfrak{s}) \rtimes W_\mathfrak{s}$. As ψ_e is the identity on $\mathbb{C}(T_\mathfrak{s})$ and $W_\mathfrak{s}$ acts faithfully on $T_\mathfrak{s}$, ψ_e must send any $w \in W_\mathfrak{s}$ to $\theta_w w$ for some $\theta_w \in \mathbb{C}(T_\mathfrak{s})^\times$.

For a simple reflection $s_\alpha \in W(R_\mathfrak{s})$ there are unique $f_1, f_2 \in \mathbb{C}(T_\mathfrak{s})$ such that $T'_{s_\alpha} = f_1 s_\alpha + f_2$, see [Lus3, 5.1.(a)]. Then

$$\psi(T'_{s_\alpha}) = f_1 \psi_e(s_\alpha) + f_2 = f_1 \theta_{s_\alpha} s_\alpha + f_2,$$

so by the invertibility of ψ we must have

$$\theta_{s_\alpha} \in \mathcal{O}(T_\mathfrak{s})^\times = \mathbb{C}^\times \times X^*(T_\mathfrak{s}).$$

Write $\theta_{s_\alpha} = z\theta_x$ with $z \in \mathbb{C}^\times$ and $x \in X^*(T_\mathfrak{s})$. (We write θ_x to emphasize that we regard x as an element of $\mathcal{O}(T_\mathfrak{s})$.) Then

$$1 = s_\alpha^2 = \psi_e(s_\alpha)^2 = (z\theta_x s_\alpha)^2 = z^2 \theta_x \theta_{s_\alpha(x)} s_\alpha^2 = z^2 \theta_{x + s_\alpha(x)}.$$

Hence $z = \pm 1$ and $s_\alpha(x) = -x$, which implies $x \in \mathbb{Z} h_\alpha^\vee$. (Here we do not use (3.12), in the sense that we do not replace B_{e_τ} even when that is possible.) For every $x \in \mathbb{Z} h_\alpha^\vee$, $f_1 \theta_x s_\alpha + f_2$ satisfies the same quadratic equation as T'_{s_α}, that follows from a computation in $\mathbb{C}(T_\mathfrak{s})$ which uses that $\theta_x s_\alpha$ is a reflection in the same direction as s_α. On the other hand $-f_1 \theta_x s_\alpha + f_2$ does not satisfy that quadratic relation, so $z = 1$ and

$$\psi_e(s_\alpha) = \theta_{n_\alpha h_\alpha^\vee} s_\alpha \qquad \text{for some } n_\alpha \in \mathbb{Z}. \tag{3.17}$$

Let $\alpha^\sharp \in R_{\mathfrak{s}^\vee}$ be the coroot associated to $h_\alpha^\vee \in R_\mathfrak{s}$. In the algebra $\mathbb{C}[\mathrm{Hom}(\mathbb{Z}R_{\mathfrak{s}^\vee}, \mathbb{Z}) \rtimes W(R_{\mathfrak{s}^\vee})]$, the element (3.17) can be rewritten as

$$\psi_e(s_\alpha) = \theta_y s_\alpha \theta_{-y} \quad \text{when } \langle y, \alpha^\sharp \rangle = n_\alpha.$$

Guided by this formula we define $y \in \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}R_{\mathfrak{s}^\vee}, \mathbb{Z})$ by $\langle y, \alpha^\sharp \rangle = n_\alpha$ for all simple coroots $\alpha^\sharp$. Embed $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}R_{\mathfrak{s}^\vee}, \mathbb{Z})$ in $\mathbb{Q}R_\mathfrak{s}$ and form the lattice

$$X_e := X^*(T_\mathfrak{s}) + \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}R_{\mathfrak{s}^\vee}, \mathbb{Z}) \subset X^*(T_\mathfrak{s}) \otimes_\mathbb{Z} \mathbb{Q}.$$

Then ψ_e extends to the automorphism of $\mathbb{C}[X_e \rtimes W(R_\mathfrak{s})]$, given by conjugation with θ_y. Hence ψ is also conjugation with θ_y, at least on $\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, \lambda^*, q_F^{1/2})$. For $y \in X^*(T_\mathfrak{s})$ that is simply an inner automorphism, which accounts for (ii).

There are only few other possible y. For each τ with $e_\tau > 0$, we have a direct summand

$$(\mathbb{Z}^{e_\tau}, R_{\mathfrak{s},\tau}, \mathbb{Z}^{e_\tau}, R_{\mathfrak{s}^\vee,\tau}) \quad \text{of} \quad \mathcal{R}_\mathfrak{s},$$

where $R_{\mathfrak{s},\tau}$ has type $A_{e_\tau - 1}, B_{e_\tau}$ or D_{e_τ}. For type $A_{e_\tau - 1}$, $\mathbb{Z}^{e_\tau}$ surjects onto $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}R_{\mathfrak{s}^\vee,\tau}, \mathbb{Z})$. Otherwise $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}R_{\mathfrak{s}^\vee,\tau}, \mathbb{Z})$ is spanned by $\mathbb{Z}^{e_\tau}$ and $y = (1, 1, \ldots, 1)/2$. Conjugation by θ_y on $\mathbb{C}[\mathbb{Z}^{e_\tau} \rtimes W(B_{e_\tau})]$ sends s_β to $h_\beta^\vee s_\beta$ and fixes the other simple reflections. When $R_{\mathfrak{s},\tau} \cong D_{e_\rho}$, this gives an automorphism of $\mathcal{H}(D_{e_\rho}, q_F^{1/2}) \rtimes \langle s_\beta \rangle$ and of $\mathcal{H}(\mathfrak{s})^{\mathrm{op}}$.

However, when $R_{\mathfrak{s},\tau}$ has type B_{e_ρ} conjugation by θ_y only extends to an automorphism of $\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, \lambda^*, q_F^{1/2})$ if $q_\beta^* = 1$, because the q-parameters $q_\beta q_\beta^*$ of s_β and $q_\beta (q_\beta^*)^{-1}$ of $h_\beta^\vee s_\beta$ need to be equal for such an automorphism. That gives the choices for ψ described in (iv). Notice that this excludes the cases C_{e_τ} that could arise via (3.12).

It remains to investigate automorphisms ψ of $\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_\mathfrak{s}$ that restrict to the identity on $\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, \lambda^*, q_F^{1/2})$. As above we deduce that for each $\gamma \in \Gamma_\mathfrak{s}$ there exist $z \in \{\pm 1\}$ and $x \in X^*(T_\mathfrak{s})$ such that $\psi(\gamma) = z\theta_x \gamma$. Just like conjugation by γ, conjugation by $\psi(\gamma)$ is a product of diagram automorphisms of D_{e_τ} on

$$\psi\big(\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, \lambda^*, q_F^{1/2})\big) = \mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, \lambda^*, q_F^{1/2}).$$

Hence $z\theta_x$ must lie in the centre of $\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, \lambda^*, q_F^{1/2})$, which means that $\langle x, \alpha^\sharp \rangle = 0$ for every coroot $\alpha^\sharp$. Looking at the rank of $R_{\mathfrak{s},\tau}$, we see that x lives only in the $\mathbb{Z}^{e_\tau}$ for which $R_{\mathfrak{s},\tau} \cong A_{e_\tau - 1}$. The part of x in the associated direct summand of $X^*(T_\mathfrak{s})$ is a multiple of $(1, 1, \ldots, 1)$. In particular θ_x commutes with γ. As γ has finite order in the finite group $\Gamma_\mathfrak{s}$:

$$1 = \gamma^{\mathrm{ord}\gamma} = \psi(\gamma)^{\mathrm{ord}\gamma} = (z\theta_x\gamma)^{\mathrm{ord}\gamma} = z^{\mathrm{ord}\gamma}\theta_x^{\mathrm{ord}\gamma}\gamma^{\mathrm{ord}\gamma} = z^{\mathrm{ord}\gamma}\theta_{\mathrm{ord}(\gamma)x}.$$

This implies that $\mathrm{ord}(\gamma)x = 0$ and $x = 0, \psi(\gamma) = \pm\gamma$. We deduce that there exists a character $\epsilon : \Gamma_\mathfrak{s} \to \{\pm 1\}$ such that $\psi(\gamma) = \epsilon(\gamma)\gamma$.
(b) This follows from part (a) and (3.14). $\square$

3.1. **Versions for G^+.**

There also exists a version of Theorem 3.3 for G^+. Let $\mathcal{L}^+ = \mathrm{Z}_{\mathcal{G}^+}(\mathrm{Z}(\mathcal{L})^\circ)$ be the F-Levi subgroup of $\mathcal{G}^+$ with identity component $\mathcal{L}$. It has the same shape:

$$L^+ = G_{n_-}^+ \times \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_k}(F). \tag{3.18}$$

Usually $[L^+ : L] = [G^+ : G]$, but there are exceptions to that. Recall from (1.13) that $G_{n_-} \subset GL(V_-)$ for some linear subspace $V_- \subset V$. If (and only if) $V_- = \{0\}$, we have $G_{n_-}^+ = G_{n_-} = \{e\}$ and $L^+ = L$. This can happen for the split groups $O(V)$ and $\mathrm{GSpin}(V)$ with $\dim V$ even, even though $G^+ \neq G$ for those groups.

The whole theory behind $\mathcal{H}(\mathfrak{s}^\vee, z)$ [AMS1, AMS2, AMS3] was written for possibly disconnected complex reductive groups, so it applies to G^+. The set of cuspidal Bernstein components in $\Phi_{\mathrm{e}}(L^+)$ is

$$\Phi_{\mathrm{cusp}}(L^+)/(\mathrm{Z}(L^+)^{\mathbf{I}_F,\circ})_{\mathbf{W}_F} = \Phi_{\mathrm{cusp}}(L^+)/\mathrm{Z}(L)^\circ.$$

An element in there is the same as an element $(\phi, \epsilon) \in \Phi_{\mathrm{cusp}}(L)/\mathrm{Z}(L)^\circ$ together with an extension of $\epsilon \in \mathrm{Irr}(\mathcal{S}_\phi)$ to $\epsilon^+ \in \mathrm{Irr}(\mathcal{S}_\phi^+)$. Let $\mathfrak{s}^{+\vee}$ denote the Bernstein component determined by (ϕ, ϵ^+), and similarly without the $+$. We note that there is a canonical bijection

$$\Phi_{\mathrm{e}}(L)^{\mathfrak{s}^\vee} \to \Phi_{\mathrm{e}}(L^+)^{\mathfrak{s}^{+\vee}} : (z\phi, \epsilon) \mapsto (z\phi, \epsilon^+).$$

The same arguments as in Paragraph 1.2 shows that

$$\mathcal{H}(\mathfrak{s}^{+\vee}, z) = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, z) \rtimes \Gamma_{\mathfrak{s}^\vee}^+. \tag{3.19}$$

The arguments in Paragraph 2.2 and for Theorem 3.1 lead to a canonical injection from the set of Bernstein components of $\mathrm{Irr}(G^+)$ to the set of Bernstein components in $\Phi_{\mathrm{e}}(G^+)$, say $\mathfrak{s}^+ \mapsto \mathfrak{s}^{+\vee}$. It relates to Theorem 3.1 by $\mathrm{Res}_G^{G^+}$, as in the proof of Theorem 2.4. On the level of representations and enhanced L-parameters of L^+, by Theorems 2.4 and 2.5 each instance $\mathfrak{s}^+ \mapsto \mathfrak{s}^{+\vee}$ comes a bijection

$$\mathrm{Irr}(L^+)^{\mathfrak{s}_L^+} \cong T_{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}^\vee} \cong \Phi_{\mathrm{e}}(L^+)^{\mathfrak{s}_L^{+\vee}}. \tag{3.20}$$

As justified by (3.20), we will sometimes write $T_{\mathfrak{s}^+}$ for $T_{\mathfrak{s}}$, or $T_{\mathfrak{s}^{+\vee}}$ for $T_{\mathfrak{s}^\vee}$.

The theory used to construct and analyse $\mathcal{H}(\mathfrak{s})$ is not known for arbitrary disconnected reductive groups. For $O(V)$ and $\mathrm{GPin}(V)$ (the only disconnected instances of G^+) we can work it out by hand though. First we need a good progenerator $\Pi_{\mathfrak{s}_L^+}$ for $\mathrm{Rep}(L^+)^{\mathfrak{s}_L^+}$ with $\mathfrak{s}_L = [L,\sigma]_L$. We start from $\Pi_{\mathfrak{s}_L} = \mathrm{ind}_{L^1}^L(\sigma_1)$, where L^1 is the subgroup of L generated by all compact subgroups and σ_1 is an irreducible constituent of $\mathrm{Res}^L_{L^1}\sigma$. We distinguish two cases.

Suppose first that $\mathrm{Out}(\mathcal{G}_{n_-})$ does not stabilize $\mathfrak{s}_L$, or that $L^+ = L$. Then $\mathrm{ind}_L^{L^+}(\sigma')$ is irreducible for all $\sigma' \in \mathrm{Irr}(L)^{\mathfrak{s}_L}$, and

$$\mathrm{ind}_L^{L^+}(\Pi_{\mathfrak{s}_L}) = \mathrm{ind}_{L^1}^{L^+}(\sigma_1) =: \Pi_{\mathfrak{s}_L^+} \tag{3.21}$$

is a progenerator of $\mathrm{Rep}(L^+)^{\mathfrak{s}_L^+}$ for the same reasons as for $\Pi_{\mathfrak{s}_L}$. If $L^+ = L$, then clearly $\mathrm{End}_{L^+}(\Pi_{\mathfrak{s}_L^+}) = \mathrm{End}_L(\Pi_{\mathfrak{s}_L})$. Otherwise, since L is normal in L^+,

$$\mathrm{Res}_L^{L^+}\mathrm{ind}_L^{L^+}(\Pi_{\mathfrak{s}_L}) = \Pi_{\mathfrak{s}_L} \oplus l \cdot \Pi_{\mathfrak{s}_L} = \Pi_{\mathfrak{s}_L} \oplus \Pi_{\mathfrak{s}'_L}$$

where $l \in L^+ \setminus L$ and $\mathfrak{s}'_L = l \cdot \mathfrak{s}_L$. Further, by Frobenius reciprocity

$$\mathrm{End}_{L^+}\big(\mathrm{ind}_L^{L^+}\Pi_{\mathfrak{s}_L}\big) \cong \mathrm{Hom}_L\big(\Pi_{\mathfrak{s}_L}, \mathrm{ind}_L^{L^+}\Pi_{\mathfrak{s}_L}\big) \cong \mathrm{Hom}_L\big(\Pi_{\mathfrak{s}_L}, \Pi_{\mathfrak{s}_L} \oplus \Pi_{\mathfrak{s}'_L}\big). \tag{3.22}$$

By the Bernstein decomposition of $\mathrm{Rep}(L)$ this equals $\mathrm{End}_L(\Pi_{\mathfrak{s}_L})$, which by (3.7) is naturally isomorphic with $\mathcal{O}(T_{\mathfrak{s}})$.

Suppose now that $\mathrm{Out}(\mathcal{G}_{n_-})$ stabilizes $\mathfrak{s}_L$ and $L^+ \neq L$. Since $X_{\mathrm{nr}}(G_{n_-}) = \{1\}$, $\mathrm{Out}(\mathcal{G}_{n_-})$ stabilizes every $\sigma' \in \mathrm{Irr}(L)^{\mathfrak{s}_L}$. Clifford theory tells us that σ extends in two ways to a representation of L^+, say σ^+ and σ^-. For an unramified character $\chi \in X_{\mathrm{nr}}(L^+) \cong X_{\mathrm{nr}}(L)$ we put

$$(\sigma \otimes \chi)^+ = \sigma^+ \otimes \chi \quad \text{and} \quad (\sigma \otimes \chi)^- = \sigma^- \otimes \chi.$$

This yields two Bernstein components $\mathrm{Irr}(L^+)^{\mathfrak{s}_L^+} = X_{\mathrm{nr}}(L^+)\sigma^+$ and $\mathrm{Irr}(L^+)^{\mathfrak{s}_L^-} = X_{\mathrm{nr}}(L^+)\sigma^-$, both naturally in bijection with $\mathrm{Irr}(L)^{\mathfrak{s}_L}$. We note that $\mathfrak{s}_L^+$ and $\mathfrak{s}_L^-$ are in different $\mathrm{N}_{G^+}(L^+)$-orbits, because they are inequivalent on $G^+_{n_-}$ and $\mathrm{N}_{G^+}(L^+)/G^+_{n_-}$ only adjusts $\mathrm{Irr}_{\mathrm{cusp}}(L^+)$ on the type GL factors of L^+. The Bernstein decomposition for L^+ enables us to write

$$\mathrm{ind}_L^{L^+}(\Pi_{\mathfrak{s}_L}) = \Pi_{\mathfrak{s}_L^+} \oplus \Pi_{\mathfrak{s}_L^-} \quad \text{with } \Pi_{\mathfrak{s}_L^\pm} \in \mathrm{Rep}(L^+)^{\mathfrak{s}_L^\pm}. \tag{3.23}$$

Then $\Pi_{\mathfrak{s}_L^+}$ is a progenerator of $\mathrm{Rep}(L^+)^{\mathfrak{s}_L^+}$ and its restriction to L is just $\mathrm{ind}_{L^1}^L(\sigma) = \Pi_{\mathfrak{s}_L}$. All the elements of $\mathcal{O}(T_{\mathfrak{s}})$ determine L^+-endomorphisms of $\mathrm{ind}_{L^{+1}}^{L^+}(\sigma^+)$, so

$$\mathcal{O}(T_{\mathfrak{s}}) = \mathrm{End}_L(\Pi_{\mathfrak{s}_L}) = \mathrm{End}_{L^+}(\Pi_{\mathfrak{s}_L^+}).$$

In both above cases we constructed a progenerator $\Pi_{\mathfrak{s}_L^+}$ of $\mathrm{Rep}(L^+)^{\mathfrak{s}_L^+}$, with L^+-endomorphism algebra $\mathcal{O}(T_\mathfrak{s})$. We define

$$\Pi_{\mathfrak{s}^+} = I_{P^+}^{G^+}(\Pi_{\mathfrak{s}_L^+}),$$

where P^+ is the semidirect product of L^+ and the unipotent radical of P.

Proposition 3.4. *The representation $\Pi_{\mathfrak{s}^+}$ is a canonical progenerator of $\mathrm{Rep}(G^+)^{\mathfrak{s}^+}$.*

Induction from G to G^+ gives an injective algebra homomorphism $\mathrm{End}_G(\Pi_\mathfrak{s}) \to \mathrm{End}_{G^+}(\Pi^{\mathfrak{s}^+})$, which is bijective when $\mathrm{Out}(\mathcal{G}_{n_-})\mathfrak{s}_L = \mathfrak{s}_L$.

Proof. The L^+-representation $\Pi_{\mathfrak{s}_L^+}$ is canonical because in both cases (3.21) and (3.23) it arises naturally from the L-representation $\Pi_{\mathfrak{s}_L}$, which we already knew is canonical (3.1).

Suppose that $\mathrm{Out}(\mathcal{G}_{n_-})\mathfrak{s}_L \neq \mathfrak{s}_L$, or that $L^+ = L$. Then

$$\Pi_{\mathfrak{s}^+} = I_{P^+}^{G^+}\big(\mathrm{ind}_L^{L^+}(\Pi_{\mathfrak{s}_L})\big) = \mathrm{ind}_G^{G^+}\big(I_P^G(\Pi_{\mathfrak{s}_L})\big) = \mathrm{ind}_G^{G^+}(\Pi_\mathfrak{s}). \tag{3.24}$$

Now $\mathrm{ind}_G^{G^+}$ yields an algebra homomorphism

$$\mathrm{End}_G(\Pi_\mathfrak{s}) \to \mathrm{End}_{G^+}(\mathrm{ind}_G^{G^+}\Pi_\mathfrak{s}) = \mathrm{End}_{G^+}(\Pi_{\mathfrak{s}^+}), \tag{3.25}$$

which is injective because $\Pi_\mathfrak{s} \subset \mathrm{Res}_G^{G^+}\Pi_{\mathfrak{s}^+}$. As G is open in G^+, $\mathrm{ind}_G^{G^+}$ preserves projectivity. Moreover G has finite index in G^+, so (3.24) shows that $\Pi_{\mathfrak{s}^+}$ is finitely generated and projective. For any nonzero $\tau \in \mathrm{Rep}(G)^{\mathfrak{s}^+}$, the part of $\mathrm{Res}_G^{G^+}\tau$ in $\mathrm{Rep}(G)^\mathfrak{s}$ generates τ so is nonzero. Hence

$$\mathrm{Hom}_{G^+}(\Pi_{\mathfrak{s}^+}, \tau) = \mathrm{Hom}_G(\Pi_\mathfrak{s}, \tau) \neq 0,$$

which shows that $\Pi_{\mathfrak{s}^+}$ generates $\mathrm{Rep}(G^+)^{\mathfrak{s}^+}$.

Next we suppose that $\mathrm{Out}(\mathcal{G}_{n_-})\mathfrak{s}_L = \mathfrak{s}_L$ and $L^+ \neq L$. Then

$$\mathrm{Res}_G^{G^+}\Pi_{\mathfrak{s}^+} = I_P^G(\Pi_{\mathfrak{s}_L^+}\big|_L) = I_P^G(\Pi_{\mathfrak{s}_L}) = \Pi_\mathfrak{s}, \tag{3.26}$$

$$\begin{aligned} \mathrm{ind}_G^{G^+}(\Pi_\mathfrak{s}) &= I_{P^+}^{G^+}\big(\mathrm{ind}_L^{L^+}(\Pi_{\mathfrak{s}_L})\big) = I_{P^+}^{G^+}\big(\Pi_{\mathfrak{s}_L^+} \oplus \mathrm{ind}_{L^{+1}}^{L^+}(\sigma^-)\big) \\ &= I_{P^+}^{G^+}(\Pi_{\mathfrak{s}_L^+}) \oplus I_{P^+}^{G^+}\big(\mathrm{ind}_{L^{+1}}^{L^+}(\sigma^-)\big) = \Pi_{\mathfrak{s}^+} \oplus I_{P^+}^{G^+}(\Pi_{\mathfrak{s}_L^-}). \end{aligned} \tag{3.27}$$

Since $\mathfrak{s}_L^+$ and $\mathfrak{s}_L^-$ are in different $\mathrm{N}_{G^+}(L^+)$-orbits, $\mathfrak{s}^+ \neq \mathfrak{s}^- = [L^+, \sigma^-]_{G^+}$. By the Bernstein decomposition $\mathrm{Rep}(G^+)^{\mathfrak{s}^+}$ and $\mathrm{Rep}(G^+)_{\mathfrak{s}^-}$ are orthogonal subcategories of $\mathrm{Rep}(G^+)$, so

$$\mathrm{End}_{G^+}(\mathrm{ind}_G^{G^+}(\Pi_\mathfrak{s}) = \mathrm{End}_{G^+}\big(\Pi_{\mathfrak{s}^+} \oplus \Pi_{\mathfrak{s}^-}\big) = \mathrm{End}_{G^+}(\Pi_{\mathfrak{s}^+}) \oplus \mathrm{End}_{G^+}(\Pi_{\mathfrak{s}^-}). \tag{3.28}$$

From (3.28), $\mathrm{ind}_G^{G^+}$ and (3.26) we obtain algebra homomorphisms

$$\mathrm{End}_G(\Pi_\mathfrak{s}) \to \mathrm{End}_{G^+}(\Pi_{\mathfrak{s}^+}) \to \mathrm{End}_G(\Pi_{\mathfrak{s}^+}) = \mathrm{End}_G(\Pi_\mathfrak{s}). \tag{3.29}$$

The composition of these homomorphisms is the identity and $\mathrm{End}_{G^+}(\Pi_{\mathfrak{s}^+})$ is naturally a subalgebra of $\mathrm{End}_G(\Pi_{\mathfrak{s}^+})$, from which we conclude that (3.29) consists of isomorphisms.

By the same argument as in the first part, $\mathrm{ind}_G^{G^+}(\Pi_\mathfrak{s})$ is finitely generated and projective. In view of (3.27), so is its direct summand $\Pi_{\mathfrak{s}^+}$. Let $\tau \in \mathrm{Rep}(G^+)_{\mathfrak{s}^+}$ be nonzero. By (3.27)

$$\mathrm{Hom}_{G^+}(\Pi_{\mathfrak{s}^+}, \tau) = \mathrm{Hom}_{G^+}(\Pi_{\mathfrak{s}^+} \oplus \Pi_{\mathfrak{s}^-}, \tau) = \mathrm{Hom}_{G^+}\big(\mathrm{ind}_G^{G^+}\Pi_\mathfrak{s}, \tau\big) = \mathrm{Hom}_G(\Pi_\mathfrak{s}, \tau).$$

As we already saw above, the right hand side is nonzero. Therefore $\Pi_{\mathfrak{s}^+}$ is indeed a progenerator of $\mathrm{Rep}(G^+)_{\mathfrak{s}^+}$. $\square$

We define $\mathcal{H}(\mathfrak{s}^+) = \mathrm{End}_{G^+}(\Pi_{\mathfrak{s}^+})$, then Proposition 3.4 shows that there is an equivalences of categories

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}}) & \xrightarrow{\sim} & \mathrm{Rep}(G)^{\mathfrak{s}^+} \\ V & \mapsto & V \otimes_{\mathcal{H}(\mathfrak{s}^+)} \Pi_{\mathfrak{s}^+} \end{array} . \tag{3.30}$$

Proposition 3.5. *There exists an algebra isomorphism*

$$\mathcal{H}(\mathfrak{s}^+) \cong \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2}) = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma^+_{\mathfrak{s}^\vee} \cong \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})^{\mathrm{op}}.$$

It extends the isomorphism $\mathcal{O}(T_\mathfrak{s}) \cong \mathcal{O}(T_{\mathfrak{s}^\vee})$ induced by (3.20) and is canonical up to the operations (ii),(iii),(iv) in Theorem 3.3.

Proof. With the progenerators $\Pi_{\mathfrak{s}^+}$ at hand, the paper [Sol4] also applies to G^+. Therefore all the arguments in Section 3 remain valid. The only difference with the proof of Theorem 3.3 is that we do not have to replace $W^+_\mathfrak{s}$ by $W_\mathfrak{s}$ any more. $\square$

From the above proof we see that in the description of Proposition 3.5 the map $\mathcal{H}(\mathfrak{s}) \to \mathcal{H}(\mathfrak{s}^+)$ from Proposition 3.4 becomes just the inclusion

$$\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_{\mathfrak{s}^\vee} \longrightarrow \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma^+_{\mathfrak{s}^\vee}. \tag{3.31}$$

Remark. In Proposition 4.6 we will fix choices for (iii) and (iv) from Proposition 3.5, depending only on a Whittaker datum for the quasi-split inner form of G. That will make the isomorphisms in Proposition 3.5 canonical up to inner automorphisms. Via (3.31), that also determines choices for (iii) and (iv) in Theorem 3.3.

Lemma 3.6. (*a*) *Suppose that $\mathrm{Out}(\mathcal{G}_{n_-})\mathfrak{s}_L = \mathfrak{s}_L$. Then the restriction map $\mathrm{Rep}(G^+)^{\mathfrak{s}^+} \to \mathrm{Rep}(G)^{\mathfrak{s}}$ is an equivalence of categories.*

(*b*) *Suppose that $\mathrm{Out}(\mathcal{G}_{n_-})\mathfrak{s}_L \neq \mathfrak{s}_L$ and that all the direct factors $\mathrm{GL}_m(F)$ of L have m even. Then $\mathrm{ind}_G^{G^+} : \mathrm{Rep}(G)^{\mathfrak{s}} \to \mathrm{Rep}(G^+)^{\mathfrak{s}^+}$ is an equivalence of categories.*

(*c*) *In the remaining cases $\mathrm{Rep}(G)^{\mathfrak{s}}$ and $\mathrm{Rep}(G^+)^{\mathfrak{s}^+}$ are not naturally equivalent.*

Proof. (a) Via (3.26) and (3.31), the restriction is induced by the algebra homomorphism $\mathcal{H}(\mathfrak{s}) \to \mathcal{H}(\mathfrak{s}^+)$. In Proposition 3.4 we saw that it is an isomorphism.
(b) The second condition implies that $\mathrm{N}_{G^+}(L^+)/L^+ \cong N_G(L)/L$. Hence $\Gamma^+_\mathfrak{s} = \Gamma_\mathfrak{s}$, which together with (3.31) means that the map $\mathcal{H}(\mathfrak{s}) \to \mathcal{H}(\mathfrak{s}^+)$ from Proposition 3.4 is an algebra isomorphism. That yields equivalences of categories

$$\begin{array}{ccccccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \leftrightarrow & \mathrm{Mod}(\mathcal{H}(\mathfrak{s})^{\mathrm{op}}) & \leftrightarrow & \mathrm{Mod}(\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}}) & \leftrightarrow & \mathrm{Rep}(G^+)^{\mathfrak{s}^+} \\ V \otimes_{\mathcal{H}(\mathfrak{s})} \Pi_\mathfrak{s} & \leftarrow\!\shortmid & V & \leftrightarrow & V^+ & \mapsto & V^+ \otimes_{\mathcal{H}(\mathfrak{s}^+)} \Pi_{\mathfrak{s}^+} \end{array} . \tag{3.32}$$

By the first condition, (3.24) holds. Hence $V \otimes_{\mathcal{H}(\mathfrak{s})} \Pi_\mathfrak{s}$ is mapped by (3.32) to

$$V^+ \otimes_{\mathcal{H}(\mathfrak{s}^+)} \Pi_{\mathfrak{s}^+} = V \otimes_{\mathcal{H}(\mathfrak{s})} \mathrm{ind}_G^{G^+}(\Pi_\mathfrak{s}) = \mathrm{ind}_G^{G^+}\big(V \otimes_{\mathcal{H}(\mathfrak{s})} \Pi_\mathfrak{s}\big).$$

(c) The assumption says that L has a direct factor $\mathrm{GL}_m(F)$ with m odd, and that $\mathrm{Out}(\mathcal{G}_{n_-})\mathfrak{s}_L = \{\mathfrak{s}_L, \mathfrak{s}'_L\}$ with $\mathfrak{s}'_L = l^- \cdot \mathfrak{s}_L \neq \mathfrak{s}_L$ for any $l^- \in G^+_{n_-} \setminus G_{n_-}$. Consider an element $s_\alpha \in N_{G^+}(L^+)$ which acts in this factor $\mathrm{GL}_m(F)$ by $g \mapsto \hat{J} g^{-T} \hat{J}^{-1}$ and on L as in (3.4). Then $\det(s_\alpha) = -1$ because m is odd, so $s_\alpha L \notin W_\mathfrak{s}$. On the other hand $s_\alpha l^-$ stabilizes $\mathfrak{s}^+_L$, so $s_\alpha l^- L^+ = s_\alpha L^+ \in W^+_\mathfrak{s}$. Thus $W_\mathfrak{s} \neq W^+_\mathfrak{s}$, which by (3.31) means that the inclusion $\mathcal{H}(\mathfrak{s}) \to \mathcal{H}(\mathfrak{s}^+)$ is not an isomorphism. $\square$

3.2. **Langlands parameters via Hecke algebras.**

Let $\mathfrak{s} = [L, \sigma]_G$ be an inertial equivalence class for G. Recall the canonical progenerator $\Pi_{\mathfrak{s}}$ and the equivalence of categories

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, ?) : \mathrm{Rep}(G)^{\mathfrak{s}} \xrightarrow{\sim} \mathrm{Mod}(\mathcal{H}(\mathfrak{s})^{\mathrm{op}}) = \mathrm{Mod}(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{\mathrm{op}}) \tag{3.33}$$

from (3.2). Let us fix an isomorphism as in Theorem 3.3.a. That and (3.15) induce equivalences of categories

$$\mathrm{Mod}(\mathcal{H}(\mathfrak{s})^{\mathrm{op}}) \cong \mathrm{Mod}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})^{\mathrm{op}}\big) \cong \mathrm{Mod}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big). \tag{3.34}$$

From (3.33) and (3.34) (or (3.30) and Proposition 3.5 for G^+) we obtain equivalences of categories

$$\begin{array}{lll} \mathrm{Rep}(G)^{\mathfrak{s}} & \cong & \mathrm{Mod}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big), \\ \mathrm{Rep}(G^+)^{\mathfrak{s}^+} & \cong & \mathrm{Mod}\big(\mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})\big). \end{array} \tag{3.35}$$

It was shown in [AMS3, Theorem 3.18] that there is a canonical bijection

$$\mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big) \longleftrightarrow \Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee}, \tag{3.36}$$

and similarly for G^+. This proceeds via reduction to the graded Hecke algebras mentioned around (1.19), which are then studied in terms of varieties of Langlands parameters, perverse sheaves and equivariant homology [AMS2]. Disconnected complex reductive groups and the associated Hecke algebras are an integral part of [AMS2, AMS3], and therefore (3.36) works in the same way for G^+ as for G. Following [AMS3], we denote the image of $(\phi, \epsilon) \in \Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee}$ under (3.36) by

$$\bar{M}(\phi, \epsilon, q_F^{1/2}) \in \mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big).$$

Remark. In (3.36) ϵ is a representation of $\mathcal{S}_\phi$, a group in which all elements have order at most two. Therefore ϵ may identified with its contragredient $\epsilon^\vee$. By construction [AMS2, AMS3] $\bar{M}(\phi, \epsilon, q_F^{1/2})$ is the unique irreducible quotient of a standard $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-module

$$\bar{E}(\phi, \epsilon, q_F^{1/2}) = \mathrm{Hom}_{\mathcal{S}_\phi}\big(\epsilon, \bar{E}(\phi, q_F^{1/2})\big). \tag{3.37}$$

Reinterpreting ϵ as $\epsilon^\vee$, that becomes the irreducible quotient of

$$\mathrm{Hom}_{\mathcal{S}_\phi}\big(\epsilon^\vee, \bar{E}(\phi, q_F^{1/2})\big) = \big(\epsilon \otimes \bar{E}(\phi, q_F^{1/2})\big)^{\mathcal{S}_\phi}.$$

This might be the most natural setup for comparison with endoscopic methods, that is indicated for instance by [MoRe, §2] and [Kal2, Definition 2.7.6].

Let $\bar{M}(\phi, \epsilon, q_F^{1/2})^{\mathrm{op}}$ be $\bar{M}(\phi, \epsilon^\vee, q_F^{1/2})$ considered as irreducible right $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-module via (3.15). Via Theorem 3.3.a it corresponds to an irreducible right $\mathcal{H}(\mathfrak{s})$-module, to which we can apply (3.33).

Theorem 3.7. *The maps* (3.33), (3.34) *and* (3.36) *induce a bijection*

$$\begin{array}{ccc} \mathrm{Irr}(G)^{\mathfrak{s}} & \longleftrightarrow & \Phi_{\mathrm{e}}(G)^{\mathfrak{s}^\vee} \\ \pi(\phi, \epsilon) & \leftarrow\!\shortmid & (\phi, \epsilon) \end{array}.$$

It satisfies the following properties:

(*a*) *The cuspidal support maps form a commutative diagram*

$$\begin{array}{ccc} \mathrm{Irr}(G)^{\mathfrak s} & \longleftrightarrow & \Phi_{\mathrm e}(G)^{\mathfrak s^\vee} \\ \downarrow \mathrm{Sc} & & \downarrow \mathrm{Sc} \\ \mathrm{Irr}(L)^{\mathfrak s_L} / W_{\mathfrak s} & \longleftrightarrow & \Phi_{\mathrm{cusp}}(L)^{\mathfrak s^\vee} / W^{\mathfrak s^\vee} \end{array} .$$

In particular (ϕ,ϵ) *is cuspidal if and only if* $\pi(\phi,\epsilon)$ *is supercuspidal.*

(*b*) $\pi(\phi,\epsilon)$ *is essentially square-integrable if and only if* ϕ *is discrete.*

(*c*) $\pi(\phi,\epsilon)$ *is tempered if and only if* ϕ *is bounded.*

(*d*) *For any* $\chi \in X_{\mathrm{nr}}(G)$*, corresponding to* $\hat\chi \in (\mathrm{Z}(G^\vee)^{\mathbf{I}_F,\circ})_{\mathbf{W}_F}$*, there is a canonical isomorphism* $\pi(\hat\chi\phi,\epsilon) = \chi \otimes \pi(\phi,\epsilon)$*.*

(*e*) *The* $\mathrm{Z}(G)_s$*-character of* $\pi(\phi,\epsilon)$ *equals the character of* $\mathrm{Z}(G)_s$ *determined by the image of* ϕ *in* $\Phi(\mathrm{Z}(G)_s)$*.*

(*f*) *Suppose that Theorem 3.3 can be made canonical, up to* $\mathrm{Out}(\mathcal G)$ *and inner automorphisms of the involved algebras. Then the above bijection is canonical up to* $\mathrm{Out}(\mathcal G)$*.*

All the above statements also hold with G^+ *instead of* G*. Then part (f) relies on a canonical version of Proposition 3.5, and we can omit* $\mathrm{Out}(\mathcal G)$*.*

Remark. Parts (b)–(e) were already predicted in [Bor, §10]. In fact Borel formulated more general versions of (d) and (e), which in principle can also be checked in our setup. We refrain from taking that up here, because it will boil down to properties of endoscopy which fall outside the scope of this paper. The canonicity requirements in part (f) will be established in Proposition 4.6.

Proof. (a) The central character of $\bar M(\phi,\epsilon,q_F^{1/2})$ is described in [AMS3, Theorem 3.18.a]. It lies in $T_{\mathfrak s^\vee}/W_{\mathfrak s^\vee}$ and by construction equals $W_{\mathfrak s^\vee}\mathrm{Sc}(\phi,\epsilon)$. Similarly the central character of $\mathrm{Hom}_G(\Pi_{\mathfrak s},\pi(\phi,\epsilon)) \in \mathrm{Irr}(\mathcal H(\mathfrak s)^{\mathrm{op}})$ lies in $T_{\mathfrak s}/W_{\mathfrak s}$ and by [Sol2, Condition 4.1 and Lemma 6.1] it equals $W_{\mathfrak s}\mathrm{Sc}(\pi(\phi,\epsilon))$.
(b) By [Sol2, Theorem 4.9.a] the map (3.33) respects temperedness. The equivalence (3.34) does so as well, because by Proposition 3.2 the isomorphism in Theorem 3.3.b preserves the notion of positive roots (which determines the conditions for temperedness, see e.g. [Sol2, p. 215]). By [AMS3, Theorem 3.18.c], under the map (3.36) temperedness of irreducible representations corresponds to boundedness of (enhanced) L-parameters.
(c) This is similar to part (b), now we use [Sol2, Theorem 4.9 and Proposition 4.10], Proposition 3.2 and [AMS3, Theorem 3.18.d].
(d) This follows from [Sol2, Lemma 4.3.c] and [AMS3, Theorem 3.18.e].
(e) First we reduce to the cuspidal case. Clearly $\pi(\phi,\epsilon)$ and $\mathrm{Sc}(\pi(\phi,\epsilon))$ have the same $\mathrm{Z}(G)$-character. Recall that $Z(G)_s{}^\vee = G^\vee/G^\vee{}_{\mathrm{der}} \cong \mathbb C^\times$. The quotient map $G^\vee \to Z(G)_s{}^\vee$ is the similitude character $\mu_G^\vee$, so the image of ϕ in $\Phi(Z(G)_s)$ is $\mu_G^\vee \circ \phi$. The cuspidal support map for enhanced L-parameters only changes things in $G^\vee{}_{\mathrm{der}}$ (and modifies the enhancements), so $\mu_G^\vee \circ \phi = \mu_G^\vee \circ \phi_c$ where $\mathrm{Sc}(\phi,\epsilon) = (\phi_c,\epsilon_c)$. In view of part (a), (ϕ_c,ϵ_c) is the enhanced L-parameter of $\mathrm{Sc}(\pi(\phi,\epsilon)) =: \pi_c$.

The GL-factors of $L^\vee$ lie in $G^\vee{}_{\mathrm{der}}$, so they are contained in the kernel of $\mu_G^\vee$. Hence $\mu_G^\vee \circ \phi$ depends only on the component of ϕ_c in $G^\vee_{n_-}$, let us call the latter ϕ_-. On the other hand, $Z(G)_s$ is contained in the factor G_{n_-} of L, so the $Z(G)_s$-character of π_c depends only on the component of π_c in G_{n_-}, say $\pi_- \in \mathrm{Irr}_{\mathrm{cusp}}(G_{n_-})$.

It remains to compare the $\mathrm{Z}(G)_s$-character ν_{π_-} of π_- with $\mu_G^\vee \circ \phi_-$. Those agree by (2.8).

(f) This holds because it can be checked at every step in the construction. The progenerator $\Pi_{\mathfrak s}$ and the bijections (3.33), (3.36) are always canonical, and now by assumption (3.34) is canonical up to $\mathrm{Out}(\mathcal G)$.

The proof for G^+ is basically the same. To get the bijection we use (3.30) and Proposition 3.5 instead of (3.33) and (3.34). Although in [Sol2] the group $\mathcal G$ is connected, the parts that we use work just as well for $\mathcal G^+$. For parts (d) and (e) it is helpful to note that

$$X_{\mathrm{nr}}(G) \cong X_{\mathrm{nr}}(G^+), \quad \mathrm{Z}(G^{+\vee}) = \mathrm{Z}(G^\vee) \quad \text{and} \quad \mathrm{Z}(\mathcal G) = \mathrm{Z}(\mathcal G^+). \qquad \square$$

Recall from Theorem 3.1 that the map $\mathfrak s \mapsto \mathfrak s^\vee$ between sets of Bernstein components for G and $G^\vee$ is injective. From that, (3.35) and Theorem 3.7 we conclude:

Corollary 3.8. *Let $\Omega^\vee(G)$ be the image of the map from Theorem 3.1 in the set of Bernstein components for $\Phi_{\mathrm e}(G)$, and define $\Omega^\vee(G^+)$ analogously. Theorem 3.1 and (3.35) provide equivalences of categories*

$$\begin{array}{lllll} \mathrm{Rep}(G) & \to & \prod_{\mathfrak s^\vee \in \Omega^\vee(G)} \mathrm{Mod}\big(\mathcal H(\mathfrak s^\vee, q_F^{1/2})\big) & = & \mathrm{Mod}\Big(\bigoplus_{\mathfrak s^\vee \in \Omega^\vee(G)} \mathcal H(\mathfrak s^\vee, q_F^{1/2})\Big), \\ \mathrm{Rep}(G^+) & \to & \prod_{\mathfrak s^{+\vee} \in \Omega^\vee(G^+)} \mathrm{Mod}\big(\mathcal H(\mathfrak s^{+\vee}, q_F^{1/2})\big) & = & \mathrm{Mod}\Big(\bigoplus_{\mathfrak s^{+\vee} \in \Omega^\vee(G^+)} \mathcal H(\mathfrak s^{+\vee}, q_F^{1/2})\Big). \end{array}$$

For irreducible representations the equivalences of categories and (3.36) provide injections, which are unions of instances of Theorem 3.7:

$$\begin{array}{lll} \mathrm{Irr}(G) & \to & \bigsqcup_{\mathfrak s^\vee \in \Omega^\vee(G)} \Phi_{\mathrm e}(G)^{\mathfrak s^\vee}, \\ \mathrm{Irr}(G^+) & \to & \bigsqcup_{\mathfrak s^{+\vee} \in \Omega^\vee(G^+)} \Phi_{\mathrm e}(G^+)^{\mathfrak s^{+\vee}}. \end{array}$$

The image of the parametrization maps in Corollary 3.8 is a union of Bernstein components of enhanced L-parameters. Surjectivity on the cuspidal level in Theorem 2.1.d would imply surjectivity in Corollary 3.8, then $\Omega^\vee(G)$ would be the set of all Bernstein components in $\Phi_{\mathrm e}(G)$. That is known when F is a p-adic field, from [Art] and [MoRe]. When F is a local function field, that surjectivity has been shown for symplectic and for split special orthogonal groups, assuming $p > 2$ [GaVa]. By Proposition 2.3, the Hypothesis 2.2 suffices to obtain such surjectivity.

Suppose now that $M \subset G$ is a Levi subgroup which contains L. It is a direct product of a group of the same type as G and of factors $\mathrm{GL}_m(F)$, so all the previous results apply just as well to L. Then $\mathcal H(\mathfrak s_M) = \mathrm{End}_M(\Pi_{\mathfrak s_M})$ embeds in $\mathcal H(\mathfrak s)$ via normalized parabolic induction and $\mathcal H(\mathfrak s_M^\vee, q_F^{1/2})$ embeds naturally in $\mathcal H(\mathfrak s^\vee, q_F^{1/2})$. As isomorphism

$$\mathcal H(\mathfrak s_M)^{\mathrm{op}} \cong \mathcal H(\mathfrak s_M^\vee, q_F^{1/2}) \tag{3.38}$$

we can simply take the restriction of $\mathcal H(\mathfrak s)^{\mathrm{op}} \cong \mathcal H(\mathfrak s^\vee, q_F^{1/2})$ from Theorem 3.3.b. The same works for $M^+ \subset G^+$, using Paragraph 3.1. In this setting we can compare the equivalences of categories (3.35) and their analogues for M, M^+, using normalized parabolic induction.

Let $\bar E(\phi, \epsilon, q_F^{1/2})$ be the standard $\mathcal H(\mathfrak s^\vee, q_F^{1/2})$-module associated to (ϕ, ϵ) in [AMS3, §2.2 and Theorem 3.18]. By definition $\bar M(\phi, \epsilon, q_F^{1/2})$ is the unique irreducible quotient ("Langlands quotient") of $\bar E(\phi, \epsilon, q_F^{1/2})$. We let $\pi_{st}(\phi, \epsilon)$ be the image of $\bar E(\phi, \epsilon, q_F^{1/2})$

under (3.35), and we use analogous notations for G^+, M, M^+, with superscripts M or $+$. Let us point out that for bounded ϕ (and in fact for almost all ϕ):

$$\bar{E}(\phi, \epsilon, q_F^{1/2}) = \bar{M}(\phi, \epsilon, q_F^{1/2}) \quad \text{and} \quad \pi_{st}(\phi, \epsilon) = \pi(\phi, \epsilon).$$

Theorem 3.9. *Let $(\phi, \epsilon^M) \in \Phi_{\mathrm{e}}(M)^{\mathfrak{s}_M}$ be bounded, or a twist of a bounded parameter by an element of $\mathrm{Z}(M^\vee)$ which is positive with respect to $M^\vee B^\vee$ in the sense of* [AMS2, Appendix A]*. Then*

$$I_{MU}^G\big(\pi_{st}(\phi, \epsilon^M)\big) \cong \bigoplus\nolimits_\epsilon \mathrm{Hom}_{\mathcal{S}_\phi^M}(\epsilon^M, \epsilon) \otimes \pi_{st}(\phi, \epsilon),$$

where the sum runs over all $\epsilon \in \mathrm{Irr}(\mathcal{S}_\phi)$ with $\mathrm{Sc}(\phi, \epsilon) = \mathrm{Sc}(\phi, \epsilon^M)$. The same holds for $M^+ \subset G^+$.

Proof. By [AMS3, Lemma 3.19.a] this holds for $\bar{E}(\phi, \epsilon^M, q_F^{1/2})$ and $\mathrm{ind}_{\mathcal{H}(\mathfrak{s}_M^\vee, q_F^{1/2})}^{\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})}$. We note that the condition in [AMS3, Lemma 3.19.a] is fulfilled by [AMS2, Proposition A.3] and the assumed properties of ϕ. Via (3.34) and (3.38) we obtain the corresponding statement for modules of $\mathcal{H}(\mathfrak{s})^{\mathrm{op}}$ and $\mathcal{H}(\mathfrak{s}_M)^{\mathrm{op}}$. By [Sol2, Condition 4.1 and Lemma 6.1] the equivalences (3.33) commute with normalized parabolic induction, which enables us to transfer the statement to representations of G and M. The same proof works for $M^+ \subset G^+$. □

4. Comparison of Langlands parameters

In this section we will compare the enhanced L-parameters for G obtained via the endoscopic methods of Arthur and Mœglin with the enhanced L-parameters associated to irreducible $\mathcal{G}(F)$-representations in Theorem 3.7. Although endoscopy only seems to be available when F is a p-adic field, in Paragraph 2.1 we showed how the resulting parametrization can be transferred to classical groups over local function fields. That requires Hypothesis 2.2 (which we hope to lift in the future). Then Mœglin's constructions to find enhanced L-parameters make sense for any classical group over a non-archimedean local field. Since that applies to G^+ rather than G, we will focus on G^+-representations in this paragraph.

We will compare them with our method via Hecke algebras, increasing the classes of representations under consideration step by step. For supercuspidal representations the enhanced L-parameters in Theorem 3.7 are by definition equal to those constructed in Theorems 2.1 and 2.4. The relation between the discrete series and the supercuspidal representations of classical groups is due to Mœglin and Tadić [Mœ1, MoTa], also proven with different methods by Kim and Matić [KiMa].

4.1. Cuspidal supports of essentially square-integrable representations.

There are cuspidal support maps both for irreducible G-representations and for $\Phi_{\mathrm{e}}(G)$. Recall from Theorem 3.7.a that these maps commute with the assignment of enhanced L-parameters via Hecke algebras. We want to check that the same holds for Mœglin's parameters of discrete series representations. (Since the cuspidal support maps commute with tensoring by unramified characters, that implies the same statement for essentially square-integrable representations.) The initial steps to determine the cuspidal support of $(\phi, \epsilon) \in \Phi_{\mathrm{e}}(G)$ are:

- Replace (ϕ, ϵ) by $\phi|_{\mathbf{W}_F}$ and $(\phi|_{\mathrm{SL}_2(\mathbb{C})}, \epsilon)$, where $\phi(\mathrm{SL}_2(\mathbb{C}))$ lies in $H := \mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi(\mathbf{W}_F))$ and
$$\mathcal{S}_\phi = \pi_0(\mathrm{Z}_{G^\vee{}_{\mathrm{der}}}(\phi)) = \pi_0\big(\mathrm{Z}_H(\phi|_{\mathrm{SL}_2(\mathbb{C})})\big).$$
- From $(\phi|_{\mathrm{SL}_2(\mathbb{C})}, \epsilon)$ we extract the triple
$$s_\phi = \phi\begin{pmatrix} q_F^{1/2} & 0 \\ 0 & q_F^{-1/2}\end{pmatrix},\ u_\phi = \phi\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix},\ \epsilon \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_H(s_\phi, u_\phi))\big).$$
Such triples can be regarded as H-valued enhanced L-parameters which are trivial on $\mathbf{W}_F$, and that provides a notion of cuspidal support for such triples. Up to H-conjugacy the triple $(s_\phi, u_\phi, \epsilon)$ contains precisely the same information as $(\phi|_{\mathrm{SL}_2(\mathbb{C})}, \epsilon)$.
- The cuspidal support of $(s_\phi, u_\phi, \epsilon)$, in the group H, is another triple (t, v, ϵ_c) with $t \in H$ conjugate to $s_\phi, v \in H$ unipotent, $tvt^{-1} = v^{q_F}$ and $\epsilon_c \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_H(t, v))\big)$.
- The cuspidal support of (ϕ, ϵ) is an enhanced L-parameter (ϕ_c, ϵ_c) reconstructed from $(\phi|_{\mathbf{W}_F}, t, v, \epsilon_c)$, so with $\phi_c\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right) = v$ and $\phi_c\Big(w, \begin{pmatrix} q_F^{1/2} & 0 \\ 0 & q_F^{-1/2}\end{pmatrix}\Big) = \phi(w)t$ for any arithmetic Frobenius element $w \in \mathbf{W}_F$.

We work this out further for discrete enhanced L-parameters of G^+. (That is a little easier than for G, and yields basically the same information.) From (1.11) we know that $H = \mathrm{Z}_{G^{+\vee}{}_{\mathrm{der}}}(\phi(\mathbf{W}_F))$ is a direct product of orthogonal and symplectic groups over $\mathbb{C}$. To complete the above characterization of $\mathrm{Sc}(\phi, \epsilon)$, it suffices to describe the cuspidal support for triples (s, u, ϵ) in $\mathrm{O}_n(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$. For that we use the detailed analysis from [Lus1] and [Mou, §5]. Fortunately, it turns out that there are only very few possibilities for the cuspidal supports [Lus1, §10].

Symplectic case

Take a Levi subgroup $L_d = \mathrm{Sp}_{d(d+1)}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})^{n-d(d+1)/2}$ of $\mathrm{Sp}_{2n}(\mathbb{C})$ and let $u_d \in \mathrm{Sp}_{d(d+1)}$ be a unipotent element with Jordan blocks of sizes $\{2, 4, \ldots, 2d\}$. Take any semisimple element $s \in L_d$ with $s u_d s^{-1} = u^{q_F}$. Then $\pi_0\big(\mathrm{Z}_{\mathrm{Sp}_{2n}(\mathbb{C})}(s, u_d)\big) \cong \mathbb{F}_2^d$ with basis $\{z_2, z_4, \ldots z_{2d}\}$, and $\epsilon_d(z_{2j}) = (-1)^j$ gives a cuspidal triple (s, u_d, ϵ_d).

Given a triple (s, u, ϵ), the only options for $\mathrm{Sc}(s, u, \epsilon)$ are (s, u_d, ϵ_d) with $d \in \mathbb{Z}_{\geq 0}$. We write
$$d' = \begin{cases} d+1 & \text{if } d \text{ is even,} \\ -d & \text{if } d \text{ is odd.} \end{cases}$$
In [Lus1, §12] the cuspidal support of (u, ϵ) is computed via this number d', which is called the defect of (u, ϵ). Assume for simplicity that all Jordan blocks of u have different size $i_1, i_2, \ldots, i_r$ which are even (this is the case if (s, u) comes from a discrete L-parameter). Write $\pi_0\big(\mathrm{Z}_{\mathrm{Sp}_{2n}(\mathbb{C})}(s, u)\big) = \mathbb{F}_2^r$ with basis $\{z_{i_1}, z_{i_2}, \ldots, z_{i_r}\}$. If r is even, we define a new $\tilde{\epsilon}$ by adding $i_0 = 0$ with $\epsilon'(z_0) = 1$, apart from that $\epsilon' = \epsilon$. Then the advanced combinatorics in [Lus1, §11] entails that $d' = \sum_j (-1)^{j+r} \epsilon'(z_{i_j}) \in 1 + 2\mathbb{Z}$. Hence
$$d = \begin{cases} -1 + \sum_j (-1)^{j+r} \epsilon'(z_{i_j}) & \text{if } d' > 0, \\ -\sum_j (-1)^{j+r} \epsilon'(z_{i_j}) & \text{if } d' < 0. \end{cases} \tag{4.1}$$

Orthogonal case

Take a Levi subgroup $L_d = \mathrm{O}_{d^2}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})^{(n-d^2)/2}$ of $\mathrm{O}_n(\mathbb{C})$ (so with $d \equiv n \bmod 2$)

and let $u_d \in \mathrm{O}_{d^2}(\mathbb{C})$ be a unipotent element with Jordan blocks of sizes $(1, 3, \ldots, 2d-1)$. Let $s \in L_d$ be semisimple such that $s u_d s^{-1} = u_d^{q_F}$. Then $\pi_0(\mathrm{Z}_{\mathrm{O}_{d^2}(\mathbb{C})}(s, u_d)) \cong \mathbb{F}_2^d$ with basis $\{z_1, z_3, \ldots, z_{2d-1}\}$ and $\epsilon_d(z_{2j-1}) = (-1)^j$ and $-\epsilon_d$ give two cuspidal triples $(s, u_d, \pm\epsilon_d)$.

Given a triple (s, u, ϵ) for $\mathrm{O}_n(\mathbb{C})$, the options for $\mathrm{Sc}(s, u, \epsilon)$ are $(s, u_d, \pm\epsilon_d)$ with $d \in \mathbb{Z}_{\geq 0}$ of the same parity as n. In this case d is the defect of (u, ϵ) [Lus1, §13]. Suppose that all Jordan blocks of u have different sizes $1_1, i_2, \ldots, i_r$, which are all odd (as for discrete L-parameters). Then [Lus1, §13] entails that

$$d = \big| \sum\nolimits_j (-1)^j \epsilon(z_{i_j}) \big|. \tag{4.2}$$

By that and [AMS1], $\mathrm{Sc}(s, u, \epsilon) = (s, u_d, \pm\epsilon_d)$ where the sign is determined by $\pm\epsilon_d(z) = \epsilon(z)$ for some $z \in \mathrm{O}_{d^2}(\mathbb{C}) \setminus \mathrm{SO}_{d^2}(\mathbb{C})$. We embed $\mathrm{O}_{d^2}(\mathbb{C})$ in $\mathrm{O}_n(\mathbb{C})$ so that the subgroup $\mathrm{O}_1(\mathbb{C}) \subset \mathrm{Z}_{\mathrm{O}_{d^2}(\mathbb{C})}(u_d)$, which comes from the Jordan block of size 1, is contained in a subgroup $\mathrm{O}_{i_m}(\mathbb{C}) \subset \mathrm{Z}_{\mathrm{O}_n(\mathbb{C})}(u)$ which comes from a Jordan block of size i_m. Then we take $z = z_1$, and we find (using that i_m is odd)

$$\pm\epsilon_d(z_1) = \epsilon(z_1) = \epsilon(z_1)^{i_m} = \epsilon(z_{i_m}). \tag{4.3}$$

This determines the sign, and thus fixes $\mathrm{Sc}(s, u, \epsilon)$.

Proposition 4.1. *Mœglin's parametrization of the discrete series of G^+ is compatible with the cuspidal support maps, in the following sense. For a discrete series representation $\pi \in \mathrm{Irr}(G^+)$ with $\mathrm{Sc}(\pi) \in \mathrm{Irr}(L^+)$, $\mathrm{Sc}(\phi_\pi, \epsilon_\pi)$ is $\mathrm{N}_{G^{+\vee}}(L^{+\vee})$-conjugate to $(\phi_{\mathrm{Sc}(\pi)}, \epsilon_{\mathrm{Sc}(\pi)})$.*

Proof. In [Mœ1] the cuspidal support of π is studied in relation with ϕ_π and ϵ_π. The Mœglin parameter of $\mathrm{Sc}(\pi)$ is obtained via a recursive procedure, whose important steps are mentioned on [Mœ1, p. 147].

Suppose first that $a, a' \in \mathrm{Jord}_\rho(\pi)$ are adjacent (that is, no b inbetween a and a' belongs to $\mathrm{Jord}_\rho(\pi)$) and that $\epsilon_\pi(\rho, a) = \epsilon_\pi(\rho, a')$. Then $\{(\rho, a), (\rho, a')\}$ can be removed from $\mathrm{Jord}(\pi)$, and the new $(\mathrm{Jord}', \epsilon')$ corresponds to a discrete series representation with the same cuspidal support as π (apart from $(a + a')d_\rho$ extra factors $\mathrm{GL}_1(\mathbb{C})$ in the Levi subgroup from $\mathrm{Sc}(\pi)$). This enables us to reduce to the cases where ϵ_π is alternated in the sense that $\epsilon_\pi(\rho, a) = -\epsilon_\pi(\rho, a')$ whenever $a, a' \in \mathrm{Jord}_\rho(\pi)$ are adjacent.

Suppose now that ϵ_π is alternated.

(i) If $\mathrm{Jord}_\rho(\pi)$ consists of even numbers a and $\epsilon_\pi(\rho, a) = -1$ for the minimal such a, then $\mathrm{Jord}_\rho(\mathrm{Sc}(\pi)) = \{2, 4, \ldots, 2d\}$ with $d = |\mathrm{Jord}_\rho(\pi)|$ and $\epsilon_{\mathrm{Sc}(\pi)}(\rho, 2a) = (-1)^a$.
(ii) If $\mathrm{Jord}_\rho(\pi)$ consists of even numbers a and $\epsilon_\pi(\rho, a) = 1$ for the minimal such a, $\mathrm{Jord}_\rho(\mathrm{Sc}(\pi)) = \{2, 4, \ldots, 2d\}$ with $d = |\mathrm{Jord}_\rho(\pi)| - 1$ and $\epsilon_{\mathrm{Sc}(\pi)}(\rho, 2a) = (-1)^a$.
(iii) If $\mathrm{Jord}_\rho(\pi)$ consists of odd numbers, then $\mathrm{Jord}_\rho(\mathrm{Sc}(\pi)) = \{1, 3, \ldots, 2d-1\}$ where $d = |\mathrm{Jord}_\rho(\pi)|$ and $\epsilon_{\mathrm{Sc}(\pi)}(\rho, 1) = \epsilon(\rho, a)$ for the minimal $a \in \mathrm{Jord}_\rho(\pi)$. This last property is implicit on [Mœ1, p. 147], which mentions that here ϵ does not change if we pass from π to $\mathrm{Sc}(\pi)$.

If we now compute the above numbers d in terms of the original ϵ_π, we recover precisely (4.1) and (4.2). In case (iii) we can embed $\mathrm{O}_{d^2}(\mathbb{C})$ in $\mathrm{O}_n(\mathbb{C})$ such that the part $P_{2a-1} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}_{2a-1}(\mathbb{C})$ of $\mathrm{Jord}_\rho(\mathrm{Sc}(\pi))$ lands in the subgroup $\mathrm{O}_{2i_a-1}(\mathbb{C}) \subset$

$\mathrm{O}_n(\mathbb{C})$ that contains the image of the part P_{2i_a-1} of $\mathrm{Jord}_\rho(\pi)$. Then the aforementioned property $\epsilon_{\mathrm{Sc}(\pi)}(\rho, 1) = \epsilon(\rho, a)$ becomes (4.3). □

4.2. **Jordan blocks of discrete series representations.**

First we check that the Jordan blocks of a discrete series representation π of $G^+ = G_n^+$ can be read off directly from the enhanced L-parameter assigned to it by Theorem 3.7.

Lemma 4.2. *Let $(\phi, \epsilon) \in \Phi_{\mathrm{e}}(G^+)$ be bounded and discrete, such that $\pi(\phi, \epsilon)$ is defined in Theorem 3.7. Then* $\mathrm{Jord}(\pi(\phi, \epsilon))$ *corresponds to* $\mathrm{Jord}(\phi)$ *under the local Langlands correspondence for general linear groups.*

Proof. Let $\rho \in \mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}_{d_\rho}(F))$ with $\rho \cong \rho^\vee \otimes \nu_{\pi(\phi,\epsilon)}$, a condition which by (2.1) is fulfilled by all Jordan blocks of $\pi(\phi, \epsilon)$. Recall that a pair (ρ, a) belongs to $\mathrm{Jord}(\pi(\phi, \epsilon))$ if and only if

- $\delta(\rho, a) \times \pi(\phi, \epsilon)$ is irreducible and
- $\delta(\rho, a') \times \pi(\phi, \epsilon)$ is reducible for some $a' \in a + 2\mathbb{Z}$.

Let $\tau \in \mathrm{Irr}(\mathbf{W}_F)$ be the L-parameter of ρ. Then $\tau \cong \tau^\vee \otimes \mu_G^\vee \circ \phi$ by (2.7) and Theorem 3.7.e, as needed for $\mathrm{Jord}(\phi)$ by (1.6). We write

$$\psi = \tau \otimes P_a \times \phi = (\tau \oplus \tau^\vee \otimes \mu_G^\vee \circ \phi) \otimes P_a \oplus \phi,$$

where in the middle we work in $\mathrm{GL}_m(F) \times G_n^+$ and on the right in G_{n+m}^+. Theorem 3.9 tells us that

$$\delta(\rho, a) \times \pi(\phi, \epsilon) = \bigoplus\nolimits_\eta \mathrm{Hom}_{\mathcal{S}_\phi^+}(\epsilon, \eta) \otimes \pi(\psi, \eta), \tag{4.4}$$

where the sum runs over all $\eta \in \mathcal{S}_\psi^+$ with $\mathrm{Sc}(\psi, \eta) = \mathrm{Sc}(\psi, \epsilon)$. The groups $\mathcal{S}_\phi^+$ and $\mathcal{S}_\psi^+$ can be compared with (1.5).

(i) When $\mathrm{sgn}(\tau \otimes P_a) \neq \mathrm{sgn}({G^\vee}_{\mathrm{der}})$: $\mathcal{S}_\psi^+ = \mathcal{S}_\phi^+$ and (4.4) is always reducible.

(ii) When $\mathrm{sgn}(\tau \otimes P_a) = \mathrm{sgn}({G^\vee}_{\mathrm{der}})$ and $(\tau, a) \in \mathrm{Jord}(\phi)$: again $\mathcal{S}_\psi^+ = \mathcal{S}_\phi^+$ and (4.4) is reducible.

(iii) When $\mathrm{sgn}(\tau \otimes P_a) = \mathrm{sgn}({G^\vee}_{\mathrm{der}})$ and $(\tau, a) \notin \mathrm{Jord}(\phi)$: $\mathcal{S}_\psi^+ = \mathcal{S}_\phi^+ \times \{1, z_{\tau,a}\}$. Then $\mathrm{Sc}(\psi, \eta) = \mathrm{Sc}(\psi, \epsilon)$ for every extension η of ϵ to $\mathcal{S}_\psi^+$, because $\tau \otimes P_a$ occurs with even multiplicity in ψ and hence does not influence the cuspidal support. In this case (4.4) is a direct sum of two inequivalent irreducible representations.

We compare this with the aforementioned characterization of $\mathrm{Jord}(\pi(\phi, \epsilon))$. The reducibility of $\delta(\rho, a') \times \pi(\phi, \epsilon)$ rules out case (i), and $\delta(\rho, a) \times \pi(\phi, \epsilon)$ is reducible in case (ii) but not in case (iii). We conclude that $(\rho, a) \in \mathrm{Jord}(\pi(\phi, \epsilon))$ if and only if $(\tau, a) \in \mathrm{Jord}(\phi)$. □

Recall that the parametrization of the discrete series in Theorem 2.1 involves the Jordan blocks of ϕ and a character $\epsilon_\pi : \mathcal{S}_\pi \to \{\pm 1\}$. To facilitate a comparison with our Hecke algebra methods, we revisit Mœglin's construction of ϵ_π [Mœ1] and we show that it shares some properties with the constructions behind Theorem 3.7.

Let $(\phi, \epsilon) \in \Phi_{\mathrm{e}}(G^+)^{\mathfrak{s}^{+\vee}}$ be discrete and bounded, and let $\pi(\phi, \epsilon)$ be the discrete series representation of $G^+ = G_n^+$ associated to it by Theorem 3.7. Recall that $\mathcal{S}_\phi^+$ is the $\mathbb{F}_2$-vector space with basis $\{z_{\tau,a} : (\tau, a) \in \mathrm{Jord}(\phi)\}$. For such a τ we let ρ be the corresponding representation of $\mathrm{GL}_{d_\rho}(F)$ and we write $\epsilon(z_{\rho,a}) = \epsilon(z_{\tau,a})$. Here we use that by Lemma 4.2 the Jordan blocks of ϕ and of $\pi(\phi, \epsilon)$ are matched via the local Langlands correspondence for general linear groups.

Proposition 4.3. *Let $a > a' \in \mathrm{Jord}_\rho(\pi(\phi,\epsilon))$ be adjacent.*

(*a*) *$\epsilon(z_{\rho,a}) = \epsilon(z_{\rho,a'})$ if and only if $\pi(\phi,\epsilon)$ embeds in $\delta(\rho,(a-1)/2,(1-a')/2) \times \tilde\pi$ for some discrete series representation $\tilde\pi$ of $G^+_{n-d_\rho(a+a')/2}$. Moreover, in this case*

$$\mathrm{Jord}(\tilde\pi) = \mathrm{Jord}(\pi(\phi,\epsilon)) \setminus \{(\rho,a),(\rho,a')\}$$

and $\tilde\pi = \pi(\tilde\phi,\tilde\epsilon)$ where $\tilde\epsilon = \epsilon|_{\mathcal{S}_{\tilde\phi}}$.

(*b*) *Suppose that a_- is the minimal element of $\mathrm{Jord}_\tau(\phi)$ and that it is even. Then part (a) also holds with $a = a_-$, $a' = 0$, provided we put $\epsilon(z_{\rho,0}) = 1$.*

Proof. (a) Suppose that $\pi(\phi,\epsilon)$ is a subrepresentation of $\delta(\rho,(a-1)/2,(1-a')/2) \times \tilde\pi$. We write $M = \mathrm{GL}_{d_\rho(a+a')/2} \times G_{n-d_\rho(a+a')/2}$, so that MU is a parabolic subgroup of G_n with Levi factor M. From Theorem 3.7 we get $\tilde\pi = \pi(\tilde\phi,\tilde\epsilon)$ for some discrete bounded $\tilde\phi \in \Phi(M^+)$. The L-parameter of

$$\delta(\rho,(a-1)/2,(1-a')/2) \quad \text{is} \quad \tau \otimes P_{(a+a')/2} \otimes |\cdot|^{(a-a')/4},$$

and $|\cdot|^{(a-a')/4}$ is in positive position with respect to M^+U. Thus Theorem 3.9 is applicable, and it says that

$$\begin{aligned}\delta(\rho,(a-1)/2,(1-a')/2) \times \tilde\pi &= I_{M^+U}^{G^+} \pi\big(\tau \otimes P_{(a+a')/2} \otimes |\cdot|^{(a-a')/4} \times \tilde\phi, \tilde\epsilon\big) \\ &= \bigoplus\nolimits_{\epsilon'} \mathrm{Hom}_{\mathcal{S}_\phi^{M+}}(\tilde\epsilon,\epsilon') \otimes \pi\big(P_{(a+a')/2} \otimes |\cdot|^{(a-a')/4} \times \tilde\phi, \epsilon'\big),\end{aligned}$$

where the sum runs over all $\epsilon' \in \mathrm{Irr}(\mathcal{S}_\phi^+)$ with

$$\mathrm{Sc}(\phi,\epsilon) = \mathrm{Sc}\big(\tau \otimes P_{(a+a')/2} \otimes |\cdot|^{(a-a')/4} \times \tilde\phi, \epsilon'\big).$$

It follows that ϕ is $G^{+\vee}$-conjugate to $P_{(a+a')/2} \otimes |\cdot|^{(a-a')/4} \times \tilde\phi$ and that $\mathrm{Hom}_{\mathcal{S}_\phi^{M+}}(\tilde\epsilon,\epsilon)$ is nonzero. We deduce that

$$\mathcal{S}_\phi^+ = \mathcal{S}_{\tilde\phi}^+ \times \langle z_{\tau,a}, z_{\tau',a}\rangle \tag{4.5}$$

and that $\tilde\epsilon = \epsilon|_{\mathcal{S}_{\tilde\phi}^+}$. Our assumption entails that $\pi(\phi,\epsilon)$ and $\delta(\rho,(a-1)/2,(1-a')/2) \times \tilde\pi$ have the same cuspidal support. Now Theorem 3.7.a and the formulas (4.1) and (4.2) for the cuspidal support of enhanced L-parameters show that $\epsilon(z_{\rho,a}) = \epsilon(z_{\rho,a'})$.

Conversely, suppose that $\epsilon(z_{\rho,a}) = \epsilon(z_{\rho,a'})$. Write ϕ as L-parameter $\phi_{a,a'} \times \tilde\phi$ for M. Then (4.5) holds and we can take $\tilde\epsilon = \epsilon|_{\mathcal{S}_{\tilde\phi}^+}$. The L-parameter $\phi_{a,a'} \in \Phi(\mathrm{GL}_{(a+a')/2}(F))$ is discrete and its cuspidal support consists of terms $\tau|\cdot|^r$ with $r \in \mathbb{R}$. Hence $\phi_{a,a'} = \tau \otimes P_{(a+a')/2} \otimes |\cdot|^r$ for some $r \in \mathbb{R}$. Embedding in $\Phi(G_n)$ and comparing with the shape of ϕ we find

$$P_{(a+a')/2} \otimes |\cdot|^r \oplus P_{(a+a')/2} \otimes |\cdot|^{-r} = P_a \oplus P_{a'}, \tag{4.6}$$

or at least up to conjugation in $\mathrm{GL}_{a+a'}(\mathbb{C})$. That entails $r = (a-a')/4$, from which we deduce that

$$\pi(\phi_{a,a'}) = \delta(\rho,(a+a')/2) \otimes |\cdot|^{(a-a')/4} = \delta(\rho,(a-1)/2,(1-a')/2).$$

By Theorem 3.9

$$I_{M^+U}^{G^+} \pi\big(\delta(\rho,(a-1)/2,(1-a')/2) \boxtimes \pi(\tilde\phi,\tilde\epsilon)\big) = \bigoplus\nolimits_{\epsilon'} \mathrm{Hom}_{\mathcal{S}_\phi^{M+}}(\tilde\epsilon,\epsilon') \otimes \pi(\phi,\epsilon'), \tag{4.7}$$

where the sum runs over all $\epsilon' \in \mathrm{Irr}(\mathcal{S}_\phi^+)$ with

$$\mathrm{Sc}(\phi, \epsilon') = \mathrm{Sc}\big(\phi_{a,a'} \times \tilde\phi, \tilde\epsilon\big). = \mathrm{Sc}(\phi, \epsilon).$$

We note that in (4.7) we may use irreducible representations instead of the standard modules from Theorem 3.9, because the latter are irreducible (since ϕ and $\tilde\phi$ are bounded and $\phi_{a,a'}$ is a twist of a discrete parameter by an unramified character).

To get a nonzero contribution to (4.7), $\epsilon'|_{\mathcal{S}_{\tilde\phi}^+}$ must equal $\tilde\epsilon = \epsilon|_{\mathcal{S}_{\tilde\phi}^+}$. Then we see from (4.1) and (4.2) that $\epsilon'(z_{\rho,a}) = \epsilon'(z_{\rho,a'})$. In other words, the only nontrivial contributions to (4.7) come from ϵ and one other ϵ', and it reduces to

$$I_{M^+U}^{G^+} \pi\big(\delta(\rho, (a-1)/2, (1-a')/2) \boxtimes \pi(\tilde\phi, \tilde\epsilon)\big) = \pi(\phi, \epsilon) \oplus \pi(\phi, \epsilon'). \tag{4.8}$$

In particular $\pi(\phi, \epsilon)$ embeds in the left hand side of (4.8), which can be written as

$$\delta(\rho, (a-1)/2, (1-a')/2) \times \pi(\tilde\phi, \tilde\epsilon).$$

As $\tilde\phi$ is discrete and bounded, Theorem 3.7.b,c guarantees that $\pi(\tilde\phi, \tilde\epsilon)$ belongs to the discrete series.

That proves the equivalence. The description of $\mathrm{Jord}(\tilde\pi)$ occurs at various places in the above arguments, it is seen most clearly from (4.6).

(b) This can be shown in the same way as part (a). Notice that a_- needs to be even to make sense of the $\mathrm{SL}_2(\mathbb{C})$-representation $P_{a_-/2}$. $\square$

It was shown in [Mœ1, Proposition 5.3 and Lemme 5.4] that the Mœglin parameters of discrete series representations also satisfy Proposition 4.3.

Next we zoom in on a particular class defined in [Mœ1, §1], completely positive discrete series representations. By [Mœ1, Proposition 5.3], among the discrete series these are precisely the π for which ϵ_π is alternated:

$$\epsilon(z_{\rho,a}) = -\epsilon(z_{\rho,a'}) \text{ for adjacent } a, a' \in \mathrm{Jord}_\rho(\pi). \tag{4.9}$$

A few useful properties of such representations follow directly from our description of the cuspidal support maps.

Corollary 4.4. *Let π be a completely positive discrete series representation of G^+.*
(a) π is uniquely determined by $\mathrm{Jord}(\pi)$ and $\mathrm{Sc}(\pi)$.
(b) $\mathrm{Jord}_\rho(\pi) \neq \emptyset$ if and only if $\mathrm{Jord}_\rho(\mathrm{Sc}(\pi)) \neq \emptyset$.

Proof. (a) Under the condition (4.9) we see from (4.1), (4.2) and (4.3) that ϵ_π is uniquely determined by ϕ_π and $\mathrm{Sc}(\phi_\pi, \epsilon_\pi)$. Combining that with Theorem 2.1 and Proposition 4.1, we see that π is uniquely determined by $\mathrm{Jord}(\pi)$ and $\mathrm{Sc}(\pi)$.
(b) This follows from (4.1) and (4.2): under the condition (4.9) these numbers d cannot be 0. $\square$

Lemma 4.5. *Let π be a completely positive discrete series representation of G^+ and let (ϕ_π, ϵ_π) be its Mœglin parameter. Then the G^+-representation π' attached to (ϕ_π, ϵ_π) by Theorem 3.7 is isomorphic to π.*

Proof. By Theorem 3.7.b,c π' is discrete series and from Lemma 4.2 we know that $\mathrm{Jord}(\pi')$ and $\mathrm{Jord}(\pi)$ both correspond to $\mathrm{Jord}(\phi_\pi)$, so π and π' have precisely the same Jordan blocks. By Theorem 3.7.a and Proposition 4.1 both $\mathrm{Sc}(\pi)$ and $\mathrm{Sc}(\pi')$ have enhanced L-parameter $\mathrm{Sc}(\phi_\pi, \epsilon_\pi)$, so $\mathrm{Sc}(\pi) \cong \mathrm{Sc}(\pi')$.
By (4.9) and Proposition 4.3.a π' cannot be embedded in $\delta(\rho, (a-1)/2, (1-a')/2) \times \tilde\pi$ for adjacent $a > a' \in \mathrm{Jord}_\rho(\pi) = \mathrm{Jord}_\rho(\pi')$ and a discrete series representation $\tilde\pi$.

Then [Mœ1, §5] entails that π' is a completely positive discrete series representation. Now Corollary 4.4.a shows that $\pi \cong \pi'$. $\square$

4.3. **Intertwining operators for discrete series representations.**
For general discrete series representations π, Proposition 4.3 achieves a kind of reduction to the completely positive instances without changing cuspidal supports. Indeed, when π is not completely positive, Proposition 4.3 allows us to replace it by $\tilde{\pi}$, to which we can apply the same considerations. By recursion that yields a completely positive discrete series representation that we denote π^+. In the process some direct factors of $\mathcal{S}_\phi$ are removed, so we lose information about ϵ_π. Most values of ϵ_π can be reconstructed from data for π^+, but not all. For the missing ones we will need to study certain normalized intertwining operators.

Suppose that $(\rho, a) \in \mathrm{Jord}(\pi)$ with a odd and that $\mathrm{Jord}_\rho(\pi^+)$ is empty. Such ρ provide the only parts of ϵ_π that cannot be recovered from ϵ_{π^+}. We note the L-parameters of such ρ are precisely the $\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi$ for which $\ell_\tau = 0 < e_\tau$.

By Corollary 4.4.b we may equally well assume that $\mathrm{Jord}_\rho(\mathrm{Sc}(\pi))$ is empty. Then Proposition 4.3 leaves two possibilities for ϵ_π on $\mathrm{Jord}_\rho(\pi)$, distinguished by $\epsilon_\pi(z_{\rho,a_-})$ where $a_- = \min(\mathrm{Jord}_\rho(\pi))$. The characterization of $\epsilon_\pi(z_{\rho,a_-}) \in \{\pm 1\}$ from [Mœ1, §6.1.1] involves several steps, which we recall next. Write

$$\mathrm{Sc}(\pi) = \sigma_1 \boxtimes \cdots \boxtimes \sigma_d \boxtimes \sigma_- \in \mathrm{Irr}(L^+),$$

where $\sigma_i \in \mathrm{Irr}(\mathrm{GL}_{n_i}(F))$ and $\sigma_- \in \mathrm{Irr}(G^+_{n_-})$. Then σ_- is the partial cuspidal support of π, as used in [Mœ1, (1)]. There is a holomorphic family of intertwining operators

$$J(s_\beta, \rho\nu^b \times \sigma_-) \in \mathrm{Hom}_{G^+_{n_- + d_\rho}}(\rho\nu^b \times \sigma_-, \rho\nu^{-b} \times \sigma_-), \tag{4.10}$$

where $b \in \mathbb{C}$ and $\nu(g) = |\det(g)|_F$. This family can be normalized via a choice of an intertwining operator in the case $b = 0$, an element

$$J(s_\beta, \rho \times \sigma_-) \in \mathrm{End}_{G^+_{n_- + d_\rho}}(\rho \times \sigma_-). \tag{4.11}$$

There are two possibilities which square to the identity, we choose the one from [Mœ1, §6.1.2] and [Art, §2.4], which is determined by compatibility with endoscopy and depends on a Whittaker datum for the quasi-split inner form of G.

Remark. The normalization of the operators (4.11) will make Theorem 3.3 and Proposition 3.5 canonical (up to inner automorphisms).

From (4.10) one obtains a family of intertwining operators

$$J(s_\beta \times s_\beta, \rho\nu^{b_1} \times \rho\nu^{b_2} \times \sigma_-) : \rho\nu^{b_1} \times \rho\nu^{b_2} \times \sigma_- \to \rho\nu^{-b_1} \times \rho\nu^{-b_2} \times \sigma_-, \tag{4.12}$$

which reduces to (4.10) (tensored with the identity on one of the $\rho\nu^{b_i}$) upon applying normalized Jacquet restriction. The same works with more factors $\rho\nu^{b_i}$.

In $\mathrm{GL}_{2d_\rho}(F)$, the element s_{12} that exchanges the two blocks of $\mathrm{GL}_{d_\rho}(F) \times \mathrm{GL}_{d_\rho}(F)$ induces an intertwining operator

$$J(s_{12}, \rho\nu^{b_1} \times \rho\nu^{b_2} \times \sigma_-) \in \mathrm{Hom}_{G^+_{n_- + 2d_\rho}}(\rho\nu^{b_1} \times \rho\nu^{b_2} \times \sigma_-, \rho\nu^{b_2} \times \rho\nu^{b_1} \times \sigma_-), \tag{4.13}$$

where $b_1, b_2 \in \mathbb{C}$. We normalize it so that it depends holomorphically on $b_1 - b_2$ and becomes the identity when $b_1 = b_2$.

Let $e \in \mathbb{N}$ be odd and let $b_1, b_2, \ldots, b_{(e-1)/2} \in \mathbb{C}$. The order two permutation

$$w_e := (s_\beta, s_\beta, \ldots, s_\beta) \circ (1\ e)(2\ e{-}1) \cdots ((e{-}1)/2\ (e{+}3)/2)$$

belongs to the Weyl group $W(B_e)$. The composition of the corresponding operators (4.12) and (4.13) yields an intertwining operator

$$J(w_e, \rho\nu^{b_1} \times \cdots \times \rho\nu^{b_{(e-1)/2}} \times \rho \times \rho\nu^{-b_{(e-1)/2}} \times \cdots \times \rho\nu^{-b_1} \times \sigma_-), \tag{4.14}$$

from the indicated $G^+_{n_-+ed_\rho}$-representation to itself. The upshot of [Mœ1, p. 176] is that the holomorphic family (with variables b_i) of intertwining operators (4.14) can be normalized so that each operator (4.14) squares to the identity and they reduce to (4.12) in the special case $b_i = 0$ for all i. All these intertwining operators are unique up to scalars, so our conditions leave just the choice of a sign, which in turn is determined by $J(s_\beta, \rho \times \sigma_-)$.

Pick $a \in \mathrm{Jord}_\rho(\pi) \setminus \{a_-\}$ with $\epsilon(z_{\rho,a}) = \epsilon(z_{\rho,a_-})$, and embed π in

$$\delta(\rho,(a-1)/2,(1-a_-)/2) \times \tilde\pi \quad \subset \quad \delta(\rho,(a-1)/2,(1+a_-)/2) \times \delta(\rho,a_-) \times \tilde\pi \tag{4.15}$$

for a discrete series representation $\tilde\pi$ of $G^+_{n-(a+a_-)/2}$. Embed the right hand side in

$$\delta(\rho,(a-1)/2,(1+a_-)/2) \times \mathrm{Ind}\,\mathrm{Sc}(\delta(\rho,a_-)) \times \mathrm{Ind}\,\mathrm{Sc}(\tilde\pi), \tag{4.16}$$

where Ind stands for normalized parabolic induction. We note that σ_- is a factor of $\mathrm{Sc}(\tilde\pi)$ and that

$$\mathrm{Ind}\,\mathrm{Sc}(\delta(\rho,a_-)) = \rho\nu^{(a_--1)/2} \times \cdots \times \rho\nu \times \rho \times \rho\nu^{-1} \times \cdots \times \rho\nu^{(1-a_-)/2},$$

which fits with (4.14). Then (4.14) and the identity on the other factors of (4.16) induce a self-intertwining operator of (4.16). That operator can be restricted to (4.15) and thus yields a normalized intertwining operator

$$N(\rho,a_-) \in \mathrm{End}_{G^+}\big(\delta(\rho,(a-1)/2,(1+a_-)/2) \times \delta(\rho,a_-) \times \tilde\pi\big).$$

Then $\epsilon_\pi(z_{\rho,a_-})$ is the scalar by which $N(\rho,a_-)$ acts on π, or equivalently

$$\pi \text{ is fixed pointwise by } \epsilon_\pi(z_{\rho,a_-}) N(\rho,a_-). \tag{4.17}$$

We emphasize that the one choice of $J(s_\beta, \rho \times \sigma_-)$ determines a normalization for (potentially) many instances of (4.17).

In the Hecke algebras $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ and $\mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$ we also have intertwining operators, they come from the underlying geometric setup [AMS2]. In the general setting of Theorem 3.7, the way an enhancement ϵ of ϕ helps to find the irreducible representation $\pi(\phi,\epsilon)$ is by applying $\mathrm{Hom}_{\mathcal{S}_\phi}(\epsilon, ?)$ to a standard module $\pi(\phi,\mathfrak{s}^\vee)$ constructed from ϕ and the cuspidal support.

In the case at hand, for $\pi = \pi(\phi,\epsilon)$ we have

$$\mathrm{Hom}_{\mathcal{S}^+_{\tilde\phi}}(\epsilon, \pi(\phi,\mathfrak{s}^{+\vee})) = \delta(\rho,(a-1)/2,(1-a_-)/2) \times \tilde\pi,$$

where $\mathcal{S}^+_{\tilde\phi}$ comes from $\tilde\pi$. When τ is the L-parameter of ρ, the geometric setup provides a canonical action of $\{1, z_{\tau,a_-}\}$ on this representation, by a G^+-intertwining operator that we denote $N(\tau,a_-)$. That means

$$\begin{aligned} \pi = \pi(\phi,\epsilon) &= \mathrm{Hom}_{\mathcal{S}^+_{\tilde\phi}}(\epsilon, \pi(\phi,\mathfrak{s}^{+\vee})) = \mathrm{Hom}_{\langle z_{\tau,a_-}\rangle}(\epsilon, \delta(\rho,(a-1)/2,(1-a_-)/2) \times \tilde\pi) \\ &= \{\text{fixed points of } \epsilon(z_{\tau,a-}) N(\tau,a_-) \text{ in } \delta(\rho,(a-1)/2,(1-a_-)/2) \times \tilde\pi\}. \qquad (4.18) \end{aligned}$$

In contrast with $J(s_\beta, \rho \times \sigma_-)$, these intertwining operators for Hecke algebra representations do not have to be normalized, they arise naturally. The only freedom we have is that from Theorem 3.3, which we will use next. Let $J(s_\beta, \tau \times \phi_-)$ be the canonical intertwining operator associated to s_β and the Hecke algebra representation corresponding to $\rho \times \sigma_-$ via Proposition 3.5. (We suppress ϵ_{σ_-} from this notation.)

Proposition 4.6. *Let $\mathfrak{s}^+ = [L^+, \sigma]_{G^+}$ be an arbitrary inertial equivalence class for G^+. Fix a Whittaker datum for the quasi-split inner form of G, so that the intertwining operators (4.11) are determined.*

The isomorphism $\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$ from Proposition 3.5 can be chosen such that the following holds. For every $\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_{\phi_\sigma}$ with $e_\tau > 0 = \ell_\tau$, the intertwining operators $J(s_\beta, \tau \times \phi_-)$ and $J(s_\beta, \rho \times \sigma_-)$ from (4.11) agree via the appropriate equivalences of categories from (3.35) induced by the chosen Hecke algebra isomorphism.

A Hecke algebra isomorphism with these properties is canonical up to conjugation by elements of $\mathcal{O}(T_{\mathfrak{s}^+})^\times$.

Proof. Let ρ' be an unramified twist of ρ such that $\rho' \cong \rho'^{\vee} \otimes \nu_\pi$ and $\rho' \not\cong \rho$. Because ρ' influences the structure of $\mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$ in the part coming from the same irreducible root system as ρ, we have to consider ρ and ρ' simultaneously. Let τ and τ' be the L-parameters of respectively ρ and ρ'.

The case $\ell_{\tau'} > 0$.
From (1.14) we know that the relevant tensor factor of $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee,\mathrm{der}}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma^+_{\mathfrak{s}^\vee}$ is an affine Hecke algebra $\mathcal{H}_{\rho'}$ with underlying root datum $(\mathbb{Z}^{e_{\rho'}}, B_{e_{\rho'}}, \mathbb{Z}^{e_{\rho'}}, C_{e_{\rho'}})$. The base point of $T_{\mathfrak{s}^\vee,\rho}$ for $\mathcal{H}_{\rho'}$ comes from ρ', and $\rho^{\boxtimes e_{\rho'}}$ is related to this basepoint by an order two element of the associated complex torus. The condition $e_\tau > 0 = \ell_\tau$ entails that $\ell_\rho = 0, a_\rho = -1$ and $q^*_\beta = 1$ for the short roots β of B_{e_ρ}.

Then Proposition 3.5 allows us to replace s_β by $h^\vee_\beta s_\beta$ in the isomorphism

$$\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2}).$$

The representation $\rho \times \sigma_-$ does not appear directly in this framework, but it does so via a short detour. Pick $\chi \in \mathrm{Irr}(\mathbb{Z}^{e_{\rho'}})$ such that the values $\chi_i := \chi(e_i) \in \mathbb{C}^\times$ are in generic position, except that $\chi_{e_{\rho'}} = -1$. We identify χ_i with an unramified character of $\mathrm{GL}_{d_\rho}(F)$, unique up to $X_{\mathrm{nr}}(\mathrm{GL}_{d_\rho}(F), \rho)$. The $\mathcal{H}_{\rho'}$-representation $\mathrm{ind}_{\mathbb{C}[\mathbb{Z}^{e_{\rho'}}]}^{\mathcal{H}_{\rho'}} \mathbb{C}_\chi$ corresponds to

$$\rho' \otimes \chi_1 \times \cdots \times \rho' \otimes \chi_{e_{\rho'}-1} \times \rho \times \sigma_-. \tag{4.19}$$

The decomposition of this representation in irreducibles is governed by the component group $\mathcal{S}_\chi$ of the L-parameter, which in this case is just $\langle s_\beta \rangle$, acting on the last coordinate.

Things become more transparent if we complete the centre of $\mathcal{H}_{\rho'}$ at its maximal ideal associated to $W(B_{e_\rho})\chi$, as in [Lus3, §7]. That also gives a completion of $\mathcal{H}_{\rho'}$. Recall [Lus3, 10.3.(a)] that such a completion functor does not change the category of modules which admit the central character $W(B_{e_\rho})\chi$. By [Lus3, Theorem 8.6], this completion of $\mathcal{H}_{\rho'}$ is Morita equivalent with the completion (also determined by χ) of the simpler extended affine Hecke algebra $\mathbb{C}[\mathbb{Z}^{e_{\rho'}}] \rtimes \mathcal{S}_\chi$. Moreover, by

[Sol1, Theorem 2.1.2.c] this gives rise to a natural equivalence between the category of $\mathbb{C}[\mathbb{Z}^{e_{\rho'}}] \rtimes \mathcal{S}_\chi$-modules with central character χ and the category of $\mathcal{H}_{\rho'}$-modules with central character $W(B_{e_\rho})\chi$. Via that equivalence, our induced representation becomes

$$\mathrm{ind}_{\mathbb{C}[\mathbb{Z}^{e_{\rho'}}]}^{\mathbb{C}[\mathbb{Z}^{e_{\rho'}}] \rtimes \langle s_\beta \rangle} \mathbb{C}_\chi . \tag{4.20}$$

Since $\chi(h_\beta^\vee) = -1$, the automorphism which exchanges s_β and $h_\beta^\vee s_\beta$ affects the action of s_β on (4.20) by multiplication with -1. As a consequence the canonical intertwining operator from s_β on (4.20), or equivalently on $\mathrm{ind}_{\mathbb{C}[\mathbb{Z}^{e_{\rho'}}]}^{\mathcal{H}_{\rho'}} \mathbb{C}_\chi$ or (4.19), is adjusted by a factor by -1 by the replacement $s_\beta \mapsto h_\beta^\vee s_\beta$.

The intertwining operator on (4.20) associated with s_β is induced by the intertwining operator from s_β on the representation

$$\mathrm{ind}_{\mathbb{C}[\mathbb{Z}]}^{\mathbb{C}[\mathbb{Z}] \rtimes \langle s_\beta \rangle} \mathbb{C}_{-1}$$

of the smaller algebra $\mathbb{C}[\mathbb{Z}] \rtimes \langle s_\beta \rangle$. By [Lus3, Theorem 8.6] and [Sol1, Theorem 2.1.2] the completion of that algebra (with respect to the central character $\langle s_\beta \rangle(-1) = -1$) is naturally isomorphic to the completion (also with respect to the central character -1) of $\mathcal{H}({\mathfrak{s}'}^\vee, q_F^{1/2})$, where $(\tau \times \phi_-, \epsilon_{\sigma_-}) \in \Phi(G^+_{n_- + d_\rho})^{{\mathfrak{s}'}^\vee}$. In this way the intertwining operator for s_β on (4.20) is related to the intertwining operator $J(s_\beta, \tau \times \phi_-)$, and multiplying the former by -1 entails that the latter is also multiplied by -1.

As $J(s_\beta, \rho \times \sigma_-)$ is a priori unique up to a factor -1, it follows that we can match it with $J(s_\beta, \tau \times \phi_-)$ under the appropriate Hecke algebra isomorphism by making the (unique) correct choice for the image of $T_{s_\beta} \in \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$ in $\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}}$.

The case $\ell_{\tau'} = 0$.
Here we need to take both $J(s_\beta, \tau \times \sigma_-)$ and $J(s_\beta, \tau' \times \sigma_-)$ into account. The relevant tensor factor of $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee,\mathrm{der}}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma^+_{\mathfrak{s}^\vee}$ is of the form

$$\mathcal{H}_\rho = \mathcal{H}(\mathcal{R}_{D_\rho}, q_F^{1/2}) \rtimes \mathrm{Out}(D_{e_\rho}),$$

where $\mathcal{R}_{D_m} = (\mathbb{Z}^m, D_m, \mathbb{Z}^m, D_m)$. As basepoint of the underlying torus we take $\rho^{\boxtimes e_\rho}$. We can modify the isomorphism $\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$ in four ways on this tensor factor. Namely, write $\mathrm{Out}(D_{e_\rho}) = \langle s_\beta \rangle$ with β a short root in $B_{e_\rho} \supset D_{e_\rho}$. As the image of $s_\beta \in \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$ in $\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}}$ we may take $-s_\beta, h_\beta^\vee s_\beta, -h_\beta^\vee s_\beta$ or s_β.

Like in the previous case, we can study the representations of $\mathcal{H}_\rho$ induced from characters χ of $\mathbb{Z}^{e_\rho}$. We pick the first $e_\rho - 1$ coordinates of χ generically in $\mathbb{C}^\times$, and take $\chi_{e_\rho} = \pm 1$. That corresponds to

$$\rho \otimes \chi_1 \times \cdots \times \rho \otimes \chi_{e_{\rho'}-1} \times \rho \otimes \pm 1 \times \sigma_-,$$

where $\rho \otimes 1 = \rho$ and $\rho \otimes -1 = \rho'$. We can complete $\mathcal{H}_\rho$ with respect to the maximal ideal of $Z(\mathcal{H}_\rho)$ associated to $W(B_{e_\rho})\chi$. By [Lus3, Theorem 8.6] that the completion is Morita equivalent to the completion (with respect to the central character $\langle s_\beta \rangle \chi = \chi$) of the simpler extended affine Hecke algebra $\mathbb{C}[\mathbb{Z}^{e_{\rho'}}] \rtimes \langle s_\beta \rangle$. Moreover [Sol1, Theorem 2.1.2.c] provides a natural equivalence between the modules with central character $W(B_{e_\rho})\chi$ or χ of these two algebras. Thus we transfer the issue to modules for $\mathbb{C}[\mathbb{Z}^{e_{\rho'}}] \rtimes \langle s_\beta \rangle$.

When $\chi_{e_\rho} = 1$, the intertwining operator for s_β on

$$\mathrm{ind}_{\mathbb{C}[\mathbb{Z}^{e_\rho}]}^{\mathbb{C}[\mathbb{Z}^{e_\rho}] \rtimes \langle s_\beta \rangle} \mathbb{C}_\chi \tag{4.21}$$

is induced from the intertwining operator on $\mathrm{ind}_{\mathbb{C}[\mathbb{Z}]}^{\mathbb{C}[\mathbb{Z}] \rtimes \langle s_\beta \rangle} \mathbb{C}_1$. The completion (with respect to the central character χ) of the algebra $\mathbb{C}[\mathbb{Z}] \rtimes \langle s_\beta \rangle$ is naturally isomorphic to the completion (with respect to the central character 1, which corresponds to the basepoint τ) of the affine Hecke algebra for the Bernstein component containing $(\tau \times \phi_-, \sigma_-)$. Thus the intertwining operator on (4.21) is essentially $J(s_\beta, \tau \times \phi_-)$. Via an instance of (3.35) for a Bernstein component of $\mathrm{Irr}(G^+_{n_- + d_\rho})$, the latter corresponds to $\pm J(s_\beta, \rho \times \sigma_-)$. Possibly adjusting the isomorphism $\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$ so that s_β goes to $-s_\beta$, we can match $J(s_\beta, \tau \times \phi_-)$ and $J(s_\beta, \rho \times \sigma_-)$.

When $\chi_{e_\rho} = -1$, the situation is similar, but now (4.21) is induced from $\mathrm{ind}_{\mathbb{C}[\mathbb{Z}]}^{\mathbb{C}[\mathbb{Z}] \rtimes \langle s_\beta \rangle} \mathbb{C}_{-1}$, which comes from $(\tau' \times \phi_-, \epsilon_{\sigma_-})$. Here the intertwining operator from s_β on (4.21) is essentially $J(s_\beta, \tau' \times \phi_-)$. Via the same instance of (3.35) as above, this operator corresponds to $\pm J(s_\beta, \rho' \times \sigma_-)$. We can still adjust the isomorphism $\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$ by composition with the automorphism that sends s_β to $h_\beta^\vee s_\beta$. That multiplies $J(s_\beta, \tau' \times \phi_-)$ with -1, while perserving $J(s_\beta, \tau \times \phi_-)$. Thus, by a suitable choice we can arrange that $J(s_\beta, \tau' \times \phi_-)$ corresponds to $J(s_\beta, \rho' \times \sigma_-)$, without disturbing the previous normalization. In total we have a unique choice (out of four) for the image of s_β under the algebra isomorphism, such that both relevant pairs of intertwining operators match up.

With the above choices, for $\ell_{\tau'} \geq 0$ and for all relevant ρ, we managed to fulfill the conditions imposed in the statement. To this end we exploited the freedom provided by the operations (iii) and (iv) in Theorem 3.3 and Proposition 3.5, and we fixed choices for all instances of (iii) and (iv). These are canonical because the intertwining operators (4.11) are determined by the Whittaker datum for the quasi-split inner form of G, which is part of our input. In view of Proposition 3.5, this renders our Hecke algebra isomorphism canonical up to conjugation by elements of $\mathcal{O}(T_{\mathfrak{s}^+})^\times$. □

Applying Proposition 4.6, we can match many more intertwining operators between G^+-representations with intertwining operators between $\mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$-modules.

Lemma 4.7. *Choose an isomorphism $\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$ as in Proposition 4.6. For every discrete series representation $\pi \in \mathrm{Irr}(G^+)^{\mathfrak{s}^+}$ and every $(\rho, a_-) \in \mathrm{Jord}(\pi)$ with a_- minimal and odd and $\mathrm{Jord}_\rho(\mathrm{Sc}(\pi))$ empty, the intertwining operators $N(\rho, a_-)$ and $N(\tau, a_-)$ from (4.17) and (4.18) coincide on the representation $\delta(\rho, (a-1)/2, (1-a_-)/2) \times \tilde{\pi}$ from (4.15).*

Proof. After (4.11) we described how $J(s_\beta, \rho \times \sigma_-)$ determines $N(\rho, a_-)$. By Proposition 4.6, $J(s_\beta, \rho \times \sigma_-)$ corresponds to $J(s_\beta, \tau \times \phi_-)$, so it remains to check that the latter determines $N(\tau, a_-)$ in the same way.

The constructions around (4.15) and (4.16) work analogously for modules of Hecke algebras, which reduces our task to comparing

$$J(w_e, \rho\nu^{b_1} \times \cdots \times \rho\nu^{b_{(e-1)/2}} \times \rho \times \rho\nu^{-b_{(e-1)/2}} \times \cdots \times \rho\nu^{-b_1} \times \sigma_-) \tag{4.22}$$

from (4.14) with its version for the appropriate Hecke algebra $\mathcal{H}(\mathfrak{s}'^\vee, q_F^{1/2})$. Recall that $\nu^b \in X_{\mathrm{nr}}(\mathrm{GL}_{d_\rho}(F))$ corresponds to the central element $q_F^b \in \mathrm{GL}_{d_\rho}(\mathbb{C})$, and then $\rho\nu^b$ corresponds to $q_F^b \tau$. In the geometric setup from [AMS2], given w_e there is a canonical intertwining operator

$$J\Big(w_e, \mathrm{ind}_{\mathcal{O}(T_{\mathfrak{s}'^\vee})}^{\mathcal{H}(\mathfrak{s}'^\vee, q_F^{1/2})} \pi\big(q_F^{b_1}\tau \times \cdots \times q_F^{b_{(e-1)/2}}\tau \times \tau \times \cdots \times q_F^{-b_1}\tau \times \phi_-, \epsilon_{\sigma_-}\big)\Big), \tag{4.23}$$

from the indicated module to itself. (Here the symbols $\times$ refer to an L-parameter with values in a direct product of groups, not to parabolic induction.) This operator has order 2, and it comes as a member of an algebraic family parametrized by $b_i \in \mathbb{R}$. When all b_i are equal to 0, the permutation $(1\ e)(2\ e{-}1)\cdots((e{-}1)/2\ (e{+}3)/2)$ lies in the connected component of the centralizer group of the L-parameter in (4.23), and the canonical intertwining operator associated to that permutation is just the identity. Hence for $b_i = 0$ the operator (4.23) reduces to

$$J\Big(s_\beta \times \cdots \times s_\beta, \mathrm{ind}_{\mathcal{O}(T_{\mathfrak{s}'^\vee})}^{\mathcal{H}(\mathfrak{s}'^\vee, q_F^{1/2})} \pi\big(\tau \times \cdots \times \tau \times \phi_-, \epsilon_{\sigma_-}\big)\Big). \tag{4.24}$$

This operator is induced by $J(s_\beta, \tau \times \phi_-)$ on each of the $e = e_\rho$ coordinates, in the following sense. Upon completion of $\mathcal{H}(\mathfrak{s}'^\vee, q_F^{1/2})$ with respect to the central character associated to $\tau \times \cdots \times \tau \times \phi_-$ (like in the proof of Proposition 4.6) we obtain an e-fold tensor product of modules

$$\mathrm{ind}_{\mathbb{C}[\mathbb{Z}]}^{\mathbb{C}[\mathbb{Z}] \rtimes \langle s_\beta \rangle} \pi(\tau \times \phi_-).$$

Then (4.24) can be identified with the e-fold tensor product of the operators $J(s_\beta, \tau \times \phi_-)$ on these modules. This is the same procedure as in (4.12), so Proposition 4.6 guarantees that (4.24) agrees with (4.12) for $b_i = 0$ and the correct number of factors. Since all instances of (4.23) square to the identity and they are part of a continuous family, all these instances are fixed when we know (4.24). That is completely analogous to the situation in (4.14). Therefore (4.22) and (4.23) agree via a Hecke algebra isomorphism as in Proposition 4.6. $\square$

After all these preparations, we are ready to compare the two parametrizations of arbitrary discrete series representations of G^+.

Proposition 4.8. *Choose a Hecke algebra isomorphism $\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$ as in Proposition 4.6. Let $(\phi, \epsilon) \in \Phi_{\mathrm{e}}(G^+)^{\mathfrak{s}^{+\vee}}$ be discrete and let $\pi \in \mathrm{Irr}(G^+)$ be associated with (ϕ, ϵ) by Theorems 2.1 and 2.5. Then $\pi(\phi, \epsilon)$ is isomorphic to π.*

Proof. Write ϕ as $z_\phi \phi_b$ with $z_\phi \in \mathrm{Z}(G^\vee)$ and $\phi_b \in \Phi(G)$ bounded and discrete. Let $\chi_\phi \in X_{\mathrm{nr}}(G^+)$ correspond to z_ϕ. By Theorem 2.5 $\pi = \chi_\phi \otimes \pi_b$ where π_b corresponds to (ϕ_b, ϵ). On the other hand Theorem 3.7.d says that

$$\pi(\phi, \epsilon) = \pi(z_\phi \phi_b, \epsilon) = \chi_\phi \otimes \pi(\phi_b, \epsilon).$$

Therefore it suffices to prove the proposition under the additional assumption that ϕ is bounded.

Applying Proposition 4.3 repeatedly, we find that $\pi(\phi, \epsilon)$ embeds in

$$\prod\nolimits_{\rho, a, a'} \delta(\rho, (a-1)/2, (1-a')/2) \times \pi(\tilde{\phi}, \tilde{\epsilon}), \tag{4.25}$$

where $\mathrm{Jord}(\tilde{\phi}) \subset \mathrm{Jord}(\phi)$ and $\tilde{\epsilon} = \epsilon|_{\mathcal{S}^+_{\tilde{\phi}}}$ is alternated in the sense of (4.9). Here the product runs over some triples with $\epsilon(z_{\rho,a}) = \epsilon(z_{\rho,a'})$, not necessarily all such triples. Similarly, by [Mœ1, §5] π embeds in

$$\prod\nolimits_{\rho,a,a'} \delta(\rho,(a-1)/2,(1-a')/2) \times \tilde{\pi} \tag{4.26}$$

with $\mathrm{Jord}(\tilde{\pi}) \subset \mathrm{Jord}(\pi)$ and $\epsilon_{\tilde{\pi}} = \epsilon|_{\mathcal{S}^+_{\tilde{\pi}}}$ alternated. By Lemma 4.5 both $\pi(\tilde{\phi},\tilde{\epsilon})$ and $\tilde{\pi}$ are completely positive discrete series representations. Further $\tilde{\pi}$ and $\pi(\tilde{\phi},\tilde{\epsilon})$ have the same Jordan blocks, because both are obtained from $\mathrm{Jord}(\pi) = \mathrm{Jord}(\pi(\phi,\epsilon))$ by removing the pairs $(\rho,a),(\rho,a')$ that appear in the product. By Theorem 3.7.a and Proposition 4.1, $\pi(\phi,\epsilon)$ and π have the same supercuspidal support. From (4.25) and (4.26) we see that $\pi(\tilde{\phi},\tilde{\epsilon})$ and $\tilde{\pi}$ also have the same supercuspidal support. With Corollary 4.4 we deduce that $\tilde{\pi} \cong \pi(\tilde{\phi},\tilde{\epsilon})$.

Thus both $\pi(\phi,\epsilon)$ and π are subrepresentations of (4.25), which is isomorphic to (4.26). By Theorem 3.9, (4.25) is a direct sum of precisely $[\mathcal{S}_\phi : \mathcal{S}_{\tilde{\phi}}]^{1/2}$ subrepresentations, which are mutually inequivalent. Every factor $\delta(\rho,(a-1)/2,(1-a')/2)$ doubles the number of constituents, because

$$\delta(\rho,(a-1)/2,(1-a')/2) \times \pi(\tilde{\phi},\tilde{\epsilon}) \tag{4.27}$$

has length two. We can distinguish three classes of ρ's:

Case (i). When $\mathrm{Jord}_\rho(\tilde{\pi})$ is nonempty, Proposition 4.3.a determines which summands must be picked to get π. (This works also for Mœglin's parametrization, by [Mœ1, §5].) Namely, start with $(\rho,b) \in \mathrm{Jord}(\tilde{\pi})$ and an adjacent $(\rho,a) \in \mathrm{Jord}(\pi) \setminus \mathrm{Jord}(\tilde{\pi})$. Then Proposition 4.3.a imposes a condition (recall that ϵ was given). Next, take $(\rho,a') \in \mathrm{Jord}(\pi) \setminus \mathrm{Jord}(\tilde{\pi})$ adjacent to (ρ,a). Of the two choices for a subrepresentation of (4.27), one fulfills the previous condition and one does not (that is another consequence of Proposition 4.3.a). Proceeding in this way, now with $\tilde{\phi} \setminus \{(\rho,a),(\rho,a')\}$ in the role of $\tilde{\phi}$, we discover step by step how to pick the right constituent of

$$\delta(\rho,(a''-1)/2,(1-a''')/2) \times \pi(\tilde{\phi},\tilde{\epsilon}) \tag{4.28}$$

for other $a'',a''' \in \mathrm{Jord}(\pi) \setminus \mathrm{Jord}(\tilde{\pi})$ as well.

Case (ii). Suppose that $\mathrm{Jord}_\rho(\tilde{\pi})$ is empty and that $\mathrm{Jord}_\rho(\pi)$ consists of even numbers. In this case we may take $a' = 0$, set $\epsilon(\rho,0) = 1$ and use Proposition 4.3.b. As in the previous case, ϵ determines which constituents of (4.27) and (4.5) must chosen to enable an embedding of π.

Case (iii). Suppose that $\mathrm{Jord}_\rho(\tilde{\pi})$ is empty and that $\mathrm{Jord}_\rho(\pi)$ consists of odd numbers. By Proposition 4.6 and Lemma 4.7, our two parametrizations involve the same constituent of $\delta(\rho,(b-1)/2,(1-a_-)/2) \times \pi(\tilde{\phi},\tilde{\epsilon})$, where b is the smallest $a \in \mathrm{Jord}_\rho(\pi) \setminus \{a_-\}$ such that $\epsilon(z_{\rho,a}) = \epsilon(z_{\rho,a_-})$. Once we know that, the method from the previous cases tells us which constituent of (4.28) we have to take, for any adjacent $a'',a''' \in \mathrm{Jord}_\rho(\pi)$.

Hence π and $\pi(\phi,\epsilon)$ are obtained from (4.25) by taking the same constituents of (4.28) in all cases, so $\pi \cong \pi(\phi,\epsilon)$. □

4.4. Tempered representations.

Consider a bounded L-parameter $\phi \in \Phi(G)$. Recall from (1.4) and (1.5) that we can decompose $(\phi, \mathbb{C}^{2n})$ as

$$\bigoplus\nolimits_{\psi \in I^\pm} N_\psi \otimes V_\psi \oplus \bigoplus\nolimits_{\psi \in I^0} N_\psi \otimes (V_\psi \oplus V_\psi^\vee), \tag{4.29}$$

where N_ψ is a multiplicity space and $V_\psi^\vee$ is endowed with the representation $\psi^\vee \otimes \mu_G^\vee \circ \phi$. There exists a Levi subgroup L of G, unique up to conjugation, such that ϕ factors through $\Phi(L)$ and defines a discrete L-parameter for L. Every factor $\mathrm{GL}_m(F)$ of L appears in G as

$$\{(A, B) \in \mathrm{GL}_m(F) \times \mathrm{GL}_m(F) : B = JA^{-T} J^{-1}\}.$$

The same goes for $L^\vee$ and $G^\vee$. Hence every $\psi \in I^\pm$ which appears with multiplicity μ in $\phi|_{\mathrm{GL}_m(\mathbb{C})}$, accounts for multiplicity 2μ in (4.29). In view of (1.7), the part of ϕ with image in the factor $G^\vee_{n_-}$ of $L^\vee$ is precisely $\prod_{\psi \in I^+ : \dim N_\psi \mathrm{odd}} \psi$, while the part of ϕ in the type GL factors of $L^\vee$ is

$$\bigoplus\nolimits_{\psi \in I^\pm} \lfloor \dim(N_\psi)/2 \rfloor \psi \;\oplus\; \bigoplus\nolimits_{\psi \in I^0} \dim(N_\psi) \psi.$$

For $\psi \in I^0$ this involves a choice of ψ or $\psi^\vee \otimes \mu_G^\vee \circ \phi$, but that hardly matters because both will appear equally often when we pass to $G^\vee$. For the component groups of ϕ it is a bit easier to work with G^+ and L^+, so we consider ϕ as element of $\Phi(G^+)$ and as $\phi_L \in \Phi(L^+)$. By these we mean just $\Phi(G)$ and $\Phi(L)$, only with component groups of ϕ or ϕ_L computed in $G^{\vee +}$ or $L^{\vee +}$. In the description of $\mathcal{S}_\phi$ following (1.5), passing to G^+ replaces $\mathcal{S}_\phi$ by $\mathcal{S}_\phi^+$, which means that we forget the determinant condition "S" on $Z_{{G^\vee}_{\mathrm{der}}}(\phi)$. Thus

$$\begin{aligned}
\mathrm{Z}_{{L^{+\vee}}_{\mathrm{der}}}(\phi_L) &\cong \prod_{\psi \in I^+ : \dim N_\psi \text{ odd}} \mathrm{O}_1(\mathbb{C}) \times \prod_{\psi \in I^\pm} \mathrm{GL}_{\lfloor \dim(N_\psi)/2 \rfloor}(\mathbb{C}) \times \prod_{\psi \in I^0} \mathrm{GL}(N_\psi), \\
\mathcal{S}_{\phi_L}^+ &\cong \prod_{\psi \in I^+ : \dim N_\psi \text{ odd}} \langle z_\psi \rangle, \\
\mathcal{S}_\phi^+ &= \prod_{\psi \in I^+ : N_\psi \neq 0} \langle z_\psi \rangle \;=\; \mathcal{S}_{\phi_L}^+ \times \prod_{\psi \in I^+ : \dim(N_\psi) \in 2\mathbb{Z}_{>0}} \langle z_\psi \rangle \;=:\; \mathcal{S}_{\phi_L}^+ \times \mathcal{S}_{\phi/\phi_L}^+.
\end{aligned}$$

Let us fix $\epsilon_L \in \mathrm{Irr}(\mathcal{S}_{\phi_L}^+)$ such that (ϕ_L, ϵ_L) belongs to the image in of the parametrization map in Theorem 3.7 for L^+. It gives a discrete series representation $\pi(\phi_L, \epsilon_L) \in \mathrm{Irr}(L^+)$, which by Proposition 4.8 is the same for the endoscopic method as for the Hecke algebra method. By Theorem 3.9, $I_{L^+U}^{G^+} \pi(\phi_L, \epsilon_L)$ has precisely $|\mathcal{S}_{\phi/\phi_L}^+|$ irreducible direct summands, which are mutually inequivalent and indexed by

$$\big\{ \epsilon \in \mathrm{Irr}(\mathcal{S}_\phi^+) : \epsilon|_{\mathcal{S}_{\phi_L}^+} = \epsilon_L \big\} \cong \mathrm{Irr}(\mathcal{S}_{\phi/\phi_L}^+).$$

The same conclusion was obtained in [MoTa, Theorem 13.1]. One part of the constructions behind Theorem 3.7 in [AMS2, AMS3] is

$$\pi(\phi, \epsilon) = \mathrm{Hom}_{\mathcal{S}_{\phi/\phi_L}^+} \big(\epsilon|_{\mathcal{S}_{\phi/\phi_L}^+}, I_{L^+U}^{G^+} \pi(\phi_L, \epsilon_L) \big). \tag{4.30}$$

This goes back to [AMS3, (1.17)], and from there it is transfered via various Hecke algebras in [AMS3] to an analogue for $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$, see (3.37). Using the equivalences of categories (3.34) and (3.33), one arrives at (4.30). Here the action of $\mathcal{S}^+_{\phi/\phi_L}$ comes from intertwining operators

$$N(z_\psi, \phi_L, \epsilon_L) \in \mathrm{End}_{G^+}\big(I_{L^+U}^{G^+} \pi(\phi_L, \epsilon_L)\big),$$

one for each generator z_ψ of $\mathcal{S}^+_{\phi/\phi_L}$.

On the other hand, an irreducible tempered G^+-representation $\pi(\phi)_\epsilon$ is constructed with endoscopy in [MoRe, §3.6], and it is checked that $I_{L^+U}^{G^+} \pi(\phi_L, \epsilon_L)$ (called σ in [MoRe]) decomposes as

$$\bigoplus\nolimits_{\epsilon \in \mathrm{Irr}(\mathcal{S}^+_\phi) \,:\, \epsilon|_{\mathcal{S}^+_{\phi_L}} = \epsilon_L} \pi(\phi)_\epsilon .$$

This decomposition can be achieved with suitable intertwining operators that make $\mathcal{S}^+_{\phi/\phi_L}$ act on $I_{L^+U}^{G^+} \pi(\phi_L, \epsilon_L)$ and are normalized in a way that is compatible with the endoscopic methods in [MoRe]. The appropriate normalization stems from [Art, §2.3] and involves L-functions and ϵ-factors. Unfortunately, it becomes untractable in the setting of Hecke algebras. Nevertheless, we can say more concretely that, for every $\psi \in I^+$ with $\dim N_\psi \in 2\mathbb{Z}_{>0}$ and $\pi(\psi) = \delta(\rho, a)$, there is a normalized intertwining operator

$$N\big(z_{\rho,a}, \pi(\phi_L, \epsilon_L)\big) \in \mathrm{End}_{G^+}\big(I_{L^+U}^{G^+} \pi(\phi_L, \epsilon_L)\big)$$

which squares to the identity. From [MoRe, §2] we see that

$$\begin{aligned} (4.31) \quad \pi(\phi)_\epsilon = \big(\epsilon|_{\mathcal{S}^+_{\phi/\phi_L}} \otimes I_{L^+U}^{G^+} \pi(\phi_L, \epsilon_L)\big)^{\mathcal{S}^+_{\phi/\phi_L}} = \\ \{\text{fixed points of the operators } \epsilon(z_\psi) N(z_\psi, \pi(\phi_L, \epsilon_L)) \text{ with } \dim N_\psi \in 2\mathbb{Z}_{>0}\}. \end{aligned}$$

Lemma 4.9. *Pick an inertial equivalence class $\mathfrak{s}^+$ for G^+ and choose an isomorphism $\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$ as in Proposition 4.6.*

For every bounded $(\phi, \epsilon) \in \Phi_{\mathrm{e}}(G^+)^{\mathfrak{s}^{+\vee}}$ and every $\psi \in I^+ \cap \mathrm{Jord}(\phi)$ with $\pi(\psi) = (\rho, a)$, the intertwining operators

$$N\big(z_{\rho,a}, \pi(\phi_L, \epsilon_L)\big) \quad \textit{and} \quad N(z_\psi, \phi_L, \epsilon_L)$$

agree via the Hecke algebra isomorphism.

Proof. We need to distinguish a few cases.

First we suppose that $\dim N_\psi$ is odd. Then ψ appears in the factor $G^{+\vee}_{n_-}$ of L^+, and the two intertwining operators of G^+-representations under consideration are induced by the analogous intertwining operators of $G^+_{n_-}$-representations. The latter two agree by Lemma 4.7.

Now we suppose $\dim N_\psi$ that is even and that $\psi = \tau \otimes P_a$ with $\tau \in \mathrm{Irr}(\mathbf{W}_F)^+_\phi$. Here a is odd because $\psi \in I^+$. The same arguments as for Lemma 4.7 show that $N\big(z_{\rho,a}, \pi(\phi_L, \epsilon_L)\big)$ and $N(z_\psi, \phi_L, \epsilon_L)$ agree, because $N(s_\beta, \rho \times \sigma_-)$ and $N(s_\beta, \tau \times \phi_-)$ agree by Proposition 4.6.

Finally we suppose that $\dim N_\psi$ is even and that $\psi = \tau \otimes P_a$ with $\tau \in \mathrm{Irr}(\mathbf{W}_F)^-_\phi$. Now a is even because $\psi \in I^+$. In this case we do not know whether $N(s_\beta, \rho \times \sigma_-)$ and $N(s_\beta, \tau \times \phi_-)$ match via the Hecke algebra isomorphism. But both are unique up to scalars and square to the identity, so the agree up to a factor ± 1.

Write $e = ad_\rho$. Motivated by (4.16), we want to compare the operators

$$N\big(w_e, \mathrm{Ind}\,\mathrm{Sc}(\delta(\rho,a)) \times \sigma_-\big) \quad \text{and} \quad N(w_e, \mathrm{Sc}(\tau \otimes P_a) \times \phi_-), \tag{4.32}$$

where the right hand side is an abbreviation of (4.23). From the remarks after (4.14) we know that the former is determined (via a continuous deformation) by the intertwining operator

$$N((s_\beta \times \cdots \times s_\beta), \rho \times \cdots \times \rho \times \sigma_-), \tag{4.33}$$

where s_β and ρ both appear a times. For each such factor ρ, we get a contribution which is induced by $N(s_\beta, \rho \times \sigma_-)$.

Similarly, in the proof of Lemma 4.7 we saw that $N(w_e, \mathrm{Sc}(\tau \otimes P_a) \times \phi_-)$ is determined in the same way by (4.24) and $N(s_\beta, \tau \times \phi_-)$. Writing

$$N(s_\beta, \rho \times \sigma_-) = \pm N(s_\beta, \tau \times \phi_-),$$

it follows that, via the appropriate Hecke algebra isomorphism, (4.33) and (4.24) agree up to a factor $(\pm 1)^a$. Since a is even they really agree, and so do the two sides of (4.32). We note that

$$N\big(z_{\rho,a}, \pi(\phi_L, \epsilon_L)\big) \quad \text{and} \quad N(z_\psi, \phi_L, \epsilon_L)$$

are induced by (4.32), on both sides in the same way as in (4.16), so with the identity on factors not involved in (4.32). We combine that with the above analysis of (4.32) to establish the lemma in this case. $\square$

From Proposition 4.8, Lemma 4.9, (4.30) and (4.31) we conclude:

Corollary 4.10. *In the setting of Proposition 4.6, let $(\phi, \epsilon) \in \Phi_{\mathrm{e}}(G^+)^{\mathfrak{s}^+}$ be bounded. Then $\pi(\phi, \epsilon) \in \mathrm{Irr}(G^+)$ from Theorem 3.7 is isomorphic with the tempered representation $\pi(\phi)_\epsilon$ from* [MoRe].

Together with Theorem 3.7.d, Corollary 4.10 implies that

$$\pi(\hat{\chi}\phi)_\epsilon \cong \chi \otimes \pi(\phi)_\epsilon \tag{4.34}$$

for all bounded $(\phi, \epsilon) \in \Phi_{\mathrm{e}}(G^+)$ and all unitary $\chi \in X_{\mathrm{nr}}(G^+)$.

4.5. **Irreducible smooth representations.**

With the Langlands classification [Ren, Théorème VII.4.2] one can construct and parametrize all irreducible smooth representations of a reductive p-adic group in terms of the irreducible tempered representations of its Levi subgroups. Although the Langlands classification usually applies to connected reductive groups, it can be extended to some disconnected reductive groups [BaJa, Theorem 4.2 and Remark 4.2]. For our G^+ with the Levi subgroups $L^+ = \mathrm{Z}_{\mathcal{G}^+}(\mathrm{Z}(\mathcal{L})^\circ)(F)$, the formulation is the almost same as for connected reductive groups, at least when $[G^+ : G] = [L^+ : L]$ and we limit ourselves to G^+-representations from standard modules that are parabolically induced from L^+-representations. Indeed, from [BaJa, Introduction] one sees that in those cases the Langlands data are triples (P^+, τ, ν) where $P^+ = L^+ U_P$ is a parabolic subgroup of G^+, $\tau \in \mathrm{Irr}(L^+)$ is tempered and $\nu \in \mathrm{Hom}(L^+, \mathbb{R}_{>0})$ is in strictly positive position with respect to P^+. As usual $I_{P^+}^{G^+}(\tau \otimes \nu)$ has a unique irreducible quotient $\pi(P^+, \tau, \nu)$, and that sets up a bijection between the G^+-conjugacy classes of such Langlands data and the associated subset of $\mathrm{Irr}(G^+)$.

There also exists a Langlands classification for (enhanced) L-parameters [SiZi], which is similar. For every $\phi \in \Phi(G)$ there exists a parabolic subgroup $P^\vee = L^\vee U_P^\vee$ of $G^\vee$, a bounded $\phi_b \in \Phi(L)$ and $\hat\nu \in H^1(\mathbf{W}_F/\mathbf{I}_F, Z(L^\vee))$, strictly positive with respect to $P^\vee$, such that $\phi = \phi_v \hat\nu$ in $\Phi(G)$. Moreover, by [SiZi, Theorem 4.6] this yields a bijection between $\Phi(G)$ and the $G^\vee$-conjugacy classes of such triples $(P^\vee, \phi_b, \hat\nu)$. Of course we can pass to a bijection between $G^{\vee +}$-conjugacy classes in $\Phi(G)$ and of triples $(P^\vee, \phi_b, \hat\nu)$. Further, by [SiZi, Proposition 7.3]

$$\mathcal{S}_\phi \cong \mathcal{S}_{\phi_b \hat\nu} = \mathcal{S}_{\phi_b},$$

where $\mathcal{S}_{\phi_b \hat\nu}$ and $\mathcal{S}_{\phi_b}$ are computed in $L^{\vee +}$. By the same argument, this extends to an isomorphism

$$\mathcal{S}_\phi^+ \cong \mathcal{S}_{\phi_b \hat\nu}^+ \quad \text{when } [G^+ : G] = [L^+ : L].$$

Then the Langlands classification from [SiZi] sends $(\phi, \epsilon) \in \Phi_{\mathrm{e}}(G)$ to $(P^\vee, \phi_b, \hat\nu, \epsilon)$ with $(\phi_b, \epsilon) \in \Phi_{\mathrm{e}}(L^+)$.

In the exceptional cases where $L^+ = L$ but $G^+ \neq G$, this does not work so nicely. Then $I_P^G(\tau \otimes \nu)$ is indecomposable, but $I_P^{G^+}(\tau \otimes \nu)$ is a direct sum of one or two indecomposable G^+-representations. For Langlands parameters, the analogous issue is that $\mathcal{S}_{\phi_b}^+ = \mathcal{S}_{\phi_b}$ has index one or two in $\mathcal{S}_\phi^+$. To handle those cases in similar style, we resort to the method from [ABPS1].

Theorem 4.11. *With the above versions of the Langlands classification, one can canonically extend the parametrization of irreducible tempered G^+-representations from* [MoTa, MoRe] *to a parametrization of* $\mathrm{Irr}(G^+)$.

Proof. **Case (i): $[G^+ : G] = [L^+ : L]$.**
The above Langlands classifications show us what to do: to $(\phi_b \hat\nu, \epsilon)$ we associate the unique irreducible quotient of $I_{P^+}^{G^+}(\pi_{L^+}(\phi_b)_\epsilon \otimes \nu)$.

From now on we suppose that $L^+ = L$ but $G^+ \neq G$. Consider Langlands data $(P^\vee, \phi_b, \nu, \epsilon)$ for $\Phi(G)$. Let Y be the variety of $\chi \in X_{\mathrm{nr}}(L)$ such that $\mathcal{S}_{\phi_b \hat\chi}^+ \cong \mathcal{S}_{\phi_b \hat\nu}^+$. It is a coset of a complex algebraic subtorus of $X_{\mathrm{nr}}(L)$, with finitely many cosets of subtori of smaller dimension removed, see [ABPS1, §3]. It intersects the maximal compact subgroup of $X_{\mathrm{nr}}(L)$, so we can pick a unitary $\chi \in Y$.
Case (ii): $\mathcal{S}_{\phi_b \hat\nu}^+ = \mathcal{S}_{\phi_b \hat\nu}$ for $\phi_b \hat\nu$ as element of $\Phi(G)$.
By Corollary 4.10 we may use Theorem 3.7 for $(\phi_b \hat\chi, \epsilon)$. Comparing Theorem 3.7 for G and G^+, we see that the condition of this case implies that $\mathrm{ind}_G^{G^+}$ preserves the irreducibility of $\pi_G(\phi_b \hat\chi)_\epsilon$. By Theorem 3.9 and (4.34) for L:

$$\pi(\phi_b \hat\nu)_\epsilon = \mathrm{ind}_G^{G^+} \pi_G(\phi_b \hat\chi)_\epsilon = \mathrm{ind}_G^{G^+} I_P^G(\pi_L(\phi_b \hat\chi)_\epsilon) = I_P^{G^+}(\pi_L(\phi_b)_\epsilon \otimes \chi) = I_P^{G^+} \pi(\phi_b \hat\chi, \epsilon).$$

Now $\pi(\phi_b \hat\chi, \epsilon) = \pi(\phi_b, \epsilon) \otimes \chi$ is irreducible for any $\chi \in Y$ (not necessarily unitary) and Theorem 3.7 for L shows that $I_P^{G^+}(\pi(\phi_b)_\epsilon) \otimes \chi) = \pi_{st}(\phi_b \hat\chi, \epsilon)$ has a unique irreducible quotient. That means that we are back in the setting of the Langlands classification. As in case (i), it imposes that to $(\phi_b \hat\nu, \epsilon)$ one must associated the irreducible quotient of $I_P^{G^+}(\pi(\phi_b)_\epsilon) \otimes \nu)$.
Case (iii): $\mathcal{S}_{\phi_b \hat\nu}^+ \neq \mathcal{S}_{\phi_b \hat\nu}$ for $\phi_b \hat\nu$ as element of $\Phi(G)$.

Let ϵ_G be the restriction of ϵ to $\mathcal{S}_{\phi_b \hat\nu}$ and write $\mathrm{ind}_{\mathcal{S}_{\phi_b \hat\nu}}^{\mathcal{S}_{\phi_b \hat\nu}^+} \epsilon_G = \epsilon \oplus \epsilon'$. Theorem 3.7 for G^+ and G, in combination with the condition of the case, show that $\mathrm{Res}_G^{G^+} \pi(\phi_b \hat\chi)_\epsilon$

is irreducible and equal to $\pi_G(\phi_b \hat\chi)_{\epsilon_G}$. By Theorem 3.9

$$I_P^{G^+}(\pi_L(\phi_b)_{\epsilon_G} \otimes \chi) = \mathrm{ind}_G^{G^+} I_P^G \pi_L(\phi_b \hat\chi)_{\epsilon_G} = \mathrm{ind}_G^{G^+} \pi_G(\phi_b \hat\chi)_{\epsilon_G} = \pi(\phi_b \hat\chi)_\epsilon \oplus \pi(\phi_b \hat\chi)_{\epsilon'}.$$

For arbitrary $\chi \in Y$, Theorems 3.7 and 3.9 entail that

$$I_P^{G^+}(\pi_L(\phi_b)_{\epsilon_G} \otimes \chi) = \pi_{st}(\phi_b \hat\chi)_\epsilon \oplus \pi_{st}(\phi_b \hat\chi)_{\epsilon'},$$

where both standard G^+-representations $\pi_{st}(\phi_b \hat\chi)_{\epsilon^{(\prime)}}$ have a unique irreducible quotient. The Langlands classification decrees that to $(\phi_b \hat\nu, \epsilon)$ we must associate one of the irreducible quotients of $I_P^{G^+}(\pi_L(\phi_b)_{\epsilon_G} \otimes \chi)$. But only $\pi_{st}(\phi_b \hat\chi)_\epsilon$ fits in a continuous family of G^+-representations $\chi \mapsto \pi_{st}(\phi_b \hat\chi)_\epsilon$ which for unitary $\chi \in Y$ recovers $\pi(\phi_b \hat\chi)_\epsilon$, so we must send $(\phi_b \hat\nu, \epsilon)$ to the irreducible quotient of that standard representation. $\square$

We next result collects the conclusions from Section 4.

Theorem 4.12. *Let $\mathfrak{s}^+$ be an inertial equivalence class for G^+ and fix a Whittaker datum for the quasi-split inner form of G. There exists an algebra isomorphism $\mathcal{H}(\mathfrak{s}^+)^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^{+\vee}, q_F^{1/2})$, canonical up to conjugation by elements of $\mathcal{O}(T_{\mathfrak{s}^+})^\times$, such that the following holds.*

For each $(\phi, \epsilon) \in \Phi_{\mathrm{e}}(G^+)^{\mathfrak{s}^{+\vee}}$, the G^+-representation $\pi(\phi, \epsilon)$ constructed via Hecke algebras in Theorem 3.7 is isomorphic to the G^+-representation associated to (ϕ, ϵ) by [MoRe] and the Langlands classification.

Proof. As before, the Hecke algebra isomorphism and its canonicity property comes from Propositions 3.5 and 4.6.

We claim that the extensions of [MoRe] by means of the Langlands classification, as in Theorem 4.11, always sends $(\phi_b \hat\nu, \epsilon) \in \Phi_{\mathrm{e}}(G^+)$ to the unique irreducible quotient of $\pi_{st}(\phi_b \hat\nu, \epsilon)$. Indeed, the proof of Theorem 4.11 states that explicitly in the cases (ii) and (iii). In case (i) it follows because Theorem 3.9 shows that

$$I_{P^+}^{G^+}(\pi_{L^+}(\phi_b)_\epsilon \otimes \nu) = I_{P^+}^{G^+}(\pi_{L^+}(\phi_b \hat\nu, \epsilon)) = \pi_{st}(\phi_b \hat\nu, \epsilon).$$

On the other hand, by construction (see the lines before Theorem 3.9), $\pi(\phi_b \hat\nu, \epsilon)$ is also the irreducible quotient of $\pi_{st}(\phi_b \hat\nu, \epsilon)$. $\square$

5. Unitary groups

In this section we discuss how the setup and the statements in Sections 1–4 can be adjusted, so that the arguments and the results hold for unitary groups. Most of this can be found in [Mœ2] and [Hei4, §C]. We prefer to use the convenient description of L-parameters for unitary groups from [GGP].

Let E/F be a separable quadratic extension. Let V be a finite dimensional E-vector space endowed with an Hermitian form. Recall that the unitary group $\mathrm{U}(V)$ is a reductive algebraic F-group, an outer form of $\mathrm{GL}_{\dim V}$. The classification of pure inner twists reads:

- When $\dim V = 2n$, there is one quasi-split group $\mathrm{U}_{2n}(E/F)$ and one pure inner twist $\mathrm{U}'_{2n}(E/F)$, which is not quasi-split.
- When $\dim V = 2n+1$, there is a quasi-split group $\mathrm{U}_{2n+1}(E/F)$, associated to a Hermitian form with discriminant 1. There is an isomorphic but different form $\mathrm{U}'_{2n+1}(E/F)$, which is associated to an Hermitian form whose discriminant is nontrivial in $F^\times / N_{E/F}(E^\times)$.

The complex dual group of $\mathrm{U}_m(E/F)$ and $\mathrm{U}'_m(E/F)$ is $\mathrm{GL}_m(\mathbb{C})$. The group $\mathbf{W}_F/\mathbf{W}_E = \mathrm{Gal}(E/F)$ acts on $\mathrm{GL}_m(\mathbb{C})$ by the outer automorphism

$$A \mapsto J_m A^{-T} J_m^{-1},$$

where $-T$ denotes inverse transpose and J_m is the anti-diagonal $m\times m$-matrix whose with on the anti-diagonal alternating 1 and -1. We use a compressed form of the Langlands dual group:

$${}^L\mathrm{U}_m(E/F) = {}^L\mathrm{U}'_m(E/F) = \mathrm{GL}_m(\mathbb{C}) \rtimes \mathbf{W}_F/\mathbf{W}_E.$$

Modifications in Paragraph 1.1.
According to [GGP, Theorem 8.1], any L-parameter ϕ for $\mathrm{U}(V)$ is determined (up to $\mathrm{U}(V)^\vee$-conjugacy) by its restriction to $\mathbf{W}_E \times SL_2(\mathbb{C})$, which we denote Φ_e. This Φ_e is a conjugate-dual representation, which means that $\Phi_\mathrm{e}^\vee$ is isomorphic to $s \cdot \Phi_\mathrm{e}$ for any $s \in \mathbf{W}_F \setminus \mathbf{W}_E$. Moreover Φ_e is conjugate-orthogonal (sign +1) if $\dim V$ is odd and conjugate-symplectic (sign -1) if $\dim V$ is even. That provides a bijection from $\Phi(U(V))$ to the isomorphism classes of conjugate-dual representations of $\mathbf{W}_E$ with sign $(-1)^{\dim V - 1}$. For consistency we define $\mathrm{sgn}(\mathrm{U}(V)^\vee) = (-1)^{\dim V -1}$.

Conversely, let a conjugate-dual m-dimensional representation Φ_e of $\mathbf{W}_E \times \mathrm{SL}_2(\mathbb{C})$ with sign $(-1)^{m-1}$ be given. Then one can determine

$$\phi\colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C}) \rtimes \mathbf{W}_F/\mathbf{W}_E \tag{5.1}$$

up to conjugacy by requiring that $\phi(\mathbf{W}_F \setminus \mathbf{W}_E)$ consists of elements s (in the non-identity component) such that $s \cdot \Phi_\mathrm{e}$ is equivalent with $\Phi_\mathrm{e}^\vee$. We abbreviate this operation to $\Phi_\mathrm{e} \mapsto \mathrm{ind}_{\mathbf{W}_E}^{\mathbf{W}_F} \Phi_\mathrm{e}$.

It is natural to relate the centralizer group of ϕ (computed in $\mathrm{U}(V)^\vee$) to a suitable centralizer group of Φ_e. To this end we recall from [GGP] that ϕ determines an explicit bilinear form B_ϕ on $\mathbb{C}^m$, with respect to which Φ_e is conjugate-dual. By [GGP, Theorem 8.1.iii]

$$\begin{aligned}
&\mathrm{Z}_{\mathrm{U}(V)^\vee}(\phi) = \mathrm{Z}_{\mathrm{Aut}(B_\phi)}(\Phi_\mathrm{e}),\\
&\mathrm{Aut}(B_\phi) = \big\{ g \in \mathrm{GL}_m(\mathbb{C}) : B_\phi(gv, gv') = B_\phi(v,v') \; \forall v, v' \in \mathbb{C}^m \big\}.
\end{aligned}$$

From [GGP, §4] one sees that $\mathrm{Z}_{\mathrm{Aut}(B_\phi)}(\Phi_\mathrm{e})$ behaves exactly like $\mathrm{Z}_{G^\vee{}^+_{\mathrm{der}}}(\phi)$ in the case of general spin groups. More explicitly, $\mathrm{Z}_{\mathrm{U}(V)^\vee}(\phi)$ and $Z_{\mathrm{Aut}(B_\phi)}(\phi)$ are given by (1.5) and (1.11), we only have to omit the S (for det = 1) from those formulas.

Modifications in Paragraph 1.2.
The standard Levi subgroups of $G_n = U(V)$ are of the form

$$L = G_{n_-} \times \mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E)$$

with $G_{n_-} = U(V')$ of the same type as G_n and $\dim V - \dim V' = 2(n_1 + \cdots + n_k)$. Similarly

$${}^L L = {}^L G_{n_-} \times \mathrm{ind}_{\mathbf{W}_E}^{\mathbf{W}_F} \big(\mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_k}(\mathbb{C}) \big).$$

By Shapiro's lemma, $\Phi(L)$ is naturally in bijection with

$$\Phi(G_{n_-}) \times \Phi(\mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E)),$$

which by [GGP, Theorem 8.1] can be regarded as a set of conjugacy classes of homomorphisms with domain $\mathbf{W}_E \times \mathrm{SL}_2(\mathbb{C})$. Accordingly, the centralizer of $\phi \in$

$\Phi(L)$ can be computed as the centralizer of Φ_{e} in

$$L_E^\vee := \mathrm{Aut}(B_\phi) \times \mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_k}(\mathbb{C}).$$

We write

$$\mathcal{S}_\phi = \mathcal{S}_{\Phi_{\mathrm{e}}} = \pi_0\big(Z_{L^\vee}(\phi)\big) = \pi_0\big(Z_{L_E^\vee}(\Phi_{\mathrm{e}})\big).$$

The cuspidal support [AMS1] of $(\phi, \epsilon) \in \Phi(G)$ can be computed via

$$Z_{G^\vee}(\phi(\mathbf{W}_F)) = Z_{\mathrm{Aut}(B_\phi)}(\Phi_{\mathrm{e}}(\mathbf{W}_E)).$$

This implies that

$$\mathrm{Sc}(\phi, \epsilon) = \mathrm{Sc}\big(\mathrm{ind}_{\mathbf{W}_E}^{\mathbf{W}_F} \Phi_{\mathrm{e}}, \epsilon\big) = \mathrm{ind}_{\mathbf{W}_E}^{\mathbf{W}_F}\big(\mathrm{Sc}(\phi, \epsilon)\big),$$

where $\mathrm{ind}_{\mathbf{W}_E}^{\mathbf{W}_F}$ does not change the enhancements.

As a consequence, everything in Paragraph 1.2 can be carried out for unitary groups, with Φ_{e} and $L_E^\vee$ instead of ϕ and $L^\vee$. However, the results are not always precisely as before. We have to distinguish two cases, depending on the ramification of $\mathrm{U}(V)$, that is, the ramification of E/F.

Suppose first that E/F is ramified. We take a Frobenius element of $\mathbf{W}_E$ also as Frobenius element of $\mathbf{W}_F$, and we pick a representative for $\mathbf{W}_F/\mathbf{W}_E$ in $\mathbf{I}_F$. Then $\mathrm{Res}_{\mathbf{I}_E}^{\mathbf{W}_E}$ and $\mathrm{Res}_{\mathbf{I}_F}^{\mathbf{W}_F}$ are compatible with $\phi \mapsto \Phi_{\mathrm{e}}$ and $\mathrm{ind}_{\mathbf{W}_E}^{\mathbf{W}_F}$. Hence the calculations in Paragraph 1.2 produce the correct results for $\mathrm{U}(V)$. We only have to remember to omit the centre $\mathbb{C}^\times$ of $\mathrm{GSpin}(V)^\vee$ and the S for det $= 1$, like we needed to do for symplectic groups.

Next we suppose that E/F is unramified. Then $\mathbf{I}_E = \mathbf{I}_F$ and as Frobenius element of $\mathbf{W}_F$ we take the square of a Frobenius element of $\mathbf{W}_E$. In contrast to the ramified case, the impact on Paragraph 1.2 is substantial.

For $\tau \in \mathrm{Irr}(\mathbf{W}_E)_\phi^\pm$, there is still a unique (up to isomorphism) unramified twist $\tau' = \tau \otimes \chi$ which is conjugate-dual and not isomorphic to τ. However, in contrast to before τ' and τ always have different signs [Sol5, Proposition 4.10.b]. We order τ, τ' so that $\ell_\tau \geq \ell_{\tau'}$ and if $\ell_\tau = \ell_{\tau'} = 0$ then $a_\tau \geq a_{\tau'}$.

The next change occurs in (1.21), there

$$\begin{aligned} T_{\mathfrak{s}^\vee} &= T/\big(\prod\nolimits_j \mathrm{Z}(\mathrm{GL}_{n_j}(\mathbb{C}))_{\mathrm{ind}_{\mathbf{W}_E}^{\mathbf{W}_F}\phi_j}\big) \\ &= \prod\nolimits_\tau \big(\mathbb{C}^\times / \mathrm{Z}(\mathrm{GL}_{n_j}(\mathbb{C}))_{\mathrm{ind}_{\mathbf{W}_E}^{\mathbf{W}_F}\phi_j}\big)^{e_\tau} = \prod\nolimits_\tau T_{\mathfrak{s}^\vee,\tau}, \end{aligned}$$

with the latter two products running over $\mathrm{Irr}'(\mathbf{W}_E)_\phi^\pm \cup \mathrm{Irr}(\mathbf{W}_E)_\phi^0$. We note that

$$\big|\mathrm{Z}(\mathrm{GL}_{n_j}(\mathbb{C}))_{\mathrm{ind}_{\mathbf{W}_E}^{\mathbf{W}_F}\phi_j}\big| = 2\, |\mathrm{Z}(\mathrm{GL}_{d_\tau}(\mathbb{C}))_\tau| = 2t_\tau.$$

In particular

$$X^*(T_{\mathfrak{s}^\vee,\tau}) = 2t_\tau\big(X^*(T) \cap \mathbb{Q}X^*(T_{\mathfrak{s}^\vee,\tau})\big).$$

Further (1.22) becomes

$$\begin{aligned} J = Z_{G^\vee}(\phi(\mathbf{I}_F)) &= \prod\nolimits_\tau G^\vee_{\phi(\mathbf{I}_E),\tau} \\ &= \prod_{\tau \in \mathrm{Irr}'(\mathbf{W}_E)_\phi^\pm} \mathrm{GL}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau} \times \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_E)_\phi^0} \mathrm{GL}_{e_\tau}(\mathbb{C})^{2t_\tau}. \end{aligned}$$

As a consequence (1.25) has to be modified in the cases $\tau \in \mathrm{Irr}(\mathbf{W}_E)^+_\phi$, now it reads

$$R(G^\vee_{\phi(\mathbf{I}_E),\tau} T, T) = \begin{cases} C_{e_\tau} & \ell_\tau + \ell_{\tau'} = 0 \\ BC_{e_\tau} & \ell_\tau + \ell_{\tau'} > 0 \end{cases}.$$

In view of the new shape of J, its maximal torus given in (1.26) becomes

$$T_J = \prod_\tau T_{J,\tau}^{t_\tau} = \prod_{\tau \in \mathrm{Irr}'(\mathbf{W}_E)^\pm_\phi} \big((\mathbb{C}^\times)^{2e_\tau + \ell_\tau + \ell_{\tau'}} \big)^{t_\tau} \times \prod_{\tau \in \mathrm{Irr}(\mathbf{W}_E)^0_\phi} \big((\mathbb{C}^\times)^{e_\tau} \big)^{2t_\tau}.$$

The computation of m_α for $\alpha \in R(J,T)_{\mathrm{red}}$ after (1.21) also changes for unramified unitary groups. For $\tau \in \mathrm{Irr}(\mathbf{W}_E)^0_\phi$, the root system $R(G^\vee_{\phi(\mathbf{I}_E),\tau} T, T)$ has $2t_\tau$ irreducible components, all of type $A_{e_\tau - 1}$ and permuted cyclically by Frob_F. Hence m_α equals $2t_\tau m'_\alpha$, and the same argument as before shows that $m'_\alpha = 1$.

When $\tau \in \mathrm{Irr}'(\mathbf{W}_E)^\pm_\phi$, the root system $R(G^\vee_{\phi(\mathbf{I}_E),\tau} T, T)$ has t_τ irreducible components. They are of type $A_{2e_\tau + \ell_\tau + \ell_{\tau'}}$ and Frob_F permutes them cyclically, so $m_\alpha = t_\tau m'_\alpha$. Here the computation of m'_α proceeds analogously to in Paragraph 1.2 for the cases $\tau \in \mathrm{Irr}(\mathbf{W}_F)^{+-}_\phi$. We conclude that $m_\alpha = 2t_\tau$ unless $\ell_\tau + \ell_{\tau'} = 0$ and $\alpha \in C_{e_\tau}$ is long, then $m_\alpha = t_\tau$.

From this we obtain the root systems $R_{\mathfrak{s}^\vee,\tau}$ whose union is $R_{\mathfrak{s}^\vee}$. For $\tau \in \mathrm{Irr}(\mathbf{W}_E)^0_\phi$ we obtain $A_{e_\tau - 1} \subset X^*(T_{\mathfrak{s}^\vee,\tau})$ as before. For $\tau \in \mathrm{Irr}'(\mathbf{W}_E)^\pm_\phi$ with $\ell_\tau + \ell_{\tau'} > 0$ we get $2t_\tau B_{e_\tau} \subset 2t_\tau X^*(T)$, which can be identified with B_{e_τ} in $X^*(T_{\mathfrak{s}^\vee,\tau})$. For $\tau \in \mathrm{Irr}'(\mathbf{W}_E)^\pm_\phi$ with $\ell_\tau + \ell_{\tau'} = 0$ we obtain $2t_\tau D_{e_\tau} \cup t_\tau (C_{e_\tau} \setminus D_{e_\tau})$ in $2t_\tau X^*(T)$, which identifies with B_{e_τ} in $X^*(T_{\mathfrak{s}^\vee,\tau})$.

The root datum for the affine Hecke algebra decomposes nicely:

$$\mathcal{R}_{\mathfrak{s}^\vee} = \bigoplus_\tau \mathcal{R}_{\mathfrak{s}^\vee,\tau} = \bigoplus_\tau \big(X^*(T_{\mathfrak{s}^\vee,\tau}), R_{\mathfrak{s}^\vee,\tau}, X_*(T_{\mathfrak{s}^\vee,\tau}), R^\vee_{\mathfrak{s}^\vee} \big).$$

The calculation of the parameter functions λ, λ^* (following the method in [AMS3, §3.3]) leads to the following modified version of Table 2:

TABLE 3. Data from $\mathcal{R}_{\mathfrak{s}^\vee}$ for each τ

a_τ	$a_{\tau'}$	$X^*(T_{\mathfrak{s}^\vee,\tau})$	$R_{\mathfrak{s}^\vee,\tau}$	$\lambda(\alpha)$	$\lambda(\beta)$	$\lambda^*(\beta)$
0	-1	$\mathbb{Z}^{e_\tau}$	B_{e_τ}	$2t_\tau$	t_τ	t_τ
≥ 1	≥ -1	$\mathbb{Z}^{e_\tau}$	B_{e_τ}	$2t_\tau$	$t_\tau(a_\tau + a_{\tau'} + 2)$	$t_\tau(a_\tau - a_{\tau'})$
$\mathrm{Irr}(\mathbf{W}_E)^0_\phi$		$\mathbb{Z}^{e_\tau}$	$A_{e_\tau - 1}$	$2t_\tau$	$--$	$--$

Here the first line is an instance of the second line, we mention it separately because it comes from the exceptional case $\ell_\tau + \ell_{\tau'} = 0$ discussed above. We note that in all lines of Table 3 $W(R_{\mathfrak{s}^\vee,\tau})$ is the full group $W_{\mathfrak{s}^\vee}$, so $\Gamma^{(+)}_{\mathfrak{s}^\vee}$ is trivial and can be omitted from the table.

Modifications in Section 2.

Most of the necessary adjustments, as well as a proof of Theorem 2.1.c,d for unitary groups, can be found in [Mœ2]. Let us spell out the significant changes.

The Jordan blocks of a discrete series representation π of $G = U(V)$ are based on unitary supercuspidal representations ρ of $\mathrm{GL}_m(E)$. Instead of (2.1), they have to be conjugate-dual: $\rho \cong \bar{\rho}^\vee$, where the bar indicates composing a representation with the natural action of $\mathrm{Gal}(E/F)$ on $\mathrm{U}(V)$.

Although there exist outer automorphisms of unitary groups, we should not involve them like for $\mathrm{SO}(V)$ and $\mathrm{GSpin}(V)$, because here $G^+ = G$. Rather, we should just replace $\mathrm{Out}(\mathcal{G})$ by the trivial group everywhere. Then all results in Section 2 hold for unitary groups (except Theorem 2.5 which is specific for general spin groups).

Modifications in Section 3.
No further adjustments are needed, everything works in the above setup. The groups $\Gamma_{\mathfrak{s}}, \Gamma^+_{\mathfrak{s}}, \Gamma_{\mathfrak{s}^\vee}, \Gamma^+_{\mathfrak{s}^\vee}$ are trivial, so all considerations about those are superfluous for unitary groups. Also, as $G^+ = G$ the material in Paragraph 3.1 becomes trivial.

Modifications in Section 4.
There is only one small change, when $\mathrm{U}(V)$ is unramified. In the proof of Proposition 4.6 the case $\ell_{\tau'} = 0$ can be treated just as $\ell_{\tau'} > 0$, because by Table 3 the relevant Hecke algebra has a root datum of type B_{e_τ} with parameters such that $\lambda(\beta) = \lambda^*(\beta) > 0$.

Acknowledgements. We thank the referees for their helpful reports.

References

[Art] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, Colloquium Publications **61**, Amer. Math. Soc., 2013.

[AsSh] M. Asgari and F. Shahidi, "Generic transfer for general spin groups", Duke Math. J. **132.1** (2006), 137–190.

[AGIKMS] H. Atobe, W.T. Gan, A. Ichino, T. Kaletha, A. Mínguez and S.W. Shin, "Local intertwining relations and co-tempered A-packets of classical groups", arXiv:2410.13504, 2024.

[ABPS1] A.-M. Aubert, P.F. Baum, R.J. Plymen and M. Solleveld "On the local Langlands correspondence for non-tempered representations", Münster J. Math. **7** (2014) 27–50.

[ABPS2] ———, "The local Langlands correspondence for inner forms of SL_n", Res. Math. Sci. **3.32** (2016), DOI: 10.1186/s40687-016-0079-4.

[ABPS3] ———, "The principal series of p-adic groups with disconnected centre", Proc. London Math. Soc. **114.5** (2017), 798–854.

[AMS1] A.-M. Aubert, A. Moussaoui and M. Solleveld, "Generalizations of the Springer correspondence and cuspidal Langlands parameters", Manus. Math. **157** (2018), 121–192.

[AMS2] ———, "Graded Hecke algebras for disconnected reductive groups", pp. 23–84 in: *Geometric aspects of the trace formula*, W. Müller, S. W. Shin, N. Templier (eds.), Simons Symposia, Springer, 2018.

[AMS3] ———, "Affine Hecke algebras for Langlands parameters", arXiv:1701.03593v6, 2024.

[BaJa] D. Ban and C. Jantzen, "The Langlands classification for non-connected p-adic groups", Israel J. Math. **126** (2001), 239–261.

[Bor] A. Borel, "Automorphic L-functions", Proc. Symp. Pure Math **33.2** (1979), 27–61.

[BuKu] C.J. Bushnell and P.C. Kutzko, *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, Ann. Math. Studies **129**, Princeton University Press, Princeton NJ, 1993.

[Del] P. Deligne, "Les corps locaux de caractéristique p, limites de corps locaux de caractéristique 0", pp. 119–157 in: *Représentations des groupes réductifs sur un corps local*, Travaux en cours, Hermann, 1984.

[GGP] W.T. Gan, B. Gross and D. Prasad, "Symplectic local root numbers, central critical L-values and restriction problems in the representation theory of classical groups", Astérisque **346** (2012), 1–109.

[GaLo] W.T. Gan and L. Lomelí, "Globalization of supercuspidal representations over function fields and applications", J. Eur. Math. Soc. **20** (2018), 2813-–2858.

[Gan] R. Ganapathy, "A Hecke algebra isomorphism over close local fields", Pacific J. Math. **319.2**, 307–332. *Corrigendum:* Pacific J. Math. **330.2** (2024), 389–398.

[GaVa] R. Ganapathy and S. Varma, "On the local Langlands correspondence for split classical groups over local function fields", J. Inst. Math. Jussieu **16.5** (2017), 987–1074

[Hai] T.J. Haines, "The stable Bernstein center and test functions for Shimura varieties", pp. 118–186 in: *Automorphic forms and Galois representations*, London Math. Soc. Lecture Note Ser. **415**, Cambridge University Press, 2014.

[HaTa] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Ann. Math. Studies **151**, Princeton University Press, Princeton, NJ, 2001.

[Hei1] V. Heiermann, "Orbites unipotentes et pôles d'ordre maximal de la function μ de Harish-Chandra", Canad. J. Math. **58.6** (2006), 1203–1228.

[Hei2] ———, "Paramètres de Langlands et algèbres d'entrelacement", Int. Math. Res. Not. **191** (2009).

[Hei3] ———, "Opérateurs d'entrelacement et algèbres de Hecke avec paramètres d'un groupe réductif p-adique - le cas des groupes classiques", Selecta Math. **17.3** (2011), 713–756.

[Hei4] ———, "Local Langlands correspondence for classical groups and affine Hecke algebras", Math. Z. **287** (2017), 1029–1052.

[Hen] G. Henniart, "Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique", Invent. Math. **139**, 439–455, 2000.

[Kal1] T. Kaletha, "Rigid inner forms of real and p-adic groups", Ann. Math. **184.2** (2016), 559–632.

[Kal2] T. Kaletha, "Supercuspidal L-packets", arXiv:1912.03274v2, 2021.

[KMSW] T. Kaletha, A. Minguez, S.-W. Shin and P.-J. White, "Endoscopic Classification of Representations: Inner Forms of Unitary Groups", arXiv:1409.3731v3, 2014.

[KiMa] Y. Kim and I. Matić, "Discrete series of odd general spin groups", Monatsh Math **203** (2024), 419–473.

[KSS] R. Kurinczuk, D. Skodlerack and S. Stevens, "Endo-parameters for p-adic classical groups", Invent. Math. **223.2** (2021), 597–723.

[LRS] G. Laumon, M. Rapoport and U. Stuhler, "$\mathcal{D}$-elliptic sheaves and the Langlands correspondence", Invent. Math. **113** (1993), 217–238.

[Lus1] G. Lusztig, "Intersection cohomology complexes on a reductive group", Invent. Math. **75.2** (1984), 205–272.

[Lus2] ———, "Cuspidal local systems and graded Hecke algebras", Publ. Math. Inst. Hautes Études Sci. **67** (1988), 145–202.

[Lus3] ———, "Affine Hecke algebras and their graded version", J. Amer. Math. Soc **2.3** (1989), 599–635.

[MiSt] M. Miyauchi and S. Stevens, "Semisimple types for p-adic classical groups", Math. Ann. **358** (2014), 257–288.

[Mœ1] C. Mœglin, "Sur la classification des séries discrètes des groupes classiques p-adiques; paramètre de Langlands et exhaustivité", J. Eur. Math. Soc. **4** (2002) 143–200.

[Mœ2] ———, "Classification et changement de base pour les séries discrètes des groupes unitaires p-adiques", Pacific J. Math. **233** (2007), 159–204.

[Mœ3] ———, "Multiplicité 1 dans les paquets d'Arthur aux places p-adiques", pp. 333–374 in: *On certain L-functions*, Clay Math. Proc. **13**, Amer. Math. Soc., 2011.

[Mœ4] ———, "Paquets stables des séries discrètes accessibles par endoscopie tordue: leur paramètre de Langlands", Contemp. Math. **614** (2014), 295-–336.

[MoRe] C. Mœglin and D. Renard, "Sur les paquets d'Arthur des groupes classiques et unitaires non quasi-déployés", pp. 341–361 in: *Relative Aspects in Representation Theory, Langlands Functoriality and Automorphic Forms*, Lecture Notes in Math. **2221**, 2018.

[MoTa] C. Mœglin and M. Tadić, "Construction of discrete series for classical p-adic groups", J. Amer. Math. Soc. **15.3** (2002), 715–786.

[MoWa] C. Mœglin and J.-L. Waldspurger, *Stabilisation de la formule des traces tordue*, Progress in Mathematics, **316** & **317**, Birkhäuser/Springer, Cham, 2016.

[Mok] C.P. Mok, "Endoscopic classification of representations of quasi-split unitary groups I", Mem. Amer. Math. Soc. **1108**, vol. **235** (2015).

[Mou] A. Moussaoui, "Centre de Bernstein enrichi pour les groupes classiques", Represent. Theory **21** (2017), 172–246.

[MoPr] A. Moy and G. Prasad, "Jacquet functors and unrefined minimal K-types", Comment. Math. Helvetici **71** (1996), 98–121.

[Ren] D. Renard, "Représentations des groupes réductifs p-adiques", Cours spécialisés **17**, Société Mathématique de France, 2010.

[Sch] P. Scholze, "The local Langlands correspondence for GL_n over p-adic fields", Invent. Math. **192** (2013), 663–715.

[SiZi] A.J. Silberger and E.-W. Zink, "Langlands classification for L-parameters", J. Algebra **511** (2018), 299–357.

[Sol1] M. Solleveld, "On the classification of irreducible representations of affine Hecke algebras with unequal parameters", Represent. Th. **16** (2012), 1–87.

[Sol2] ______ "On completions of Hecke algebras", pp. 207–262 in: *Representations of Reductive p-adic Groups*, A.-M. Aubert, M. Mishra, A. Roche, S. Spallone (eds.), Progress in Mathematics **328**, Birkhäuser, 2019.

[Sol3] ______, "A local Langlands correspondence for unipotent representations", Amer. J. Math. **145.3** (2023), 673–719.

[Sol4] ______, "Endomorphism algebras and Hecke algebras for reductive p-adic groups", J. Algebra **606** (2022), 371–470, and correction in arXiv:2005.07899v3 (2023).

[Sol5] ______, "Parameters of Hecke algebras for Bernstein components of p-adic groups", Indag. Math. **36.1** (2025), 124–170.

[Sol6] ______, "Graded Hecke algebras, constructible sheaves and the p-adic Kazhdan–Lusztig conjecture", J. Algebra **667** (2025), 865–910.

[Xu] B. Xu, "On the cuspidal support of discrete series for p-adic quasisplit $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$", Manus. Math. **154** (2017), 441-–502.

Sorbonne Université and Université Paris Cité, CNRS, IMJ-PRG, F-75005 Paris, France

Email address: anne-marie.aubert@imj-prg.fr

Laboratoire de Mathématiques et Applications, Université de Poitiers, 11 Boulevard Marie et Pierre Curie, Bâtiment H3 - TSA 61125, 86073 Poitiers Cedex 9, France

Email address: ahmed.moussaoui@math.univ-poitiers.fr

IMAPP, Radboud Universiteit Nijmegen, Heyendaalseweg 135, 6525AJ Nijmegen, the Netherlands

Email address: m.solleveld@science.ru.nl