HECKE ALGEBRAS FOR INNER FORMS OF p-ADIC SPECIAL LINEAR GROUPS

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Abstract. Let F be a non-archimedean local field and let $G^\sharp$ be the group of F-rational points of an inner form of SL_n. We study Hecke algebras for all Bernstein components of $G^\sharp$, via restriction from an inner form G of $\mathrm{GL}_n(F)$.

For any packet of L-indistinguishable Bernstein components, we exhibit an explicit algebra whose module category is equivalent to the associated category of complex smooth $G^\sharp$-representations. This algebra comes from an idempotent in the full Hecke algebra of $G^\sharp$, and the idempotent is derived from a type for G. We show that the Hecke algebras for Bernstein components of $G^\sharp$ are similar to affine Hecke algebras of type A, yet in many cases are not Morita equivalent to any crossed product of an affine Hecke algebra with a finite group.

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Introduction

Let F be a non-archimedean local field and let D be a division algebra, of dimension d^2 over its centre F. Then $G = \mathrm{GL}_m(D)$ is the group of F-rational points of an inner form of GL_n, where $n = md$. We will say simply that G is an inner form of $\mathrm{GL}_n(F)$. Its derived group $G^\sharp$, the kernel of the reduced norm map $G \to F^\times$, is an inner form of $\mathrm{SL}_n(F)$. Every inner form of $\mathrm{SL}_n(F)$ looks like this.

Since the appearance of the important paper [16] there has been a surge of interest in these groups, cf. [10, 11, 3]. In this paper we continue our investigations of the (complex) representation theory of inner forms of $\mathrm{SL}_n(F)$. Following the Bushnell–Kutzko approach [7], we study algebras associated to idempotents in the Hecke algebra of $G^\sharp$. The main idea of this approach is to understand Bernstein components for a reductive p-adic group better by constructing types and making the ensuing Hecke algebras explicit.

It turns out that for the groups under consideration, while it is really hard to find types, the appropriate Hecke algebras are accessible via different techniques. Our starting point is the construction of types for all Bernstein components of G by Sécherre–Stevens [23, 24]. We consider the Hecke algebra of such a type, which is described in [22]. In several steps we modify this algebra to one whose module category is equivalent to a union of some Bernstein blocks for $G^\sharp$. Let us discuss our strategy and our main result.

We fix a parabolic subgroup $P \subset G$ with Levi factor L. A supercuspidal L-representation ω gives an inertial equivalence class $\mathfrak{s} = [L, \omega]_G$. Let $\mathrm{Rep}^{\mathfrak{s}}(G)$ denote the corresponding Bernstein block of the category of smooth complex G-representations, and let $\mathrm{Irr}^{\mathfrak{s}}(G)$ denote the set of irreducible objects in $\mathrm{Rep}^{\mathfrak{s}}(G)$. Let $\mathrm{Irr}^{\mathfrak{s}}(G^\sharp)$ be the set of irreducible $G^\sharp$-representations that are subquotients of $\mathrm{Res}^G_{G^\sharp}(\pi)$ for some $\pi \in \mathrm{Irr}^{\mathfrak{s}}(G)$. We define $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp)$ as the collection of $G^\sharp$-representations all whose irreducible subquotients lie in $\mathrm{Irr}^{\mathfrak{s}}(G^\sharp)$. We want to investigate the category $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp)$. It is a product of finitely many Bernstein blocks for $G^\sharp$:

$$\mathrm{Rep}^{\mathfrak{s}}(G^\sharp) = \prod\nolimits_{\mathfrak{t}^\sharp \prec \mathfrak{s}} \mathrm{Rep}^{\mathfrak{t}^\sharp}(G^\sharp). \tag{1}$$

We note that the Bernstein components $\mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp)$ which are subordinate to one $\mathfrak{s}$ form precisely one class of L-indistinguishable components: every L-packet for $G^\sharp$ which intersects one of them intersects them all.

The structure of $\mathrm{Rep}^{\mathfrak{s}}(G)$ is largely determined by the torus $T_{\mathfrak{s}}$ and the finite group $W_{\mathfrak{s}}$ associated by Bernstein to $\mathfrak{s}$. Recall that the Bernstein torus of $\mathfrak{s}$ is

$$T_{\mathfrak{s}} = \{\omega \otimes \chi \mid \chi \in X_{\mathrm{nr}}(L)\} \subset \mathrm{Irr}(L),$$

where $X_{\mathrm{nr}}(L)$ denotes the group of unramified characters of L. The finite group $W_{\mathfrak{s}}$ equals $\mathrm{N}_M(L)/L$ for a suitable Levi subgroup $M \subset G$ containing L. For this particular reductive p-adic group $W_{\mathfrak{s}}$ is always a Weyl group (in fact a direct product of symmetric groups), but for $G^\sharp$ more general finite groups are needed. We also have to take into account that we are dealing with several Bernstein components simultaneously.

Let $\mathcal{H}(G)$ be the Hecke algebra of G and $\mathcal{H}(G)^{\mathfrak{s}}$ its two-sided ideal corresponding to the Bernstein block $\mathrm{Rep}^{\mathfrak{s}}(G)$. Similarly, let $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ be the two-sided ideal of

$\mathcal{H}(G^\sharp)$ corresponding to $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp)$. Then

$$\mathcal{H}(G^\sharp)^{\mathfrak{s}} = \prod\nolimits_{\mathfrak{t}^\sharp \prec \mathfrak{s}} \mathcal{H}(G^\sharp)^{\mathfrak{t}^\sharp}.$$

Of course we would like to determine $\mathcal{H}(G^\sharp)^{\mathfrak{t}^\sharp}$, but it turns out that $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ is much easier to study. So our main goal is an explicit description of $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ up to Morita equivalence. From that the subalgebras $\mathcal{H}(G^\sharp)^{\mathfrak{t}^\sharp}$ can in principle be extracted, as maximal indecomposable subalgebras. We note that sometimes $\mathcal{H}(G^\sharp)^{\mathfrak{s}} = \mathcal{H}(G^\sharp)^{\mathfrak{t}^\sharp}$, see Examples 5.3, 5.5.

From [23] we know that there exists a simple type (K, λ) for $[L, \omega]_M$. By [24] it has a G-cover (K_G, λ_G). We denote the associated central idempotent of $\mathcal{H}(K)$ by e_λ, and similarly for other irreducible representations. There is an affine Hecke algebra $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$, a tensor product of affine Hecke algebras of type GL_e, such that

$$e_{\lambda_G} \mathcal{H}(G) e_{\lambda_G} \cong e_\lambda \mathcal{H}(M) e_\lambda \cong \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\lambda), \tag{2}$$

and these algebras are Morita equivalent with $\mathcal{H}(G)^{\mathfrak{s}}$.

An important role in the restrictions of representations from $\mathcal{H}(G)^{\mathfrak{s}}$ to $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ is played by the group

$$X^G(\mathfrak{s}) := \{\gamma \in \mathrm{Irr}(G/G^\sharp Z(G)) \mid \gamma \otimes I_P^G(\omega) \in \mathrm{Rep}^{\mathfrak{s}}(G)\}.$$

It acts on $\mathcal{H}(G)$ by pointwise multiplication of functions $G \to \mathbb{C}$. For the restriction process we need an idempotent that is invariant under $X^G(\mathfrak{s})$. To that end we replace λ_G by the sum of the representations $\gamma \otimes \lambda_G$ with $\gamma \in X^G(\mathfrak{s})$, which we call μ_G. Then (2) remains valid with μ instead of λ, but of course V_μ is reducible as a representation of K.

Let $e_{\mu_{G^\sharp}} \in \mathcal{H}(G^\sharp)$ be the restriction of $e_{\mu_G} : G \to \mathbb{C}$ to $G^\sharp$. Up to a scalar factor it is also the restriction of e_{λ_G} to $G^\sharp$. We normalize the Haar measures in such a way that $e_{\mu_{G^\sharp}}$ is idempotent. For any $G^\sharp$-representation V, $e_{\mu_{G^\sharp}} V$ is the space of vectors in V on which $K_G \cap G^\sharp$ acts as some multiple of the (reducible) representation $\lambda_G|_{K_G \cap G^\sharp}$.

Then $e_{\mu_{G^\sharp}} \mathcal{H}(G^\sharp) e_{\mu_{G^\sharp}}$ is a nice subalgebra of $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$, but in general it is not Morita equivalent with $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$. There is only an equivalence between the module category of $e_{\mu_{G^\sharp}} \mathcal{H}(G^\sharp) e_{\mu_{G^\sharp}}$ and $\prod_{\mathfrak{t}^\sharp} \mathrm{Rep}^{\mathfrak{t}^\sharp}(G^\sharp)$, where $\mathfrak{t}^\sharp$ runs over some, but not necessarily all, inertial equivalence classes $\prec \mathfrak{s}$. To see the entire category $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp)$ we need finitely many isomorphic but mutually orthogonal algebras

$$a e_{\mu_{G^\sharp}} a^{-1} \mathcal{H}(G^\sharp) a e_{\mu_{G^\sharp}} a^{-1} \text{ with } a \in G.$$

To formulate our main result precisely, we need also the groups

$$\begin{aligned}
X^L(\mathfrak{s}) &= \{\gamma \in \mathrm{Irr}(L/L^\sharp Z(G)) \mid \gamma \otimes \omega \in [L, \omega]_L\}, \\
W_{\mathfrak{s}}^\sharp &= \{w \in \mathrm{N}_G(L) \mid \exists \gamma \in \mathrm{Irr}(L/L^\sharp Z(G)) : w(\gamma \otimes \omega) \in [L, \omega]_L\}, \\
\mathfrak{R}_{\mathfrak{s}}^\sharp &= W_{\mathfrak{s}}^\sharp \cap \mathrm{N}_G(P \cap M)/L, \\
X^L(\omega, V_\mu) &= \{\gamma \in \mathrm{Irr}(L/L^\sharp) \mid \begin{array}{l} \text{there exists an } L\text{-isomorphism } \omega \to \omega \otimes \gamma^{-1} \\ \text{which induces the identity on } V_\mu\}. \end{array}
\end{aligned}$$

Here $L^\sharp = L \cap G^\sharp$, so

$$L/L^\sharp \cong G/G^\sharp \cong F^\times .$$

We observe that $\mathfrak{R}^\sharp_\mathfrak{s}$ is naturally isomorphic to $X^G(\mathfrak{s})/X^L(\mathfrak{s})$, and that $W^\sharp_\mathfrak{s} = W_\mathfrak{s} \rtimes \mathfrak{R}^\sharp_\mathfrak{s}$ (see Lemmas 2.4 and 2.3). One can regard $W^\sharp_\mathfrak{s}$ as the Bernstein group for $\mathrm{Rep}^\mathfrak{s}(G^\sharp)$.

Theorem 1. [see Theorem 4.15.]
The algebra $\mathcal{H}(G^\sharp)^\mathfrak{s}$ is Morita equivalent with a direct sum of $|X^L(\omega, V_\mu)|$ copies of $e_{\mu_{G^\sharp}} \mathcal{H}(G^\sharp) e_{\mu_{G^\sharp}}$. The latter algebra is isomorphic with

$$\big(\mathcal{H}(T^\sharp_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu) \big)^{X^L(\mathfrak{s})/X^L(\omega, V_\mu) X_{\mathrm{nr}}(L/L^\sharp Z(G))} \rtimes \mathfrak{R}^\sharp_\mathfrak{s},$$

where $T^\sharp_\mathfrak{s} = T_\mathfrak{s} / X_{\mathrm{nr}}(L/L^\sharp)$. The actions of the groups $X^L(\mathfrak{s})$ and $\mathfrak{R}^\sharp_\mathfrak{s}$ come from automorphisms of $T_\mathfrak{s} \rtimes W_\mathfrak{s}$ and projective transformations of V_μ.

The projective actions of $X^L(\mathfrak{s})$ and $\mathfrak{R}^\sharp_\mathfrak{s}$ on V_μ are always linear in the split case $G = \mathrm{GL}_n(F)$, but not in general, see Examples 5.4 and 5.5.

Contrary to what one might expect from Theorem 1, the Bernstein torus $T_{\mathfrak{t}^\sharp}$ for $\mathrm{Rep}^{\mathfrak{t}^\sharp}(G^\sharp)$ is not always $T^\sharp_\mathfrak{s}$, see Example 5.2. In general one has to divide by a finite subgroup of $T^\sharp_\mathfrak{s}$ coming from $X^L(\mathfrak{s})$. It is possible that $W_{\mathfrak{t}^\sharp}$ (with $\mathfrak{t}^\sharp \prec \mathfrak{s}$) is strictly larger than $W_\mathfrak{s}$, and that it acts on $T_{\mathfrak{t}^\sharp}$ without fixed points, see Examples 5.1 and 5.3.

Of course the above has already been done for $\mathrm{SL}_n(F)$ itself, see [5, 6, 13, 14]. Indeed, for $\mathrm{SL}_n(F)$ our work has a large intersection with these papers. But the split case is substantially easier than the non-split case, for example because every irreducible representation of $\mathrm{GL}_n(F)$ restricts to a representation of $\mathrm{SL}_n(F)$ without multiplicities. Therefore our methods are necessarily different from those of Bushnell–Kutzko and Goldberg–Roche, even if our proofs are considered only for $\mathrm{SL}_n(F)$.

It is interesting to compare Theorem 1 for $\mathrm{SL}_n(F)$ with the main results of [14]. Our description of the Hecke algebras is more explicit, thanks to considering the entire packet $\mathrm{Rep}^\mathfrak{s}(G^\sharp)$ of Bernstein blocks simultaneously. In [14, §11] some 2-cocycle pops up in the Hecke algebras, which Goldberg–Roche expect to be trivial. From Theorem 1 one can deduce that it is indeed trivial, see Remark 4.16

Now we describe the contents of the paper in more detail. We start Section 2 with recalling a few results about restriction of representations from G to $G^\sharp$. Then we discuss what happens when one restricts an entire Bernstein component of representations at once. We introduce and study several finite groups which will be used throughout.

It turns out to be advantageous to restrict from G to $G^\sharp$ in two steps, via $G^\sharp Z(G)$. This intermediate group is of finite index in G if the characteristic of F does not divide n. Otherwise $[G : G^\sharp Z(G)] = \infty$ but, when studying only $\mathrm{Rep}^\mathfrak{s}(G)$, one can apply the same techniques as for a group extension of finite index. Restriction from $G^\sharp Z(G)$ to $G^\sharp$ is straightforward, so everything comes down to understanding the decomposition of representations and Bernstein components of G upon restriction to $G^\sharp Z(G)$.

For any subgroup $H \subset G$ we write $H^\sharp = H \cap G^\sharp$. The correct analogue of $W_\mathfrak{s}$ for $\mathrm{Rep}^\mathfrak{s}(G)$ combines Weyl groups and characters of the Levi subgroup L that are trivial on $L^\sharp Z(G)$:

$$\mathrm{Stab}(\mathfrak{s}) := \{(w,\gamma) \in W(G,L) \times \mathrm{Irr}(L/L^\sharp Z(G)) \mid w(\gamma \otimes \omega) \in [L,\omega]_L\}.$$

This group acts naturally on L-representations by

$$((w,\gamma)\pi)(l) = w(\gamma \otimes \pi)(l) := \gamma(l)\pi(w^{-1}lw).$$

From another angle $\mathrm{Stab}(\mathfrak{s})$ can be considered as the generalization, for the inertial class $\mathfrak{s}$, of some groups associated to a single G-representation in [10, 11]. Its relevance is confirmed by the following result.

Theorem 2. [see Theorem 2.7.]
Let $\chi_1, \chi_2 \in X_{\mathrm{nr}}(L)$. The following are equivalent:

(i) $\mathrm{Res}^G_{G^\sharp Z(G)}(I_P^G(\omega\otimes\chi_1))$ *and* $\mathrm{Res}^G_{G^\sharp Z(G)}(I_P^G(\omega\otimes\chi_2))$ *have a common irreducible subquotient;*
(ii) $\mathrm{Res}^G_{G^\sharp Z(G)}(I_P^G(\omega\otimes\chi_1))$ *and* $\mathrm{Res}^G_{G^\sharp Z(G)}(I_P^G(\omega\otimes\chi_2))$ *have the same irreducible constituents, counted with multiplicity;*
(iii) $\omega\otimes\chi_1$ *and* $\omega\otimes\chi_2$ *belong to the same* $\mathrm{Stab}(\mathfrak{s})$*-orbit.*

The proof of Theorem 1 uses almost the entire paper. It contains four chains of arguments, in Sections 3 and 4, which are largely independent:

- The main idea (see Lemmas 3.10 and 3.11) consists of Morita equivalences

$$\begin{aligned} \mathcal{H}(G^\sharp)^{\mathfrak{s}} &\sim_M (\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})X_{\mathrm{nr}}(G)}, \\ \mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}} &\sim_M (\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})}. \end{aligned} \tag{3}$$

  This enables us to reduce the study of $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ (up to Morita equivalence) to $(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})}$. The analogue of Theorem 1 for $G^\sharp Z(G)$ is Theorem 4.13.
- A technically complicated step is the construction of an idempotent $e^{\mathfrak{s}}_M \in \mathcal{H}(M)$ (in Lemmas 3.3 and 3.4), which is well-suited for restriction from M to $M^\sharp$. It relies on the conjugacy of the types $(K, w(\lambda)\otimes\gamma)$ with $(w,\gamma) \in \mathrm{Stab}(\mathfrak{s})$, studied in Proposition 3.1. With this idempotent we get a Morita equivalence (Proposition 3.9)

$$(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})} \sim_M (e^{\mathfrak{s}}_M \mathcal{H}(M) e^{\mathfrak{s}}_M)^{X_L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}. \tag{4}$$

  This allows us to perform many calculations entirely in M, which is easier than in G. We exhibit an idempotent $e^\sharp_{\lambda_G}$, larger than e_{λ_G}, which in Proposition 3.15 is used to improve (4) to an isomorphism

$$e^\sharp_{\lambda_G} (\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})} e^\sharp_{\lambda_G} \cong (e^{\mathfrak{s}}_M \mathcal{H}(M) e^{\mathfrak{s}}_M)^{X_L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}.$$

- To reveal the structure of $(e^{\mathfrak{s}}_M \mathcal{H}(M) e^{\mathfrak{s}}_M)^{X_L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}$ we first study (in subsection 4.1) the Hecke algebras associated to the types for $[L,\omega]_L$ and $[L,\omega]_M$ constructed in [22]. They are tensor products of affine Hecke algebras of type GL_e with a matrix algebra. Obviously this part relies very much on the work of Sécherre. These considerations culminate in Theorem 4.5, which describes the Hecke algebras associated to relevant larger idempotents, in similar terms. We make the action of $X^G(\mathfrak{s})$ on these algebras explicit in Lemmas 4.8 and 4.11.

- We would like to construct types for $G^\sharp Z(G)$ and for $G^\sharp$, whose associated Hecke algebras are as described in Theorems 4.13 and 1. In Theorems 3.16 and 3.17 we take a step towards this goal, by constructing idempotents $e^\sharp_{\lambda_{G^\sharp Z(G)}} \in \mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ and $e^\sharp_{\lambda_{G^\sharp}} \in \mathcal{H}(G^\sharp)^{\mathfrak{s}}$ which see the correct module categories and have the desired Hecke algebras. In fact these idempotents are just the restrictions of $e^\sharp_{\lambda_G} : G \to \mathbb{C}$ to $G^\sharp Z(G)$ and $G^\sharp$, respectively.

  However, we encounter serious obstructions to types in $G^\sharp$. The main problem is that sometimes types $(K, \lambda \otimes \gamma)$ for $[L, \omega]_M$ are conjugate in M but not in any compact subgroup of M, see Remark 4.7 and Examples 5.6–5.8.

Interestingly, some of the algebras that turn up do not look like affine Hecke algebras. In the literature there was hitherto (to the best of our knowledge) only one example of a Hecke algebra of a type which was not Morita equivalent to a crossed product of an affine Hecke algebra with a finite group, namely [14, §11.8]. But in several cases of Theorem 1 the part $\mathrm{End}_{\mathbb{C}}(V_\mu)$ plays an essential role, and it cannot be removed via some equivalence. Hence these algebras are further away from affine Hecke algebras than any previously known Hecke algebras related to types. See especially Example 5.5.

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1. Notations and conventions

We start with some generalities, to fix the notations. Good sources for the material in this section are [20, 7].

Let G be a connected reductive group over a local non-archimedean field. All our representations are tacitly assumed to be smooth and over the complex numbers. We write $\mathrm{Rep}(G)$ for the category of such G-representations and $\mathrm{Irr}(G)$ for the collection of isomorphism classes of irreducible representations therein.

Let P be a parabolic subgroup of G with Levi factor L. The "Weyl" group of L is $W(G, L) = N_G(L)/L$. It acts on equivalence classes of L-representations π by

$$(w \cdot \pi)(g) = \pi(\bar{w} g \bar{w}^{-1}),$$

where $\bar{w} \in N_G(L)$ is a chosen representative for $w \in W(G, L)$. We write

$$W_\pi = \{w \in W(G, L) \mid w \cdot \pi \cong \pi\}.$$

Let ω be an irreducible supercuspidal L-representation. The inertial equivalence class $\mathfrak{s} = [L, \omega]_G$ gives rise to a category of smooth G-representations $\mathrm{Rep}^{\mathfrak{s}}(G)$ and a subset $\mathrm{Irr}^{\mathfrak{s}}(G) \subset \mathrm{Irr}(G)$. Write $X_{\mathrm{nr}}(L)$ for the group of unramified characters $L \to \mathbb{C}^\times$. Then $\mathrm{Irr}^{\mathfrak{s}}(G)$ consists of all irreducible irreducible constituents of the parabolically induced representations $I_P^G(\omega \otimes \chi)$ with $\chi \in X_{\mathrm{nr}}(L)$. We note that I_P^G always means normalized, smooth parabolic induction from L via P to G.

The set $\mathrm{Irr}^{\mathfrak{s}_L}(L)$ with $\mathfrak{s}_L = [L, \omega]_L$ can be described explicitly, namely by

$$X_{\mathrm{nr}}(L, \omega) = \{\chi \in X_{\mathrm{nr}}(L) : \omega \otimes \chi \cong \omega\}, \tag{5}$$

$$\mathrm{Irr}^{\mathfrak{s}_L}(L) = \{\omega \otimes \chi : \chi \in X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L, \omega)\}. \tag{6}$$

Several objects are attached to the Bernstein component $\mathrm{Irr}^{\mathfrak{s}}(G)$ of $\mathrm{Irr}(G)$ [4]. Firstly, there is the torus

$$T_{\mathfrak{s}} := X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\omega),$$

which is homeomorphic to $\mathrm{Irr}^{\mathfrak{s}_L}(L)$. Secondly, we have the groups

$$\begin{aligned} N_G(\mathfrak{s}_L) =& \{g \in N_G(L) \mid g\cdot\omega \in \mathrm{Irr}^{\mathfrak{s}_L}(L)\} \\ =& \{g \in N_G(L) \mid g\cdot [L,\omega]_L = [L,\omega]_L\}, \\ W_{\mathfrak{s}} :=& \{w \in W(G,L) \mid w\cdot\omega \in \mathrm{Irr}^{\mathfrak{s}_L}(L)\} = N_G(\mathfrak{s}_L)/L. \end{aligned}$$

Of course $T_{\mathfrak{s}}$ and $W_{\mathfrak{s}}$ are only determined up to isomorphism by $\mathfrak{s}$, actually they depend on $\mathfrak{s}_L$. To cope with this, we tacitly assume that $\mathfrak{s}_L$ is known when talking about $\mathfrak{s}$.

The choice of $\omega \in \mathrm{Irr}^{\mathfrak{s}_L}(L)$ fixes a bijection $T_{\mathfrak{s}} \to \mathrm{Irr}^{\mathfrak{s}_L}(L)$, and via this bijection the action of $W_{\mathfrak{s}}$ on $\mathrm{Irr}^{\mathfrak{s}_L}(L)$ is transferred to $T_{\mathfrak{s}}$. The finite group $W_{\mathfrak{s}}$ can be thought of as the "Weyl group" of $\mathfrak{s}$, although in general it is not generated by reflections.

Let $C_c^\infty(G)$ be the vector space of compactly supported locally constant functions $G \to \mathbb{C}$. The choice of a Haar measure on G determines a convolution product $*$ on $C_c^\infty(G)$. The algebra $(C_c^\infty(G), *)$ is known as the Hecke algebra $\mathcal{H}(G)$. There is an equivalence between $\mathrm{Rep}(G)$ and the category $\mathrm{Mod}(\mathcal{H}(G))$ of $\mathcal{H}(G)$-modules V such that $\mathcal{H}(G)\cdot V = V$. We denote the collection of inertial equivalence classes for G by $\mathfrak{B}(G)$. The Bernstein decomposition

$$\mathrm{Rep}(G) = \prod\nolimits_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Rep}^{\mathfrak{s}}(G)$$

induces a factorization in two-sided ideals

$$\mathcal{H}(G) = \prod\nolimits_{\mathfrak{s}\in\mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}.$$

Let K be a compact open subgroup of K and let (λ, V_λ) be an irreducible K-representation. Let $e_\lambda \in \mathcal{H}(K)$ be the associated central idempotent and write

$$\mathrm{Rep}^\lambda(G) = \{(\pi, V) \in \mathrm{Rep}(G) \mid \mathcal{H}(G)e_\lambda \cdot V = V\}.$$

Clearly $e_\lambda\mathcal{H}(G)e_\lambda$ is a subalgebra of $\mathcal{H}(G)$, and $V \mapsto e_\lambda\cdot V$ defines a functor from $\mathrm{Rep}(G)$ to $\mathrm{Mod}(e_\lambda\mathcal{H}(G)e_\lambda)$. By [7, Proposition 3.3] this functor restricts to an equivalence of categories $\mathrm{Rep}^\lambda(G) \to \mathrm{Mod}(e_\lambda\mathcal{H}(G)e_\lambda)$ if and only if $\mathrm{Rep}^\lambda(G)$ is closed under taking G-subquotients. Moreover, in that case there are finitely many inertial equivalence classes $\mathfrak{s}_1, \ldots \mathfrak{s}_\kappa$ such that

$$\mathrm{Rep}^\lambda(G) = \mathrm{Rep}^{\mathfrak{s}_1}(G) \times \cdots \times \mathrm{Rep}^{\mathfrak{s}_\kappa}(G).$$

One calls (K, λ) a type for $\{\mathfrak{s}_1, \ldots, \mathfrak{s}_\kappa\}$, or an $\mathfrak{s}_1$-type if $\kappa = 1$.

To a type (K, λ) one associates the algebra

$$\begin{aligned} \mathcal{H}(G,\lambda) := \{f : G \to \mathrm{End}_{\mathbb{C}}(V_\lambda^\vee) \mid\ & \mathrm{supp}(f) \text{ is compact}, \\ & f(k_1 g k_2) = \lambda^\vee(k_1) f(g) \lambda^\vee(k_2)\ \forall g \in G, k_1, k_2 \in K\}. \end{aligned}$$

Here $(\lambda^\vee, V_\lambda^\vee)$ is the contragredient of (λ, V_λ) and the product is convolution of functions. By [7, (2.12)] there is a canonical isomorphism

$$e_\lambda\mathcal{H}(G)e_\lambda \cong \mathcal{H}(G,\lambda) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(V_\lambda). \tag{7}$$

From now on we discuss things that are specific for $G = \mathrm{GL}_m(D)$, where D is a central simple F-algebra. We write $\dim_F(D) = d^2$. Every Levi subgroup L of G

is isomorphic to $\prod_j \mathrm{GL}_{\tilde m_j}(D)$ for some $\tilde m_j \in \mathbb{N}$ with $\sum_j \tilde m_j = m$. Hence every irreducible L-representation ω can be written as $\otimes_j \tilde\omega_j$ with $\tilde\omega_j \in \mathrm{Irr}(\mathrm{GL}_{\tilde m_j}(D))$. Then ω is supercuspidal if and only if every $\tilde\omega_j$ is so. As above, we assume that this is the case. Replacing (L,ω) by an inertially equivalent pair allows us to make the following simplifying assumptions:

Conditions 1.1.

- if $\tilde m_i = \tilde m_j$ and $[\mathrm{GL}_{\tilde m_j}(D), \tilde\omega_i]_{\mathrm{GL}_{\tilde m_j}(D)} = [\mathrm{GL}_{\tilde m_j}(D), \tilde\omega_j]_{\mathrm{GL}_{\tilde m_j}(D)}$, then $\tilde\omega_i = \tilde\omega_j$;
- $\omega = \prod_i \omega_i^{\otimes e_i}$, such that ω_i and ω_j are not inertially equivalent if $i \neq j$;
- $L = \prod_i L_i^{e_i} = \prod_i \mathrm{GL}_{m_i}(D)^{e_i}$, embedded diagonally in $\mathrm{GL}_m(D)$ such that factors L_i with the same (m_i, e_i) are in subsequent positions;
- as representatives for the elements of $W(G,L)$ we take permutation matrices;
- P is the parabolic subgroup of G generated by L and the upper triangular matrices;
- if $m_i = m_j, e_i = e_j$ and ω_i is isomorphic to $\omega_j \otimes \gamma$ for some character γ of $\mathrm{GL}_{m_i}(D)$, then $\omega_i = \omega_j \otimes \gamma\chi$ for some $\chi \in X_{\mathrm{nr}}(\mathrm{GL}_{m_i}(D))$.

We remark that these conditions are natural generalizations of [14, §1.2] to our setting. Most of the time we will not need the conditions for stating the results, but they are useful in many proofs. Under Conditions 1.1 we define

$$M = \prod_i M_i = \prod_i Z_G\big(\prod_{j\neq i} L_j^{e_j}\big) = \prod_i \mathrm{GL}_{m_i e_i}(D), \tag{8}$$

a Levi subgroup of G containing L. For $\mathfrak{s} = [L,\omega]_G$ we have

$$W_{\mathfrak{s}} = W(M,L) = N_M(L)/L = \prod\nolimits_i N_{M_i}(L_i^{e_i})/L_i^{e_i} \cong \prod\nolimits_i S_{e_i}, \tag{9}$$

a direct product of symmetric groups. Writing $\mathfrak{s}_i = [L_i,\omega_i]_{L_i}$, the torus associated to $\mathfrak{s}$ becomes

$$T_{\mathfrak{s}} = \prod\nolimits_i (T_{\mathfrak{s}_i})^{e_i}, \tag{10}$$

$$T_{\mathfrak{s}_i} = X_{\mathrm{nr}}(L_i)/X_{\mathrm{nr}}(L_i,\omega_i). \tag{11}$$

By our choice of representatives for $W(G,L)$, $\omega_i^{\otimes e_i}$ is stable under $N_{M_i}(L_i^{e_i})/L_i^{e_i} \cong S_{e_i}$. The action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$ is just permuting coordinates in the standard way and

$$W_{\mathfrak{s}} = W_\omega. \tag{12}$$

2. Restricting representations

2.1. Restriction to the derived group.

We will study the restriction of representations of $G = \mathrm{GL}_m(D)$ to its derived group $G^\sharp = \mathrm{GL}_m(D)_{\mathrm{der}}$. For subgroups $H \subset G$ we will write

$$H^\sharp = H \cap G^\sharp.$$

Recall that the reduced norm map Nrd: $M_m(D) \to F$ induces a group isomorphism

$$\mathrm{Nrd} : G/G^\sharp \to F^\times.$$

We start with some important relations between representations of G and $G^\sharp$, which were proven both by Tadić and by Bushnell–Kutzko.

Proposition 2.1. (*a*) *Every irreducible representation of $G^\sharp$ appears in an irreducible representation of G.*

(*b*) *For $\pi, \pi' \in \mathrm{Irr}(G)$ the following are equivalent:*
  (i) $\mathrm{Res}^G_{G^\sharp}(\pi)$ *and* $\mathrm{Res}^G_{G^\sharp}(\pi')$ *have a common irreducible subquotient;*
  (ii) $\mathrm{Res}^G_{G^\sharp}(\pi) \cong \mathrm{Res}^G_{G^\sharp}(\pi')$;
  (iii) *there is a* $\gamma \in \mathrm{Irr}(G/G^\sharp)$ *such that* $\pi' \cong \pi \otimes \gamma$.

(*c*) *The restriction of $(\pi, V) \in \mathrm{Irr}(G)$ to $G^\sharp$ is a finite direct sum of irreducible $G^\sharp$-representations, each one appearing with the same multiplicity.*

(*d*) *Let (π', V') be an irreducible $G^\sharp$-subrepresentation of (π, V). Then the stabilizer in G of V' is a open, normal, finite index subgroup $H_\pi \subset G$ which contains $G^\sharp$ and the centre of G.*

Proof. All these results can be found in [28, §2], where they are in fact shown for any reductive group over a local non-archimedean field. For $G = \mathrm{GL}_n(F)$, these statements were proven in [5, Propositions 1.7 and 1.17] and [6, Proposition 1.5]. The proofs in [5, 6] also apply to $G = \mathrm{GL}_m(D)$. □

Let $\pi \in \mathrm{Irr}(G)$. By Proposition 2.1.d

$$\mathrm{End}_{G^\sharp}(V) = \mathrm{End}_{H_\pi}(V), \tag{13}$$

which allows us to use [16, Chapter 2] and [12, Section 2] (which is needed for [16]). We put

$$X^G(\pi) := \{\gamma \in \mathrm{Irr}(G/G^\sharp) \mid \pi \otimes \gamma \cong \pi\}.$$

As worked out in [16, Chapter 2], this group governs the reducibility of $\mathrm{Res}^G_{G^\sharp}(\pi)$. (We will use this definition of $X^G(\pi)$ more generally if $\pi \in \mathrm{Rep}(G)$ admits a central character.) By (13) every element of $X^G(\pi)$ is trivial on H_π, so $X^G(\pi)$ is finite. Via the local Langlands correspondence for G, the group $X^G(\pi)$ corresponds to the geometric R-group of the L-packet for $G^\sharp$ obtained from $\mathrm{Res}^G_{G^\sharp}(\pi)$, see [3, §3]. We note that

$$X^G(\pi) \cap X_{\mathrm{nr}}(G) = X_{\mathrm{nr}}(G, \pi).$$

For every $\gamma \in X^G(\pi)$ there exists a nonzero intertwining operator

$$I(\gamma, \pi) \in \mathrm{Hom}_G(\pi \otimes \gamma, \pi) = \mathrm{Hom}_G(\pi, \pi \otimes \gamma^{-1}), \tag{14}$$

which is unique up to a scalar. As $G^\sharp \subset \ker(\gamma)$, $I(\gamma, \pi)$ can also be considered as an element of $\mathrm{End}_{G^\sharp}(\pi)$. As such, these operators determine a 2-cocycle κ_π by

$$I(\gamma, \pi) \circ I(\gamma', \pi) = \kappa_\pi(\gamma, \gamma') I(\gamma\gamma', \pi). \tag{15}$$

By [16, Lemma 2.4] they span the $G^\sharp$-intertwining algebra of π:

$$\mathrm{End}_{G^\sharp}(\mathrm{Res}^G_{G^\sharp}\pi) \cong \mathbb{C}[X^G(\pi), \kappa_\pi], \tag{16}$$

where the right hand side denotes the twisted group algebra of $X^G(\pi)$. By [16, Corollary 2.10]

$$\mathrm{Res}^G_{G^\sharp}\pi \cong \bigoplus_{\rho \in \mathrm{Irr}(\mathbb{C}[X^G(\pi), \kappa_\pi])} \mathrm{Hom}_{\mathbb{C}[X^G(\pi), \kappa_\pi]}(\rho, \pi) \otimes \rho \tag{17}$$

as representations of $G^\sharp \times X^G(\pi)$.

Let P be a parabolic subgroup of G with Levi factor L. The inclusion $L \to G$ induces isomorphisms

$$L/L^\sharp \to G/G^\sharp \cong F^\times. \tag{18}$$

Let $\omega \in \mathrm{Irr}(L)$ be supercuspidal and unitary. Using (18) we can identify the $G^\sharp$-representations

$$\mathrm{Res}^G_{G^\sharp}(I_P^G(\omega)) \quad \text{and} \quad I_{P^\sharp}^{G^\sharp}(\mathrm{Res}^L_{L^\sharp}(\omega)),$$

which yields an inclusion $X^L(\omega) \to X^G(I_P^G(\omega))$. Every intertwining operator $I(\gamma,\omega)$ for $\gamma \in X^L(\omega)$ induces an intertwining operator

$$I(\gamma, I_P^G(\omega)) := I_P^G(I(\gamma,\omega)) \in \mathrm{Hom}_G(\gamma \otimes I_P^G(\omega), I_P^G(\omega)), \tag{19}$$

for γ as an element of $X^G(I_P^G(\omega))$. We warn that, even though $X^L(\omega)$ is finite abelian and ω is supercuspidal, it is still possible that the 2-cocycle κ_ω is nontrivial and that

$$\mathrm{End}_{L^\sharp}(\mathrm{Res}^L_{L^\sharp}(\omega)) \cong \mathbb{C}[X^L(\omega), \kappa_\omega]$$

is noncommutative, see [10, Example 6.3.3].

We introduce the groups

$$W_\omega^\sharp = \{w \in W(G,L) \mid \exists \gamma \in \mathrm{Irr}(L/L^\sharp) \mid w \cdot (\gamma \otimes \omega) \cong \omega\}, \tag{20}$$

$$\mathrm{Stab}(\omega) = \{(w,\gamma) \in W(G,L) \times \mathrm{Irr}(L/L^\sharp) \mid w \cdot (\gamma \otimes \omega) \cong \omega\}. \tag{21}$$

Notice that the actions of $W(G,L)$ and $\mathrm{Irr}(L/L^\sharp)$ on $\mathrm{Irr}(L)$ commute because every element of $\mathrm{Irr}(L/L^\sharp)$ extends to a character of G which is trivial on the derived subgroup of G. Clearly $W_\omega \times X^L(\omega)$ is a normal subgroup of $\mathrm{Stab}(\omega)$ and there is a short exact sequence

$$1 \to X^L(\omega) \to \mathrm{Stab}(\omega) \to W_\omega^\sharp \to 1. \tag{22}$$

By [10, Proposition 6.2.2] the projection of $\mathrm{Stab}(\omega)$ on the second coordinate gives rise to a short exact sequence

$$1 \to W_\omega \to \mathrm{Stab}(\omega) \to X^G(I_P^G(\omega)) \to 1 \tag{23}$$

and the group

$$\mathfrak{R}_\omega^\sharp := \mathrm{Stab}(\omega)/(W_\omega \times X^L(\omega)) \cong X^G(I_P^G(\omega))/X^L(\omega) \cong W_\omega^\sharp/W_\omega \tag{24}$$

is naturally isomorphic to the "dual R-group" of the L-packet for $G^\sharp$ obtained from $\mathrm{Res}^G_{G^\sharp}(I_P^G(\omega))$. We remark that, with the method of Lemma 2.3.c, it is also possible to realize $\mathfrak{R}_\omega^\sharp$ as a subgroup of $\mathrm{Stab}(\omega)$.

When ω is unitary, the G-representation $I_P^G(\omega)$ is unitary, and hence completely reducible as $G^\sharp$-representation. In this case (17) shows that the intertwining operators associated to elements of $\mathrm{Stab}(\omega)$ span $\mathrm{End}_{G^\sharp}(I_P^G(\omega))$. By [2, Theorem 1.6.a] that holds more generally for $I_P^G(\omega \otimes \chi)$ when χ is in Langlands position with respect to P.

The group $\mathrm{Stab}(\omega)$ also acts on the set of irreducible $L^\sharp$-representations appearing in $\mathrm{Res}^L_{L^\sharp}(\omega)$. For an irreducible subrepresentation $\sigma^\sharp$ of $\mathrm{Res}^L_{L^\sharp}(\omega)$ [10, Proposition 6.2.3] says that

$$W_\omega \subset W_{\sigma^\sharp} \subset W_\omega^\sharp \tag{25}$$

and that the analytic R-group of $I_{P^\sharp}^{G^\sharp}(\sigma^\sharp)$ is

$$\mathfrak{R}_{\sigma^\sharp} := W_{\sigma^\sharp}/W_\omega,$$

the stabilizer of $\sigma^\sharp$ in $\mathfrak{R}_\omega^\sharp$. It is possible that $W_{\sigma^\sharp} \neq W_\omega^\sharp$ and $\mathfrak{R}_{\sigma^\sharp} \neq \mathfrak{R}_\omega^\sharp$, see [10, Example 6.3.4].

In view of [2, Section 1] the above results remain valid if $\omega \in \mathrm{Irr}(L)$ is assumed to be supercuspidal but not necessarily unitary. Just one modification is required: if $I_P^G(\omega)$ is reducible, one should consider the L-packet for $G^\sharp$ obtained from the (unique) Langlands constituent of $I_P^G(\omega)$.

2.2. **Restriction of Bernstein components.**

Next we study the restriction of an entire Bernstein component $\mathrm{Irr}^{\mathfrak s}(G)$ to $G^\sharp$. Let $\mathrm{Irr}^{\mathfrak s}(G^\sharp)$ be the set of irreducible $G^\sharp$-representations that are subquotients of $\mathrm{Res}^G_{G^\sharp}(\pi)$ for some $\pi \in \mathrm{Irr}^{\mathfrak s}(G)$.

Lemma 2.2. $\mathrm{Irr}^{\mathfrak s}(G^\sharp)$ *is a union of finitely many Bernstein components for* $G^\sharp$*.*

Proof. Consider any $\pi^\sharp \in \mathrm{Irr}^{\mathfrak s}(G^\sharp)$. It is a subquotient of

$$\mathrm{Res}^G_{G^\sharp}(I_P^G(\omega \otimes \chi_1)) = I_{P^\sharp}^{G^\sharp}(\mathrm{Res}^L_{L^\sharp}(\omega \otimes \chi_1))$$

for some $\chi_1 \in X_{\mathrm{nr}}(L)$. Choose an irreducible summand σ_1 of the supercuspidal $L^\sharp$-representation $\mathrm{Res}^L_{L^\sharp}(\omega \otimes \chi_1)$, such that $\pi^\sharp$ is a subquotient of $I_{P^\sharp}^{G^\sharp}(\sigma_1)$. Then $\pi^\sharp$ lies in the Bernstein component $\mathrm{Irr}^{[L^\sharp,\sigma_1]_{G^\sharp}}(G^\sharp)$. Any unramified character of $\chi_2^\sharp$ of $L^\sharp$ lifts to an unramified character of L, say χ_2. Now

$$I_{P^\sharp}^{G^\sharp}(\mathrm{Res}^L_{L^\sharp}(\sigma_1) \otimes \chi_2^\sharp) \subset I_{P^\sharp}^{G^\sharp}(\mathrm{Res}^L_{L^\sharp}(\omega \otimes \chi_1\chi_2)) = \mathrm{Res}^G_{G^\sharp}(I_P^G(\omega \otimes \chi_1\chi_2)),$$

which shows that all irreducible subquotients of $I_{P^\sharp}^{G^\sharp}(\mathrm{Res}^L_{L^\sharp}(\sigma_1) \otimes \chi_2^\sharp)$ belong to $\mathrm{Irr}^{\mathfrak s}(G^\sharp)$. It follows that $\mathrm{Irr}^{[L^\sharp,\sigma_1]_{G^\sharp}}(G^\sharp) \subset \mathrm{Irr}^{\mathfrak s}(G^\sharp)$.

The above also shows that any inertial equivalence class $\mathfrak t^\sharp$ with $\mathrm{Irr}^{\mathfrak t^\sharp}(G^\sharp) \subset \mathrm{Irr}^{\mathfrak s}(G^\sharp)$ must be of the form

$$\mathfrak t^\sharp = [L^\sharp, \sigma_2]_{G^\sharp} \tag{26}$$

for some irreducible constituent σ_2 of $\mathrm{Res}^L_{L^\sharp}(\omega \otimes \chi_2)$. So up to an unramified twist σ_2 is an irreducible constituent of $\mathrm{Res}^L_{L^\sharp}(\omega)$. Now Proposition 2.1.c shows that there are only finitely many possibilities for $\mathfrak t^\sharp$. $\square$

Motivated by this lemma, we write $\mathfrak t^\sharp \prec \mathfrak s$ if $\mathrm{Irr}^{\mathfrak t^\sharp}(G^\sharp) \subset \mathrm{Irr}^{\mathfrak s}(G^\sharp)$. In other words,

$$\mathrm{Irr}^{\mathfrak s}(G^\sharp) = \bigcup\nolimits_{\mathfrak t^\sharp \prec \mathfrak s} \mathrm{Irr}^{\mathfrak t^\sharp}(G^\sharp). \tag{27}$$

The last part of the proof of Lemma 2.2 shows that every $\mathfrak t^\sharp \prec \mathfrak s$ is of the form $[L^\sharp, \sigma^\sharp]_{G^\sharp}$ for some irreducible constituent $\sigma^\sharp$ of $\mathrm{Res}^L_{L^\sharp}(\omega)$. Recall from (17) that constituents $\sigma^\sharp$ as above are parametrized by irreducible representations of the twisted group algebra $\mathbb C[X^L(\omega), \kappa_\omega]$. However, non-isomorphic $\sigma^\sharp$ may give rise to the same inertial equivalence class $\mathfrak t^\sharp$ for $G^\sharp$. It is quite difficult to determine the Bernstein tori $T_{\mathfrak t^\sharp}$ precisely.

The finite group associated by Bernstein to $\mathfrak t^\sharp = [L^\sharp, \sigma^\sharp]_{G^\sharp}$ is its stabilizer in $W(G,L) = W(G^\sharp, L^\sharp)$:

$$W_{\mathfrak t^\sharp} = \{ w \in W(G,L) \mid w \cdot \sigma^\sharp \in [L^\sharp, \sigma^\sharp]_{L^\sharp} \}.$$

As the different $\sigma^\sharp$ are M-conjugate, they all produce the same group $W_{\mathfrak t^\sharp}$. So it depends only on $[M,\sigma]_M$. It is quite possible that $W_{\mathfrak t^\sharp}$ is strictly larger than $W_{\mathfrak s}$, we already saw this in Example 5.1. First steps to study such cases were sketched (for $\mathrm{SL}_n(F)$) in [6, §9].

For every $w \in W_{\mathfrak{t}^\sharp}$, Proposition 2.1.b (for $L^\sharp$) guarantees the existence of a $\gamma \in \mathrm{Irr}(L/L^\sharp)$ such that $w(\omega) \otimes \gamma \in [L,\omega]_L$. Up to an unramified character γ has to be trivial on $Z(G)$, so we may take $\gamma \in \mathrm{Irr}(L/L^\sharp Z(G))$. Hence $W_{\mathfrak{t}^\sharp}$ is contained in the group

$$W_{\mathfrak{s}}^\sharp := \{w \in W(G,L) \mid \exists \gamma \in \mathrm{Irr}(L/L^\sharp Z(G)) \text{ such that } w(\gamma \otimes \omega) \in [L,\omega]_L\}$$

By (25) the subgroup W_ω fixes every $[L^\sharp,\sigma^\sharp]_{L^\sharp}$, so $W_\omega \subset W_{\mathfrak{t}^\sharp}$ and

$$\mathrm{Stab}_{W_{\mathfrak{s}}^\sharp / W_\omega}([L^\sharp,\sigma^\sharp]_{L^\sharp}) = W_{\mathfrak{t}^\sharp}/W_\omega \supset W_{\sigma^\sharp}/W_\omega. \tag{28}$$

Lemma 2.3. *(a) $W_{\mathfrak{s}}$ is a normal subgroup of $W_{\mathfrak{s}}^\sharp$ which fixes all the $[L^\sharp,\sigma^\sharp]_{L^\sharp}$ with $[L^\sharp,\sigma^\sharp]_{G^\sharp} \prec \mathfrak{s}$.*

(b) $W_{\mathfrak{t}^\sharp}$ is the normal subgroup of $W_{\mathfrak{s}}^\sharp$ consisting of all elements that fix every $[L^\sharp,\sigma^\sharp]_{L^\sharp}$ with $[L^\sharp,\sigma^\sharp]_{G^\sharp} \prec \mathfrak{s}$. In particular it contains $W_{\mathfrak{s}}$.

(c) There exist subgroups $\mathfrak{R}_{\mathfrak{s}}^\sharp \subset W_{\mathfrak{s}}^\sharp$, $\mathfrak{R}_{\mathfrak{t}^\sharp} \subset W_{\mathfrak{t}^\sharp}$ and $\mathfrak{R}_\omega^\sharp \subset W_\omega^\sharp$ such that $W_{\mathfrak{s}}^\sharp = W_{\mathfrak{s}} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp$, $W_{\mathfrak{t}^\sharp} = W_{\mathfrak{s}} \rtimes \mathfrak{R}_{\mathfrak{t}^\sharp}$ and $W_\omega^\sharp = W_\omega \rtimes \mathfrak{R}_\omega^\sharp$.

Proof. (a) We may use the conditions 1.1. Then $W_\omega = W_{\mathfrak{s}}$. For any $\sigma^\sharp$ as above the root systems $R_{\sigma^\sharp}$ and R_ω are equal by [10, Lemma 3.4.1]. With (12) we get

$$W_{\mathfrak{s}} = W_\omega = W(R_\omega) = W(R_{\sigma^\sharp}) \subset W_{\sigma^\sharp},$$

showing that $W_{\mathfrak{s}}$ fixes $\sigma^\sharp$ and $[L^\sharp,\sigma^\sharp]_{L^\sharp}$.

(b) As we observed above, we can arrange that every $\sigma^\sharp$ is a subquotient of $\mathrm{Res}^L_{L^\sharp}(\omega)$. Since all constituents of this representation are associate under L and $W(G,L)$ acts trivially on $L/L^\sharp$, $W_{\mathfrak{t}^\sharp}$ fixes every possible $[L^\sharp,\sigma^\sharp]_{L^\sharp}$. This gives the description of $W_{\mathfrak{t}^\sharp}$. Because all the $[L^\sharp,\sigma^\sharp]_{L^\sharp}$ together form a $W_{\mathfrak{s}}^\sharp$-space, $W_{\mathfrak{t}^\sharp}$ is normal in that group.

(c) In the special case $G^\sharp = \mathrm{SL}_n(F)$, this was proven for $W_{\mathfrak{t}^\sharp}$ in [14, Proposition 2.3]. Our proof is a generalization of that in [14].

Recall the description of $M = \prod_i M_i$ and $W_{\mathfrak{s}}$ from equations (8) and (9). We note that $P \cap M$ is a parabolic subgroup of M containing L, and that the group $W(M,L) = W_{\mathfrak{s}}$ acts simply transitively on the collection of such parabolic subgroups. This implies that

$$\mathfrak{R}_{\mathfrak{s}}^\sharp := W_{\mathfrak{s}}^\sharp \cap N_G(P \cap M)/L \tag{29}$$

is a complement to $W_{\mathfrak{s}}$ in $W_{\mathfrak{s}}^\sharp$. For $W_{\mathfrak{t}^\sharp}$ (28) shows that we may take

$$\mathfrak{R}_{\mathfrak{t}^\sharp} := W_{\mathfrak{t}^\sharp} \cap N_G(P \cap M)/L. \tag{30}$$

Similarly, for $W_\omega^\sharp$ (24) leads us to

$$\mathfrak{R}_\omega^\sharp := W_\omega^\sharp \cap N_G(P \cap M)/L. \tag{31}$$

□

As analogues of $X^L(\omega), X^G(I_P^G(\omega))$ and $\mathrm{Stab}(\omega)$ for $\mathfrak{s} = [L,\omega]_G$ we introduce

$$X^L(\mathfrak{s}) = \{\gamma \in \mathrm{Irr}(L/L^\sharp Z(G)) \mid \gamma \otimes \omega \in [L,\omega]_L\}, \tag{32}$$

$$X^G(\mathfrak{s}) = \{\gamma \in \mathrm{Irr}(G/G^\sharp Z(G)) \mid \gamma \otimes I_P^G(\omega) \in \mathfrak{s}\}, \tag{33}$$

$$\mathrm{Stab}(\mathfrak{s}) = \{(w,\gamma) \in W(G,L) \times \mathrm{Irr}(L/L^\sharp Z(G)) \mid w(\gamma \otimes \omega) \in [L,\omega]_L\}. \tag{34}$$

Notice that $\mathrm{Stab}(\mathfrak{s})$ contains $\mathrm{Stab}(\omega \otimes \chi)$ for every $\chi \in X_{\mathrm{nr}}(L/L^\sharp)$. It is easy to see that $W_{\mathfrak{s}} \times X^L(\mathfrak{s})$ is a normal subgroup of $\mathrm{Stab}(\mathfrak{s})$ and that there are short exact sequences

$$1 \to X^L(\mathfrak{s}) \to \mathrm{Stab}(\mathfrak{s}) \to W_{\mathfrak{s}}^\sharp \to 1, \tag{35}$$

$$1 \to X^L(\mathfrak{s}) \times W_{\mathfrak{s}} \to \mathrm{Stab}(\mathfrak{s}) \to W_{\mathfrak{s}}^\sharp / W_{\mathfrak{s}} \cong \mathfrak{R}_{\mathfrak{s}}^\sharp \to 1. \tag{36}$$

Furthermore we define

$$\mathrm{Stab}(\mathfrak{s}, P \cap M) = \{(w,\gamma) \in \mathrm{Stab}(\mathfrak{s}) \mid w \in N_G(P \cap M)/L\}.$$

The reduced norm map induces isomorphisms

$$L/L^\sharp Z(G) \to G/G^\sharp Z(G) \to F^\times / \mathrm{Nrd}(Z(G)). \tag{37}$$

The right hand side is an abelian group of exponent md, but it is not necessarily finite, see (39).

Lemma 2.4. *(a)* $\mathrm{Stab}(\mathfrak{s}) = \mathrm{Stab}(\mathfrak{s}, P \cap M) \ltimes W_{\mathfrak{s}}$.
(b) The projection of $\mathrm{Stab}(\mathfrak{s})$ *on the second coordinate gives a group isomorphism*

$$\mathrm{Stab}(\mathfrak{s}, P \cap M) \cong \mathrm{Stab}(\mathfrak{s})/W_{\mathfrak{s}} \to X^G(\mathfrak{s}).$$

(c) The groups $X^L(\mathfrak{s}), X^G(\mathfrak{s})$ *and* $\mathrm{Stab}(\mathfrak{s})$ *are finite.*
(d) There are natural isomorphisms

$$X^G(\mathfrak{s})/X^L(\mathfrak{s}) \cong \mathrm{Stab}(\mathfrak{s}, P \cap M)/X^L(\mathfrak{s}) \cong \mathfrak{R}_{\mathfrak{s}}^\sharp.$$

Proof. (a) This can be shown in the same way as Lemma 2.3.c.
(b) If $(w,\gamma) \in \mathrm{Stab}(\mathfrak{s})$, then

$$\gamma \otimes I_P^G(\omega) \cong \gamma \otimes I_P^G(w \cdot \omega) \cong I_P^G(\gamma \otimes w \cdot \omega) \cong I_P^G(w(\gamma \otimes \omega)) \in \mathfrak{s},$$

so $\gamma \in X^G(\mathfrak{s})$. Conversely, if $\gamma \in X^G(\mathfrak{s})$, then $I_P^G(\omega \otimes \gamma) \in \mathfrak{s}$. Hence $\omega \otimes \gamma \in w^{-1} \cdot [L,\omega]_L = [L, w^{-1} \cdot \omega]_L$ for some $w \in W(G,L)$, and $(w,\gamma) \in \mathrm{Stab}(\mathfrak{s})$.

As $W(G,L)$ and $\mathrm{Irr}(L/L^\sharp)$ commute, the projection map $\mathrm{Stab}(\mathfrak{s}) \to X^G(\mathfrak{s})$ is a group homomorphism. In view of (12), we may assume that ω is such that $W_{\mathfrak{s}} = W_\omega$. Then the kernel of this group homomorphism is $W_\omega = W_{\mathfrak{s}}$.
(c) Suppose that $\omega \otimes \gamma \cong \omega \otimes \chi$ for some $\chi \in X_{\mathrm{nr}}(L)$. Then χ is trivial on $Z(G)$ and $\gamma^{-1}\chi \in X^L(\omega)$. We already know from Proposition 2.1 and (13) that $X^L(\omega)$ finite. Hence $(\gamma^{-1}\chi)^{|X^L(\omega)|} = 1$ and

$$\chi^{|X^L(\omega)} = \gamma^{-|X^L(\omega)} \in \mathrm{Irr}(L/L^\sharp Z(G)).$$

By (37) $L/L^\sharp Z(G)$ is a group of exponent md, so $\chi^{md|X^L(\omega)|} = \gamma^{-md|X^L(\omega)} = 1$. Thus there are only finitely many possibilities for χ, and we can conclude that $X^L(\mathfrak{s})$ is finite.

If $(w,\gamma),(w,\gamma') \in \mathrm{Stab}(\mathfrak{s})$, then $(w,\gamma)^{-1}(w,\gamma') = \gamma^{-1}\gamma' \in X^L(\mathfrak{s})$. As $W(G,L)$ and $X^L(\mathfrak{s})$ are finite, this shows that $\mathrm{Stab}(\mathfrak{s})$ is also finite. Now $X^G(\mathfrak{s})$ is finite by part (b).
(d) This follows from (36) and part (b). $\square$

2.3. **The intermediate group.**

For some calculations it is beneficial to do the restriction of representations from G to $G^\sharp$ in two steps, via the intermediate group $G^\sharp Z(G)$. This is a central extension of $G^\sharp$, so

$$\mathrm{End}_{G^\sharp}(\pi) = \mathrm{End}_{G^\sharp Z(G)}(\pi) \tag{38}$$

for all representations π of G or $G^\sharp Z(G)$ that admit a central character. In particular $\mathrm{Res}^{G^\sharp Z(G)}_{G^\sharp}$ preserves irreducibility of representations. The centre of G is

$$Z(G) = G \cap Z(M_m(D)) = G \cap F \cdot I_m = F^\times I_m. \tag{39}$$

Recall that $\dim_F(D) = d^2$. Since $\mathrm{Nrd}(zI_m) = z^{md}$ for $z \in F^\times$,

$$\mathrm{Nrd}(Z(G)) = F^{\times md},$$

the group of md-th powers in $F^\times$. Hence $G/G^\sharp Z(G)$ is an abelian group and all its elements have order dividing md. In case $\mathrm{char}(F)$ is positive and divides md, $G^\sharp Z(G)$ is closed but not open in G. Otherwise it is closed, open and of finite index in G. However, $G^\sharp Z(G)$ is never Zariski-closed in G.

The intersection of $G^\sharp$ and $Z(G)$ is the finite group $\{zI_m \mid z \in F^\times, z^{md} = 1\}$, so

$$G^\sharp Z(G) \cong (G^\sharp \times Z(G)) / \{(zI_m, z^{-1}) \mid z \in F^\times, z^{md} = 1\}. \tag{40}$$

As $G^\sharp \times Z(G)$ is a connected reductive algebraic group over F, this shows that $G^\sharp Z(G)$ is one as well. But this algebraic structure is not induced from the enveloping group G. The inflation functor $\mathrm{Rep}(G^\sharp Z(G)) \to \mathrm{Rep}(G^\sharp \times Z(G))$ identifies $\mathrm{Rep}(G^\sharp Z(G))$ with

$$\{\pi \in \mathrm{Rep}(G^\sharp \times Z(G)) \mid \pi(zI_m, z^{-1}) = 1 \, \forall z \in F^\times \text{ with } z^{md} = 1\}.$$

Lemma 2.5. (*a*) *Every irreducible $G^\sharp$-representation can be lifted to an irreducible representation of $G^\sharp Z(G)$.*

(*b*) *All fibers of*

$$\mathrm{Res}^{G^\sharp Z(G)}_{G^\sharp} : \mathrm{Irr}(G^\sharp Z(G)) \to \mathrm{Irr}(G^\sharp)$$

are homeomorphic to $\mathrm{Irr}(F^{\times md})$.

Proof. (a) Any $\pi^\sharp \in \mathrm{Irr}(G^\sharp)$ determines a character χ_{md} of the central subgroup $\{z \in F^\times \mid z^{md} = 1\}$. Since there are only finitely many md-th roots of unity in F, χ_{md} can be lifted to a character χ of $F^\times$. Then $\pi^\sharp \otimes \chi$ is a representation of $G^\sharp \times Z(G)$ that descends to $G^\sharp Z(G)$.
(b) This follows from the proof of part (a) and the short exact sequence

$$1 \to G^\sharp \to G^\sharp Z(G) \xrightarrow{\mathrm{Nrd}} F^{\times md} \to 1. \tag{41}$$

$\square$

More explicitly, $\chi \in \mathrm{Irr}(F^{\times md})$ acts freely on $\mathrm{Irr}(G^\sharp Z(G))$ by retraction to $\bar{\chi} \in \mathrm{Irr}(G^\sharp Z(G))$ and tensoring representations of $G^\sharp Z(G)$ with $\bar{\chi}$.

For any totally disconnected group H we define $X_{\mathrm{nr}}(H)$ as the collection of smooth characters which are trivial on every compact subgroup of H. Then

$$X_{\mathrm{nr}}(G^\sharp Z(G)) \cong X_{\mathrm{nr}}(F^{\times md}) \cong X_{\mathrm{nr}}(F^{\times md} / \mathfrak{o}_F \cap F^{\times md}) = X_{\mathrm{nr}}(\varpi_F^{md\mathbb{Z}}) \cong \mathbb{C}^\times,$$

for any uniformizer ϖ_F in the ring of integers $\mathfrak{o}_F$.

It follows that the preimage of $\mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp)$ in $\mathrm{Irr}(G^\sharp Z(G))$ consists of countably many Bernstein components $\mathrm{Irr}^{\mathfrak{t}}(G^\sharp Z(G))$, each one homeomorphic to

$$X_{\mathrm{nr}}(F^{\times md}) \times \mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp) \cong X_{\mathrm{nr}}(G^\sharp Z(G)) \times \mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp) \cong \mathbb{C}^\times \times \mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp).$$

Two such components differ from each other by a ramified character of $F^{\times md}$, or equivalently by a character of $\mathrm{Nrd}(\mathfrak{o}_F^\times \cdot 1_m)$.

In comparison, every Bernstein component $\mathrm{Irr}^{\mathfrak{t}}(G^\sharp Z(G))$ projects onto a single Bernstein component for $G^\sharp$, say $\mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp)$. All the fibers of

$$\mathrm{Res}^{G^\sharp Z(G)}_{G^\sharp} : \mathrm{Irr}^{\mathfrak{t}}(G^\sharp Z(G)) \to \mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp) \tag{42}$$

are homeomorphic to $X_{\mathrm{nr}}(Z(G)) = X_{\mathrm{nr}}(F^{\times md}) \cong \mathbb{C}^\times$. In particular

$$T_{\mathfrak{t}^\sharp} = T_{\mathfrak{t}} / X_{\mathrm{nr}}(\mathrm{Nrd}(Z(G))). \tag{43}$$

Lemma 2.6. *The finite groups associated to* $\mathfrak{t}$ *and* $\mathfrak{t}^\sharp$ *are equal:* $W_{\mathfrak{t}} = W_{\mathfrak{t}^\sharp}$.

Proof. As we observed above, $\mathfrak{t}^\sharp$ is the only Bernstein component involved in the restriction of $\mathrm{Irr}^{\mathfrak{t}}(G^\sharp Z(G))$ to $G^\sharp$. Hence $W^{\mathfrak{t}} \subset W^{\mathfrak{t}^\sharp}$. Conversely, if $w \in W(L^\sharp)$ stabilizes $\mathfrak{t}^\sharp$, then it stabilizes the set of Bernstein components $\mathrm{Irr}^{\mathfrak{t}'}(G^\sharp Z(G))$ which project onto $\mathrm{Irr}^{\mathfrak{t}^\sharp}(G^\sharp)$. But any such $\mathfrak{t}'$ differs from $\mathfrak{t}$ only by a ramified character of $Z(G)$. Since conjugation by elements of $N_G(L)$ does not affect $Z(G)$, $w(\mathfrak{t})$ cannot be another $\mathfrak{t}'$, and so $w \in W_{\mathfrak{t}}$. $\square$

The above provides a complete picture of $\mathrm{Res}^{G^\sharp Z(G)}_{G^\sharp}$, so we can focus on $\mathrm{Res}^G_{G^\sharp Z(G)}$. Although $[G : G^\sharp Z(G)]$ is sometimes infinite (e.g. if $\mathrm{char}(F)$ divides md), only finitely many characters of $G/G^\sharp Z(G)$ occur in relation to a fixed Bernstein component. This follows from Lemma 2.4.c and makes it possible to treat $G^\sharp Z(G) \subset G$ as a group extension of finite degree.

Given an inertial equivalence class $\mathfrak{s} = [L, \omega]_G$, we define $\mathrm{Irr}^{\mathfrak{s}}(G^\sharp Z(G))$ as the set of all elements of $\mathrm{Irr}(G^\sharp Z(G))$ that can be obtained as a subquotient of $\mathrm{Res}^G_{G^\sharp Z(G)}(\pi)$ for some $\pi \in \mathrm{Irr}^{\mathfrak{s}}(G)$. We also define $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp Z(G))$ as the collection of $G^\sharp Z(G)$-representations all whose irreducible subquotients lie in $\mathrm{Irr}^{\mathfrak{s}}(G^\sharp Z(G))$. It follows from Lemma 2.2 and the above that $\mathrm{Irr}^{\mathfrak{s}}(G^\sharp Z(G))$ is a union of finitely many Bernstein components $\mathfrak{t}$ for $G^\sharp Z(G)$. We denote this relation between $\mathfrak{s}$ and $\mathfrak{t}$ by $\mathfrak{t} \prec \mathfrak{s}$. Thus

$$\mathrm{Irr}^{\mathfrak{s}}(G^\sharp Z(G)) = \bigcup\nolimits_{\mathfrak{t} \prec \mathfrak{s}} \mathrm{Irr}^{\mathfrak{t}}(G^\sharp Z(G)). \tag{44}$$

All the reducibility of G-representations caused by restricting them to $G^\sharp$ can already be observed by restricting them to $G^\sharp Z(G)$. In view of (38) and Lemma 2.6, all our results on $\mathrm{Res}^G_{G^\sharp}$ remain valid if we replace everywhere $G^\sharp$ by $G^\sharp Z(G)$ and $L^\sharp$ by $L^\sharp Z(G)$.

In view of (16), intertwining operators associated to $\mathrm{Stab}(\mathfrak{s})$ span $\mathrm{End}_{G^\sharp Z(G)}(I_P^G(\omega \otimes \chi))$ whenever $\chi \in X_{\mathrm{nr}}(L)$ is unitary. With results of Harish-Chandra we will show that even more is true. For $w \in W(G, L)$ let

$$J(w, I_P^G(\omega \otimes \chi))) \in \mathrm{Hom}_G(I_P^G(\omega \otimes \chi), I_P^G(w(\omega \otimes \chi))) \tag{45}$$

be the intertwining operator constructed in [25, §5.5.1] and [29, §V.3]. We recall that it is rational as a function of $\chi \in X_{\mathrm{nr}}(L)$ and that it is regular and invertible if

χ is unitary. In contrast to the intertwining operators below, (45) can be normalized in a canonical way.

For $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}, P \cap M)$ there exists a $\chi' \in X_{\mathrm{nr}}(L)$, unique up to $X_{\mathrm{nr}}(L, \omega)$, such that $w(\omega \otimes \chi\gamma) \cong \omega \otimes \chi'$. Choose a nonzero

$$J(\gamma, \omega \otimes \chi) \in \mathrm{Hom}_L(\omega \otimes \chi, w^{-1}(\omega \otimes \chi'\gamma^{-1})). \tag{46}$$

In view of Lemma 2.4.b γ determines w, so this is unambigous and determines $J(\gamma, \omega \otimes \chi)$ up to a scalar. For unramified γ we have $\chi' = \chi\gamma$, but nevertheless $J(\gamma, \omega \otimes \chi)$ need not be a scalar multiple of identity. The reason lies in the difference between $T_{\mathfrak{s}}$ and $X_{\mathrm{nr}}(L)$, we refer to [29, §V] for more background.

Parabolic induction produces

$$J(\gamma, I_P^G(\omega \otimes \chi)) := I_P^G(J(\gamma, \omega \otimes \chi)) \in \mathrm{Hom}_G(I_P^G(\omega \otimes \chi), I_P^G(w^{-1}(\omega \otimes \chi'\gamma^{-1}))). \tag{47}$$

For $w'(w, \gamma) \in \mathrm{Stab}(\mathfrak{s})$ with $w' \in W_{\mathfrak{s}}$ and $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}, P \cap M)$ we define

$$\begin{aligned} J(w'(w, \gamma), I_P^G(\omega \otimes \chi)) :=& \\ J(w', I_P^G(\omega \otimes \chi'\gamma^{-1})) \circ J(w, I_P^G&(w^{-1}(\omega \otimes \chi'\gamma^{-1}))) \circ J(\gamma, I_P^G(\omega \otimes \chi)). \end{aligned} \tag{48}$$

By construction this lies both in $\mathrm{Hom}_G(I_P^G(\omega \otimes \chi), I_P^G(w'(\omega \otimes \chi'\gamma^{-1})))$ and in $\mathrm{Hom}_{G^\sharp Z(G)}(I_P^G(\omega \otimes \chi), I_P^G(w'(\omega \otimes \chi')))$. We remark that the map from $\mathrm{Stab}(\mathfrak{s})$ to intertwining operators (48) is not always multiplicative, some 2-cocycle with values in $\mathbb{C}^\times$ might be involved. However, in view of the canonical normalization of (45),

$$W_{\mathfrak{s}} \ni w' \mapsto \{J(w', I_P^G(\omega \otimes \chi)) \mid \chi \in X_{\mathrm{nr}}(L)\} \tag{49}$$

is a group homomorphism.

The following result is the main justification for introducing $\mathrm{Stab}(\mathfrak{s})$ in (32).

Theorem 2.7. *Let $\omega \in \mathrm{Irr}(L)$ be supercuspidal and let $\chi_1, \chi_2 \in X_{\mathrm{nr}}(L)$ be unramified characters.*

(*a*) *The following are equivalent:*
- (i) $\mathrm{Res}^G_{G^\sharp Z(G)}(I_P^G(\omega \otimes \chi_1))$ *and* $\mathrm{Res}^G_{G^\sharp Z(G)}(I_P^G(\omega \otimes \chi_2))$ *have a common irreducible subquotient;*
- (ii) $\mathrm{Res}^G_{G^\sharp Z(G)}(I_P^G(\omega \otimes \chi_1))$ *and* $\mathrm{Res}^G_{G^\sharp Z(G)}(I_P^G(\omega \otimes \chi_2))$ *have the same irreducible constituents, counted with multiplicity;*
- (iii) $\omega \otimes \chi_1$ *and* $\omega \otimes \chi_2$ *belong to the same* $\mathrm{Stab}(\mathfrak{s})$*-orbit.*

(*b*) *If $\omega \otimes \chi_1$ and $\omega \otimes \chi_2$ are unitary, then $\mathrm{Hom}_{G^\sharp Z(G)}(I_P^G(\omega \otimes \chi_1), I_P^G(\omega \otimes \chi_2))$ is spanned by intertwining operators $J((w, \gamma), I_P^G(\omega \otimes \chi_1))$ with $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s})$ and $w(\omega \otimes \chi_1\gamma) \cong \omega \otimes \chi_2$.*

Proof. First we assume that $\omega \otimes \chi_1$ and $\omega \otimes \chi_2$ are unitary. By Harish-Chandra's Plancherel isomorphism [29] and the commuting algebra theorem [25, Theorem 5.5.3.2], the theorem is true for G, with $W_{\mathfrak{s}}$ instead of $\mathrm{Stab}(\mathfrak{s})$. More generally, for any tempered $\rho_1, \rho_2 \in \mathrm{Irr}(L)$, $\mathrm{Hom}_G(I_P^G(\rho_1), I_P^G(\rho_2))$ is spanned by intertwining operators $J(w, I_P^G(\rho_1))$ with $w \in W(G, L)$ and $w \cdot \rho_1 \cong \rho_2$.

For $\pi_1, \pi_2 \in \mathrm{Irr}(G)$ Proposition 2.1.b says that $\mathrm{Res}^G_{G^\sharp Z(G)}(\pi_1)$ and $\mathrm{Res}^G_{G^\sharp Z(G)}(\pi_2)$ are isomorphic if $\pi_2 \cong \pi_1 \otimes \gamma$ for some $\gamma \in \mathrm{Irr}(G/G^\sharp Z(G))$, and have no common irreducible subquotients otherwise. Together with (16) this implies that

$$\mathrm{Hom}_{G^\sharp Z(G)}(I_P^G(\omega \otimes \chi_1), I_P^G(\omega \otimes \chi_2))$$

is spanned by intertwining operators $J((w,\gamma), I_P^G(\omega \otimes \chi_1))$ with $w(\omega \otimes \chi_1 \gamma) \cong \omega \otimes \chi_2$. Such pairs (w,γ) automatically belong to $\mathrm{Stab}(\mathfrak{s})$. Since both factors of $J((w,\gamma), I_P^G(\omega \otimes \chi_1))$ are bijective, the equivalence of (i), (ii) and (iii) follows. This proves (b) and (a) in the unitary case.

Now we allow χ_1 and χ_2 to be non-unitary. Assume (i). From Proposition 2.1.b we obtain a $\gamma \in \mathrm{Irr}(G/G^\sharp Z(G))$ such that $I_P^G(\omega \otimes \chi_1 \gamma)$ and $I_P^G(\omega \otimes \chi_2)$ have a common irreducible quotient. The theory of the Bernstein centre for G [4] implies that $\omega \otimes \chi_1 \gamma$ and $\omega \otimes \chi_2$ are isomorphic via an element $w \in W(G,L)$. Then $(w,\gamma) \in \mathrm{Stab}(\mathfrak{s})$, so (iii) holds.

Suppose now that $w(\omega \otimes \chi_1 \gamma) \cong \omega \otimes \chi_2$ for some $(w,\gamma) \in \mathrm{Stab}(\mathfrak{s})$ and consider the map

$$\mathcal{H}(G^\sharp Z(G)) \times X_{\mathrm{nr}}(L) \to \mathbb{C} : (f,\chi) \mapsto \mathrm{tr}(f, I_P^G(\omega \otimes \chi)) - \mathrm{tr}(f, I_P^G(w(\omega \otimes \gamma\chi))). \tag{50}$$

It is well-defined since $\mathrm{Res}^G_{G^\sharp} I_P^G(\omega \otimes \chi)$ has finite length, by Proposition 2.1.c. By what we proved above, the value is 0 whenever χ is unitary. But for a fixed $f \in \mathcal{H}(G)$ this is a rational function of $\chi \in X_{\mathrm{nr}}(L)$, and the unitary characters are Zariski-dense in $X_{\mathrm{nr}}(L)$. Therefore (50) is identically 0, which shows that $(I_P^G(\omega \otimes \chi))$ and $I_P^G(w(\omega \otimes \gamma\chi))$ have the same trace. By Proposition 2.1.c these $G^\sharp$-representations have finite length, so by [9, 2.3.3] their irreducible constituents (and multiplicities) are determined by their traces. Thus (iii) implies (ii), which obviously implies (i). $\square$

3. Morita equivalences

Let $\mathfrak{s} = [L,\omega]_G$. We want to analyse the two-sided ideal $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ of $\mathcal{H}(G^\sharp Z(G))$ associated to the category of representations $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp Z(G))$ introduced in Subsection 2.3. In this section we will transform these algebras to more manageable forms by means of Morita equivalences.

We note that by (44) we can regard $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ as a finite direct sum of ideals associated to one Bernstein component:

$$\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}} = \bigoplus\nolimits_{\mathfrak{t} \prec \mathfrak{s}} \mathcal{H}(G^\sharp Z(G))^{\mathfrak{t}}. \tag{51}$$

Recall that the abelian group $\mathrm{Irr}(G/G^\sharp)$ acts on $\mathcal{H}(G)$ by

$$(\chi \cdot f)(g) = \chi(g) f(g).$$

We also introduce an alternative action of $\gamma \in \mathrm{Irr}(G/G^\sharp)$ on $\mathcal{H}(G)$ (and on similar algebras):

$$\alpha_\gamma(f) = \gamma^{-1} \cdot f. \tag{52}$$

Obviously these two actions have the same invariants. An advantage of the latter lies in the induced action on representations:

$$\alpha_\gamma(\pi) = \pi \circ \alpha_\gamma^{-1} = \pi \otimes \gamma.$$

Suppose for the moment that the characteristic of F does not divide md, so that $G/G^\sharp Z(G)$ is finite. Then there are canonical isomorphisms

$$\begin{aligned} \bigoplus\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)/\sim} \mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}} &\cong \mathcal{H}(G^\sharp Z(G)) \\ &\cong \mathcal{H}(G)^{\mathrm{Irr}(G/G^\sharp Z(G))} \cong \bigoplus\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)/\sim} (\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})}, \end{aligned} \tag{53}$$

where $\mathfrak{s} \sim \mathfrak{s}'$ if and only if they differ by a character of $G/G^\sharp Z(G)$.

Unfortunately this is not true if $\mathrm{char}(F)$ does divide md. In that case there are no nonzero $\mathrm{Irr}(G/G^\sharp Z(G))$-invariant elements in $\mathcal{H}(G)$, because such elements could not be locally constant as functions on G. In Subsection 3.3 we will return to this point and show that it remains valid as a Morita equivalence.

Throughout this section we assume that the Conditions 1.1 are in force.

3.1. **Construction of a particular idempotent.**

We would like to find a type which behaves well under restriction from G to $G^\sharp$. As this is rather complicated, we start with a simpler goal: an idempotent in $\mathcal{H}(M)$ which is suitable for restriction from M to $M^\sharp$.

Recall from [24] that there exists an $\mathfrak{s}$-type (K_G, λ_G), and that it can be constructed as a cover of a type (K, λ) for $\mathfrak{s}_M = [L, \omega]_M$. We refer to [7, Section 8] for the notion of a cover of a type. For the moment, it suffices to know that $K = K_G \cap M$ and that λ is the restriction of λ_G to K.

From (9) we know that $N_G([L, \omega]_L) \subset M$ and by Condition 1.1 PM is a parabolic subgroup with Levi factor M. In this situation [7, Theorem 12.1] says that there is an algebra isomorphism

$$e_{\lambda_G} \mathcal{H}(G) e_{\lambda_G} \cong e_\lambda \mathcal{H}(M) e_\lambda \tag{54}$$

and that the normalized parabolic induction functor

$$I_{PM}^G : \mathrm{Rep}^{\mathfrak{s}_M}(M) \to \mathrm{Rep}^{\mathfrak{s}}(G)$$

is an equivalence of categories.

By Conditions 1.1 and (8) we may assume that (K, λ) factors as

$$\begin{aligned} K &= \prod\nolimits_i (K \cap M_i) =: \prod\nolimits_i K_i, \\ (\lambda, V_\lambda) &= \big(\bigotimes\nolimits_i \lambda_i, \bigotimes\nolimits_i V_{\lambda_i} \big). \end{aligned} \tag{55}$$

Moreover we may assume that, whenever $m_i = m_j$ and ω_i and ω_j differ only by a character of $L_i / L_i^\sharp$, $K_i = K_j$ and λ_i and λ_j also differ only by a character of $K_i / (K_i \cap L_i^\sharp)$. We note that these assumptions imply that $\mathfrak{R}_\mathfrak{s}^\sharp$ normalizes K. By respectively (54), (7) and (55) there are isomorphisms

$$e_{\lambda_G} \mathcal{H}(G) e_{\lambda_G} \;\cong\; \mathcal{H}(M, \lambda) \otimes_\mathbb{C} \mathrm{End}_\mathbb{C}(V_\lambda) \;\cong\; \bigotimes\nolimits_i \mathcal{H}(M_i, \lambda_i) \otimes_\mathbb{C} \mathrm{End}_\mathbb{C}(V_{\lambda_i}). \tag{56}$$

We need more specific information about the type (K, λ) in M. To study this and the related types $(K, w(\lambda) \otimes \gamma)$ we will make ample use of the theory developed by Sécherre and Stevens [22, 23, 24].

In [22] (K, λ) arises as a cover of a $[L, \omega]_L$-type (K_L, λ_L). In particular λ is trivial on both $K \cap U$ and $K \cap \overline{U}$, where U and $\overline{U}$ are the unipotent radicals of $P \cap M$ and of the opposite parabolic subgroup of M, and λ_L is the restriction of λ to $K_L = K \cap L$.

Proposition 3.1. *We can choose the $\mathfrak{s}_M$-type (K, λ) such that, for all $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s})$, $(K, w(\lambda) \otimes \gamma)$ is conjugate to (K, λ) by an element $c_\gamma \in L$. Moreover $c_\gamma Z(L)$ lies in a compact subgroup of $L/Z(L)$ and we can arrange that c_γ depends only on the isomorphism class of $w(\lambda) \otimes \gamma \in \mathrm{Irr}(K)$.*

Remark. For $\mathrm{GL}_n(F)$ very similar results were proven in [13, §4.2], using [8].

Proof. By definition (K, λ) and $(K, w(\lambda) \otimes \gamma)$ are both types for $[L, \omega]_M$. By Conditions 1.1 and (55) they differ only by a character of $M/M^\sharp Z(G) \cong G/G^\sharp Z(G)$, which automatically lies in $X^L(w(\mathfrak{s}) \otimes \gamma) = X^L(\mathfrak{s})$. Hence it suffices to prove the proposition in the case $w = 1, \gamma \in X^L(\mathfrak{s})$. This setup implies that we consider (w, γ) only modulo isomorphism of the representations $w(\lambda) \otimes \gamma$.

In view of the factorizations $M = \prod_i M_i$ and (55), we can treat the various i's separately. Thus we may assume that $M_i = G$. To get in line with [22], we temporarily change the notation to $G = \mathrm{GL}_{m_i e}(D), L = \mathrm{GL}_{m_i}(D)^e, J_P = K_i$ and $\omega = \omega_i^{\otimes e}$. We need a type (J_P, λ_P) for $[L, \omega]_G$ with suitable properties. We will use the one constructed in [22] as a cover of a simple type $(J_i^e, \lambda_i^{\otimes e})$ for $[L, \omega]_L$. Analogously there is a cover $(J_P, \lambda_P \otimes \gamma)$ of the $[L, \omega]_L$-type

$$(J_i^e, \lambda_i^{\otimes e} \otimes \gamma) = (J_i^e, (\lambda_i \otimes \gamma)^{\otimes e}).$$

In these constructions (J_i, λ_i) and $(J_i, \lambda_i \otimes \gamma)$ are two maximal simple types for the supercuspidal inertial equivalence class $[\mathrm{GL}_{m_i}(D), \omega_i]_{\mathrm{GL}_{m_i}(D)}$. According to [24, Corollary 7.3] they are conjugate, say by $c_i \in \mathrm{GL}_{m_i}(D)$. Then $(J_i^e, \lambda_i^{\otimes e})$ and $(J_i^e, \lambda_i^{\otimes e} \otimes \gamma)$ are conjugate by

$$c_{\gamma,i} := \mathrm{diag}(c_i, c_i, \ldots, c_i) \in L.$$

Recall that P is the parabolic subgroup of G generated by L and the upper triangular matrices. Let U be the unipotent radical of P and $\overline{U}$ the unipotent radical of the parabolic subgroup opposite to P. The group J_P constructed in [22, §5.2] and [23, §5.5] admits an Iwahori decomposition

$$J_P = (J_P \cap \overline{U})(J_P \cap L)(J_P \cap U) = (H^1 \cap \overline{U})\, J_i^e\, (J \cap U). \tag{57}$$

Let us elaborate a little on the subgroups H^1 and $J_P \subset J$ of G. In [23] a certain stratum $[\mathfrak{C}, n_0, 0, \beta]$ is associated to $(\mathrm{GL}_{m_i}(D), \omega_i)$, which gives rise to compact open subgroups H_i^1 and J_i of $\mathrm{GL}_{m_i}(D)$. From this stratum Sécherre [22, 5.2.2] defines another stratum $[\mathfrak{A}, n, 0, \beta]$ associated to $(L, \omega_i^{\otimes e})$, which in the same way produces H^1 and J. The procedure entails that H^1 and J can be obtained by putting together copies of H_i^1, J_i and their radicals in block matrix form. The proofs of [24, Theorem 7.2 and Corollary 7.3] show that we can take c_i such that it normalizes J_i and H_i^1. Then it follows from the explicit relation between the above two strata that $c_{\gamma,i}$ normalizes J and H^1. Notice also that $c_{\gamma,i}$ normalizes $\overline{U}$ and U, because it lies in L. Hence $c_{\gamma,i}$ normalizes J_P and its decomposition (57).

By definition [22, 5.2.3] the representation λ_P of J_P is trivial on $J_P \cap \overline{U}$ and on $J_P \cap U$, whereas its restriction to $J_P \cap L$ equals $\lambda_i^{\otimes e}$. As c_i conjugates λ_i to $\lambda_i \otimes \gamma$, we deduce that $c_{\gamma,i}$ conjugates (J_P, λ_P) to $(J_P, \lambda_P \otimes \gamma)$.

To get back to the general case we recall that $M = \prod_i M_i$ and we put

$$c_\gamma := \prod\nolimits_i c_{\gamma,i} = \prod\nolimits_i \mathrm{diag}(c_i, c_i, \ldots, c_i). \tag{58}$$

It remains to see that c_γ becomes a compact element in $L/Z(L)$. Since J_i is open and compact, its fixed points in the semisimple Bruhat–Tits building $\mathcal{B}(\mathrm{GL}_{m_i}(D))$ form a nonempty bounded subset. Then c_i stabilizes this subset, so by the Bruhat–Tits fixed point theorem c_i fixes some point $x_i \in \mathcal{B}(\mathrm{GL}_{m_i}(D))$. But the stabilizer of x_i is a compact modulo centre subgroup, so in particular c_i is compact modulo centre. Therefore the image of c_γ in $L/Z(L)$ is a compact element. $\square$

In the above proof it is also possible to replace (J_P, λ_P) by a sound simple type in the sense of [24]. Indeed, the group J from [22, §5] is generated by J_P and $J \cap \overline{U}$, so it is also normalized by $c_{\gamma,i}$. By [22, Proposition 5.4]

$$(J, \mathrm{Ind}_{J_P}^J(\lambda_P)) \quad \text{and} \quad (J, \mathrm{Ind}_{J_P}^J(\lambda_P \otimes \gamma))$$

are sound simple types. The above proof also shows that they are conjugate by $c_{\gamma,i}$. As noted in the proof of [22, Proposition 5.5], there is a canonical support preserving algebra isomorphism

$$\mathcal{H}(M_i, \lambda_P) \cong \mathcal{H}(M_i, \mathrm{Ind}_{J_P}^J(\lambda_P)). \tag{59}$$

In particular the structure theory of the Hecke algebras in [22] also applies to our types (K, λ).

We write

$$L^1 := \bigcap\nolimits_{\chi \in X_{\mathrm{nr}}(L)} \ker \chi.$$

Notice that $G^1 = \{g \in G \mid \mathrm{Nrd}(g) \in \mathfrak{o}_F^\times\}$ is the group generated by all compact subgroups of G. Hence L^1 is the group generated by all compact subgroups of L.

We fix a choice of elements $c_\gamma \in L$ as in Proposition 3.1, such that $c_\gamma \in L^1$ whenever possible. This determines subgroups

$$\begin{array}{lll} X^L(\mathfrak{s})^1 & := & \{\gamma \in X^L(\mathfrak{s}) \mid c_\gamma \in L^1\}, \\ \mathrm{Stab}(\mathfrak{s}, P \cap M)^1 & := & \{(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}, P \cap M) \mid c_\gamma \in L^1\}. \end{array} \tag{60}$$

Their relevance is that the $c_\gamma \in L^1$ can be used to construct larger $\mathfrak{s}_M$-types from (K, λ), whereas the c_γ with $\gamma \in X^L(\mathfrak{s}) \setminus X^L(\mathfrak{s})$ are unsuitable for that purpose. We remark that in the split case $G = GL_n(F)$ it is known from [6, Proposition 2.2] that one can find $c_\gamma \in L^1$ for all $\gamma \in X^L(\mathfrak{s})$.

Consider the group

$$\mathrm{Stab}(\mathfrak{s}, \lambda) = \{(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}, P \cap M) \mid w(\lambda) \otimes \gamma \cong \lambda \text{ as } K\text{-representations}\}.$$

The elements of this group are precisely the $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}, P \cap M)$ for which $e_{w(\lambda)\otimes\gamma} = e_\lambda$.

Lemma 3.2. *Projection on the first coordinate gives a short exact sequence*

$$1 \to X^L(\mathfrak{s}) \cap \mathrm{Stab}(\mathfrak{s}, \lambda) \to \mathrm{Stab}(\mathfrak{s}, \lambda) \to \mathfrak{R}_{\mathfrak{s}}^\sharp \to 1.$$

The inclusion $X^L(\mathfrak{s}) \to \mathrm{Stab}(\mathfrak{s}, P \cap M)$ *induces a group isomorphism*

$$X^L(\mathfrak{s}) / (X^L(\mathfrak{s}) \cap \mathrm{Stab}(\mathfrak{s}, \lambda)) \to \mathrm{Stab}(\mathfrak{s}, P \cap M) / \mathrm{Stab}(\mathfrak{s}, \lambda).$$

Proof. Let $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}, P \cap M)$. Then $w(\lambda) \otimes \gamma \cong \lambda \otimes \gamma'$ for some $\gamma' \in X^L(\mathfrak{s})$ and $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}, \lambda)$ if and only if $\gamma' \in \mathrm{Stab}(\mathfrak{s}, \lambda)$. Hence all the fibers of $\mathrm{Stab}(\mathfrak{s}, \lambda) \to \mathfrak{R}_{\mathfrak{s}}^\sharp$ have the same cardinality, namely $|X^L(\mathfrak{s}) \cap \mathrm{Stab}(\mathfrak{s}, \lambda)|$. The required short exact sequence follows. The asserted isomorphism of groups is a direct consequence thereof. $\square$

Motivated by Lemma 3.2 we abbreviate

$$\begin{aligned} X^L(\mathfrak{s}, \lambda) &:= X^L(\mathfrak{s}) \cap \mathrm{Stab}(\mathfrak{s}, \lambda), \\ X^L(\mathfrak{s}/\lambda) &:= X^L(\mathfrak{s}) / X^L(\mathfrak{s}, \lambda), \\ X^L(\mathfrak{s}/\lambda)^1 &:= X^L(\mathfrak{s})^1 / X^L(\mathfrak{s}, \lambda). \end{aligned} \tag{61}$$

The latter two groups are isomorphic to respectively

$$\mathrm{Stab}(\mathfrak{s}, P \cap M)/\mathrm{Stab}(\mathfrak{s}, \lambda) \text{ and } \mathrm{Stab}(\mathfrak{s}, P \cap M)^1/\mathrm{Stab}(\mathfrak{s}, \lambda).$$

By Lemma 3.2 the element $\sum_{\gamma \in X^L(\mathfrak{s}/\lambda)} e_{\lambda \otimes \gamma} \in \mathcal{H}(K)$ is well-defined and idempotent. Clearly this element is invariant under $X^L(\mathfrak{s})$, which makes it more suitable to study the restriction of $\mathrm{Rep}^{\mathfrak{s}}(G)$ to $G^\sharp$. However, in some cases this idempotent sees only a too small part of a G-representation. This will become apparent in the proof of Proposition 3.9.d. We need to replace it by a larger idempotent, for which we use the following lemma.

Lemma 3.3. *Let $(\omega, V_\omega) \in \mathrm{Irr}(L)$ be supercuspidal and write $\mathfrak{s} = [L, \omega]_G$. There exist a subgroup $H_\lambda \subset L$ and a subset $[L/H_\lambda] \subset L$ such that:*

(*a*) *$[L/H_\lambda]$ is a set of representatives for L/H_λ, where $H_\lambda \subset L$ is normal, of finite index and contains $L^\sharp Z(L)$.*

(*b*) *Every element of $[L/H_\lambda]$ commutes with $W_{\mathfrak{s}}^\sharp$ and has finite order in $L/Z(L)$.*

(*c*) *For every $\chi \in X_{\mathrm{nr}}(L)$ the space*

$$\sum\nolimits_{a \in [L/H_\lambda]} \sum\nolimits_{\gamma \in X^L(\mathfrak{s})} a e_{\lambda_L \otimes \gamma} a^{-1} V_{\omega \otimes \chi}$$

intersects every $L^\sharp$-isotypical component of $V_{\omega \otimes \chi}$ nontrivially.

(*d*) *For every $(\pi, V_\pi) \in \mathrm{Irr}^{\mathfrak{s}_M}(M)$ the space*

$$\sum\nolimits_{a \in [L/H_\lambda]} \sum\nolimits_{\gamma \in X^L(\mathfrak{s})} a e_{\lambda \otimes \gamma} a^{-1} V_\pi$$

intersects every $M^\sharp$-isotypical component of V_π nontrivially.

Proof. (a) First we identify the group H_λ. Recall the operators $I(\gamma, \omega) \in \mathrm{Hom}_L(\omega \otimes \gamma, \omega)$ from (14), with $\gamma \in X^L(\mathfrak{s}, \lambda) \cap X^L(\omega)$. Like in (17) and [16, Corollary 2.10], these provide a decomposition of $L^\sharp$-representations

$$\omega = \bigoplus_{\rho \in \mathrm{Irr}(\mathbb{C}[X^L(\mathfrak{s},\lambda) \cap X^L(\omega), \kappa_\omega])} \mathrm{Hom}_{\mathbb{C}[X^L(\mathfrak{s},\lambda) \cap X^L(\omega), \kappa_\omega]}(\rho, \omega) \otimes \rho.$$

Let us abbreviate it to

$$V_\omega = \bigoplus\nolimits_\rho V_{\omega,\rho}. \tag{62}$$

It follows from Proposition 2.1 and (17) that all the summands $V_{\omega,\rho}$ are L-conjugate and that $\mathrm{Stab}_L(V_{\omega,\rho})$ is a finite index normal subgroup which contains $L^\sharp Z(L)$. This leads to a bijection

$$\mathrm{Irr}(\mathbb{C}[X^L(\mathfrak{s}, \lambda) \cap X^L(\omega), \kappa_\omega]) \;\longleftrightarrow\; L/\mathrm{Stab}_L(V_{\omega,\rho}). \tag{63}$$

We claim that

$$\sum\nolimits_{\gamma \in X^L(\mathfrak{s})} e_{\lambda_L \otimes \gamma} V_\omega \tag{64}$$

is an irreducible representation of a subgroup $N \subset L$ that normalizes K_L. From [22, Théorème 4.6] it is known that

$$e_{\lambda_L \otimes \gamma} \mathcal{H}(L) e_{\lambda_L \otimes \gamma} \cong \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\lambda_L \otimes \gamma}), \tag{65}$$

where $\mathrm{End}_{\mathbb{C}}(V_{\lambda_L \otimes \gamma})$ corresponds to the subalgebra $e_{\lambda_L \otimes \gamma} \mathcal{H}(K) e_{\lambda_L \otimes \gamma}$. Since $(K_L, \lambda_L \otimes \gamma)$ is a type for $[L, \omega \otimes \gamma]_L$, every $e_{\lambda_L \otimes \gamma} V_\omega$ is isomorphic, as module over the right hand side of (65), to $\mathbb{C}_t \otimes V_{\lambda_L \otimes \gamma}$ for a unique $t \in T^{\mathfrak{s}}$. The action of K_L goes via $e_{\lambda_L \otimes \gamma} \mathcal{H}(K) e_{\lambda_L \otimes \gamma}$, so $e_{\lambda_L \otimes \gamma} V_\omega$ is already irreducible as K_L-representation.

Since $Z(L)$-stabilizes this vector space, it also irreducible as representation of the group $K_L Z(L)$.

These representations, with $\gamma \in X^L(\mathfrak{s}/\lambda)$ are inequivalent and permuted transitively by the elements c_γ from Proposition 3.1. Hence (64) is irreducible as a representation of the group N generated by $K_L Z(L)$ and the c_γ.

Suppose now that (64) intersects both V_{ω,ρ_1} and V_{ω,ρ_2} nontrivially. By the above claim, N contains an element that maps V_{ω,ρ_1} to V_{ω,ρ_2}. It follows that, under the bijection (63), the set of ρ's such that $V_{\omega,\rho}$ intersects (64) nontrivially corresponds to a subgroup of $L/\mathrm{Stab}_L(V_{\omega,\rho})$, say $H_\lambda/\mathrm{Stab}_L(V_{\omega,\rho})$. Because $L/\mathrm{Stab}_L(V_{\omega,\rho})$ was already finite and abelian, H_λ has the desired properties.

We note that none of the above changes if we twist ω by an unramified character of L.

(b) Recall that $L = \prod_i GL_{m_i}(D)^{e_i}$ and that the reduced norm map $D^\times \to F^\times$ is surjective. It provides a group isomorphism

$$L/H_\lambda \to F^\times/\mathrm{Nrd}(H_\lambda),$$

and $\mathrm{Nrd}(H_\lambda)$ contains $\mathrm{Nrd}(Z(L)) = F^{\times e}$ where e is the greatest common divisor of the numbers m_i. We can choose explicit representatives for L/H_λ. It suffices to use elements a whose components $a_i \in \mathrm{GL}_{m_i}(D)$ are powers of some element of the form

$$\begin{pmatrix} 0 & 1 & 0 & & \cdots & 0 \\ 0 & 0 & 1 & & \cdots & 0 \\ \vdots & \vdots & & & & \\ 0 & 0 & & \cdots & 0 & 1 \\ d_i & 0 & & \cdots & 0 & 0 \end{pmatrix} \in GL_{m_i}(D),$$

that is, the matrix of the cyclic permutation $(1 \to m_i \to m_i - 1 \to \cdots 3 \to 2)$, with one entry replaced by an element $d_i \in D^\times$. In this way we assure that a has finite order in $L/Z(L)$, at most ed.

If two factors $L_i = \mathrm{GL}_{m_i}(D)$ and $L_j = \mathrm{GL}_{m_j}(D)$ of $L = \prod_i L_i^{e_i}$ are conjugate via an element of $W_{\mathfrak{s}}^\sharp$, then $m_i = m_j$ and the corresponding supercuspidal representations ω_i and ω_j differ only by a character of $\mathrm{GL}_{m_i}(D)$, say η. As in the proof of Proposition 3.1, let (J_i, λ_i) be a simple type for (L_i, ω_i). As in (55) we use the type $(J_i, \lambda \otimes \eta)$ for (L_j, ω_j).

Given $\gamma \in X^L(\mathfrak{s}, \lambda) \cap X^L(\omega)$, we can factor

$$I(\gamma, \omega) = \prod\nolimits_i I(\gamma, \omega_i)^{\oplus e_i},$$

with $I(\gamma, \omega_i) \in \mathrm{Hom}_{L_i}(\omega_i \otimes \gamma, \omega_i)$. Here we can simply take $I(\gamma, \omega_j) = I(\gamma, \omega_i)$. Then the decomposition of V_{ω_j} in isotypical subspaces $V_{\omega_i,\rho}$ for $\mathbb{C}[X^L(\mathfrak{s}, \lambda) \cap X^L(\omega), \kappa_\omega]$, like (62) is the same as that of V_{ω_i}, and

$$V_{\omega,\rho} = \bigoplus\nolimits_i V_{\omega_i,\rho}^{\oplus e_i}.$$

Suppose now that a component a_i of a as above maps $V_{\omega_i,\rho}$ to $V_{\omega_i,\rho'}$. Then a_i also maps $V_{\omega_j,\rho}$ to $V_{\omega_j,\rho'}$, so we may take $a_j = a_i$. With this construction $a = \prod_i a_i^{\oplus e_i}$ commutes with $W_{\mathfrak{s}}^\sharp$.

We fix such a set of representatives a and denote it by

$$[L/H_\lambda] = \{a_l \mid l \in L/H_\lambda\}. \tag{66}$$

(c) Let $\chi \in X_{\mathrm{nr}}(L)$ and consider any $\rho \in \mathrm{Irr}(\mathbb{C}[X^L(\mathfrak{s},\lambda) \cap X^L(\omega), \kappa_\omega])$. By construction

$$\sum\nolimits_{a \in [L/H_\lambda]} \sum\nolimits_{\gamma \in X^L(\mathfrak{s})} a e_{\lambda_L \otimes \gamma} a^{-1} V_{\omega \otimes \chi} \tag{67}$$

intersects $V_{\omega\otimes\chi,\rho}$ nontrivially. All the idempotents $a e_{\lambda_L \otimes \gamma} a^{-1}$ are invariant under $X^L(\mathfrak{s},\lambda) \cap X^L(\omega)$ because e_{λ_L} is. Hence $a e_{\lambda_L \otimes \gamma} a^{-1} V_{\omega\otimes\chi}$ is nonzero for at least one of these idempotents. The action of $X^L(\omega)/X^L(\omega) \cap \mathrm{Stab}(\mathfrak{s},\lambda)$ permutes the idempotents $a e_{\lambda_L \otimes \gamma} a^{-1}$ faithfully, so by Frobenius reciprocity the space

$$\sum\nolimits_{\gamma \in X^L(\mathfrak{s})} a e_{\lambda_L \otimes \gamma} a^{-1} V_{\omega \otimes \chi}$$

contains all irreducible representations of $\mathbb{C}[X^L(\omega \otimes \chi), \kappa_{\omega\otimes\chi}]$ that contain ρ. As $\rho \in \mathrm{Irr}(\mathbb{C}[X^L(\mathfrak{s},\lambda) \cap X^L(\omega), \kappa_\omega])$ was arbitrary, (67) contains all irreducible representations of $\mathbb{C}[X^L(\omega \otimes \chi), \kappa_{\omega\otimes\chi}]$. With (17) (for L) this says that (67) intersects every $L^\sharp$-isotypical component of $V_{\omega\otimes\chi}$ nontrivially.
(d) Let $\pi \in \mathrm{Irr}^{\mathfrak{s}_M}(M)$ and choose $\chi \in X_{\mathrm{nr}}(L)$ such that π is a subquotient of $I_{P\cap M}^M(\omega \otimes \chi)$. Lemma 2.4.d. in combination with the equality $W(M,L) = W_{\mathfrak{s}}$ shows that

$$X^M(\pi) \subset X^L(\omega \otimes \chi) = X^L(\omega).$$

Since $(K, \lambda \otimes \gamma)$ is a type,

$$a e_{\lambda \otimes \gamma} a^{-1} V_\pi \neq 0$$

for all possible a, γ. The group $X^L(\mathfrak{s},\lambda) \cap X^M(\pi)$ effects a decomposition of the $M^\sharp$-representation V_π by means of the operators $I_{P\cap M}^M(\gamma, \omega \otimes \chi)$. Analogous to (62) we write it as

$$V_\pi = \bigoplus\nolimits_{\rho \in \mathrm{Irr}(\mathbb{C}[X^L(\mathfrak{s},\lambda) \cap X^M(\pi), \kappa_\omega])} V_{\pi,\rho}. \tag{68}$$

The construction of H_λ entails that

$$\sum\nolimits_{a \in [L/H_\lambda]} \sum\nolimits_{\gamma \in X^L(\mathfrak{s})} a e_{\lambda \otimes \gamma} a^{-1} V_\pi$$

intersects every summand $V_{\pi,\rho}$ of (68) nontrivially. Now the same argument as for part (c) shows that this space intersects every $M^\sharp$-isotypical component of V_π nontrivially. $\square$

With (61) and Lemma 3.3 we construct some additonal idempotents:

$$\begin{aligned}
e_{\mu_L} &:= \sum\nolimits_{\gamma \in X^L(\mathfrak{s}/\lambda)} e_{\lambda_L \otimes \gamma} \in \mathcal{H}(K_L), \\
e_L^{\mathfrak{s}} &:= \sum\nolimits_{a \in [L/H_\lambda]} a e_{\mu_L} a^{-1} \in \mathcal{H}(L), \\
e_\mu &:= \sum\nolimits_{\gamma \in X^L(\mathfrak{s}/\lambda)} e_{\lambda \otimes \gamma} \in \mathcal{H}(K), \\
e_M^{\mathfrak{s}} &:= \sum\nolimits_{a \in [L/H_\lambda]} a e_\mu a^{-1} \in \mathcal{H}(M).
\end{aligned} \tag{69}$$

Lemma 3.4. *The four elements in* (69) *are idempotent and* $\mathrm{Stab}(\mathfrak{s}, P\cap M)$*-invariant. Furthermore* $e_{\mu_L}, e_L^{\mathfrak{s}} \in \mathcal{H}(L)^{\mathfrak{s}_L}$ *and* $e_\mu, e_M^{\mathfrak{s}} \in \mathcal{H}(M)^{\mathfrak{s}_M}$.

Proof. We only write down the proof for the last two elements, the argument for the first two is analogous.

We already observed that the different idempotents $e_{\lambda \otimes \gamma}$ are orthogonal, so that their sum e_μ is again idempotent. We claim that $e = a_l e_{\lambda\otimes\gamma} a_l^{-1}$ and $e' = a_{l'} e_{\lambda\otimes\gamma'} a_{l'}^{-1}$ are orthogonal unless $l = l'$ and $\gamma = \gamma'$.

By construction, the images of e and e' in $\mathrm{End}_{\mathbb{C}}(V_{I_{P\cap M}^M \omega})$ are orthogonal. This remains true if we twist ω by an unramified character $\chi \in X_{\mathrm{nr}}(L)$. But the M-representations $I_{P\cap M}^M(\omega \otimes \chi)$ together generate the entire category $\mathrm{Rep}^{\mathfrak{s}_M}(M)$. Hence $ee' = e'e = 0$ on every representation in $\mathrm{Rep}^{\mathfrak{s}_M}(M)$.

Since $e_\lambda \in \mathcal{H}(M)^{\mathfrak{s}_M}$ and that algebra is stable under conjugation with elements of M and under $\mathrm{Stab}(\mathfrak{s})$ by (72), all the $a e_{\lambda\otimes\gamma} a^{-1}$ lie in $\mathcal{H}(M)^{\mathfrak{s}_M}$. Thus $e, e' \in \mathcal{H}(M)^{\mathfrak{s}_M}$, and we can conclude that they are indeed orthogonal. This implies that $e_M^{\mathfrak{s}} \in \mathcal{H}(M)^{\mathfrak{s}_M}$ is idempotent.

Since $e_{\lambda\otimes\gamma}$ is invariant under $X^L(\mathfrak{s}, \lambda)$, so is e_μ. The action of $X^L(\mathfrak{s})$ commutes with conjugation by any element of M, hence the sum over $\gamma \in X^L(\mathfrak{s}/\lambda)$ in the definition of e_μ makes it $X^L(\mathfrak{s})$-invariant.

By (55) and the last part of Proposition 3.1, e_μ is invariant under $\mathrm{Stab}(\mathfrak{s}, P \cap M)$ (but not necessarily under $W_{\mathfrak{s}}$). By Lemma 3.3.b this remains the case after conjugation by any $a \in [L/H_\lambda]$. Hence $a e_\mu a^{-1}$ and $e_M^{\mathfrak{s}}$ are also invariant under $\mathrm{Stab}(\mathfrak{s}, P \cap M)$. $\square$

We can interpret the group L/H_λ from Lemma 3.3 in a different way. Define

$$\begin{aligned} V_\mu &:= V_{\mu_L} = e_{\mu_L} V_\omega, \\ X^L(\omega, V_\mu) &:= \{\gamma \in X^L(\omega) \mid I(\gamma,\omega)|_{V_\mu} \in \mathbb{C}^\times \mathrm{id}_{V_\mu}\}. \end{aligned} \tag{70}$$

Lemma 3.5. *There is a group isomorphism $L/H_\lambda \cong \mathrm{Irr}(X^L(\omega, V_\mu))$.*

Proof. We use the notation from the proof of Lemma 3.3. Consider the twisted group algebra

$$\mathbb{C}[X^L(\mathfrak{s},\lambda) \cap X^L(\omega), \kappa_\omega]. \tag{71}$$

We noticed in (62) and (63) that all its irreducible representations have the same dimension, say δ. Let C be the subgroup of $X^L(\mathfrak{s},\lambda) \cap X^L(\omega)$ that consists of all elements γ for which $I(\gamma,\omega)$ acts as a scalar operator on $V_{\omega,\rho}$. Since all the $V_{\omega,\rho}$ are L-conjugate, this does not depend on ρ. As the dimension of (71) equals $|X^L(\mathfrak{s},\lambda) \cap X^L(\omega)|$, we find that

$$[X^L(\mathfrak{s},\lambda) \cap X^L(\omega) : C] = \delta^2 \text{ and } |C| = [L : \mathrm{Stab}_L(V_{\omega,\rho})].$$

Since C acts on every $V_{\omega,\rho}$ by a character, we can normalize the operators $I(\gamma,\omega)$ such that $\kappa_\omega|_{C\times C} = 1$. The subalgebra of (71) spanned by the $I(\gamma,\omega)$ with $\gamma \in C$ has dimension $|C|$, so every character of C appears in $V_{\omega,\rho}$ for precisely one $\rho \in \mathrm{Irr}(\mathbb{C}[X^L(\mathfrak{s},\lambda) \cap X^L(\omega), \kappa_\omega])$. Now we see from (63) that

$$C = \mathrm{Irr}(L/\mathrm{Stab}_L(V_{\omega,\rho})) \text{ and } \mathrm{Irr}(C) \cong L/\mathrm{Stab}_L(V_{\omega,\rho}).$$

Under the this isomorphism the subgroup $H_\lambda/\mathrm{Stab}_L(V_{\omega,\rho})$ corresponds to the set of character of C that occur in V_μ. That set can also be written as $\mathrm{Irr}(C/X^L(\omega, V_\mu))$. Hence the quotient

$$L/H_\lambda = (L/\mathrm{Stab}_L(V_{\omega,\rho}))\big/(H_\lambda/\mathrm{Stab}_L(V_{\omega,\rho}))$$

is isomorphic to $\mathrm{Irr}(X^L(\omega, V_\mu))$. $\Box$

3.2. **Descent to a Levi subgroup.**

Motivated by the isomorphisms (53) we focus on $(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})}$. We would like to replace it by a Morita equivalent subalgebra of $\mathcal{H}(M)^{\mathfrak{s}_M}$, where $\mathfrak{s}_M = [L, \omega]_M$ and $\mathfrak{s} = [L, \omega]_G$. However, the latter algebra is in general not stable under the action of $X^G(\mathfrak{s})$. In fact, for $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s})$ we have

$$\gamma \cdot \mathcal{H}(M)^{\mathfrak{s}_M} = \mathcal{H}(M)^{[L,\omega\otimes\gamma^{-1}]_M} = \mathcal{H}(M)^{[L,w\cdot\omega]_M} = \mathcal{H}(M)^{w(\mathfrak{s}_M)}. \tag{72}$$

Let us regard $\mathfrak{R}^\sharp_{\mathfrak{s}}$ from Lemma 2.3.c as a group of permutation matrices in G. Then it acts on M by conjugation and we can form the crossed product

$$\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}) = \mathcal{H}(M) \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}.$$

We define $\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}}$ as the two-sided ideal of $\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})$ such that

$$\mathrm{Ind}^G_{PM\rtimes\mathfrak{R}^\sharp_{\mathfrak{s}}}(V) \in \mathrm{Rep}^{\mathfrak{s}}(G) \text{ for all } V \in \mathrm{Mod}(\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}}).$$

Lemma 3.6. *The algebra $\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}}$ equals $\big(\bigoplus_{w\in\mathfrak{R}^\sharp_{\mathfrak{s}}} \mathcal{H}(M)^{w(\mathfrak{s}_M)}\big) \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}$.*

Proof. First we note that $\mathrm{Res}^{\mathcal{H}(M\rtimes\mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}}}_{\mathcal{H}(M)}(V) \in \sum_{w\in W_{\mathfrak{s}}\mathfrak{R}^\sharp_{\mathfrak{s}}} \mathrm{Mod}(\mathcal{H}(M)^{w(\mathfrak{s}_M)})$ for all eligible V, because these $w(\mathfrak{s}_M)$ are only inertial equivalence classes for M which lift to $\mathfrak{s}$. Hence

$$\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}} \subset \sum_{r\in\mathfrak{R}^\sharp_{\mathfrak{s}}} \sum_{w\in W_{\mathfrak{s}}} \mathcal{H}(M)^{w(\mathfrak{s}_M)} r.$$

The right hand side satisfies the defining property of $\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}}$, so both sides are equal. Because $W_{\mathfrak{s}} \subset M$ and $\mathcal{H}(M)^{\mathfrak{s}_M}\mathcal{H}(M)^{w(\mathfrak{s}_M)} = 0$ for $w \in \mathfrak{R}^\sharp_{\mathfrak{s}} \setminus \{1\}$, the right hand side is actually a crossed product in the asserted way. $\Box$

By (72) the algebra from Lemma 3.6 is stable under $X^G(\mathfrak{s})$. We extend the action α of $X^L(\mathfrak{s})$ on $\mathcal{H}(M)$ to $\mathrm{Stab}(\mathfrak{s})$ by

$$\alpha_{(w,\gamma)}(f) := w(\gamma^{-1} \cdot f)w^{-1}. \tag{73}$$

Given $w \in \mathfrak{R}^\sharp_{\mathfrak{s}}$, Lemma 2.4.d shows that there exists a $\gamma \in \mathrm{Irr}(L/L^\sharp Z(G))$ such that $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s})$, and that γ is unique up to $X^L(\mathfrak{s})$. Hence $w \mapsto \alpha_{(w,\gamma)}$ determines a group action of $\mathfrak{R}^\sharp_{\mathfrak{s}}$ on $(\mathcal{H}(M))^{X^L(\mathfrak{s})}$. By (72) this action stabilizes $(\mathcal{H}(M)^{\mathfrak{s}_M})^{X^L(\mathfrak{s})}$. Using this action, we can rewrite the α-invariant subalgebra of $\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}}$ conveniently:

Lemma 3.7. *There is a canonical isomorphism*

$$\big(\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}}\big)^{X^G(\mathfrak{s})} \cong (\mathcal{H}(M)^{\mathfrak{s}_M})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}.$$

Proof. Using Lemma 3.6 and the fact that $X^G(\mathfrak{s})$ fixes all elements of $\mathbb{C}[\mathfrak{R}^\sharp_{\mathfrak{s}}]$, we can rewrite

$$\big(\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}}\big)^{X^G(\mathfrak{s})} \cong \Big(\big(\bigoplus_{w\in\mathfrak{R}^\sharp_{\mathfrak{s}}} \mathcal{H}(M)^{w(\mathfrak{s}_M)}\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}\Big)^{X^G(\mathfrak{s})/X^L(\mathfrak{s})}.$$

By Lemma 2.4.d this is

$$\Big(\bigoplus_{w_1,w_2\in\mathfrak{R}^\sharp_{\mathfrak{s}}} w_1 (\mathcal{H}(M)^{\mathfrak{s}_M})^{X^L(\mathfrak{s})} w_2^{-1}\Big)^{\mathfrak{R}^\sharp_{\mathfrak{s}}} \cong \big(\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{R}^\sharp_{\mathfrak{s}}]) \otimes (\mathcal{H}(M)^{\mathfrak{s}_M})^{X^L(\mathfrak{s})}\big)^{\mathfrak{R}^\sharp_{\mathfrak{s}}}.$$

In the right hand side the action of $\mathfrak{R}_\mathfrak{s}^\sharp$ has become the regular representation on $\mathrm{End}_\mathbb{C}(\mathbb{C}[\mathfrak{R}_\mathfrak{s}^\sharp])$ tensored with the action $\alpha_{(w,\gamma)}$ as in (73). By a folklore result (see [26, Lemma A.3] for a proof) the right hand side is isomorphic to $(\mathcal{H}(M)^{\mathfrak{s}_M})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp$. □

In Proposition 3.9 we will show that the algebras from Lemma 3.7 are Morita equivalent with $(\mathcal{H}(G)^\mathfrak{s})^{X^G(\mathfrak{s})}$.

We recall from Lemma 3.4 that $e_M^\mathfrak{s}$ in (69) is $\mathrm{Stab}(\mathfrak{s}, P \cap M)$-invariant, so from (73) we obtain an action of $\mathrm{Stab}(\mathfrak{s}, P \cap M)$ on $e_M^\mathfrak{s} \mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp)^\mathfrak{s} e_M^\mathfrak{s}$.

Lemma 3.8. *The following algebras are Morita equivalent:* $\mathcal{H}(G)^\mathfrak{s}, \mathcal{H}(M)^{\mathfrak{s}_M}, \mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp)^\mathfrak{s}, e_M^\mathfrak{s} \mathcal{H}(M) e_M^\mathfrak{s}$ *and* $e_M^\mathfrak{s} \mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp)^\mathfrak{s} e_M^\mathfrak{s}$.

Proof. We will denote Morita equivalence with $\sim_M$. The Morita equivalence of $\mathcal{H}(G)^\mathfrak{s}$ and $\mathcal{H}(M)^{\mathfrak{s}_M}$ follows from the fact that $N_G(\mathfrak{s}_L) \subset M$. It is given in one direction by

$$I_{PM}^G : \mathrm{Mod}(\mathcal{H}(M)^{\mathfrak{s}_M}) = \mathrm{Rep}^{\mathfrak{s}_M}(M) \to \mathrm{Rep}^\mathfrak{s}(G) = \mathrm{Mod}(\mathcal{H}(G)^\mathfrak{s}) \tag{74}$$

and in the other direction by

$$\mathrm{pr}_{\mathfrak{s}_M} \circ r_{PM}^G : \mathrm{Rep}^\mathfrak{s}(G) \to \mathrm{Rep}^\mathfrak{s}(M) \to \mathrm{Rep}^{\mathfrak{s}_M}(M), \tag{75}$$

the normalized Jacquet restriction functor r_{PM}^G followed by projection on the factor $\mathrm{Rep}^{\mathfrak{s}_M}(M)$ of $\mathrm{Rep}^\mathfrak{s}(M)$. Lemma 3.6 shows that

$$\mathcal{H}(M)^{\mathfrak{s}_M} \sim_M \mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp)^\mathfrak{s}, \tag{76}$$

the equivalence being given by

$$\mathrm{Ind}_{\mathcal{H}(M)^{\mathfrak{s}_M}}^{\mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp)^\mathfrak{s}} = \mathrm{Ind}_M^{M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp}.$$

With the bimodules $e_M^\mathfrak{s} \mathcal{H}(M)^{\mathfrak{s}_M}$ and $\mathcal{H}(M)^{\mathfrak{s}_M} e_M^\mathfrak{s}$ we see that

$$e_M^\mathfrak{s} \mathcal{H}(M) e_M^\mathfrak{s} = e_M^\mathfrak{s} \mathcal{H}(M)^{\mathfrak{s}_M} e_M^\mathfrak{s} \sim_M \mathcal{H}(M)^{\mathfrak{s}_M} e_M^\mathfrak{s} \mathcal{H}(M)^{\mathfrak{s}_M}. \tag{77}$$

Since (K, λ) is an $\mathfrak{s}_M$-type, every module of $\mathcal{H}(M)^{\mathfrak{s}_M}$ is generated by its λ-isotypical vectors and a fortiori by the image of $e_M^\mathfrak{s}$ in such a module. Therefore

$$\mathcal{H}(M)^{\mathfrak{s}_M} e_M^\mathfrak{s} \mathcal{H}(M)^{\mathfrak{s}_M} = \mathcal{H}(M)^{\mathfrak{s}_M}.$$

The same argument, now additionally using (76), also shows that

$$\mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp)^\mathfrak{s} \sim_M e_M^\mathfrak{s} \mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp)^\mathfrak{s} e_M^\mathfrak{s}.$$ □

The above lemma serves mainly as preparation for some more involved Morita equivalences:

Proposition 3.9. *The following algebras are Morita equivalent to* $(\mathcal{H}(G)^\mathfrak{s})^{X^G(\mathfrak{s})}$*:*

(a) $(\mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp)^\mathfrak{s})^{X^G(\mathfrak{s})} \cong (\mathcal{H}(M)^{\mathfrak{s}_M})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp$;
(b) $\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes \mathrm{Stab}(\mathfrak{s}, P \cap M)$;
(c) $e_M^\mathfrak{s} \mathcal{H}(M) e_M^\mathfrak{s} \rtimes \mathrm{Stab}(\mathfrak{s}, P \cap M)$;
(d) $(e_M^\mathfrak{s} \mathcal{H}(M) e_M^\mathfrak{s})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp$.

Proof. For the definitions of the finite groups see page 12.
(a) The isomorphism between the two algebras is Lemma 3.7. Let U be the unipotent radical of PM. As discussed in [19], there are natural isomorphisms

$$\begin{aligned} I_{PM}^G(V) &\cong C_c^\infty(G/U) \otimes_{\mathcal{H}(M)} V & \qquad & V \in \mathrm{Rep}(M),\\ r_{PM}^G(W) &\cong C_c^\infty(U\backslash G) \otimes_{\mathcal{H}(G)} W & & W \in \mathrm{Rep}(G). \end{aligned}$$

For $V \in \mathrm{Rep}^{\mathfrak{s}_M}(M)$ we may just as well take the bimodule $C_c^\infty(G/U)\mathcal{H}(M)^{\mathfrak{s}_M}$, and to get (75) the bimodule $\mathcal{H}(M)^{\mathfrak{s}_M} C_c^\infty(U\backslash G)$ is suitable. But if we want to obtain modules over $\mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}}$, it is better to use the bimodules

$$\begin{aligned} C_c^\infty(G/U)^{\mathfrak{s}} &:= \bigoplus\nolimits_{w \in \mathfrak{R}_{\mathfrak{s}}^\sharp} C_c^\infty(G/U)\mathcal{H}(M)^{w(\mathfrak{s}_M)} = C_c^\infty(G/U)\mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}},\\ C_c^\infty(U\backslash G)^{\mathfrak{s}} &:= \bigoplus\nolimits_{w \in \mathfrak{R}_{\mathfrak{s}}^\sharp} \mathcal{H}(M)^{w(\mathfrak{s}_M)} C_c^\infty(U\backslash G) = \mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}} C_c^\infty(U\backslash G). \end{aligned}$$

Indeed, we can rewrite (74) as

$$\begin{aligned} I_{PM}^G(V) &\cong C_c^\infty(G/U)\mathcal{H}(M)^{\mathfrak{s}_M} \otimes_{\mathcal{H}(M)^{\mathfrak{s}_M}} V\\ &= C_c^\infty(G/U)^{\mathfrak{s}} \otimes_{\mathcal{H}(M)^{\mathfrak{s}_M}} V\\ &\cong C_c^\infty(G/U)^{\mathfrak{s}} \otimes_{\mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}}} \mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}} \otimes_{\mathcal{H}(M)^{\mathfrak{s}_M}} V\\ &\cong C_c^\infty(G/U)^{\mathfrak{s}} \otimes_{\mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}}} \mathrm{Ind}_{\mathcal{H}(M)^{\mathfrak{s}_M}}^{\mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}}} (V). \end{aligned}$$

Similarly (75) translates to

$$\begin{aligned} \mathrm{Ind}_M^{M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp} \circ \mathrm{pr}_{\mathfrak{s}_M} \circ r_{PM}^G &= \mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}} \otimes_{\mathcal{H}(M)^{\mathfrak{s}_M}} \mathcal{H}(M)^{\mathfrak{s}_M} C_c^\infty(U\backslash G) \otimes_{\mathcal{H}(G)^{\mathfrak{s}}} W\\ &= \mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}} C_c^\infty(U\backslash G) \otimes_{\mathcal{H}(G)^{\mathfrak{s}}} W\\ &= C_c^\infty(U\backslash G)^{\mathfrak{s}} \otimes_{\mathcal{H}(G)^{\mathfrak{s}}} W. \end{aligned}$$

These calculations entail that the bimodules $C_c^\infty(G/U)^{\mathfrak{s}}$ and $C_c^\infty(U\backslash G)^{\mathfrak{s}}$ implement

$$\mathcal{H}(G)^{\mathfrak{s}} \sim_M \mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}}. \tag{78}$$

These bimodules are naturally endowed with an action of $X^G(\mathfrak{s})$, by pointwise multiplication of functions $G \to \mathbb{C}$. This action is obviously compatible with the group actions on $\mathcal{H}(G)^{\mathfrak{s}}$ and $\mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}}$, in the sense that

$$\gamma \cdot (f_1 f_2) = (\gamma \cdot f_1)(\gamma \cdot f_2) \quad \text{and} \quad \gamma \cdot (f_2 f_3) = (\gamma \cdot f_2)(\gamma \cdot f_3)$$

for $\gamma \in X^G(\mathfrak{s}), f_1 \in \mathcal{H}(G)^{\mathfrak{s}}, f_2 \in C_c^\infty(G/U)^{\mathfrak{s}}, f_3 \in \mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}}$. Hence we may restrict (78) to functions supported on $\bigcap_{\gamma \in X^G(\mathfrak{s})} \ker \gamma$, and we obtain

$$\begin{aligned} (C_c^\infty(G/U)^{\mathfrak{s}})^{X^G(\mathfrak{s})} \otimes_{(\mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}})^{X^G(\mathfrak{s})}} (C_c^\infty(U\backslash G)^{\mathfrak{s}})^{X^G(\mathfrak{s})} &\cong (\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})},\\ (C_c^\infty(U\backslash G)^{\mathfrak{s}})^{X^G(\mathfrak{s})} \otimes_{(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})}} (C_c^\infty(G/U)^{\mathfrak{s}})^{X^G(\mathfrak{s})} &\cong (\mathcal{H}(M \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}})^{X^G(\mathfrak{s})}. \end{aligned} \tag{79}$$

(b) Consider the idempotent

$$p = |X^L(\mathfrak{s})|^{-1} \sum\nolimits_{\gamma \in X^L(\mathfrak{s})} \gamma \;\in\; \mathbb{C}[X^L(\mathfrak{s})].$$

It is easy to see that the map

$$(\mathcal{H}(M)^{\mathfrak{s}_M})^{X^L(\mathfrak{s})} \to p(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}))p \;:\; a \mapsto pap$$

is an isomorphism of algebras [26, Lemma A.2]. Therefore $(\mathcal{H}(M)^{\mathfrak{s}_M})^{X^L(\mathfrak{s})}$ is Morita equivalent with $(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}))p(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}))$, via the bimodules $p(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}))$ and $(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}))p$. Suppose that

$$(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}))p(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s})) \subsetneq \mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}).$$

Then the quotient algebra

$$\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}) / (\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}))p(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}))$$

is nonzero. This algebra is a direct limit of unital algebras, so it has an irreducible module V on which it does not act as zero. We can regard V as an irreducible $\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s})$-module with $pV = 0$. For any $\pi \in \mathrm{Irr}^{\mathfrak{s}_M}(M)$ we have $X^M(\pi) \subset X^L(\omega)$ since $W(M, L) = W_{\mathfrak{s}}$ and by Lemma 2.4.d. By (17) and (19) the decomposition of V_π over $M^\sharp Z(G)$ is governed by $\mathbb{C}[X^M(\pi), \kappa_\omega]$. Now Clifford theory (see for example [27, Appendix A]) says that, for any $\rho \in \mathrm{Irr}(\mathbb{C}[X^M(\pi), \kappa_\omega])$,

$$\mathrm{Ind}_{\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^M(\pi)}^{\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s})} (V_\pi \otimes \rho^\vee)$$

is an irreducible module over $\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s})$. Moreover every irreducible $\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s})$-module is of this form, so we may take it as V. But by (17)

$$\rho \text{ appears in } V_\pi. \tag{80}$$

Hence $V_\pi \otimes \rho^\vee$ has nonzero $X^L(\omega)$-invariant vectors and $pV \neq 0$. This contradiction shows that

$$(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}))p(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s})) = \mathcal{H}(M)^{\mathfrak{s}_M} \rtimes X^L(\mathfrak{s}). \tag{81}$$

Recall from Lemma 3.7 that

$$(\mathcal{H}(M^{\mathfrak{s}_M} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp)^{\mathfrak{s}})^{X^G(\mathfrak{s})} \cong (\mathcal{H}(M)^{\mathfrak{s}_M})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp = p(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes \mathrm{Stab}(\mathfrak{s}, P \cap M))p.$$

The bimodules $p(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes \mathrm{Stab}(\mathfrak{s}, P \cap M))$ and $(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes \mathrm{Stab}(\mathfrak{s}, P \cap M))p$ make it Morita equivalent with

$$(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes \mathrm{Stab}(\mathfrak{s}, P \cap M))p(\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes \mathrm{Stab}(\mathfrak{s}, P \cap M)),$$

which by (81) equals $\mathcal{H}(M)^{\mathfrak{s}_M} \rtimes \mathrm{Stab}(\mathfrak{s}, P \cap M)$.
(c) This follows from (77) and Lemma 3.4, upon applying $\rtimes \mathrm{Stab}(\mathfrak{s}, P \cap M)$ everywhere.
(d) First we want to show that

$$(e_M^{\mathfrak{s}} \mathcal{H}(M) e_M^{\mathfrak{s}})^{X^L(\mathfrak{s})} \sim_M e_M^{\mathfrak{s}} \mathcal{H}(M) e_M^{\mathfrak{s}} \rtimes X^L(\mathfrak{s}). \tag{82}$$

To this end we use the same argument as in part (b), only with $e_M^{\mathfrak{s}} \mathcal{H}(M) e_M^{\mathfrak{s}}$ instead of $\mathcal{H}(M)^{\mathfrak{s}_M}$. Everything goes fine until (80). The corresponding statement in the present setting would be that every irreducible module of $\mathbb{C}[X^M(\pi), \kappa_\pi]$ appears in $e_M^{\mathfrak{s}} V_\pi$. By 17 this is equivalent to saying that $e_M^{\mathfrak{s}} V_\pi$ intersects every $M^\sharp$-isotypical component of V_π nontrivially, which is exactly Lemma 3.3.d. Therefore this version of (80) does hold. The analogue of (81) is now valid, and establishes (82). The bimodules for this Morita equivalence are

$$p(e_M^{\mathfrak{s}} \mathcal{H}(M) e_M^{\mathfrak{s}} \rtimes X^L(\mathfrak{s})) \text{ and } (e_M^{\mathfrak{s}} \mathcal{H}(M) e_M^{\mathfrak{s}} \rtimes X^L(\mathfrak{s}))p.$$

The same argument as after (81) makes clear how this implies the required Morita equivalence

$$e_M^{\mathfrak{s}} \mathcal{H}(M) e_M^{\mathfrak{s}} \rtimes \mathrm{Stab}(\mathfrak{s}, P \cap M) \sim_M (e_M^{\mathfrak{s}} \mathcal{H}(M) e_M^{\mathfrak{s}})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp. \qquad \square$$

From the above proof one can extract bimodules for the Morita equivalence

$$(e_M^{\mathfrak s}\mathcal H(M)e_M^{\mathfrak s})^{X^L(\mathfrak s)} \rtimes \mathfrak R_{\mathfrak s}^\sharp \sim_M (\mathcal H(M)^{\mathfrak s_M})^{X^L(\mathfrak s)} \rtimes \mathfrak R_{\mathfrak s}^\sharp, \tag{83}$$

namely

$$(\mathcal H(M)^{\mathfrak s_M} e_M^{\mathfrak s})^{X^L(\mathfrak s)} \rtimes \mathfrak R_{\mathfrak s}^\sharp \quad\text{and}\quad (e_M^{\mathfrak s}\mathcal H(M)^{\mathfrak s_M})^{X^L(\mathfrak s)} \rtimes \mathfrak R_{\mathfrak s}^\sharp. \tag{84}$$

It seems complicated to prove directly that these are Morita bimodules, without the detour via parts (b) and (c) of Proposition 3.9.

3.3. **Passage to the derived group.**

We study how Hecke algebras for $G^\sharp$ and for $G^\sharp Z(G)$ can be replaced by Morita equivalent algebras built from $\mathcal H(G)$. In the last results of this subsection we will show that a Morita equivalent subalgebra $\mathcal H(G^\sharp)^{\mathfrak s}$ is isomorphic to subalgebras of $\mathcal H(G)^{\mathfrak s}$ and of $\mathcal H(M \rtimes \mathfrak R_{\mathfrak s}^\sharp)^{\mathfrak s}$.

Lemma 3.10. *The algebra $\mathcal H(G^\sharp Z(G))^{\mathfrak s}$ is Morita equivalent with $(\mathcal H(G)^{\mathfrak s})^{X^G(\mathfrak s)}$.*

Proof. Let $\mathfrak o_D$ be the ring of integers of D. Let C_l be the l-th congruence subgroup of $\mathrm{GL}_m(\mathfrak o_D)$, and put $C_l' = C_l \cap G^\sharp Z(G)$. The group $G^\sharp Z(G) C_l$ is of finite index in G, because $\mathrm{Nrd}(G^\sharp Z(G) C_l)$ contains both $F^{\times md}$ and an open neighborhood $\mathrm{Nrd}(C_l)$ of $1 \in F^\times$. By Lemma 2.4.c we can choose l so large, that every element of $X^G(\mathfrak s)$ is trivial on C_l and that all representations in $\mathrm{Rep}^{\mathfrak s}(G)$ have nonzero C_l-invariant vectors. Let $e_{C_l} \in \mathcal H(C_l)$ be the central idempotent associated to the trivial representation of C_l. It is known from [4, §3] that (C_l,triv) is a type, so the algebra

$$\mathcal H(G, C_l)^{\mathfrak s} = e_{C_l}\mathcal H(G)^{\mathfrak s} e_{C_l}$$

of C_l-biinvariant functions in $\mathcal H(G)^{\mathfrak s}$ is Morita equivalent with $\mathcal H(G)^{\mathfrak s}$. The Morita bimodules are $e_{C_l}\mathcal H(G)^{\mathfrak s}$ and $\mathcal H(G)^{\mathfrak s} e_{C_l}$. Since $X^G(\mathfrak s)$ fixes e_{C_l}, these bimodules carry an $X^G(\mathfrak s)$-action, which clearly is compatible with the actions on $\mathcal H(G)^{\mathfrak s}$ and $\mathcal H(G, C_l)^{\mathfrak s}$. We can restrict the equations which make them Morita bimodules to the subspaces of functions $G \to \mathbb C$ supported on $\bigcap_{\gamma \in X^G(\mathfrak s)} \ker \gamma$. We find that the bimodules $(e_{C_l}\mathcal H(G)^{\mathfrak s})^{X^G(\mathfrak s)}$ and $(\mathcal H(G)^{\mathfrak s} e_{C_l})^{X^G(\mathfrak s)}$ provide a Morita equivalence between

$$(\mathcal H(G)^{\mathfrak s})^{X^G(\mathfrak s)} \quad\text{and}\quad (\mathcal H(G, C_l)^{\mathfrak s})^{X^G(\mathfrak s)}. \tag{85}$$

We saw in (44) that $\mathrm{Irr}^{\mathfrak s}(G^\sharp Z(G))$ is a union of Bernstein components, in fact a finite union by Lemma 2.2. Hence we may assume that every representation in $\mathrm{Irr}^{\mathfrak s}(G^\sharp Z(G))$ contains nonzero C_l'-invariant vectors. As (C_l',triv) is a type, that suffices for a Morita equivalence between

$$\mathcal H(G^\sharp Z(G))^{\mathfrak s} \quad\text{and}\quad \mathcal H(G^\sharp Z(G), C_l')^{\mathfrak s}. \tag{86}$$

We may assume that the Haar measures on G and on $G^\sharp Z(G)$ are chosen such that C_l and C_l' get the same volume. Then the natural injection

$$C_l' \setminus G^\sharp Z(G) / C_l' \to C_l \setminus G / C_l$$

provides an injective algebra homomorphism

$$\mathcal H(G^\sharp Z(G), C_l') \to \mathcal H(G, C_l), \tag{87}$$

whose image consists of the $\mathrm{Irr}(G/G^\sharp Z(G)C_l)$-invariants in $\mathcal{H}(G,C_l)$. Let $\mathfrak{B}(G)_l$ be the set of inertial equivalence classes for G corresponding to the category of G-representations that are generated by their C_l-invariant vectors. The finite group $\mathrm{Irr}(G/G^\sharp Z(G)C_l)$ acts on it, and we denote the set of orbits by $\mathfrak{B}(G)_l/\sim$. Now we can write

$$\bigoplus\nolimits_{\mathfrak{s}\in\mathfrak{B}(G)_l/\sim} \mathcal{H}(G^\sharp Z(G),C'_l)^{\mathfrak{s}} = \mathcal{H}(G^\sharp Z(G),C'_l) \cong \\ \mathcal{H}(G,C_l)^{\mathrm{Irr}(G/G^\sharp Z(G)C_l)} = \bigoplus\nolimits_{\mathfrak{s}\in\mathfrak{B}(G)_l/\sim} (\mathcal{H}(G,C_l)^{\mathfrak{s}})^{X^G(\mathfrak{s})}.$$

By considering the factors corresponding to one $\mathfrak{s}$ on both sides we obtain an isomorphism

$$\mathcal{H}(G^\sharp Z(G),C'_l)^{\mathfrak{s}} \cong (\mathcal{H}(G,C_l)^{\mathfrak{s}})^{X^G(\mathfrak{s})}.$$

To conclude, we combine this with (85) and (86). $\square$

The Morita equivalences in parts (a) and (d) of Proposition 3.9, for algebras associated to $G^\sharp Z(G)$, have analogues for $G^\sharp$. For parts (b) and (c), which involve crossed products by $\mathrm{Stab}(\mathfrak{s},P\cap M)$, this is not clear.

Lemma 3.11. *The algebra $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ is Morita equivalent with*

$$(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})X_{\mathrm{nr}}(G)} \quad \text{and with} \quad (\mathcal{H}(M)^{\mathfrak{s}})^{X^L(\mathfrak{s})X_{\mathrm{nr}}(G)} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}.$$

Proof. By (42) we have

$$\mathcal{H}(G^\sharp)^{\mathfrak{s}} \cong (\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}})^{X_{\mathrm{nr}}(Z(G))}. \tag{88}$$

As $X_{\mathrm{nr}}(G/Z(G)) \subset X^L(\mathfrak{s}) \subset X^G(\mathfrak{s})$, every $\chi \in X_{\mathrm{nr}}(Z(G))$ extends in a unique way to a character of $\mathcal{H}(G)^{X^G(\mathfrak{s})}$. In other words, we can identify

$$X_{\mathrm{nr}}(Z(G)) = X_{\mathrm{nr}}(G) \text{ in } \mathrm{Irr}(G/G^\sharp)/X^G(\mathfrak{s}). \tag{89}$$

All the bimodules involved in (85) and (86) carry a compatible action of (89). We can restrict the proofs of these Morita equivalences to smooth functions supported on $G^1 = \bigcap_{\chi\in X_{\mathrm{nr}}(G)} \ker\chi$. That leads to a Morita equivalence

$$(\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}})^{X_{\mathrm{nr}}(Z(G))} \sim_M (\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})X_{\mathrm{nr}}(G)}.$$

Let us take another look at the Morita equivalence (78), between $\mathcal{H}(G)^{\mathfrak{s}}$ and $\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}}$. The argument between (78) and (79) also works with $X^G(\mathfrak{s})X_{\mathrm{nr}}(G)$ instead of $X^G(\mathfrak{s})$, and provides a Morita equivalence

$$(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})X_{\mathrm{nr}}(G)} \sim_M (\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}})^{X^G(\mathfrak{s})X_{\mathrm{nr}}(G)}. \tag{90}$$

The isomorphism in Lemma 3.7 is $X_{\mathrm{nr}}(G)$-equivariant, so it restricts to

$$(\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}})^{X^G(\mathfrak{s})X_{\mathrm{nr}}(G)} \cong (\mathcal{H}(M)^{\mathfrak{s}_M})^{X^L(\mathfrak{s})X_{\mathrm{nr}}(G)} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}. \qquad \square$$

We would like to formulate a version Lemma 3.11 with idempotents in $\mathcal{H}(G)$ and $\mathcal{H}(M)$. Consider the types $(K_G, \lambda_G \otimes \gamma)$ for $\gamma \in X^G(\mathfrak{s})$.

Lemma 3.12. *Let $\gamma,\gamma' \in X^G(\mathfrak{s})$.*

(a) The K_G-representations $\lambda_G \otimes \gamma$ and $\lambda_G \otimes \gamma'$ are equivalent if and only if $\gamma^{-1}\gamma' \in X^L(\mathfrak{s},\lambda)$.

(b) For any $a,a' \in M$ the idempotents $a e_{\lambda_G\otimes\gamma} a^{-1}$ and $a' e_{\lambda_G\otimes\gamma'} (a')^{-1}$ are orthogonal if $\gamma^{-1}\gamma' \in X^G(\mathfrak{s}) \setminus X^L(\mathfrak{s})$.

Proof. (a) Suppose first that $\gamma^{-1}\gamma' \in X^G(\mathfrak{s}) \setminus X^L(\mathfrak{s})$. Then (K,λ) and $(K,\lambda\otimes\gamma)$ are types for different Bernstein components of M, so λ and $\lambda\otimes\gamma$ are not equivalent. As both λ_G and γ are trivial on $K_G \cap U$ and on $K_G \cap \overline{U}$, this implies that λ_G and $\lambda_G \otimes \gamma$ are not equivalent either.

Now suppose that $\gamma \in X^L(\mathfrak{s})$. By the definition of $\mathrm{Stab}(\mathfrak{s},\lambda)$, the K-representations λ and $\lambda\otimes\gamma$ are equivalent if and only if $\gamma \in \mathrm{Stab}(\mathfrak{s},\lambda)$. By the same argument as above, this statement can be lifted to λ_G and $\lambda_G \otimes \gamma$.
(b) Consider the idempotents $a e_{\lambda\otimes\gamma} a^{-1}$ and $a' e_{\lambda\otimes\gamma'} (a')^{-1}$ in $\mathcal{H}(M)$. They belong to the subalgebras $\mathcal{H}(M)^{\mathfrak{s}_M\otimes\gamma}$ and $\mathcal{H}(M)^{\mathfrak{s}_M\otimes\gamma'}$, respectively. Since $X^L(\mathfrak{s}) = X^M(\mathfrak{s})$ and $\gamma X^L(\mathfrak{s}) \neq \gamma' X^L(\mathfrak{s})$, these are two orthogonal ideals of $\mathcal{H}(L)$. In particular the two above idempotents are orthogonal.

Let $\langle K_G \cap U\rangle$ denote the idempotent, in the multiplier algebra of $\mathcal{H}(G)$, which corresponds to averaging over the group $K_G \cap U$. Then

$$
\begin{aligned}
a e_{\lambda_G\otimes\gamma} a^{-1} &= a \langle K_G \cap U\rangle \langle K_G \cap \overline{U}\rangle e_{\lambda\otimes\gamma} a^{-1} \\
&= \langle a (K_G \cap U) a^{-1}\rangle \langle a (K_G \cap \overline{U}) a^{-1}\rangle a e_{\lambda\otimes\gamma} a^{-1}
\end{aligned}
$$

Similarly

$$
a' e_{\lambda_G\otimes\gamma} (a')^{-1} = a e_{\lambda\otimes\gamma'} a^{-1} \langle a (K_G \cap U) a^{-1}\rangle \langle a (K_G \cap \overline{U}) a^{-1}\rangle
$$

Now we see from the earlier orthogonality result that

$$
a e_{\lambda_G\otimes\gamma} a^{-1} a' e_{\lambda_G\otimes\gamma'} (a')^{-1} = 0. \qquad \square
$$

Generalizing (61) we define

$$
X^G(\mathfrak{s}/\lambda) = X^G(\mathfrak{s}) / (X^L(\mathfrak{s}) \cap \mathrm{Stab}(\mathfrak{s},\lambda)).
$$

By Lemma 3.12 the elements

$$
\begin{aligned}
e_{\mu_G} &:= \sum\nolimits_{\gamma \in X^G(\mathfrak{s}/\lambda)} e_{\lambda_G\otimes\gamma} \in \mathcal{H}(G), \\
e^\sharp_{\lambda_G} &:= \sum\nolimits_{a \in [L/H_\lambda]} a e_{\mu_G} a^{-1} \in \mathcal{H}(G), \qquad (91)\\
e^\sharp_{\lambda} &:= \sum\nolimits_{a \in [L/H_\lambda]} \sum\nolimits_{\gamma \in X^G(\mathfrak{s}/\lambda)} a e_{\lambda\otimes\gamma} a^{-1} \in \mathcal{H}(M)
\end{aligned}
$$

are idempotent. We will show that the latter two idempotents see precisely the categories of representations of G and $G^\sharp$ (resp. M and $M^\sharp$) associated to $\mathfrak{s}$. However, in general they do not come from a type, for the elements $a \in [L/H_\lambda]$ and $c_\gamma \in L$ need not lie in a compact subgroup of G.

Lemma 3.13. *Let $(\pi, V_\pi) \in \mathrm{Irr}^{\mathfrak{s}}(G)$. Then $e^\sharp_{\lambda_G} V_\pi$ intersects every $G^\sharp$-isotypical component of V_π nontrivially.*

Proof. The twisted group algebra $\mathbb{C}[X^G(\pi),\kappa_\pi]$ acts on V_π via intertwining operators. In view of (17), we have to show that $e^\sharp_{\lambda_G} V_\pi$ intersects the ρ-isotypical part of V_π nontrivially, for every $\rho \in \mathrm{Irr}(\mathbb{C}[X^G(\pi),\kappa_\pi])$.

Choose $\chi \in X_{\mathrm{nr}}(L)$ such that π is a subquotient of $I_P^G(\omega\otimes\chi)$. Then

$$
X^G(\pi) \cap X^L(\mathfrak{s},\lambda) \subset X^L(\omega\otimes\chi).
$$

As observed in the proof of Lemma 3.3.d, every irreducible representation of $\mathbb{C}[X^G(\pi) \cap X^L(\mathfrak{s},\lambda), \kappa_{\omega\otimes\chi}]$ appears in $e^\sharp_{\lambda_G} V_\pi$. The idempotents

$$
\{ a e_{\lambda_G\otimes\gamma} a^{-1} : a \in [L/H_\lambda], \gamma \in X^G(\mathfrak{s}) \} \qquad (92)
$$

are invariant under $X^L(\mathfrak{s},\lambda)$ because e_{λ_G} is, and they are mutually orthogonal by Lemma 3.12. As these idempotents sum to $e^\sharp_{\lambda_G}$, it follows that every every irreducible representation of $\mathbb{C}[X^G(\pi)\cap X^L(\mathfrak{s},\lambda),\kappa_{\omega\otimes\chi}]$ already appears in one subspace $ae_{\lambda_G\otimes\gamma}a^{-1}V_\pi$. The quotient group $X^G(\pi)/X^G(\pi)\cap X^L(\mathfrak{s},\lambda)$ permutes the set of idempotents (92) faithfully. With Frobenius reciprocity we conclude that every irreducible representation of $\mathbb{C}[X^G(\pi),\kappa_\pi]$ appears in $e^\sharp_{\lambda_G}V_\pi$. $\square$

Lemma 3.14. *(a) The algebras $e^\sharp_{\lambda_G}\mathcal{H}(G)^{X^G(\mathfrak{s})}e^\sharp_{\lambda_G} = (e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G})^{X^G(\mathfrak{s})}$ and $(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})}$ are Morita equivalent.*
(b) $(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})X_{\mathrm{nr}}(G)} \sim_M (e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G})^{X^G(\mathfrak{s})X_{\mathrm{nr}}(G)}$.

Proof. (a) Because all the types $(K_G,\lambda_G\otimes\gamma)$ are for the same Bernstein component $\mathfrak{s}$, the idempotent $e^\sharp_{\lambda_G}$ sees precisely the category of representations $\mathrm{Rep}^{\mathfrak{s}}(G)$. Therefore the bimodules $e^\sharp_{\lambda_G}\mathcal{H}(G)$ and $\mathcal{H}(G)e^\sharp_{\lambda_G}$ implement a Morita equivalence

$$\mathcal{H}(G)^{\mathfrak{s}} \sim_M e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G}. \tag{93}$$

The same reasoning as in parts (b) and (c) of Proposition 3.9 establishes Morita equivalences

$$(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})} \sim_M \mathcal{H}(G)^{\mathfrak{s}} \rtimes X^G(\mathfrak{s}) \sim_M (e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G}) \rtimes X^G(\mathfrak{s}).$$

To get from the right hand side to $(e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G})^{X^G(\mathfrak{s})}$ we follow the proof of Proposition 3.9.d. This is justified by Lemma 3.13.
(b) The above argument also shows that the Morita equivalence of part (a) is implemented by the bimodules

$$e^\sharp_{\lambda_G}\mathcal{H}(G)^{X^G(\mathfrak{s})} \quad \text{and} \quad \mathcal{H}(G)^{X^G(\mathfrak{s})}e^\sharp_{\lambda_G}. \tag{94}$$

These bimodules are endowed with actions of $X_{\mathrm{nr}}(G)$. Taking invariants under these group actions amounts to considering only functions supported on G^1. We note that $e^\sharp_{\lambda_G}$ is supported on G^1 and that this is a normal subgroup of G. Therefore the equations that make (94) Morita bimodules restrict to analogous equations for functions supported on G^1, which provides the desired Morita equivalence. $\square$

Recall the idempotents $e_\mu, e^{\mathfrak{s}}_M$ from (69) and $e^\sharp_{\lambda_G}, e^\sharp_\lambda, e_{\mu_G}$ from (91).

Proposition 3.15. *There are algebra isomorphisms*
(a) $e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G} \cong e^\sharp_\lambda\mathcal{H}(M\rtimes\mathfrak{R}^\sharp_{\mathfrak{s}})e^\sharp_\lambda$,
(b) $(e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G})^{X^G(\mathfrak{s})} \cong (e^\sharp_\lambda\mathcal{H}(M\rtimes\mathfrak{R}^\sharp_{\mathfrak{s}})e^\sharp_\lambda)^{X^G(\mathfrak{s})} \cong (e^{\mathfrak{s}}_M\mathcal{H}(M)e^{\mathfrak{s}}_M)^{X^L(\mathfrak{s})}\rtimes\mathfrak{R}^\sharp_{\mathfrak{s}}$,
(c) between the three algebras of $X_{\mathrm{nr}}(G)$-invariants in (b).

Moreover the isomorphisms in (b) and (c) can be chosen such that, for every $a_1,a_2\in[L/H_\lambda]$, they restrict to linear bijections

$$(a_1e_{\mu_G}\mathcal{H}(G)e_{\mu_G}a_2^{-1})^{X^G(\mathfrak{s})} \longleftrightarrow (a_1e_\mu\mathcal{H}(M)e_\mu a_2^{-1})^{X^L(\mathfrak{s})}\rtimes\mathfrak{R}^\sharp_{\mathfrak{s}},$$
$$(a_1e_{\mu_G}\mathcal{H}(G)e_{\mu_G}a_2^{-1})^{X^G(\mathfrak{s})X_{\mathrm{nr}}(G)} \longleftrightarrow (a_1e_\mu\mathcal{H}(M)e_\mu a_2^{-1})^{X^L(\mathfrak{s})X_{\mathrm{nr}}(G)}\rtimes\mathfrak{R}^\sharp_{\mathfrak{s}}.$$

Proof. For any $\gamma \in X^L(\mathfrak{s})$ and $w \in \mathfrak{R}_\mathfrak{s}^\sharp$ there exists a $\gamma' \in X^G(\mathfrak{s})$ such that $w(\lambda \otimes \gamma) \cong \lambda \otimes \gamma'$ as representations of K. Hence

$$e_\lambda^\sharp \mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp) e_\lambda^\sharp \cap \mathcal{H}(M)^{w(\mathfrak{s}_M)} = w e_M^\mathfrak{s} w^{-1} \mathcal{H}(M) w e_M^\mathfrak{s} w^{-1},$$

$$e_\lambda^\sharp \mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp) e_\lambda^\sharp = \big(\bigoplus\nolimits_{w \in \mathfrak{R}_\mathfrak{s}^\sharp} w e_M^\mathfrak{s} \mathcal{H}(M) e_M^\mathfrak{s} w^{-1} \big) \rtimes \mathfrak{R}_\mathfrak{s}^\sharp. \tag{95}$$

We note that the right hand side of (95) is isomorphic to

$$e_M^\mathfrak{s} \mathcal{H}(M) e_M^\mathfrak{s} \otimes \mathrm{End}_\mathbb{C}(\mathbb{C}\mathfrak{R}_\mathfrak{s}^\sharp) \tag{96}$$

The equality (95) also shows that

$$(e_\lambda^\sharp \mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp) e_\lambda^\sharp)^{X^G(\mathfrak{s})} = \big(\bigoplus\nolimits_{w \in \mathfrak{R}_\mathfrak{s}^\sharp} (w e_M^\mathfrak{s} \mathcal{H}(M) e_M^\mathfrak{s} w^{-1})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp \big)^{X^G(\mathfrak{s})/X^L(\mathfrak{s})}.$$

We can apply the argument from the proof of Lemma 3.7 to the right hand side, which gives a canonical isomorphism

$$(e_\lambda^\sharp \mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp) e_\lambda^\sharp)^{X^G(\mathfrak{s})} \cong (e_M^\mathfrak{s} \mathcal{H}(M) e_M^\mathfrak{s})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp. \tag{97}$$

Notice that for $a \in L/H_\lambda$ the idempotents $a e_\mu a^{-1}$ and $a \sum_{\gamma \in X^G(\mathfrak{s}/\lambda)} e_{\lambda \otimes \gamma} a^{-1}$ are invariant under $\mathrm{Stab}(\mathfrak{s}, P \cap M)$ and $X^G(\mathfrak{s})$, respectively. Hence we can write
(98)

$$(e_{\lambda_G}^\sharp \mathcal{H}(G) e_{\lambda_G}^\sharp)^{X^G(\mathfrak{s})} = \bigoplus\nolimits_{a_1, a_2 \in [L/H_\lambda]} (a_1 e_{\mu_G} \mathcal{H}(G) e_{\mu_G} a_2^{-1})^{X^G(\mathfrak{s})},$$

$$(e_M^\mathfrak{s} \mathcal{H}(M) e_M^\mathfrak{s})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp = \bigoplus\nolimits_{a_1, a_2 \in [L/H_\lambda]} (a_1 e^\mu \mathcal{H}(M) e_\mu a_2^{-1})^{X^L(\mathfrak{s}) X_{\mathrm{nr}}(G)} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp.$$

It is clear from the proof of Lemma 3.7 that the isomorphism (97) respects these decompositions. Moreover (97) is equivariant with respect to the actions of $X_{\mathrm{nr}}(G)$, so it restricts to

$$(e_\lambda^\sharp \mathcal{H}(M \rtimes \mathfrak{R}_\mathfrak{s}^\sharp) e_\lambda^\sharp)^{X^G(\mathfrak{s}) X_{\mathrm{nr}}(G)} \cong (e_M^\mathfrak{s} \mathcal{H}(M) e_M^\mathfrak{s})^{X^L(\mathfrak{s}) X_{\mathrm{nr}}(G} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp.$$

We have proved the second isomorphism of part (b) and of part (c).

For every $\gamma \in X^G(\mathfrak{s}/\lambda)$ and $a \in [L/H_\lambda]$ one has

$$a e_{\lambda_G \otimes \gamma} a^{-1} \mathcal{H}(G) a e_{\lambda_G \otimes \gamma} a^{-1} = a \alpha_\gamma^{-1} (e_{\lambda_G \otimes \gamma} \mathcal{H}(G) e_{\lambda_G \otimes \gamma}) a^{-1} \cong e_{\lambda_G \otimes \gamma} \mathcal{H}(G) e_{\lambda_G \otimes \gamma}.$$

By Lemma 3.12 these are mutually orthogonal subalgebras of $e_{\lambda_G}^\sharp \mathcal{H}(G) e_{\lambda_G}^\sharp$. The inclusion

$$a e_{\lambda_G \otimes \gamma} a^{-1} \mathcal{H}(G) a e_{\lambda_G \otimes \gamma} a^{-1} \to e_{\lambda_G}^\sharp \mathcal{H}(G) e_{\lambda_G}^\sharp$$

is a Morita equivalence and for all $V \in \mathrm{Rep}(G)^\mathfrak{s}$:

$$e_{\lambda_G}^\sharp V = \bigoplus_{a \in [L/H_\lambda]} \bigoplus_{\gamma \in X^G(\mathfrak{s}/\lambda)} a e_{\lambda_G \otimes \gamma} a^{-1} V.$$

It follows that the $a e_{\lambda_G \otimes \gamma} a^{-1}$ form the idempotent matrix units in some subalgebra $M_n(\mathbb{C}) \subset e_{\lambda_G}^\sharp \mathcal{H}(G) e_{\lambda_G}^\sharp$, and that

$$e_{\lambda_G}^\sharp \mathcal{H}(G) e_{\lambda_G}^\sharp \cong e_{\lambda_G} \mathcal{H}(G) e_{\lambda_G} \otimes M_n(\mathbb{C})$$

where $n = |L/H_\lambda| \, |X^G(\mathfrak{s}/\lambda)|$.

The same argument shows that

$$e_M^\mathfrak{s} \mathcal{H}(M) e_M^\mathfrak{s} \cong e_\lambda \mathcal{H}(M) e_\lambda \otimes M_{n'}(\mathbb{C}),$$

where $n' = |L/H_\lambda|\,|X^L(\mathfrak{s}/\lambda)|$. Since (K_G, λ_G) is a cover of (K, λ),

$$e_\lambda \mathcal{H}(M) e_\lambda \cong e_{\lambda_G} \mathcal{H}(G) e_{\lambda_G}.$$

By Lemma 3.12

$$n'\,|\mathfrak{R}^\sharp_{\mathfrak{s}}| = |L/H_\lambda|\,|X^L(\mathfrak{s}/\lambda)|\,|\mathfrak{R}^\sharp_{\mathfrak{s}}| = |L/H_\lambda|\,|X^G(\mathfrak{s}/\lambda)| = n.$$

With (96) we deduce that

$$e^\sharp_\lambda \mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}) e^\sharp_\lambda \cong e_{\lambda_G} \mathcal{H}(G) e_{\lambda_G} \otimes M_n(\mathbb{C}) \cong e^\sharp_{\lambda_G} \mathcal{H}(G) e^\sharp_{\lambda_G}, \tag{99}$$

proving part (a). It entails from (78) that

$$e^\sharp_{\lambda_G} C_c^\infty(G/U) e^\sharp_\lambda \quad \text{and} \quad e^\sharp_\lambda C_c^\infty(U\backslash G) e^\sharp_{\lambda_G} \tag{100}$$

are bimodules for a Morita equivalence

$$e^\sharp_\lambda \mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}) e^\sharp_\lambda \sim_M e^\sharp_{\lambda_G} \mathcal{H}(G) e^\sharp_{\lambda_G}. \tag{101}$$

But by (99) these algebras are isomorphic, so the bimodules are free of rank 1 over both algebras.

Similarly, it follows from (79) that

$$(C_c^\infty(G/U)^{\mathfrak{s}})^{X^G(\mathfrak{s})} \quad \text{and} \quad (C_c^\infty(U\backslash G)^{\mathfrak{s}})^{X^G(\mathfrak{s})} \tag{102}$$

are bimodules for a Morita equivalence between $(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})}$ and $(\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}})^{X^G(\mathfrak{s})}$.

By Lemma 3.14, Proposition 3.9 and (97) there is a chain of Morita equivalences

$$\begin{aligned} (e^\sharp_{\lambda_G} \mathcal{H}(G) e^\sharp_{\lambda_G})^{X^G(\mathfrak{s})} \sim_M (\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})} \sim_M (\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}})^{X^G(\mathfrak{s})} \\ \sim_M (e^\sharp_\lambda \mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}) e^\sharp_\lambda)^{X^G(\mathfrak{s})}. \end{aligned} \tag{103}$$

The respective Morita bimodules are given by (94), (102) and (84). In relation to (97) we can rewrite (84) as

$$e^\sharp_\lambda (\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}})^{X^G(\mathfrak{s})} \quad \text{and} \quad (\mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}})^{\mathfrak{s}})^{X^G(\mathfrak{s})} e^\sharp_\lambda. \tag{104}$$

It follows that Morita bimodules for the composition of (103) are

$$(e^\sharp_{\lambda_G} C_c^\infty(G/U) e^\sharp_\lambda)^{X^G(\mathfrak{s})} \quad \text{and} \quad (e^\sharp_\lambda C_c^\infty(U\backslash G) e^\sharp_{\lambda_G})^{X^G(\mathfrak{s})}. \tag{105}$$

As the modules (102) are free of rank 1 over both the algebras (101), and the actions of $X^G(\mathfrak{s})$ on (105) and the involved algebras come from the action on functions $G \to \mathbb{C}$, the modules (105) are again free of rank 1 over

$$(e^\sharp_{\lambda_G} \mathcal{H}(G) e^\sharp_{\lambda_G})^{X^G(\mathfrak{s})} \quad \text{and} \quad (e^\sharp_\lambda \mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}) e^\sharp_\lambda)^{X^G(\mathfrak{s})}. \tag{106}$$

Therefore these two algebras are isomorphic. Since the idempotents $a e_{\mu_G} a^{-1}$ and $a \sum_{\gamma \in X^G(\mathfrak{s}/\lambda)} e_{\lambda \otimes \gamma} a^{-1}$ are $X^G(\mathfrak{s})$-invariant, $e^\sharp_{\lambda_G} \mathcal{H}(G) e^\sharp_{\lambda_G})^{X^G(\mathfrak{s})}$ and the bimodules (105) can be decomposed in the same way as (98). It follows that the isomorphism between the algebras in (106), as just constructed from (105), respects the decompositions indexed by $a_1, a_2 \in [L/H_\lambda]$. This settles part (b).

It remains to prove the first isomorphism of part (c), but here we encounter the problem that the isomorphism between the algebras (106) is not explicit. In particular we do not know for sure that it is equivariant with respect to $X_{\mathrm{nr}}(G)$. Nevertheless, we claim that the chain of Morita equivalences (103) remains valid upon taking $X_{\mathrm{nr}}(G)$-invariants. For the first equivalence that is Lemma 3.14.b and for the second equivalence it was checked in (90). For the third equivalence we can use the same argument as for the first, the equations making (104) Morita bimodules can be restricted to functions $G \to \mathbb{C}$ supported on $G^1 G^\sharp$. Composing these three steps, we obtain

$$(e^\sharp_{\lambda_G} \mathcal{H}(G) e^\sharp_{\lambda_G})^{X^G(\mathfrak{s}) X_{\mathrm{nr}}(G)} \sim_M (e^\sharp_\lambda \mathcal{H}(M \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}) e^\sharp_\lambda)^{X^G(\mathfrak{s}) X_{\mathrm{nr}}(G)}. \tag{107}$$

with Morita bimodules

$$(e^\sharp_{\lambda_G} C_c^\infty(G/U) e^\sharp_\lambda)^{X^G(\mathfrak{s}) X_{\mathrm{nr}}(G)} \quad \text{and} \quad (e^\sharp_\lambda C_c^\infty(U \backslash G) e^\sharp_{\lambda_G})^{X^G(\mathfrak{s}) X_{\mathrm{nr}}(G)}. \tag{108}$$

Since the modules (105) are free of rank one over the algebras (106), the modules (108) are free of rank one over both the algebras in (107). Therefore these two algebras are isomorphic. The isomorphism respects the decompositions indexed by $a_1, a_2 \in [L/H_\lambda]$ for the same reasons as in part (b). $\square$

We normalize the Haar measures on $G, G^\sharp$ and $G^\sharp Z(G)$ such that K_G and $K_G \cap G^\sharp$ and $K_G \cap G^\sharp Z(G)$ have the same volume. Consider the idempotent $e^\sharp_{\lambda_G} \in \mathcal{H}(G)$ as a function $G \to \mathbb{C}$ and let $e^\sharp_{\lambda_{G^\sharp}}$ (respectively $e^\sharp_{\lambda_{G^\sharp Z(G)}}$) be its restriction to $G^\sharp$ (respectively $G^\sharp Z(G)$). It turns out that

$$e^\sharp_{\lambda_{G^\sharp Z(G)}} \in \mathcal{H}(G^\sharp Z(G)) \quad \text{and} \quad e^\sharp_{\lambda_{G^\sharp}} \in \mathcal{H}(G^\sharp). \tag{109}$$

are idempotents. We describe the associated subalgebras of $\mathcal{H}(G^\sharp Z(G))$ (respectively $\mathcal{H}(G^\sharp)$ in two separate but analogous theorems.

Theorem 3.16. *The element $e^\sharp_{\lambda_{G^\sharp Z(G)}} \in \mathcal{H}(G^\sharp Z(G))$ is idempotent and*

$$e^\sharp_{\lambda_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G)) e^\sharp_{\lambda_{G^\sharp Z(G)}} \cong e^\sharp_{\lambda_G} \mathcal{H}(G)^{X^G(\mathfrak{s})} e^\sharp_{\lambda_G} \cong (e^{\mathfrak{s}}_M \mathcal{H}(M) e^{\mathfrak{s}}_M)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}.$$

These algebras are Morita equivalent with $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ and with $(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})}$.

Proof. Consider the l-th congruence subgroup $C_l \subset GL_m(\mathfrak{o}_D)$, as in the proof of Lemma 3.10. We choose the level l so high that all representations in $\mathrm{Rep}^{\mathfrak{s}}(G)$ have nonzero C_l-fixed vectors and that $e^\sharp_{\lambda_G}$ is C_l-biinvariant. Put $C_l' = C_l \cap G^\sharp Z(G)$. The proof of Lemma 3.10 shows that the algebra isomorphism

$$\mathcal{H}(G^\sharp Z(G), C_l')^{\mathfrak{s}} \to (\mathcal{H}(G, C_l)^{\mathfrak{s}})^{X^G(\mathfrak{s})} \tag{110}$$

coming from (87) maps $e^\sharp_{\lambda_{G^\sharp Z(G)}}$ to $e^\sharp_{\lambda_G}$. As $e^\sharp_{\lambda_G}$ is idempotent, so is $e^\sharp_{\lambda_{G^\sharp Z(G)}}$. It follows that (110) restricts to an isomorphism

$$\begin{aligned} e^\sharp_{\lambda_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G)) e^\sharp_{\lambda_{G^\sharp Z(G)}} &= e^\sharp_{\lambda_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G), C_l')^{\mathfrak{s}} e^\sharp_{\lambda_{G^\sharp Z(G)}} \to \\ & e^\sharp_{\lambda_G} (\mathcal{H}(G, C_l)^{\mathfrak{s}})^{X^G(\mathfrak{s})} e^\sharp_{\lambda_G} = e^\sharp_{\lambda_G} \mathcal{H}(G)^{X^G(\mathfrak{s})} e^\sharp_{\lambda_G}. \end{aligned} \tag{111}$$

The second isomorphism of the lemma is Proposition 3.15.b. By Lemma 3.14 these algebras are Morita equivalent with $(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s})}$, and by Lemma 3.10 also with $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$. $\square$

Theorem 3.17. *The element $e^\sharp_{\lambda_{G^\sharp}} \in \mathcal{H}(G^\sharp)$ is idempotent and*

$$e^\sharp_{\lambda_{G^\sharp}} \mathcal{H}(G^\sharp) e^\sharp_{\lambda_{G^\sharp}} \cong e^\sharp_{\lambda_G} \mathcal{H}(G)^{X^G(\mathfrak{s}) X_{\mathrm{nr}}(G)} e^\sharp_{\lambda_G} \cong (e^{\mathfrak{s}}_M \mathcal{H}(M) e^{\mathfrak{s}}_M)^{X^L(\mathfrak{s}) X_{\mathrm{nr}}(G)} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}.$$

These algebras are Morita equivalent with $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ and with $(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s}) X_{\mathrm{nr}}(G)}$.

Proof. Recall that (K_G, λ_G) is a type for the single Bernstein component $\mathfrak{s}$. The representations in $\mathrm{Rep}^{\mathfrak{s}}(G)$ contain only one character of $Z(G) \cap G^1$, so we must have

$$Z(G) \cap K_G = Z(G) \cap G^1 = \mathfrak{o}_F^\times \cdot 1_m.$$

Because λ_G is irreducible as a representation of K_G, $Z(G) \cap K_G$ acts on it by a character, say ζ_λ.

Endow $Z(G)$ with the Haar measure for which $Z(G) \cap K_G$ gets volume $|Z(G) \cap G^\sharp|$. There is an equality

$$e^\sharp_{\lambda_{G^\sharp Z(G)}} = e^\sharp_{\lambda_{G^\sharp}} e_{\zeta_\lambda}$$

of distributions on $G^\sharp Z(G)$, where e_{ζ_λ} denotes the idempotent associated to $(Z(G) \cap G^1, \zeta_\lambda)$. Then

$$e^\sharp_{\lambda_{G^\sharp}} \mathcal{H}(G^\sharp) e^\sharp_{\lambda_{G^\sharp}} \to e^\sharp_{\lambda_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G)) e^\sharp_{\lambda_{G^\sharp Z(G)}} : f \mapsto f e_{\zeta_\lambda} \tag{112}$$

is an injective algebra homomorphism with image

$$e^\sharp_{\lambda_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp (Z(G) \cap G^1)) e^\sharp_{\lambda_{G^\sharp Z(G)}}.$$

This is precisely the subalgebra of $e^\sharp_{\lambda_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G)) e^\sharp_{\lambda_{G^\sharp Z(G)}}$ which is invariant under $X_{\mathrm{nr}}(G^\sharp Z(G))$. Under the isomorphism (111) it corresponds to

$$e^\sharp_{\lambda_G} \mathcal{H}(G)^{X^G(\mathfrak{s}) X_{\mathrm{nr}}(G)} e^\sharp_{\lambda_G}.$$

That algebra is isomorphic to

$$(e^{\mathfrak{s}}_M \mathcal{H}(M) e^{\mathfrak{s}}_M)^{X^L(\mathfrak{s}) X_{\mathrm{nr}}(G)} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}$$

by Proposition 3.15.c and Morita equivalent to $(\mathcal{H}(G)^{\mathfrak{s}})^{X^G(\mathfrak{s}) X_{\mathrm{nr}}(G)}$ by Lemma 3.14.b. In Lemma 3.11 we already observed that this last algebra is Morita equivalent with $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$. $\square$

4. The structure of the Hecke algebras

4.1. **Hecke algebras for general linear groups.**

Consider the inertial equivalence class $\mathfrak{s} = [L, \omega]_G$. Via the map $\chi \mapsto \omega \otimes \chi$ we identify $T_{\mathfrak{s}}$ with the complex torus $X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L, \omega)$. This gives us the lattices $X^*(T_{\mathfrak{s}})$ and $X_*(T_{\mathfrak{s}})$ of algebraic characters and cocharacters, respectively. We emphasize that this depends on the choice of the basepoint ω of $T_{\mathfrak{s}}$. Under the Conditions 1.1, any other basepoint is the form $\omega' = \omega \otimes \chi'$ where $\chi' \in X_{\mathrm{nr}}(L)^{W_{\mathfrak{s}}}$. There is a natural isomorphism $x \mapsto x'$ from $X^*(T_{\mathfrak{s}})$ with respect to ω to $X^*(T_{\mathfrak{s}})$ with respect to ω'. As functions on $T_{\mathfrak{s}}$, it works out to

$$x'(\omega \otimes \chi) = x(\omega \otimes \chi) x(\omega \otimes \chi')^{-1}. \tag{113}$$

The inertial equivalence class $\mathfrak{s}$ comes not only with the torus $T_{\mathfrak{s}}$ and the group $W_{\mathfrak{s}}$, but also with a root system $R_{\mathfrak{s}} \subset X^*(T_{\mathfrak{s}})$, whose Weyl group is $W_{\mathfrak{s}}$. From (113) we see that a character x is independent of the choice of a basepoint of $T_{\mathfrak{s}}$ if it is invariant under $X_{\mathrm{nr}}(L)^{W_{\mathfrak{s}}}$, that is, if x lies in the lattice $\mathbb{Z}R_{\mathfrak{s}}$ spanned by $R_{\mathfrak{s}}$.

Let $\mathcal{H}(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}, q_{\mathfrak{s}})$ denote the affine Hecke algebra associated to the root datum $(X^*(T_{\mathfrak{s}}), X_*(T_{\mathfrak{s}}), R_{\mathfrak{s}}, R_{\mathfrak{s}}^\vee)$ and the parameter function $q_{\mathfrak{s}}$ as in [22]. It has a standard basis $\{[x] : x \in X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}\}$, with multiplication rules described first by Iwahori and Matsumoto [17].

We remark that here $q_\mathfrak{s}$ is not a just one real number, but a collection of parameters $q_{\mathfrak{s},i} > 0$, one for each factor M_i of M, or equivalently one for each irreducible component of the root system $R_\mathfrak{s}$. The parameter $q_\mathfrak{s}$ has a natural extension to a map

$$q_\mathfrak{s} : X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s} \to \mathbb{R}_{>0},$$

see [18, §1]. On the part of $X^*(T_\mathfrak{s})$ that is positive with respect to $P \cap M$ it can be defined as follows. Since $T_\mathfrak{s}$ is a quotient of $X_{\mathrm{nr}}(L)$, $X^*(T_\mathfrak{s})$ is naturally isomorphic to a subgroup of L/L^1. In this way $q_\mathfrak{s}$ corresponds to δ_u^{-1}, the inverse of the modular character for the action of L on the unipotent radical of $P \cap M$.

Let us recall the Bernstein presentation of an affine Hecke algebra [18, §3]. For $x \in X^*(T_\mathfrak{s})$ positive with respect to $P \cap M$ we write

$$\theta_x := q_\mathfrak{s}(x)^{-1/2}[x] = \delta_u^{1/2}(x)[x]. \tag{114}$$

The map $x \mapsto \theta_x$ can be extended in a unique way to a group homomorphism $X^*(T_\mathfrak{s}) \to \mathcal{H}(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}, q_\mathfrak{s})^\times$ [18, 2.6], for which we use the same notation. By [18, Proposition 3.7]

$$\{\theta_x[w] : x \in X^*(T_\mathfrak{s}), w \in W_\mathfrak{s}\}$$

is a basis of $\mathcal{H}(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}, q_\mathfrak{s})$. Furthermore the span of the θ_x is a subalgebra $\mathcal{A}$ isomorphic to $\mathbb{C}[X^*(T_\mathfrak{s})] \cong \mathcal{O}(T_\mathfrak{s})$ and the span of the $[w]$ with $w \in W_\mathfrak{s}$ is the Iwahori–Hecke algebra $\mathcal{H}(W_\mathfrak{s}, q_\mathfrak{s})$. The multiplication map

$$\mathcal{A} \otimes \mathcal{H}(W_\mathfrak{s}, q_\mathfrak{s}) \to \mathcal{H}(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}, q_\mathfrak{s}) \tag{115}$$

is a linear bijection. The commutation relations between these two subalgebras are known as the Bernstein–Lusztig–Zelevinsky relations. Let $\alpha \in R_\mathfrak{s}$ be a simple root, with corresponding reflection $s \in W_\mathfrak{s}$. By [18, Proposition 3.6], for any $x \in X(T_\mathfrak{s})$

$$\theta_x[s] - [s]\theta_{s(x)} = (q_\mathfrak{s}(s) - 1)(\theta_x - \theta_{s(x)})(1 - \theta_{-\alpha})^{-1} \in \mathcal{A}. \tag{116}$$

Since the elements $[s]$ generate $\mathcal{H}(W_\mathfrak{s}, q_\mathfrak{s})$, this determines the commutation relations for $\mathcal{A}$ with all $[w]$ ($w \in W_\mathfrak{s}$). It follows from (116) that

$$Z(\mathcal{H}(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}, q_\mathfrak{s})) = \mathcal{A}^{W_\mathfrak{s}}. \tag{117}$$

In view of (113), (115) and (116), we can also regard $\mathcal{H}(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}, q_\mathfrak{s})$ as the algebra whose underlying vector space is

$$\mathcal{O}(T_\mathfrak{s}) \otimes \mathcal{H}(W_\mathfrak{s}, q_\mathfrak{s}) \tag{118}$$

and whose multiplication satisfies

$$f[s] - [s](s \cdot f) = (q_\mathfrak{s}(s) - 1)(f - (s \cdot f))(1 - \theta_{-\alpha})^{-1} \qquad f \in \mathcal{O}(T_\mathfrak{s}), \tag{119}$$

with respect to the canonical action of $W_\mathfrak{s}$ on $\mathcal{O}(T_\mathfrak{s})$. The advantage is that, written in this way, the multiplication does not depend on the choice of a basepoint $\omega \in T_\mathfrak{s}$ used to define $X^*(T_\mathfrak{s})$. We will denote this interpretation of $\mathcal{H}(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}, q_\mathfrak{s})$ by $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$.

Let ϖ_D be a uniformizer of D. Consider the group of diagonal matrices in L all whose diagonal entries are powers of ϖ_D and whose components in each L_i are multiples of the identity. It can be identified with a sublattice of $X^*(X_{\mathrm{nr}}(L))$. The lattice $X^*(T_\mathfrak{s})$ can be represented in a unique way by such matrices, say by the group $\widetilde{X^*(T_\mathfrak{s})} \subset L$.

Recall that (K, λ) is a type for $\mathfrak{s}_M$ and that (K_L, λ_L) is a $\mathfrak{s}_L$-type. The next result is largely due to Sécherre [22].

Theorem 4.1. *For every* $(w,\gamma) \in \mathrm{Stab}(\mathfrak{s}, P \cap M)$ *there are isomorphisms*

$$\begin{aligned} e_{w(\lambda_L)\otimes\gamma}\mathcal{H}(L)e_{w(\lambda_L)\otimes\gamma} &\cong \mathcal{H}(L, w(\lambda_L)\otimes\gamma) \otimes \mathrm{End}_{\mathbb{C}}(V_{w(\lambda_L)\otimes\gamma}) \\ &\cong \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{w(\lambda)\otimes\gamma}), \\ e_{w(\lambda)\otimes\gamma}\mathcal{H}(M)e_{w(\lambda)\otimes\gamma} &\cong \mathcal{H}(M, w(\lambda)\otimes\gamma) \otimes \mathrm{End}_{\mathbb{C}}(V_{w(\lambda)\otimes\gamma}) \\ &\cong \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{w(\lambda)\otimes\gamma}). \end{aligned}$$

In both cases the first isomorphism is canonical, and the second depends only on the choice of the parabolic subgroup P. The support of these algebras is, respectively, $K_L\widetilde{X^*(T_{\mathfrak{s}})}K_L$ and $K\widetilde{X^*(T_{\mathfrak{s}})}W_{\mathfrak{s}}K$.

Proof. Since all the types $(K, w(\lambda)\otimes\gamma)$ have the same properties, it suffices to treat the case $(w,\gamma) = (1,1)$. The first and third isomorphisms are instances of (7). The support of the algebras was determined in [22, §4]. Sécherre also proved that the remaining isomorphisms exist, but some extra work is needed to make them canonical.

The L-representations $\omega\otimes\chi$ with $\chi \in X_{\mathrm{nr}}(L)$ paste to an algebra homomorphism

$$\mathcal{F}_L : e_{\lambda_L}\mathcal{H}(L)e_{\lambda_L} \to \mathcal{O}(X_{\mathrm{nr}}(L)) \otimes \mathrm{End}_{\mathbb{C}}(e_{\lambda_L}V_\omega), \tag{120}$$

which is injective because these are all irreducible representations in $\mathrm{Rep}^{\mathfrak{s}_L}(L)$. By [22, Théorème 4.6] $e_{\lambda_L}\mathcal{H}(L)e_{\lambda_L}$ is isomorphic to $\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\lambda_L})$. Hence

$$e_{\lambda_L}V_\omega \cong V_{\lambda_L} = V_\lambda \tag{121}$$

and (120) restricts to a canonical isomorphism

$$\mathcal{F}_L : e_{\lambda_L}\mathcal{H}(L)e_{\lambda_L} \to \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\lambda_L}). \tag{122}$$

Here $\mathcal{O}(T_{\mathfrak{s}})$ is the centre of the right hand side, so it corresponds to $\mathcal{H}(L,\lambda_L)$. Consider the isomorphism

$$e_\lambda\mathcal{H}(M)e_\lambda \cong \mathcal{H}(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\lambda). \tag{123}$$

from [22, Théorème 4.6]. It comes from $\mathcal{H}(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}, q_{\mathfrak{s}}) \cong \mathcal{H}(M,\lambda)$. We define

$$f_{x,\lambda} \in \mathcal{H}(M,\lambda) \text{ as the image of } [x] \text{ under (123).} \tag{124}$$

Because (K,λ) is a cover of (K_L,λ_L), we may use the results of [7, §7]. By [7, Corollaries 7.2 and 7.11] there exists a unique injective algebra homomorphism

$$t_{P,\lambda} : \mathcal{H}(L,\lambda_L) \to \mathcal{H}(M,\lambda) \tag{125}$$

such that the diagram

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{H}(M,\lambda)) \arrow[r] \arrow[d, "t^*_{P,\lambda}"'] & \mathrm{Rep}(M) \arrow[d, "r^M_{P\cap M}"'] \\ \mathrm{Mod}(\mathcal{H}(L,\lambda_L)) \arrow[r] & \mathrm{Rep}(L) \end{tikzcd}$$

commutes. We note that in [7] unnormalized Jacquet restriction is used, whereas we prefer the normalized version. Therefore our $t_{P,\lambda}$ equals $t_{\delta_u^{1/2}}$ in the notation of [7, §7], where δ_u denotes the modular character for the action of L on the unipotent radical of $P \cap M$.

Consider the diagram

$$
\begin{array}{ccc}
\mathcal{H}(M,\lambda) & \to & \mathcal{H}(X^*(T_{\mathfrak s}) \rtimes W_{\mathfrak s}, q_{\mathfrak s}) \cong \mathcal{H}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \\
\uparrow t_{P,\lambda} & & \uparrow i_{P,\lambda} \\
\mathcal{H}(L,\lambda_L) & \to & \mathcal{O}(T_{\mathfrak s}) \cong \mathbb{C}[X^*(T_{\mathfrak s})],
\end{array} \tag{126}
$$

where the upper map is (123) and the lower map comes from (122). The horizontal maps are isomorphisms and $t_{P,\lambda}$ is injective. We want to define the right vertical map $i_{P,\lambda}$ so that the diagram commutes.

The construction of the upper map in [22, §4] shows that it is canonical on the subalgebra of $\mathcal{H}(X^*(T_{\mathfrak s}) \rtimes W_{\mathfrak s}, q_{\mathfrak s})$ generated by the elements $[s]$ with $s \in X^*(T_{\mathfrak s}) \rtimes W_{\mathfrak s}$ a simple affine reflection. This subalgebra has a basis $\{[x] : x \in \mathbb{Z}R_{\mathfrak s} \rtimes W_{\mathfrak s}\}$, where $\mathbb{Z}R_{\mathfrak s}$ is the sublattice of $X^*(T_{\mathfrak s})$ spanned by the root system $R_{\mathfrak s}$. In particular the image $f_{x,\lambda} \in \mathcal{H}(M,\lambda)$ of $[x]$ with $x \in \mathbb{Z}R_{\mathfrak s} \rtimes W_{\mathfrak s}$ is defined canonically.

By [22, Théorème 4.6] the remaining freedom for (123) boils down to, for each factor M_i of M, the choice of a nonzero element in a one-dimensional vector space. This is equivalent to the freedom in the choice of the basepoint ω of $T_{\mathfrak s}$.

Take a $x \in X^*(T_{\mathfrak s})$ which is positive with respect to $P \cap M$, and let f_{x,λ_L} be the corresponding element of $\mathcal{H}(L,\lambda_L)$. For such elements $t_{P,\lambda}$ is described explicitly by [7, Theorem 7.2]. In our notation

$$t_{P,\lambda}(f_{x,\lambda_L}) = t_{\delta_u^{1/2}}(f_{x,\lambda_L}) = \delta_u^{1/2}(x) f_{x,\lambda}.$$

Suppose that furthermore $x \in \mathbb{Z}R_{\mathfrak s}$, considered as subset of $\mathbb{C}[X^*(T^{\mathfrak s})]$. Then its images three of the maps in (126) are canonically determined. In order that the diagram commutes, it is necessary that

$$i_{P,\lambda}(x) = \theta_x, \tag{127}$$

with θ_x as in (114). The condition (127) determines $i_{P,\lambda}(x)$ for all $x \in \mathbb{Z}R_{\mathfrak s}$. Now every way to extend $i_{P,\lambda}$ to the whole of $\mathbb{C}[X^*(T_{\mathfrak s})]$ corresponds to precisely one choice of an isomorphism (123). Thus we can normalize (123) by requiring that (127) holds for all $x \in X^*(T_{\mathfrak s})$ which are positive with respect to $P \cap M$.

In effect, we defined $i_{P,\lambda}$ to be the identity of $\mathcal{O}(T_{\mathfrak s})$ with respect to the isomorphisms

$$\mathcal{A} \cong \mathbb{C}[X^*(T_{\mathfrak s})] \cong \mathcal{O}(T_{\mathfrak s}).$$

So we turned (127) into an algebra homomorphism

$$i_{P,\lambda} : \mathcal{O}(T_{\mathfrak s}) \to \mathcal{H}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}).$$

A priori it depends on the choice of a basepoint of $T_{\mathfrak s}$, but since we use the same basepoint on both sides and by (113), any other basepoint would produce the same map $i_{P,\lambda}$. Thus (123) becomes canonical if we interpret the right hand side as $\mathcal{H}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb{C}}(V_\lambda)$. $\square$

4.2. **Projective normalizers.**

We will subject the algebra $e^{\mathfrak s}_M \mathcal{H}(M) e^{\mathfrak s}_M$ to a closer study, and describe its structure explicitly. At the same time we investigate how close e_μ and $e^{\mathfrak s}_M$ from (69) are to the idempotent of a type. A natural candidate for such a type would involve the projective normalizer of (K,λ), but unfortunately it will turn out that this is in general not sufficiently sophisticated.

Recall the groups defined in (60) and (61) and consider the vector spaces

$$\begin{aligned} V_{\mu^1} = V_{\mu_L^1} &:= \sum_{(w,\gamma)\in \mathrm{Stab}(\mathfrak{s},P\cap M)^1} e_{w(\lambda_L)\otimes\gamma} V_\omega = \sum_{\gamma\in X^L(\mathfrak{s})^1} e_{\lambda_L\otimes\gamma} V_\omega, \\ V_{\mu_L} := V_\mu &= \sum_{(w,\gamma)\in \mathrm{Stab}(\mathfrak{s},P\cap M)} e_{w(\lambda_L)\otimes\gamma} V_\omega = \sum_{\gamma\in X^L(\mathfrak{s})} e_{\lambda_L\otimes\gamma} V_\omega. \end{aligned} \tag{128}$$

They carry in a natural way representations of K_L, namely

$$\begin{aligned} \mu_L^1 &= \bigoplus_{(w,\gamma)\in \mathrm{Stab}(\mathfrak{s},P\cap M)^1/\mathrm{Stab}(\mathfrak{s},\lambda)} w(\lambda_L)\otimes\gamma = \bigoplus_{\gamma\in X^L(\mathfrak{s}/\lambda)^1} \lambda_L\otimes\gamma, \\ \mu_L &= \bigoplus_{(w,\gamma)\in \mathrm{Stab}(\mathfrak{s},P\cap M)/\mathrm{Stab}(\mathfrak{s},\lambda)} w(\lambda_L)\otimes\gamma = \bigoplus_{\gamma\in X^L(\mathfrak{s}/\lambda)} \lambda_L\otimes\gamma. \end{aligned} \tag{129}$$

We lift them to representations

$$\begin{aligned} \mu^1 &= \bigoplus\nolimits_{\gamma\in X^L(\mathfrak{s}/\lambda)^1} \lambda\otimes\gamma, \\ \mu &= \bigoplus\nolimits_{\gamma\in X^L(\mathfrak{s}/\lambda)} \lambda\otimes\gamma \end{aligned} \tag{130}$$

of K by making it trivial on $K\cap U$ and on $K\cap\overline{U}$. In particular μ_L^1 is the restriction of μ^1 to $K\cap L$. They relate to the idempotent $e_M^{\mathfrak{s}}$ by

$$\sum_{a\in[L/H_\lambda]} \sum_{\gamma\in X^L(\mathfrak{s}/\lambda)} a e_{\gamma\otimes\mu^1} a^{-1} = \sum_{a\in[L/H_\lambda]} a e_\mu a^{-1} = e_M^{\mathfrak{s}}.$$

It will turn out that $e_{\mu^1}\in\mathcal{H}(K)$ is the idempotent of a type, for a compact open subgroup of M that contains K.

The normalizer of the pair (K_L,λ_L) is

$$N(K_L,\lambda_L) := \{m\in N_L(K_L) \mid m\cdot\lambda_L \cong \lambda_L\}.$$

Lemma 4.2. $N(K_L,\lambda_L) = \widetilde{X^*(T_{\mathfrak{s}})} K_L = K_L \widetilde{X^*(T_{\mathfrak{s}})}$.

Proof. By Theorem 4.1

$$N(K_L,\lambda_L) \subset K_L \widetilde{X^*(T_{\mathfrak{s}})} K_L. \tag{131}$$

With conditions 1.1 and (55) we can be more precise. As discussed in the proof of Proposition 3.1,

$$K_L\widetilde{X^*(T_{\mathfrak{s}})} = \prod\nolimits_i \big(K_{L_i}(L_i\cap\widetilde{X^*(T_{\mathfrak{s}})})\big)^{e_i} = \prod\nolimits_i \big(K_{L_i}(D^\times 1_{L_i}\cap\widetilde{X^*(T_{\mathfrak{s}})})\big)^{e_i}. \tag{132}$$

As $\lambda_L = \bigotimes_i \lambda_{L_i}^{\otimes e_i}$, the group $N(K_L,\lambda_L)$ can be factorized similarly. Consider any element of the form

$$k_i z_i \text{ with } k_i\in K_{L_i}, z_i\in D^\times 1_{L_i}\cap\widetilde{X^*(T_{\mathfrak{s}})}. \tag{133}$$

The group K_{L_i}, called $J(\beta,\mathfrak{A})$ in [22], is made from a stratum in $L_i = \mathrm{GL}_{m_i}(D)$, and therefore it is normalized by z_i, see [22, §2.3]. Furthermore z_i belongs to the support of $e_{\lambda_i}\mathcal{H}(L_i)e_{\lambda_i}$, so it normalizes (K_{L_i},λ_i). Knowing that, we can follow the proof of Proposition 3.1 with k_iz_i in the role of c_i. It leads to the conclusion that $k_iz_i\in N(K_L,\lambda_L)$. It follows that

$$K_L\widetilde{X^*(T_{\mathfrak{s}})} = \widetilde{X^*(T_{\mathfrak{s}})}K_L \subset N(K_L,\lambda_L).$$

Combine this with (131). $\square$

Inspired by [6] we define two variants of the projective normalizer of (K_L, λ_L):

$$PN(K_L,\lambda_L) := \{m \in N_L(K_L) \mid m \cdot \lambda_L \cong \lambda_L \otimes \gamma \text{ for some } \gamma \in X^L(\mathfrak{s})\},$$
$$PN^1(K_L,\lambda_L) := PN(K_L,\lambda_L) \cap L^1.$$

Lemma 4.3. *Recall the K_L-representations μ_L^1 and μ_L from (128) and (129).*
(a) $(\mu_L^1, V_{\mu_L^1})$ extends to an irreducible representation of $PN^1(K_L,\mu_L)$.
(b) $(PN^1(K_L,\mu_L),\mu_L^1)$ is an $[L,\omega]_L$-type and

$$[PN^1(K_L,\lambda_L) : K_L] = [\mathrm{Stab}(\mathfrak{s}, P\cap M)^1 : \mathrm{Stab}(\mathfrak{s},\lambda)] = |X^L(\mathfrak{s}/\lambda)^1|.$$

(c) (μ_L, V_{μ_L}) extends to an irreducible representation of $PN(K_L,\lambda_L)$ and

$$[PN(K_L,\lambda_L) : N(K_L,\lambda_L)] = [\mathrm{Stab}(\mathfrak{s}, P\cap M) : \mathrm{Stab}(\mathfrak{s},\lambda)] = |X^L(\mathfrak{s}/\lambda)|.$$

Proof. (a) Just as in (120), there is a canonical injective algebra homomorphism

$$\mathcal{F}_L : e_{\mu_L^1}\mathcal{H}(L)e_{\mu_L^1} \to \mathcal{O}(X_{\mathrm{nr}}(L)) \otimes \mathrm{End}_{\mathbb{C}}(e_{\mu_L^1}V_\omega). \tag{134}$$

For $\gamma \in X^L(\mathfrak{s})^1$ the element $c_\gamma \in L^1$ from Proposition 3.1 maps to $\mathrm{End}_{\mathbb{C}}(V_\omega)$ by the definition of L^1. Moreover

$$e_{\lambda_L\otimes\gamma_1}\mathcal{H}(L)e_{\lambda_L\otimes\gamma_2} = c_{\gamma_1}e_{\lambda_L}\mathcal{H}(L)e_{\lambda_L}c_{\gamma_2}^{-1},$$

so by (122) the image of (134) is contained in

$$\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu_L^1}).$$

As the different idempotents $e_{\lambda_L\otimes\gamma}$ are orthogonal,

$$V_{\mu_L^1} = \bigoplus\nolimits_{\gamma\in X^L(\mathfrak{s}/\lambda)^1} e_{\lambda_L\otimes\gamma}V_\omega,$$
$$\mathcal{H}(K_L)e_{\mu_L^1} = e_{\mu_L^1}\mathcal{H}(K_L) \cong \bigoplus\nolimits_{\gamma\in X^L(\mathfrak{s}/\lambda)^1} \mathrm{End}_{\mathbb{C}}(e_{\lambda_L\otimes\gamma}V_\omega).$$

Furthermore $\mathcal{F}_L(\mathbb{C}\{c_\gamma e_{\mu_L^1} : \gamma \in X^L(\mathfrak{s}/\lambda)^1\})$ is a subspace of $\mathrm{End}_{\mathbb{C}}(V_{\mu_L^1})$ of dimension $|X^L(\mathfrak{s}/\lambda)^1|$. So by the injectivity of (134) the algebra homomorphism

$$\mathcal{F}_L : \mathcal{H}(K_L)e_{\mu_L^1} \otimes \mathbb{C}\{c_\gamma e_{\mu_L^1} : \gamma \in X^L(\mathfrak{s}/\lambda)^1\} \to \mathrm{End}_{\mathbb{C}}(V_{\mu_L^1}) \tag{135}$$

is bijective. Consider any $m \in PN^1(K_L,\lambda_L)$. It permutes the $\lambda_L \otimes \gamma$ with $\gamma \in X^L(\mathfrak{s})^1$, so it commutes with $e_{\mu_L^1}$. Also $\mathcal{F}_L(me_{\mu_L^1}) \in \mathrm{End}_{\mathbb{C}}(V_{\mu_L^1})$ because $m \in L^1$. So by the injectivity of (134) and the surjectivity of (135), $me_{\mu_L^1} = fe_{\mu_L^1}$ for some

$$f \in \mathcal{H}(K_L \cdot \{c_\gamma : \gamma \in X^L(\mathfrak{s}/\lambda)^1\}).$$

Consequently $m \in K_L \cdot \{c_\gamma : \gamma \in X^L(\mathfrak{s}/\lambda)^1\}$ and

$$PN^1(K_L,\lambda_L) = K_L \cdot \{c_\gamma : \gamma \in X^L(\mathfrak{s}/\lambda)^1\} \tag{136}$$

Now (135) shows that $V_{\mu_L^1} = e_{\mu_L^1}V_\omega$ is an irreducible representation of (136).
(b) All the pairs $(K_L, \lambda_L\otimes\gamma)$ are simple types for the same supercuspidal equivalence class $[L,\omega]_L$, so by [24, Corollary 7.3] the idempotents $e_{\lambda_L\otimes\gamma}$ are L-conjugate. (In fact e_{λ_L} and $e_{\lambda_L\otimes\gamma}$ are conjugate by the element c_γ from Proposition 3.1.) Hence the category $\mathrm{Rep}^{\mu_L^1}(L)$ equals $\mathrm{Rep}^{\lambda_L}(L)$, and $(PN^1(K_L,\lambda_L),\mu_L^1)$ is a type for this factor of $\mathrm{Rep}(L)$. The claims about the indices follow from (136).
(c) For every $\gamma \in X^L(\mathfrak{s})$ the element $c_\gamma \in L$ is unique up to $N(K_L,\lambda_L)$, so

$$PN(K_L,\lambda_L) = N(K_L,\lambda_L)\{c_\gamma \mid \gamma \in X^L(\mathfrak{s})\}. \tag{137}$$

Together with Proposition 3.1 this proves the claims about $[PN(K_L, \lambda_L) : N(K_L, \lambda_L)]$.

Part (a), and the map (134) show that μ_L^1 extends to an irreducible representation of $PN^1(K_L, \lambda)N(K_L, \lambda_L)$. The same holds for $\gamma \otimes \mu_L^1$ with $\gamma \in X^L(\mathfrak{s})$. We have

$$V_{\mu_L} = \bigoplus\nolimits_{\gamma \in X^L(\mathfrak{s}/\lambda)} V_{\gamma \otimes \mu_L^1}$$

as representations of $PN^1(K_L, \lambda_L)N(K_L, \lambda_L)$, and these subspaces are permuted transitively by the c_γ with $\gamma \in X^L(\mathfrak{s})$. This and (137) show that V_{μ_L} extends to an irreducible representation of $PN(K_L, \lambda_L)$. □

Lemma 4.3 has an analogue in M. To state it we rather start with the sets

$$\begin{aligned} PN^1(K, \lambda) &:= (K \cap \overline{U}) PN^1(K_L, \lambda_L)(K \cap U), \\ PN(K, \lambda) &:= (K \cap \overline{U}) PN(K_L, \lambda_L)(K \cap U). \end{aligned} \tag{138}$$

Lemma 4.4. (*a*) *The multiplication map*

$$(K \cap \overline{U}) \times PN^1(K_L, \lambda_L) \times (K \cap U) \to PN^1(K, \lambda)$$

is bijective and $PN^1(K, \lambda)$ is a compact open subgroup of M.

(*b*) *μ^1 extends to an irreducible $PN^1(K, \lambda)$-representation and $(PN^1(K, \lambda), \mu^1)$ is an $\mathfrak{s}_M$-type.*

(*c*) *$PN(K, \lambda)$ is a group and the multiplication map*

$$(K \cap \overline{U}) \times PN(K_L, \lambda_L) \times (K \cap U) \to PN(K, \lambda)$$

is bijective. Furthermore μ extends to an irreducible $PN(K, \lambda)$-representation.

Remark. We will show later that $PN^1(K, \lambda)$ and $PN(K, \lambda)$ are really the projective normalizers of (K, λ) in M^1 and M, respectively.

Proof. (a) By [22, Proposition 5.3] the multiplication map

$$(K \cap \overline{U}) \times (K \cap L) \times (K \cap U) \to K \tag{139}$$

is a homeomorphism. By (136) and because $c_\gamma \in L^1$ normalizes $K \cap U$ and $K \cap \overline{U}$ (see the proof of Proposition 3.1), the analogue of (139) for $PN^1(K, \lambda)$ holds as well. At the same time this shows that $PN^1(K, \lambda)$ is compact, for its three factors are. As $PN^1(K_L, \lambda_L)$ normalizes $K \cap U$ and $K \cap \overline{U}$ this factorization also proves that $PN^1(K, \lambda)$ is a group. Since K is open in M, so is the larger group $PN^1(K, \lambda)$.

(b) In view of Lemma 4.3.a and (130) we can extend μ^1 to $PN^1(K, \lambda)$ by

$$\mu^1(\overline{n} m n) := \mu_L^1(m),$$

where $\overline{n} m n$ is as in the decomposition from part (a). Then μ^1 is irreducible because μ_L^1 is. Because $(K, w(\lambda) \otimes \gamma)$ is an $\mathfrak{s}_M$-type, for each $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}, P \cap M)$, the category $\mathrm{Rep}^{\mu^1}(M)$ equals $\mathrm{Rep}^{\lambda}(M) = \mathrm{Rep}^{\mathfrak{s}_M}(M)$, and $(PN^1(K, \lambda), \mu^1)$ is an $\mathfrak{s}_M$-type.

(c) The first two claims can be shown in the same way as part (a), using (137) instead of (136). For the last assertion we employ Lemma 4.3.c and set

$$\mu(\overline{n} m n) := \mu_L(m),$$

with respect to the factorization we just established. □

Now we can determine the structure of $e_M^{\mathfrak{s}} \mathcal{H}(M) e_M^{\mathfrak{s}}$ and some related algebras.

Theorem 4.5. (a) *There exist canonical algebra isomorphisms*

$$e_{\mu^1_L}\mathcal{H}(L)e_{\mu^1_L} \cong \mathcal{H}(L,\mu^1_L) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^1_L}) \cong \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^1}).$$

The support of the left hand side is $PN^1(K_L,\lambda_L)\widetilde{X^*(T_{\mathfrak{s}})}PN^1(K_L,\lambda_L)$

(b) *Part (a) extends to algebra isomorphisms*

$$e_{\mu_L}\mathcal{H}(L)e_{\mu_L} \cong \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu),$$
$$e^{\mathfrak{s}}_L\mathcal{H}(L)e^{\mathfrak{s}}_L \cong \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \otimes M_{|L/H_\lambda|}(\mathbb{C}).$$

The support of $e_{\mu_L}\mathcal{H}(L)e_{\mu_L}$ *is* $PN(K_L,\lambda_L)\widetilde{X^*(T_{\mathfrak{s}})}PN(K_L,\lambda_L)$.

(c) *There exist algebra isomorphisms*

$$e_{\mu^1}\mathcal{H}(M)e_{\mu^1} \cong \mathcal{H}(M,\mu^1) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^1}) \cong \mathcal{H}(T_{\mathfrak{s}},W_{\mathfrak{s}},q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^1}),$$

which are canonical up to the choice of the parabolic subgroup P. The support of the left hand side is $PN^1(K,\lambda)\widetilde{X^*(T_{\mathfrak{s}})}W_{\mathfrak{s}}PN^1(K,\lambda)$.

(d) *Part (c) extends to algebra isomorphisms*

$$e_\mu\mathcal{H}(M)e_\mu \cong \mathcal{H}(T_{\mathfrak{s}},W_{\mathfrak{s}},q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu),$$
$$e^{\mathfrak{s}}_M\mathcal{H}(M)e^{\mathfrak{s}}_M \cong \mathcal{H}(T_{\mathfrak{s}},W_{\mathfrak{s}},q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \otimes M_{|L/H_\lambda|}(\mathbb{C}).$$

The support of $e_\mu\mathcal{H}(M)e_\mu$ *is* $PN(K,\lambda)\widetilde{X^*(T_{\mathfrak{s}})}W_{\mathfrak{s}}PN(K,\lambda)$.

Proof. (a) By the Morita equivalence of $e^{\mathfrak{s}}_L\mathcal{H}(L)e^{\mathfrak{s}}_L$ and $e_{\lambda_L}\mathcal{H}(L)e_{\lambda_L}$, there are isomorphisms of $e^{\mathfrak{s}}_L\mathcal{H}(L)e^{\mathfrak{s}}_L$-bimodules

(140)
$$\begin{aligned}
&e^{\mathfrak{s}}_L\mathcal{H}(L)e^{\mathfrak{s}}_L \cong e^{\mathfrak{s}}_L\mathcal{H}(L)e_{\lambda_L} \otimes_{e_{\lambda_L}\mathcal{H}(L)e_{\lambda_L}} e_{\lambda_L}\mathcal{H}(L)e^{\mathfrak{s}}_L\\
&= \bigoplus_{a_1,a_2\in[L/H_\lambda]} \bigoplus_{\gamma_1,\gamma_2\in X^L(\mathfrak{s}/\lambda)} a_1 e_{\lambda_L\otimes\gamma_1}a_1^{-1}\mathcal{H}(L)e_{\lambda_L} \otimes_{e_{\lambda_L}\mathcal{H}(L)e_{\lambda_L}} e_{\lambda_L}\mathcal{H}(L)a_2 e_{\lambda_L\otimes\gamma_2}a_2^{-1}\\
&= \bigoplus_{a_1,a_2\in[L/H_\lambda]} \bigoplus_{\gamma_1,\gamma_2\in X^L(\mathfrak{s}/\lambda)} a_1 c_{\gamma_1} e_\lambda c_{\gamma_1}^{-1} a_1^{-1}\mathcal{H}(L)e_{\lambda_L}\mathcal{H}(L)a_2 c_{\gamma_2} e_{\lambda_L} c_{\gamma_2}^{-1} a_2^{-1}\\
&= \bigoplus_{a_1,a_2\in[L/H_\lambda]} \bigoplus_{\gamma_1,\gamma_2\in X^L(\mathfrak{s}/\lambda)} a_1 c_{\gamma_1} e_{\lambda_L}\mathcal{H}(L)e_{\lambda_L} c_{\gamma_2}^{-1} a_2^{-1}.
\end{aligned}$$

Here the subalgebra $e_{\mu^1_L}\mathcal{H}(L)e_{\mu^1_L}$ corresponds to

$$\bigoplus\nolimits_{\gamma_1,\gamma_2\in X^L(\mathfrak{s}/\lambda)^1} c_{\gamma_1} e_{\lambda_L}\mathcal{H}(L)e_{\lambda_L} c_{\gamma_2}^{-1}.$$

In combination with (122) it follows that the canonical map (134) is an isomorphism

$$\mathcal{F}_L : e_{\mu^1_L}\mathcal{H}(L)e_{\mu^1_L} \to \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^1}). \tag{141}$$

This and Theorem 4.1 imply that the support of $e_{\mu^1_L}\mathcal{H}(L)e_{\mu^1_L}$ is as indicated. Notice that $\mathcal{O}(T_{\mathfrak{s}})$ is the commutant of $\mathrm{End}_{\mathbb{C}}(V_{\mu^1})$ in $\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^1})$. Hence it corresponds to $\mathcal{H}(L,\mu^1_L)$ under the canonical isomorphism

$$e_{\mu^1_L}\mathcal{H}(L)e_{\mu^1_L} \cong \mathcal{H}(L,\mu^1_L) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^1}).$$

(b) By (140)

$$e_{\mu_L}\mathcal{H}(L)e_{\mu_L} = \bigoplus_{\gamma_1,\gamma_2\in X^L(\mathfrak{s}/\lambda)} c_{\gamma_1} e_{\lambda_L}\mathcal{H}(L)e_{\lambda_L} c_{\gamma_2}^{-1},$$

and by part (a) and (137) its support is $PN(K_L,\lambda_L)\widetilde{X^*(T_{\mathfrak s})}PN(K_L,\lambda_L)$. Also by (140), we can identify $e^{\mathfrak s}_L\mathcal H(L)e^{\mathfrak s}_L$ as a vector space with

$$\bigoplus_{a_3\in[L/H_\lambda],\gamma_3\in X^L(\mathfrak s)/X^L(\mathfrak s)^1}\mathbb C a_3\gamma_3\otimes\bigoplus_{a_1\in[L/H_\lambda],\gamma_1\in X^L(\mathfrak s)/X^L(\mathfrak s)^1} a_1 e_{\mu_L^1\otimes\gamma_1}\mathcal H(L)e_{\mu_L^1\otimes\gamma_1}a_1^{-1}.$$

From (141) we get an isomorphism

$$\bigoplus_{a_1\in[L/H_\lambda],\gamma_1\in X^L(\mathfrak s)/X^L(\mathfrak s)^1} a_1 e_{\mu_L^1\otimes\gamma_1}\mathcal H(L)e_{\mu_L^1\otimes\gamma_1}a_1^{-1}\to \bigoplus_{a_1\in[L/H_\lambda],\gamma_1\in X^L(\mathfrak s)/X^L(\mathfrak s)^1}\mathcal O(T_{\mathfrak s})\otimes\mathrm{End}_{\mathbb C}(a_1V_{\mu_L^1\otimes\gamma_1}). \tag{142}$$

Recall that

$$V_{\mu_L}=e_{\mu_L}\mathcal H(L)e_{\lambda_L}\otimes_{e_{\lambda_L}\mathcal H(L)e_{\lambda_L}}V_{\lambda_L}=e_{\mu_L}\mathcal H(L)e_{\mu_L^1}\otimes_{e_{\mu_L^1}\mathcal H(L)e_{\mu_L^1}}V_{\mu_L^1}= \bigoplus_{\gamma_1\in X^L(\mathfrak s)/X^L(\mathfrak s)^1} c_{\gamma_1}e_{\mu_L^1}\otimes V_{\mu_L^1}=\bigoplus_{\gamma_1}c_{\gamma_1}V_{\mu_L^1}.$$

For $\gamma_3\in X^L(\mathfrak s)$ the choice of c_{γ_3} is unique up $N(K_L,\lambda_L)$, by Lemma 4.3.c. The particular shape (58) implies that it is in fact unique up to $N(K_L,\lambda_L)^{W_{\mathfrak s}}$, so

$$c_{\gamma_3}c_{\gamma_1}\text{ differs from }c_{\gamma_3\gamma_1}\text{ by an element of }N(K_L,\lambda_L)^{W_{\mathfrak s}}. \tag{143}$$

With Lemma 4.2 we deduce that left multiplication by c_{γ_3} defines a bijection

$$V_{\gamma_1\otimes\mu_L^1}=c_{\gamma_1}V_{\mu_L^1}\to c_{\gamma_3\gamma_1}V_{\mu_L^1}=V_{\gamma_3\gamma_1\otimes\mu_L^1}$$

which depends on $\omega\otimes\chi\in\mathrm{Irr}^{\mathfrak s_L}(L)$ in an algebraic way. More precisely,

$$c_{\gamma_3}e_{\gamma_1\otimes\mu_L^1}\in\mathcal O(T_{\mathfrak s})^{W_{\mathfrak s}}\otimes\mathrm{Hom}_{\mathbb C}(V_{\gamma_1\otimes\mu_L^1},V_{\gamma_3\gamma_1\otimes\mu_L^1}). \tag{144}$$

Consequently (142) extends to an algebra isomorphism

$$e_{\mu_L}\mathcal H(L)e_{\mu_L}\to\mathcal O(T_{\mathfrak s})\otimes\mathrm{End}_{\mathbb C}(V_{\mu_L}). \tag{145}$$

It is more difficult to see what (142) should look like for elements of $[L/H_\lambda]$. For those we use a different, inexplicit argument.

For each $a\in[L/H_\lambda]$ the inclusion

$$ae_{\mu_L}a^{-1}\mathcal H(L)ae_{\mu_L}a^{-1}\to e^{\mathfrak s}_L\mathcal H(L)e^{\mathfrak s}_L$$

is a Morita equivalence, because the idempotents $ae_{\mu_L}a^{-1}, e^{\mathfrak s}_L$ and e_{λ_L} all see exactly the same category of L-representations, namely $\mathrm{Rep}^{\mathfrak s_L}(L)$. For every $V\in\mathrm{Rep}^{\mathfrak s_L}(L)$ we have

$$e^{\mathfrak s}_L V=\bigoplus\nolimits_{a\in[L/H_\lambda]}ae_{\mu_L}a^{-1}V,$$

where all the summands have the same dimension. It follows that

$$e^{\mathfrak s}_L\mathcal H(L)e^{\mathfrak s}_L\cong e_{\mu_L}\mathcal H(L)e_{\mu_L}\otimes M_{|L/H_\lambda|}(\mathbb C).$$

By (145) the right hand side is isomorphic to

$$\mathcal O(T_{\mathfrak s})\otimes\mathrm{End}_{\mathbb C}(V_{\mu_L})\otimes M_{|L/H_\lambda|}(\mathbb C)\cong\mathcal O(T_{\mathfrak s})\otimes\mathrm{End}_{\mathbb C}(e^{\mathfrak s}_L V).$$

(c) Just like (140) there is an isomorphism of $e^{\mathfrak s}_M\mathcal H(M)e^{\mathfrak s}_M$-bimodules

$$e^{\mathfrak s}_M\mathcal H(M)e^{\mathfrak s}_M\cong\bigoplus_{a_1,a_2\in[L/H_\lambda]}\bigoplus_{\gamma_1,\gamma_2\in X^L(\mathfrak s/\lambda)}a_1c_{\gamma_1}e_\lambda\mathcal H(M)e_\lambda c_{\gamma_2}^{-1}a_2^{-1},$$

and it extends

$$e_\mu \mathcal{H}(M) e_\mu \cong \bigoplus_{\gamma_1,\gamma_2 \in X^L(\mathfrak{s}/\lambda)} c_{\gamma_1} e_\lambda \mathcal{H}(M) e_\lambda c_{\gamma_2}^{-1}.$$

For $x \in X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}$ let $f_{x,\lambda} \in e_\lambda \mathcal{H}(M) e_\lambda$ be the element that corresponds to

$$[x] \in \mathcal{H}(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}, q_\mathfrak{s}) \cong \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$$

via Theorem 4.1. The elements $f_{x,\lambda}$ commute with $e_\lambda \mathcal{H}(K) e_\lambda \cong \mathrm{End}_\mathbb{C}(V_\lambda)$. It follows that the element

$$c_\gamma f_{x,\lambda} c_\gamma^{-1} = c_\gamma e_\lambda f_{x,\lambda} e_\lambda c_\gamma^{-1} = e_{\lambda\otimes\gamma} c_\gamma f_{x,\lambda} c_\gamma^{-1} e_{\lambda\otimes\gamma} \tag{146}$$

is independent of the choice of c_γ in Proposition 3.1. As conjugation by c_γ turns the commutative diagram (126) into the corresponding diagram for $\lambda \otimes \gamma$, we have

$$c_\gamma f_{x,\lambda} c_\gamma^{-1} = f_{x,w(\lambda)\otimes\gamma}, \tag{147}$$

the image of $[x]$ in $e_{w(\lambda)\otimes\gamma} \mathcal{H}(M) e_{\lambda\otimes\gamma}$ under the canonical isomorphisms from Theorem 4.1. For every $x \in X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}$ we define

$$f_{x,\mu^1} := \sum_{\gamma \in X^L(\mathfrak{s}/\lambda)^1} c_\gamma f_{x,\lambda} c_\gamma^{-1} = \sum_{\gamma \in X^L(\mathfrak{s}/\lambda)^1} f_{x,\lambda\otimes\gamma} \in e_{\mu^1} \mathcal{H}(M) e_{\mu^1}. \tag{148}$$

By (146) f_{x,μ^1} commutes with $e_{\mu^1} \mathcal{H}(K) e_{\mu^1}$ and with the c_γ for $\gamma \in X^L(\mathfrak{s}/\lambda)^1$, so it commutes with $e_{\mu^1} \mathcal{H}(PN^1(K,\lambda)) e_{\mu^1}$. By (140) and Theorem 4.1

$$e_{\mu^1} \mathcal{H}(M) e_{\mu^1} = \bigoplus\nolimits_{x \in X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}} \mathbb{C} f_{x,\mu^1} \otimes e_{\mu^1} \mathcal{H}(PN^1(K,\lambda)) e_{\mu^1},$$

and the support of this algebra is $PN^1(K,\lambda) \widetilde{X^*(T_\mathfrak{s})} W_\mathfrak{s} PN^1(K,\lambda)$.

The orthogonality of the different idempotents $e_{\lambda\otimes\gamma}$ implies that the f_{x,μ^1} satisfy the same multiplication rules as the $f_{x,\lambda}$. Hence the span of the f_{x,μ^1} is a subalgebra of $e_{\mu^1} \mathcal{H}(M) e_{\mu^1}$ isomorphic with $\mathcal{H}(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}, q_\mathfrak{s})$. We constructed an algebra isomorphism

$$e_{\mu^1} \mathcal{H}(M) e_{\mu^1} \cong \mathcal{H}(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_{\mu^1}). \tag{149}$$

Since $\mathcal{H}(M,\mu^1)$ is the commutant of $\mathrm{End}_\mathbb{C}(V_{\mu^1})$ inside

$$\mathcal{H}(M,\mu^1) \otimes \mathrm{End}_\mathbb{C}(V_{\mu^1}) \cong e_{\mu^1} \mathcal{H}(M) e_{\mu^1}, \tag{150}$$

it corresponds to $\mathcal{H}(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}, q_\mathfrak{s})$ under the isomorphisms (149) and (150).

Tensored with the identity on $\mathrm{End}_\mathbb{C}(V_\lambda)$, $t_{P,\lambda}$ from (125) becomes a canonical injection

$$e_{\lambda_L} \mathcal{H}(L) e_{\lambda_L} \cong \mathcal{H}(L,\lambda_L) \otimes \mathrm{End}_\mathbb{C}(V_\lambda) \to \mathcal{H}(M,\lambda) \otimes \mathrm{End}_\mathbb{C}(V_\lambda) \cong e_\lambda \mathcal{H}(M) e_\lambda. \tag{151}$$

Since $t_{P,\lambda}$ and the analogous map t_{P,μ^1} for μ^1 are uniquely defined by the same property, they agree in the sense that

$$t_{P,\lambda} \otimes \mathrm{id} = t_{P,\mu^1} \otimes \mathrm{id} \text{ on } e_{\lambda_L} \mathcal{H}(L) e_{\lambda_L} \cong \mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\lambda). \tag{152}$$

Consequently the isomorphisms (149) and (141) fit in a commutative diagram

$$\begin{array}{ccc} \mathcal{H}(M,\mu^1) & \to & \mathcal{H}(X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}, q_\mathfrak{s}) \cong \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \\ \uparrow t_{P,\mu^1} & & \uparrow i_{P,\mu^1} \\ \mathcal{H}(L,\mu^1_L) & \to & \mathcal{O}(T_\mathfrak{s}) \cong \mathbb{C}[X^*(T_\mathfrak{s})], \end{array} \tag{153}$$

Here i_{P,μ^1} is defined like $i_{P,\lambda}$, see (127). In this sense (149) is canonical.
(d) Part (c) works equally well with $\gamma_1 \otimes \mu^1$ instead of μ^1. For all $\gamma_1 \in X^L(\mathfrak{s})$ together that gives a canonical isomorphism
(154)
$$\bigoplus_{\gamma_1 \in X^L(\mathfrak{s})/X^L(\mathfrak{s})^1} e_{\gamma_1 \otimes \mu^1} \mathcal{H}(M) e_{\gamma_1 \otimes \mu^1} \to \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \bigoplus_{\gamma_1 \in X^L(\mathfrak{s})/X^L(\mathfrak{s})^1} \mathrm{End}_\mathbb{C}(V_{\gamma_1 \otimes \mu^1}).$$
The formula (144) defines an element of $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)$ which commutes with $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$. Therefore we can extend (154) to isomorphisms
$$\begin{aligned} e_\mu \mathcal{H}(M) e_\mu &\to \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu), \\ e^\mathfrak{s}_M \mathcal{H}(M) e^\mathfrak{s}_M &\to \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(e^\mathfrak{s}_M V) \end{aligned}$$
in the same way as we did in the proof of part (b). The support of $e_\mu \mathcal{H}(M) e_\mu$ can also be determined as in part (b), using the support of $e_{\mu^1} \mathcal{H}(M) e_{\mu^1}$, as determined in part (c). □

We note that a vector space basis of $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \subset e_\mu \mathcal{H}(M) e_\mu$ is formed by the elements
$$f_{x,\mu} = \sum_{\gamma \in X^L(\mathfrak{s}/\lambda)} c_\gamma f_{x,\lambda} c_\gamma^{-1} = \sum_{\gamma \in X^L(\mathfrak{s}/\lambda)} f_{x,\lambda \otimes \gamma}. \tag{155}$$
We define the projective normalizer of (K, λ) in M as
$$\{g \in N_M(K) \mid g \cdot \lambda \cong \lambda \otimes \gamma \text{ for some } \gamma \in X^L(\mathfrak{s})\}. \tag{156}$$
Using the explicit information gathered in the above proof, we can show that it is none other than $PN(K, \lambda)$ as defined in (138).

Lemma 4.6. *Recall the K-representations μ and μ^1 from* (130).
(*a*) $(PN^1(K, \lambda), \mu^1)$ *is a cover of* $(PN^1(K_L, \lambda_L), \mu^1_L)$.
(*b*) $PN(K, \lambda)$ *equals the projective normalizer of* (K, λ) *in* M.
(*c*) $PN^1(K, \lambda)$ *equals the projective normalizer of* (K, λ) *in* M^1.

Proof. (a) For the definition of a cover we refer to [7, 8.1]. By Lemma 4.4 $PN^1(K, \lambda) \cap L = PN^1(K_L, \lambda_L)$ and by [22, Proposition 5.5] K admits an Iwahori decomposition with respect to any parabolic subgroup of M with Levi factor L. Hence $PN^1(K, \lambda)$ is also decomposed in this sense. The second condition for a cover says that $\mu|_{N(K_L, \mu_L)} = \mu_L$, which is true by definition. The third condition is about the existence of an invertible "strongly positive" element in $\mathcal{H}(M, \mu^1)$. By [22, Proposition 5.5] $\mathcal{H}(M, \lambda)$ contains such an element, in the notation of the proof of Theorem 4.5 it corresponds to $f_{x,\lambda}$ for a suitable $x \in X^*(T_\mathfrak{s})$. Then f_{x,μ^1} and its image in $\mathcal{H}(M, \mu^1)$ have the correct properties.
(b) By Lemma 4.4 $PN(K, \lambda)$ is contained in this normalizer.

Consider any g in the group (156). Its intertwining property entails that $e_\mu g e_\mu \in e_\mu \mathcal{H}(M) e_\mu$ has inverse $e_\mu g^{-1} e_\mu$. From Theorem 4.1 we can see what the support of $e_\mu \mathcal{H}(M) e_\mu$ is, namely
$$PN(K, \lambda) \widetilde{X^*(T_\mathfrak{s})} W_\mathfrak{s} PN(K, \lambda).$$
Possibly adjusting g from the left and from the right by an element of $PN(K, \lambda)$, we may assume that
$$g \in \widetilde{X^*(T_\mathfrak{s})} W_\mathfrak{s} \subset N_M(L).$$

Then g also normalizes (K_L, μ_L). With Conditions 1.1 we see easily that every element of $W_\mathfrak{s}$ normalizes (K_L, μ_L). Writing $g = xw$ with $x \in \widetilde{X^*(T_\mathfrak{s})}, w \in W_\mathfrak{s}$, we find that $x \in L$ normalizes (K_L, μ_L) as well. In other words

$$x \in PN(K_L, \lambda_L) \subset PN(K, \lambda).$$

It follows that $w \in W_\mathfrak{s}$ must also normalize (K, μ). By considering supports in Theorem 4.5.c, we see that

$$e_{\mu^1} w e_{\mu^1} \mapsto [w] \otimes \mathrm{End}_\mathbb{C}(V_{\mu^1}) \subset \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_{\mu^1}).$$

By (154) there exists a unique $h_1 \in \bigoplus_{\gamma_1 \in X^L(\mathfrak{s})/X^L(\mathfrak{s})^1} \mathrm{End}_\mathbb{C}(V_{\gamma_1 \otimes \mu^1})$ such that

$$e_\mu w e_\mu = \sum_{\gamma_1 \in X^L(\mathfrak{s})/X^L(\mathfrak{s})^1} e_{\gamma_1 \otimes \mu^1} w e_{\gamma_1 \otimes \mu^1} \mapsto [w] \otimes h_1 \in \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu).$$

Similarly $e_\mu w^{-1} e_\mu$ maps to $[w^{-1}] \otimes h_2$ under (154), for some $h_2 \in \mathrm{End}_\mathbb{C}(V_\mu)$. Because w lies in (156),

$$\begin{aligned} & e_\mu w e_\mu \cdot e_\mu w^{-1} e_\mu = e_\mu, \\ & [w] \otimes h_1 \cdot [w^{-1}] \otimes h_2 = [w] \cdot [w^{-1}] \otimes h_1 h_2 = [1] \otimes \mathrm{id}. \end{aligned}$$

In particular $[w][w^{-1}] \in \mathbb{C}[1]$. The multiplication rules for $\mathcal{H}(W_\mathfrak{s}, q_\mathfrak{s})$ [22, 2.5.3], applied with induction to the length of $w \in \widetilde{W_\mathfrak{s}}$, show that this is only possible if $w = 1$. Consequently $g = x \in PN(K, \lambda) \cap \widetilde{X^*(T_\mathfrak{s})}$.
(c) This follows immediately from (b). □

Remark 4.7. In Lemma 4.6 we construct a $\mathfrak{s}_M$-type with representation μ^1, but we do not succeed in finding a $\mathfrak{s}_M$-type with representation μ. The obstruction appears to be that some of the representations $\lambda \otimes \gamma$ are conjugate in M, but not via an element of M^1. Examples 5.7 and 5.8 show that this can really happen when G is not split.

Mainly for this reason we have been unable to construct types for all Bernstein components of $G^\sharp$. In the special case where all the K_G-representations $\lambda_G \otimes \gamma$ with $\gamma \in X^G(\mathfrak{s})$ are conjugate via elements of G^1, we can construct types for every Bernstein component $\mathfrak{t}^\sharp \prec \mathfrak{s}$. We did not include this in the paper because it is quite some work and it is not clear how often these extra conditions are fulfilled.

4.3. **Hecke algebras for the intermediate group.**

In (69) we constructed an idempotent $e^\mathfrak{s}_M$, using the set $[L/H_\lambda]$ from Lemma 3.3. In Proposition 3.9 and Lemma 3.10 that the algebras

$$(e^\mathfrak{s}_M \mathcal{H}(M) e^\mathfrak{s}_M)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s} \quad \text{and} \quad (e^\mathfrak{s}_M \mathcal{H}(M) e^\mathfrak{s}_M) \rtimes \mathrm{Stab}(\mathfrak{s}, P \cap M).$$

are Morita equivalent with $\mathcal{H}(G^\sharp Z(G))^\mathfrak{s}$. In Theorem 3.16 we showed that the first one is even isomorphic to a subalgebra of $\mathcal{H}(G^\sharp Z(G))^\mathfrak{s}$ determined by an idempotent. In Lemma 3.7 we saw that the actions of $X^L(\mathfrak{s})$ and $\mathfrak{R}^\sharp_\mathfrak{s}$ both come from the action α of $\mathrm{Stab}(\mathfrak{s}, P \cap M)$ defined in (73).

Lemma 4.8. *There is an equality*

$$(e^\mathfrak{s}_M \mathcal{H}(M) e^\mathfrak{s}_M)^{X^L(\mathfrak{s},\lambda)} = \bigoplus_{a \in [L/H_\lambda]} (a e_\mu a^{-1} \mathcal{H}(M) a e_\mu a^{-1})^{X^L(\mathfrak{s},\lambda)},$$

and this algebra is $\mathrm{Stab}(\mathfrak{s}, P \cap M)$*-equivariantly isomorphic to*

$$\bigoplus\nolimits_1^{|X^L(\omega,V_\mu)|} (e_\mu \mathcal{H}(M) e_\mu)^{X^L(\mathfrak{s},\lambda)}.$$

Remark. Here and below we use the notation $\bigoplus_1^n$ for the direct sum of n copies of something.

Proof. As vector spaces

$$e^{\mathfrak{s}}_M \mathcal{H}(M) e^{\mathfrak{s}}_M = \bigoplus\nolimits_{a_1,a_2 \in [L/H_\lambda]} a_1 e_\mu \mathcal{H}(M) e_\mu a_2^{-1}. \tag{157}$$

On the other hand, for any $\mathcal{H}(M)^{\mathfrak{s}_M}$-module V we have the decompositions

$$e^{\mathfrak{s}}_M V = \bigoplus_{a \in [L/H_\lambda]} a e_\mu a^{-1} V = \bigoplus_{\rho \in \mathrm{Irr}(\mathbb{C}[X^L(\mathfrak{s},\lambda) \cap X^L(\omega), \kappa_\omega])} e^{\mathfrak{s}}_M V_\rho. \tag{158}$$

Here every $a e_\mu a^{-1} V$ equals $\bigoplus_{\rho \in I_a} e^{\mathfrak{s}}_M V_\rho$ for a suitable collection I_a of ρ's. If $f \in e^{\mathfrak{s}}_M \mathcal{H}(M) e^{\mathfrak{s}}_M$ is invariant under $X^L(\mathfrak{s},\lambda) \cap X^L(\omega)$, then it commutes with the idempotents $e_\rho \in \mathbb{C}[X^L(\mathfrak{s},\lambda) \cap X^L(\omega), \kappa_\omega])$, so it stabilizes each of the subspaces $e^{\mathfrak{s}}_M V_\rho$. Therefore it also preserves the rougher decomposition $e^{\mathfrak{s}}_M V = \bigoplus_{a \in [L/H_\lambda]} a e_\mu a^{-1} V$. In view of (157), this is only possible if

$$f \in \bigoplus\nolimits_{a \in [L/H_\lambda]} a e_\mu a^{-1} \mathcal{H}(M) a e_\mu a^{-1},$$

which proves the desired equality.

By Lemma 3.3.b conjugation by $a \in [L/H_\lambda]$ gives a $\mathrm{Stab}(\mathfrak{s}, P \cap M)$-equivariant isomorphism

$$e_\mu \mathcal{H}(M) e_\mu \to a e_\mu \mathcal{H}(M) e_\mu a^{-1} = a e_\mu a^{-1} \mathcal{H}(M) a e_\mu a^{-1}.$$

By Lemma 3.5 $|L/H_\lambda| = |\mathrm{Irr}(X^L(\omega, V_\mu))|$ and this equals $|X^L(\omega, V_\mu)|$ since we dealing with an abelian group. $\square$

It turns out that the direct sum decomposition from Lemma 4.8 can already be observed on the level of subalgebras of $\mathcal{H}(G)$:

Lemma 4.9. *There are algebra isomorphisms*

$$\begin{aligned}
\bigoplus\nolimits_1^{|X^L(\omega,V_\mu)|} (e_\mu \mathcal{H}(M) e_\mu)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}} &\cong (e^{\mathfrak{s}}_M \mathcal{H}(M) e^{\mathfrak{s}}_M)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}} \\
&\cong (e^\sharp_{\lambda_G} \mathcal{H}(G) e^\sharp_{\lambda_G})^{X^G(\mathfrak{s})} = \bigoplus\nolimits_{a \in [L/H_\lambda]} (a e_{\mu_G} \mathcal{H}(G) e_{\mu_G} a^{-1})^{X^G(\mathfrak{s})} \\
&\cong \bigoplus\nolimits_1^{|X^L(\omega,V_\mu)|} (e_{\mu_G} \mathcal{H}(G) e_{\mu_G})^{X^G(\mathfrak{s})}.
\end{aligned}$$

Proof. The first isomorphism is a direct consequence of Lemma 4.8 and the second is Proposition 3.15.b. As shown in Proposition 3.15, it can be decomposed as

$$\bigoplus_{a_1,a_2 \in [L/H_\lambda]} (a_1 e_{\mu_G} \mathcal{H}(G) e_{\mu_G} a_2^{-1})^{X^G(\mathfrak{s})} \longleftrightarrow \bigoplus_{a_1,a_2 \in [L/H_\lambda]} (a_1 e_\mu \mathcal{H}(M) e_\mu a_2^{-1})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}.$$

But by Lemma 4.8 the summands with $a_1 \neq a_2$ are 0 on the right hand side, so they are also 0 on the left hand side. This proves the equality in the lemma.

The final isomorphism is given by

$$(e_{\mu_G} \mathcal{H}(G) e_{\mu_G})^{X^G(\mathfrak{s})} \to (a e_{\mu_G} \mathcal{H}(G) e_{\mu_G} a^{-1})^{X^G(\mathfrak{s})} : f \mapsto a f a^{-1}. \qquad \square$$

Recall from Theorem 3.16 that the middle algebra in Lemma 4.9 is isomorphic to $e^\sharp_{\lambda_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G)) e^\sharp_{\lambda_{G^\sharp Z(G)}}$, which is Morita equivalent with $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$. In the above direct sum decomposition also holds on this level. to formulate it, let $e_{\mu_{G^\sharp Z(G)}} \in \mathcal{H}(G^\sharp Z(G))$ be the restriction of $e_{\mu_G} : K_G \to \mathbb{C}$ to $G^\sharp Z(G) \cap K$. By our choice of Haar measures, $e_{\mu_{G^\sharp Z(G)}}$ is idempotent. See also (109).

Corollary 4.10. *There are algebra isomorphisms*

$$
\begin{aligned}
e^\sharp_{\lambda_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G)) e^\sharp_{\lambda_{G^\sharp Z(G)}} &= \bigoplus\nolimits_{a \in [L/H_\lambda]} a e_{\mu_{G^\sharp Z(G)}} a^{-1} \mathcal{H}(G^\sharp Z(G)) a e_{\mu_{G^\sharp Z(G)}} a^{-1} \\
&\cong \bigoplus\nolimits_1^{|X^L(\omega, V_\mu)|} e_{\mu_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G)) e_{\mu_{G^\sharp Z(G)}} \\
&\cong \bigoplus\nolimits_1^{|X^L(\omega, V_\mu)|} (\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}.
\end{aligned}
$$

Proof. The equality comes from Lemma 4.9 and Theorem 3.16. For all $a \in [L/H_\lambda]$ the map

$$e_{\mu_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G)) e_{\mu_{G^\sharp Z(G)}} \to a e_{\mu_{G^\sharp Z(G)}} a^{-1} \mathcal{H}(G^\sharp Z(G)) a e_{\mu_{G^\sharp Z(G)}} a^{-1} : f \mapsto a f a^{-1}$$

is an algebra isomorphism. The remaining isomorphism follows again from Lemma 4.9. $\square$

Now we analyse the action of $\mathrm{Stab}(\mathfrak{s}, P \cap M)$ on $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$ in Corollary 4.10. For every $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}, P \cap M)$ there exists a $\chi_\gamma \in X_{\mathrm{nr}}(L)$ such that

$$w(\omega) \otimes \gamma \cong \omega \otimes \chi_\gamma \in \mathrm{Irr}(L). \tag{159}$$

Here $\omega \otimes \chi_\gamma$ is unique, so χ_γ is unique up to $X_{\mathrm{nr}}(L, \omega)$. If γ itself is unramified, then $w = 1$ by Lemma 2.4.d. Therefore we may and will assume that

$$\chi_\gamma = \gamma \quad \text{if} \quad \gamma \in X_{\mathrm{nr}}(L / L^\sharp Z(G)) = X_{\mathrm{nr}}(L) \cap X^L(\mathfrak{s}). \tag{160}$$

In view of the conditions 1.1, in particular $L = \prod_i L_i^{e_i}$, we may simultaneously assume that

$$\chi_\gamma = \prod\nolimits_i \chi_{\gamma,i}^{\otimes e_i}. \tag{161}$$

Notice that with this choice $\chi_\gamma \in X_{\mathrm{nr}}(L)$ is invariant under $W_{\mathfrak{s}} = W(M, L)$. Via the map $\chi \mapsto \omega \otimes \chi$, χ_γ determines a $W_{\mathfrak{s}}$-invariant element of $T_{\mathfrak{s}}$.

Recall the bijection

$$J(\gamma, \omega \otimes \chi_\gamma^{-1}) \in \mathrm{Hom}_L(\omega \otimes \chi_\gamma^{-1}, w^{-1}(\omega) \otimes \gamma^{-1})$$

from (46). It restricts to a bijection

$$V_\mu = e_\mu V_{\omega \otimes \chi_\gamma^{-1}} \to V_\mu = e_\mu V_{w^{-1}(\omega) \otimes \gamma^{-1}}.$$

Clearly (160) enables us to take

$$J(\gamma, \omega \otimes \chi_\gamma^{-1}) = \mathrm{id}_{V_\omega} \quad \text{if} \quad \gamma \in X_{\mathrm{nr}}(L / L^\sharp Z(G)). \tag{162}$$

Recall from (70) and (14) that

$$J(\gamma, \omega \otimes \chi_\gamma^{-1})|_{V_\mu} \in \mathbb{C}^\times \mathrm{id}_{V_\mu} \quad \text{if} \quad \gamma \in X^L(\omega, V_\mu). \tag{163}$$

Lemma 4.11. *The action α of* $\mathrm{Stab}(\mathfrak{s}, P \cap M)$ *on*

$$e_\mu \mathcal{H}(M) e_\mu \cong \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)$$

preserves both tensor factors. On $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$ *it is given by*

$$\alpha_{(w,\gamma)}(\theta_x [v]) = \chi_\gamma^{-1}(x) \theta_{w(x)} [w v w^{-1}] \qquad x \in X^*(T_\mathfrak{s}), v \in W_\mathfrak{s},$$

and on $\mathrm{End}_\mathbb{C}(V_\mu)$ *by*

$$\alpha_{(w,\gamma)}(h) = J(\gamma, \omega \otimes \chi_\gamma^{-1}) \circ h \circ J(\gamma, \omega \otimes \chi_\gamma^{-1})^{-1}.$$

Furthermore $X^L(\omega, V_\mu)$ *is the subgroup of elements that act trivially.*

Remark. It is crucial that χ_γ is $W_\mathfrak{s}$-invariant and that w normalizes $P \cap M$, for otherwise the above formulae would not define an algebra automorphism of $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$. This can be seen with the Bernstein presentation, in particular (116).

Proof. By definition, for all $\chi \in X_{\mathrm{nr}}(L), f \in e_{\mu_L} \mathcal{H}(L) e_{\mu_L}$

$$\begin{aligned} (\omega \otimes \chi)(\alpha_{(w,\gamma)}(f)) &= (w^{-1}(\omega \otimes \chi\gamma^{-1}))(f) \\ &= J(\gamma, \omega \otimes \chi_\gamma^{-1}) \circ (\omega \otimes w^{-1}(\chi)\chi_\gamma^{-1})(f) \circ J(\gamma, \omega \otimes \chi_\gamma^{-1})^{-1}. \end{aligned} \tag{164}$$

For f with image in $\mathrm{End}_\mathbb{C}(V_\mu)$ the unramified characters χ and $w^{-1}(\chi)\chi_\gamma^{-1}$ are of no consequence. Thus (164) implies the asserted formula for $\alpha_{(w,\gamma)}$ on $\mathrm{End}_\mathbb{C}(V_\mu)$.

Since $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$ is the centralizer of $\mathrm{End}_\mathbb{C}(V_\mu)$ in $e_\mu \mathcal{H}(M) e_\mu$, it is also stabilized by $\mathrm{Stab}(\mathfrak{s}, P \cap M)$. By considering supports we see that

$$\alpha_{(w,\gamma)}([x]) \in \mathbb{C}[w x w^{-1}] \text{ for all } x \in X^*(T_\mathfrak{s}) \rtimes W_\mathfrak{s}. \tag{165}$$

For any simple reflection $s \in W_\mathfrak{s}$, $w s w^{-1}$ is again a simple reflection in $W_\mathfrak{s}$, because $w \in \mathfrak{R}_\mathfrak{s}^\sharp$ normalizes $P \cap M$. In $\mathcal{H}(W_\mathfrak{s}, q_\mathfrak{s})$ we have

$$([s] + 1)([s] - q_\mathfrak{s}(s)) = 0 = ([w s w^{-1}] + 1)([w s w^{-1}] - q_\mathfrak{s}(w s w^{-1})),$$

where $q_\mathfrak{s}(s), q_\mathfrak{s}(w s w^{-1}) \in \mathbb{R}_{>1}$. Since $\alpha_{(w,\gamma)}$ is an algebra automorphism, we deduce that

$$q_\mathfrak{s}(s) = q_\mathfrak{s}(w s w^{-1})$$

and that $\alpha_{(w,\gamma)}([s]) = [w s w^{-1}]$. Every $v \in W_\mathfrak{s}$ is a product of simple reflections s_i, and then $[v]$ is a product of the $[s_i]$ in the same way. Hence

$$\alpha_{(w,\gamma)}([v]) = [w v w^{-1}] \text{ for all } v \in W_\mathfrak{s}.$$

The formula (73) also defines an action α of $\mathrm{Stab}(\mathfrak{s}, P \cap M)$ on

$$e_{\mu_L} \mathcal{H}(L) e_{\mu_L} \cong \mathcal{O}(T_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu),$$

which for similar reasons stabilizes $\mathrm{End}_\mathbb{C}(V_\mu)$. Now we have two actions of $\mathrm{Stab}(\mathfrak{s}, P \cap M)$ on $\mathrm{End}_\mathbb{C}(V_\mu)$, depending on whether we considere it as a subalgebra of $e_{\mu_L} \mathcal{H}(L) e_{\mu_L}$ or of $e_\mu \mathcal{H}(M) e_\mu$. It is obvious from the definition of μ that these two actions agree.

The maps (151) and (153) lead to a canonical injection

$$i_{P,\mu} : \mathcal{O}(T_\mathfrak{s}) \to \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}),$$

which is a restriction of the map $\bigoplus_{\gamma \in X^L(\mathfrak{s})/X^L(\mathfrak{s})^1} i_{P,\mu^1 \otimes \gamma}$. Since it is canonical, $i_{P,\mu}$ commutes with the respective actions $\alpha_{(w,\gamma)}$.

Now we take $x \in X^*(T_\mathfrak{s}) \subset \mathcal{O}(T_\mathfrak{s})$ in (164). With a Morita equivalence we can replace $\omega \otimes \chi$ by the one-dimensional $\mathcal{O}(T_\mathfrak{s})$-representation $[\omega \otimes \chi]$ with character $\omega \otimes \chi \in T_\mathfrak{s}$. Then (164) becomes

$$\begin{aligned}(166) \quad [\omega \otimes \chi](\alpha_{(w,\chi)}(x)) &= [\omega \otimes \chi_\gamma^{-1} w^{-1}(\chi)](x) \\ &= [\omega \otimes w^{-1}(\chi)](\chi_\gamma^{-1}(x)x) = [\omega \otimes \chi](\chi_\gamma^{-1}(x)w(x)).\end{aligned}$$

Thus $\alpha_{(w,\gamma)}(x) = \chi_\gamma^{-1}(x)w(x)$. For $x \in X^*(T_\mathfrak{s})$ positive $w(x)$ is also positive, as w normalizes $P \cap M$. We obtain

$$\alpha_{(w,\gamma)}(\theta_x) = \alpha_{(w,\gamma)}(i_{P,\mu}(x)) = i_{P,\mu}(\alpha_{(w,\chi)}(x)) = i_{P,\mu}(\chi_\gamma^{-1}(x)w(x)) = \chi_\gamma^{-1}(x)\theta_{w(x)}.$$

Since $\alpha_{(w,\gamma)}$ is an algebra homomorphism, this implies the same formula for all $x \in X^*(T_\mathfrak{s})$.

It is clear from (163) and (164) that the group $X^L(\omega, V_\mu)$ fixes every element of $e_\mu \mathcal{H}(M) e_\mu$. Conversely, suppose that (w,γ) acts trivially. From the formulas for the action on $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s})$ we see that $w = 1$ and $\chi_\gamma \in X_{\mathrm{nr}}(L,\omega)$, so $\gamma \in X^L(\omega)$. Then we deduce from (164) that $\gamma \in X^L(\omega, V_\mu)$. $\square$

Since we have a type with idempotent e_{μ^1} but not with idempotent e_μ, we would like to reduce $(e_\mu \mathcal{H}(M) e_\mu)^{X^L(\mathfrak{s})}$ to $(e_{\mu^1} \mathcal{H}(M) e_{\mu^1})^{X^L(\mathfrak{s})}$. The next lemma solves a part of the problem, namely the elements c_γ that do not lie in L^1.

Lemma 4.12. *Let $\gamma \in X^L(\mathfrak{s})$ be such that*

$$\chi(c_\gamma) = 1 \textit{ for all } \chi \in X_{\mathrm{nr}}(L/L^\sharp Z(G)) \cap X_{\mathrm{nr}}(L,\omega).$$

Then there exists $x_\gamma \in \widetilde{X^*(T_\mathfrak{s})}$ such that $x_\gamma c_\gamma \in L^1 L^\sharp$.

Proof. Recall that every $\chi \in X_{\mathrm{nr}}(L,\omega)$ vanishes on $Z(L)$. Hence

$$X_{\mathrm{nr}}(L/L^\sharp Z(G)) \cap X_{\mathrm{nr}}(L,\omega) = X_{\mathrm{nr}}(L/L^\sharp) \cap X_{\mathrm{nr}}(L,\omega),$$

and c_γ determines a character of

$$(167) \qquad X_{\mathrm{nr}}(L/L^\sharp)/X_{\mathrm{nr}}(L/L^\sharp) \cap X_{\mathrm{nr}}(L,\omega).$$

This is a subtorus of $T_\mathfrak{s} \cong X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\omega)$, so we can find $x_\gamma \in \widetilde{X^*(T_\mathfrak{s})}$ such that x_γ^{-1} restricts to the same character of (167) as c_γ. Then

$$x_\gamma c_\gamma \in \bigcap\nolimits_{\chi \in X_{\mathrm{nr}}(L/L^\sharp)} \ker \chi = L^1 L^\sharp. \qquad \square$$

In view of Lemma 4.2 we may replace c_γ by $x_\gamma c_\gamma$ as in Lemma 4.12. From now on we assume that this has been done for all γ to which Lemma 4.12 applies. Recall that we were already assuming that $c_\gamma \in L^1$ whenever this is possible.

This gives rise to groups

$$\begin{aligned} &X^L(\mathfrak{s})^2 = \{\gamma \in X^L(\mathfrak{s}) \mid c_\gamma \in L^1 L^\sharp\}, \\ &\mathrm{Stab}(\mathfrak{s}, P \cap M)^2 = \{(w,\gamma) \in \mathrm{Stab}(\mathfrak{s}, P \cap M) \mid c_\gamma \in L^1 L^\sharp\},\end{aligned}$$

and to an idempotent

$$e_{\mu^2} = \sum_{\gamma \in X^L(\mathfrak{s})^2 / X^L(\mathfrak{s})^1} e_{\mu^1} = \sum_{\gamma \in X^L(\mathfrak{s})^2 / (X^L(\mathfrak{s}) \cap X^G(\mathfrak{s},\lambda)} e_{\lambda \otimes \gamma}.$$

The tower of groups

$$X^L(\mathfrak{s})^1 \subset X^L(\mathfrak{s})^2 \subset X^L(\mathfrak{s})$$

corresponds to a tower of K-representations

$$\mu^1 \subset \mu^2 \subset \mu,$$

where $V_{\mu^2} = \bigoplus_{\gamma \in X^L(\mathfrak{s})^2 / X^L(\mathfrak{s})^1} V_{\mu^1}$. With these objects we can refine Corollary 4.10.

Theorem 4.13. *The algebra $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ is Morita equivalent with a direct sum of $|X^L(\omega, V_\mu)|$ copies of*

$$\begin{aligned} & e_{\mu_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G)) e_{\mu_{G^\sharp Z(G)}} \cong \\ & (\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})/X^L(\omega,V_\mu)} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}} \cong \\ & (\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^2}))^{X^L(\mathfrak{s})^2/X^L(\omega,V_\mu)} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}. \end{aligned}$$

Here the actions $X^L(\mathfrak{s})$ and $\mathfrak{R}^\sharp_{\mathfrak{s}}$ come from the action of $\mathrm{Stab}(\mathfrak{s}, P \cap M)$ *described in Lemma 4.11. In particular the group $X_{\mathrm{nr}}(L/L^\sharp Z(G))$ acts only via translations on $T_{\mathfrak{s}}$.*

Proof. By Theorem 3.16 and Corollary 4.10 it suffices to establish the second isomorphism. From Lemma 4.11 we see that $X_{\mathrm{nr}}(L/L^\sharp Z(G)) \cap X_{\mathrm{nr}}(L, \omega)$ fixes the subalgebra

$$e_{\mu^2} \mathcal{H}(M) e_{\mu^2} \subset e_\mu \mathcal{H}(M) e_\mu$$

pointwise. Hence the same holds for

$$e_{\mu^2 \otimes \gamma} \mathcal{H}(M) e_{\mu^2 \otimes \gamma} = c_\gamma e_{\mu^2} \mathcal{H}(M) e_{\mu^2} c_\gamma^{-1}$$

for any $\gamma \in X^L(\mathfrak{s})$. On the other hand, by Lemma 4.12 $X_{\mathrm{nr}}(L/L^\sharp Z(G)) \cap X_{\mathrm{nr}}(L, \omega)$ does not fix any $c_\gamma \in L \setminus L^1 L^\sharp$. Therefore

$$(e_\mu \mathcal{H}(M) e_\mu)^{X^L(\mathfrak{s})} = \Big(\bigoplus_{\gamma \in X^L(\mathfrak{s})/X^L(\mathfrak{s})^2} e_{\mu^2 \otimes \gamma} \mathcal{H}(M) e_{\mu^2 \otimes \gamma} \Big)^{X^L(\mathfrak{s})} \cong \big(e_{\mu^2} \mathcal{H}(M) e_{\mu^2} \big)^{X^L(\mathfrak{s})^2}.$$

The proof of Theorem 4.5.d shows that

$$e_{\mu^2} \mathcal{H}(M) e_{\mu^2} \cong \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^2}).$$

For $(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}, P \cap M)^2$ the intertwiner $J(\gamma, \omega \otimes \chi_\gamma^{-1})$ restricts to a bijection

$$V_{\mu^2} = e_{\mu^2} V_{\omega \otimes \chi_\gamma^{-1}} \to V_{\mu^2} = e_{\mu^2} V_{w^{-1}(\omega) \otimes \gamma^{-1}}.$$

Hence the action of $\mathrm{Stab}(\mathfrak{s}, P \cap M)^2$ on $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$ described in Lemma 4.11 preserves the subalgebra $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^2})$. By equations (162) and (163) the groups $X_{\mathrm{nr}}(L/L^\sharp Z(G))$ and $X^L(\omega, V_\mu)$ act as asserted. $\square$

In combination with Lemma 3.5, the above proof makes it clear that the use of the group $L/H_\lambda \cong \mathrm{Irr}(X^L(\omega, V_\mu))$ is unavoidable. Namely, by (17) the operators $I(\gamma, \omega \otimes \chi)$ associated to $\gamma \in X^L(\omega, V_\mu)$ cause some irreducible representations of G to split upon restriction to $G^\sharp Z(G)$. But in the proof of Theorem 4.13 we saw that the same operators act trivially on

$$e_\mu \mathcal{H}(M) e_\mu \cong \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu).$$

This has to be compensated somehow, and we do so by adding a direct summand for every irreducible representation of $X^L(\omega, V_\mu)$. See also Example 5.6.

4.4. **Hecke algebras for the derived group.**

Let $\omega \in \mathrm{Irr}(L)$ be supercuspidal and let $\sigma^\sharp$ be an irreducible subquotient of $\mathrm{Res}^L_{L^\sharp}(\omega)$, or equivalently of $\mathrm{Res}^L_{L^\sharp Z(G)}(\omega)$. Consider the Bernstein torus $T_{\mathfrak{t}^\sharp}$ of $\mathfrak{t}^\sharp = [L^\sharp, \sigma^\sharp]_{G^\sharp}$ and $T_{\mathfrak{t}}$ of $\mathfrak{t} = [L^\sharp Z(G), \sigma^\sharp]_{G^\sharp Z(G)}$. Then $T_{\mathfrak{t}^\sharp}$ is the quotient of $T_{\mathfrak{t}}$ with respect to the action of $X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp)$. In turn $T_{\mathfrak{t}}$ clearly is a quotient of $X_{\mathrm{nr}}(L)$ via $\chi \to \sigma^\sharp \otimes \chi$. But it is not so obvious that it is also a quotient of $T_{\mathfrak{s}} \cong X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\omega)$, because the isomorphism $\omega \otimes \chi \cong \omega$ for $\chi \in X_{\mathrm{nr}}(L,\omega)$ might be complicated. The next result shows that this awkward scenario does not occur and that $T_{\mathfrak{t}}$ is a quotient of $T_{\mathfrak{s}}$.

Lemma 4.14. *For ω and $\sigma^\sharp$ as above, $\sigma^\sharp \otimes \chi \cong \sigma^\sharp$ for all $\chi \in X_{\mathrm{nr}}(L,\omega)$.*

Proof. Let μ_L be as in (129). By Lemma 4.3 and Theorem 4.5 $\mathcal{H}(L)^{\mathfrak{s}_L}$ is Morita equivalent with

$$e^{\mathfrak{s}}_L \mathcal{H}(L) e^{\mathfrak{s}}_L \cong \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \otimes M_{|L/H_\lambda|}(\mathbb{C}). \tag{168}$$

The analogues of Lemma 3.10 and Proposition 3.9 for L show that $\mathcal{H}(L^\sharp Z(G))^{\mathfrak{s}_L}$ is Morita equivalent with

$$e^{\mathfrak{s}}_L \mathcal{H}(L)^{X_L(\mathfrak{s})} e^{\mathfrak{s}}_L \cong \big(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \otimes M_{|L/H_\lambda|}(\mathbb{C})\big)^{X^L(\mathfrak{s})}. \tag{169}$$

Under the first Morita equivalence, ω is mapped to $e^{\mathfrak{s}}_L V_\omega$, considered as an $\mathcal{O}(T_{\mathfrak{s}})$-module with character $\omega \in T_{\mathfrak{s}}$. For every $\gamma \in X^L(\omega)$, $\omega \otimes \gamma$ is mapped to the same module of (168).

Recall the operator $I(\gamma,\omega) \in \mathrm{Hom}_L(\omega \otimes \gamma, \omega)$ from (14). It restricts to

$$I_{e^{\mathfrak{s}}_L}(\gamma,\omega) \in \mathrm{Hom}_{e^{\mathfrak{s}}_L \mathcal{H}(L) e^{\mathfrak{s}}_L}(e^{\mathfrak{s}}_L V_{\omega\otimes\chi}, e^{\mathfrak{s}}_L V_\omega).$$

Since we are dealing with Morita equivalences and by (16), these operators give rise to algebra isomorphisms

$$\mathrm{End}_{e^{\mathfrak{s}}_L \mathcal{H}(L)^{X_L(\mathfrak{s})} e^{\mathfrak{s}}_L}(e^{\mathfrak{s}}_L V_\omega) \cong \mathrm{End}_{L^\sharp}(\omega)) \cong \mathbb{C}[X^L(\omega,\kappa_\omega)]. \tag{170}$$

It follows from (17) that there exists a unique $\rho \in \mathrm{Irr}(\mathbb{C}[X^L(\omega),\kappa_\omega])$ such that $\sigma^\sharp \cong \mathrm{Hom}_{\mathbb{C}[X^L(\omega),\kappa_\omega]}(\rho,\omega)$. Then (170) implies

$$e^{\mathfrak{s}}_L V_{\sigma^\sharp} \cong \mathrm{Hom}_{\mathbb{C}[X^L(\omega),\kappa_\omega]}(\rho, e^{\mathfrak{s}}_L V_\omega).$$

But $X^L(\omega\otimes\chi) = X^L(\omega), \kappa_{\omega\otimes\chi} = \kappa_\omega$ and

$$\sigma^\sharp \otimes \chi \cong \mathrm{Hom}_{\mathbb{C}[X^L(\omega),\kappa_\omega]}(\rho, \omega\otimes\chi).$$

Moreover ω and $\omega \otimes \chi$ correspond to the same module of (168), so

$$e^{\mathfrak{s}}_L(V_{\sigma^\sharp\otimes\chi}) \cong \mathrm{Hom}_{\mathbb{C}[X^L(\omega),\kappa_\omega]}(\rho, e^{\mathfrak{s}}_L V_{\omega\otimes\chi}) = \mathrm{Hom}_{\mathbb{C}[X^L(\omega),\kappa_\omega]}(\rho, e^{\mathfrak{s}}_L V_\omega) \cong e^{\mathfrak{s}}_L V_{\sigma^\sharp}.$$

In view of the second Morita equivalence above, this implies that $V_{\sigma^\sharp\otimes\chi} \cong V_{\sigma^\sharp}$ as $L^\sharp$-representations. $\square$

Now we are finally able to give a concrete description of the Hecke algebras associated to $G^\sharp$. Let $T^\sharp_{\mathfrak{s}}$ be the restriction of $T_{\mathfrak{s}}$ to $L^\sharp$, that is,

$$T^\sharp_{\mathfrak{s}} := T_{\mathfrak{s}}/X_{\mathrm{nr}}(L/L^\sharp) = T_{\mathfrak{s}}/X_{\mathrm{nr}}(G) \cong X_{\mathrm{nr}}(L^\sharp)/X_{\mathrm{nr}}(L,\omega). \tag{171}$$

With this torus we build an affine Hecke algebra $\mathcal{H}(T_{\mathfrak{s}}^\sharp, W_{\mathfrak{s}}, q_{\mathfrak{s}})$ like in (118) and (119). Recall from (27) that there are finitely many Bernstein components $\mathfrak{t}^\sharp$ for $G^\sharp$ such that

$$\mathcal{H}(G^\sharp)^{\mathfrak{s}} = \bigoplus\nolimits_{\mathfrak{t}^\sharp \prec \mathfrak{s}} \mathcal{H}(G^\sharp)^{\mathfrak{t}^\sharp}. \tag{172}$$

By Lemma 4.14 the Bernstein torus associated to any $\mathfrak{t}^\sharp \prec \mathfrak{s}$ is a quotient of $T_{\mathfrak{s}}^\sharp$ by a finite group. However, we warn that in general $T_{\mathfrak{t}^\sharp}$ is not equal to $T_{\mathfrak{s}}^\sharp$, see Example 5.2.

Define $\kappa((w,\gamma),(w',\gamma')) \in \mathbb{C}^\times$ by

$$J(\gamma, \omega \otimes \chi) \circ J(\gamma', \omega \otimes \chi) = \kappa((w,\gamma),(w',\gamma')) J(\gamma\gamma', \omega \otimes \chi). \tag{173}$$

By the formula for $\alpha_{(w,\gamma)}(h)$ in Lemma 4.11 this determines a 2-cocycle of

$$\mathrm{Stab}(\omega \otimes \chi, P \cap M) := \mathrm{Stab}(\omega \otimes \chi) \cap \mathrm{Stab}(\mathfrak{s}, P \cap M). \tag{174}$$

We note that by (23), (24) and Lemma 2.4.b the group (174) is isomorphic to $X^G(I_P^G(\omega \otimes \chi))$, via projection on the second coordinate.

Recall the idempotent $e^\sharp_{\lambda_{G^\sharp}}$ from (109). We define $e_{\mu_{G^\sharp}} \in \mathcal{H}(G^\sharp)$ similarly, as the restriction of $e_{\mu_G} : K \to \mathbb{C}$ to $K \cap G^\sharp$. Our final and main result translates Theorem 4.13 from $G^\sharp Z(G)$ to G.

Theorem 4.15. *The algebra $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ is Morita equivalent with*

$$e^\sharp_{\lambda_{G^\sharp}} \mathcal{H}(G^\sharp) e^\sharp_{\lambda_{G^\sharp}} = \bigoplus_{a \in [L/H_\lambda]} a e_{\mu_{G^\sharp}} a^{-1} \mathcal{H}(G^\sharp) a e_{\mu_{G^\sharp}} a^{-1} \cong \bigoplus\nolimits_1^{|X^L(\omega, V_\mu)|} e_{\mu_{G^\sharp}} \mathcal{H}(G^\sharp) e_{\mu_{G^\sharp}}.$$

There are algebra isomorphisms

$$\begin{aligned} e_{\mu_{G^\sharp}} \mathcal{H}(G^\sharp) e_{\mu_{G^\sharp}} &\cong \big(\mathcal{H}(T_{\mathfrak{s}}^\sharp, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \big)^{X^L(\mathfrak{s})/X^L(\omega, V_\mu) X_{\mathrm{nr}}(L/L^\sharp Z(G))} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \\ &\cong \big(\mathcal{H}(T_{\mathfrak{s}}^\sharp, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\mu^2}) \big)^{X^L(\mathfrak{s})^2/X^L(\omega, V_\mu) X_{\mathrm{nr}}(L/L^\sharp Z(G))} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp. \end{aligned}$$

The actions of $X^L(\mathfrak{s})$ and $\mathfrak{R}_{\mathfrak{s}}^\sharp$ come from $\mathrm{Stab}(\mathfrak{s}, P \cap M)$ *via Lemma 4.11, which involves a projective action on*

$$V_\mu = \bigoplus\nolimits_{\gamma \in X^L(\mathfrak{s})/X^L(\mathfrak{s})^2} V_{\mu^2 \otimes \gamma}.$$

The restriction of the associated 2-cocycle to $\mathrm{Stab}(\omega \otimes \chi, P \cap M)$ *corresponds to the 2-cocycle $\kappa_{I_P^G(\omega \otimes \chi)}$ from* (16). *Its cohomology class is trivial if $G = \mathrm{GL}_m(D)$ is split.*

Proof. By Theorem 3.17

$$\mathcal{H}(G^\sharp)^{\mathfrak{s}} \sim_M e^\sharp_{\lambda_{G^\sharp}} \mathcal{H}(G^\sharp) e^\sharp_{\lambda_{G^\sharp}} \cong e^\sharp_{\lambda_G} \mathcal{H}(G)^{X^G(\mathfrak{s}) X_{\mathrm{nr}}(G)} e^\sharp_{\lambda_G},$$

while Theorem 3.16 tells us that

$$e^\sharp_{\lambda_{G^\sharp Z(G)}} \mathcal{H}(G^\sharp Z(G)) e^\sharp_{\lambda_{G^\sharp Z(G)}} \cong e^\sharp_{\lambda_G} \mathcal{H}(G)^{X^G(\mathfrak{s})} e^\sharp_{\lambda_G}.$$

So the second line of the Theorem follows from Corollary 4.10 upon taking invariants for $X_{\mathrm{nr}}(G)/X^G(\mathfrak{s}) \cap X_{\mathrm{nr}}(G) \cong X_{\mathrm{nr}}(G^\sharp Z(G))$. Furthermore this gives an isomorphism

$$\bigoplus\nolimits_1^{|X^L(\omega, V_\mu)|} e_{\mu_{G^\sharp}} \mathcal{H}(G^\sharp) e_{\mu_{G^\sharp}} \cong \bigoplus\nolimits_1^{|X^L(\omega, V_\mu)|} (e_\mu \mathcal{H}(M) e_\mu)^{X^L(\mathfrak{s}) X_{\mathrm{nr}}(G)} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp.$$

The proof of Lemma 4.11 can also be applied to the action of $X_{\rm nr}(G) = X_{\rm nr}(L/L^\sharp)$ on

$$e_\mu \mathcal{H}(M) e_\mu \cong \mathcal{H}(T_{\mathfrak s}, W_{\mathfrak s}, q_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\mu),$$

and it shows that $X_{\rm nr}(G)$ acts only via translations of $T_{\mathfrak s}$. The remaining action of $X^L(\mathfrak s)^2$ is trivial on $X_{\rm nr}(L/L^\sharp Z(G)) X^L(\omega, V_\mu)$ by Theorem 4.13. This gives the third isomorphism, from which the fourth follows (again using Theorem 4.13).

The definitions (14) and (48) show that the 2-cocycle κ of $\mathrm{Stab}(\omega, P \cap M)$ determined by (173) is related to (15) by

$$\kappa((w,\gamma),(w',\gamma')) = \kappa_{I_P^G(\omega \otimes \chi)}(\gamma,\gamma').$$

Let ϕ be the Langlands parameter of the Langlands quotient of $I_P^G(\omega \otimes \chi)$. Via the local Langlands correspondence $\kappa_{I_P^G(\omega\otimes\chi)}$ is related to a 2-cocycle of the component group of ϕ, see [16, Lemma 12.5] and [3, Theorem 3.1]. Hence $\kappa_{I_P^G(\omega\otimes\chi)}$ is trivial if $\mathrm{GL}_m(D)$ is split, and otherwise it reflects the Hasse invariant of D. $\square$

Remark 4.16. In the case $G^\sharp = \mathrm{SL}_n(F)$ Theorem 4.15 can be compared with [14, Theorem 11.1]. The algebra that Goldberg and Roche investigate is $\mathcal{H}(G^\sharp, \tau)$, where τ is an irreducible subrepresentation of λ_G as a representation of $PN^1(K_G, \lambda_G) \cap G^\sharp$. They show that this is a type for a single Bernstein component $\mathfrak t^\sharp \prec \mathfrak s$. Then $\mathcal{H}(G^\sharp,\tau)$ is a subalgebra of our $e_{\mu_{G^\sharp}} \mathcal{H}(G^\sharp) e_{\mu_{G^\sharp}}$, because the idempotent e_τ is smaller than $e_{\mu_{G^\sharp}}$. In our terminology, [14] shows that

$$\begin{aligned} \mathcal{H}(G^\sharp,\tau) &\cong \mathcal{H}(T_{\mathfrak t^\sharp}, W_{\mathfrak s}, q_{\mathfrak s}) \rtimes \mathfrak R_{\mathfrak t^\sharp}, \\ e_\tau \mathcal{H}(G^\sharp) e_\tau &\cong \mathcal{H}(T_{\mathfrak t^\sharp}, W_{\mathfrak s}, q_{\mathfrak s}) \rtimes \mathfrak R_{\mathfrak t^\sharp} \otimes \mathrm{End}_{\mathbb C}(V_\tau). \end{aligned}$$

Only about the part $\rtimes \mathfrak R_{\mathfrak t^\sharp}$ that Goldberg and Roche are not so sure. In [14] it is still conceivable that $\mathfrak R_{\mathfrak t^\sharp}$ (denoted C there) is only embedded in $\mathcal{H}(G^\sharp,\tau)$ as part of a twisted group algebra $\mathbb C[\mathfrak R_{\mathfrak t^\sharp}, \delta]$. With Theorem 4.15, (actually already with Proposition 3.9) we see that the 2-cocyle δ from [14, §11] is always trivial.

5. Examples

This paper is rather technical, so we think it will be helpful for the reader to see some examples. These will also make clear that in general none of the introduced objects is trivial. Most of the notations used below are defined in Subsections 2.1 and 2.2.

Example 5.1 (Weyl group in $G^\sharp$ bigger than in G)**.**

Let ζ be a ramified character of $D^\times$ of order 3 and take

$$G = \mathrm{GL}_6(D),\ L = \mathrm{GL}_1(D)^6,\ \omega = 1 \otimes 1 \otimes \zeta \otimes \zeta \otimes \zeta^2 \otimes \zeta^2.$$

For $M = \mathrm{GL}_2(D)^3,\ \mathfrak s = [L,\omega]_G$ we have

$$T_{\mathfrak s} = X_{\rm nr}(L) \cong (\mathbb C^\times)^6,\ W_{\mathfrak s} = W(M,L) \cong (S_2)^3.$$

Furthermore $X^L(\omega) = \{1\}$ and

$$X^G(I_P^G(\omega)) = \{1,\zeta,\zeta^2\},\ \mathfrak R_{\mathfrak s}^\sharp = \langle (135)(246) \rangle \subset S_6 \cong W(G,L).$$

The stabilizer of ω in $\mathrm{Stab}(\mathfrak s)$ is generated by $((135)(246),\zeta)$ and

$$\mathrm{Stab}(\mathfrak s) = \mathrm{Stab}(\omega) W_{\mathfrak s} X_{\rm nr}(G/Z(G)).$$

Let $\mathcal{H}(\mathrm{GL}_2, q)$ denote an affine Hecke algebra of type GL_2. By Theorems 4.13 and 4.15 there are Morita equivalences

$$\begin{aligned} \mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}} &\sim_M (\mathcal{H}(\mathrm{GL}_2, q)^{\otimes 3})^{X_{\mathrm{nr}}(G/Z(G))} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}, \\ \mathcal{H}(G^\sharp)^{\mathfrak{s}} &\sim_M (\mathcal{H}(\mathrm{GL}_2, q)^{\otimes 3})^{X_{\mathrm{nr}}(G)} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}. \end{aligned}$$

Now we see that $\mathfrak{s}$ gives rise to a unique inertial class $\mathfrak{t}^\sharp$ for $G^\sharp$. Hence

$$W_{\mathfrak{t}^\sharp} = W^\sharp_{\mathfrak{s}} = W_{\mathfrak{s}} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}} \supsetneq W_{\mathfrak{s}}.$$

That is, the finite group associated by Bernstein to $\mathfrak{t}^\sharp$ is strictly larger than the finite group for $\mathfrak{s}$.

Example 5.2 (Torus for $\mathfrak{s}$ in $G^\sharp$ smaller than expected)**.**
Let $\sigma_2 \in \mathrm{Irr}(\mathrm{GL}_4(F))$ be as in [21, §4]. Its interesting property is $X^{\mathrm{GL}_4(F)}(\sigma_2) = \langle \chi_0 \eta \rangle$. Here χ_0 and η are characters of $F^\times$, both of order 4, with χ_0 unramified and η (totally) ramified. There exists a similar supercuspidal representation σ_3 with $X^{\mathrm{GL}_4(F)}(\sigma_3) = \langle \chi_0^{-1} \eta \rangle$. Let

$$G = \mathrm{GL}_8(F),\ L = \mathrm{GL}_4(F)^2,\ \omega = \sigma_2 \otimes \sigma_3.$$

Then $X^L(\omega) = \{1, \eta^2 \chi_0^2\}$ and

$$\begin{aligned} X^L(\mathfrak{s}) &= \langle \eta \rangle X_{\mathrm{nr}}(L/L^\sharp Z(G)), \\ T_{\mathfrak{s}} &= X_{\mathrm{nr}}(L) \cong (\mathbb{C}^\times)^2. \end{aligned}$$

The natural guess for the torus of an inertial class $\mathfrak{t}^\sharp \prec \mathfrak{s}$ is

$$T^\sharp_{\mathfrak{s}} = X_{\mathrm{nr}}(L^\sharp) = T_{\mathfrak{s}} / X_{\mathrm{nr}}(L/L^\sharp).$$

Yet it is not correct in this example. Recall from (159) that there exists a $\chi_\eta \in X_{\mathrm{nr}}(L)$ with $\omega \otimes \eta \cong \omega \otimes \chi_\eta$. One can check that $\chi_\eta = \chi_0^{-1} \otimes \chi_0 \neq 1$ and

$$X^L(\mathfrak{s}) \neq X^L(\omega) X_{\mathrm{nr}}(L/L^\sharp Z(G)).$$

Upon restriction to $L^\sharp$, ω decomposes as a sum of two irreducibles, caused by the L-intertwining operator

$$J(\eta^2, \omega) : \omega \to \omega \otimes \eta^{-2} \chi_\eta^2 \ \cong\ \omega \otimes \eta^2 \otimes (\chi_0^{-2} \otimes \chi_0^2).$$

Let $\sigma^\sharp$ be one of them. We may assume that $J(\eta, \omega)^2 = J(\eta^2, \omega)$, so η stabilizes $\sigma^\sharp$ up to an unramified twist. Then, with $\mathfrak{t}^\sharp = [L^\sharp, \sigma^\sharp]_{G^\sharp}$:

$$\begin{aligned} T_{\mathfrak{t}} &= T_{\mathfrak{s}} / X^L(\mathfrak{s}, \sigma^\sharp) = X_{\mathrm{nr}}(L^\sharp Z(G)) / \langle \chi_0^{-1} \otimes \chi_0 \rangle, \\ T_{\mathfrak{t}^\sharp} &= X_{\mathrm{nr}}(L^\sharp) / \langle 1 \otimes \chi_0^2 \rangle. \end{aligned}$$

Example 5.3 ($W^\sharp_{\mathfrak{s}}$ acts on torus without fixed points)**.**
This is the example from [21, §4], worked out in our setup.

$$G = \mathrm{GL}_8(F),\ L = \mathrm{GL}_2(F)^2 \times \mathrm{GL}_4(F)$$

Take η, χ_0, σ_2 as in the previous example, and let $\sigma_1 \in \mathrm{Irr}(\mathrm{GL}_2(F))$ be supercuspidal, such that $X^{\mathrm{GL}_2(F)}(\sigma_1) = \{1, \eta^2\}$. Write $\gamma = \eta \chi_0$ and $\omega = \sigma_1 \otimes \gamma \sigma_1 \otimes \sigma_2$. Then

$$W_{\mathfrak{s}} = 1,\ \mathfrak{R}^\sharp_{\mathfrak{s}} = W^\sharp_{\mathfrak{s}} = W(G, L).$$

Let w be the unique nontrivial element of $W(G,L)$, it corresponds to the permutation $(12)(34) \in S_8$. In this case $X^L(\omega) = 1$ and the action of w on $T_{\mathfrak{s}}$ involves translation by γ:

$$(w,\gamma)\cdot(\sigma_1\chi_1\otimes\gamma\sigma_1\chi_2\otimes\sigma_2\chi_3) = (\gamma^2\sigma_1\chi_2\otimes\gamma\sigma_1\chi_1\otimes\gamma\sigma_2\chi_3) \cong (\chi_0^2\sigma_1\chi_2\otimes\gamma\sigma_1\chi_1\otimes\sigma_2\chi_3)$$

for χ_i unramified. Using ω as basepoint and $\chi_1,\chi_2 \in X_{\mathrm{nr}}(GL_2(F)) \cong \mathbb{C}^\times, \chi_3 \in X_{\mathrm{nr}}(GL_4(F)) \cong \mathbb{C}^\times$ as coordinates, we obtain

$$\begin{aligned} &\chi_\gamma = (\chi_0^2, 1, 1) = (-1,1,1) \in (\mathbb{C}^\times)^3, \\ &(w,\gamma)\cdot(\chi_1,\chi_2,\chi_3) = (-\chi_2,\chi_1,\chi_3). \end{aligned}$$

This is a transformation without fixed points of $T_{\mathfrak{s}} = X_{\mathrm{nr}}(L)$, and also of $X_{\mathrm{nr}}(L^\sharp)$ and of $X_{\mathrm{nr}}(L^\sharp Z(G))$. We have

$$\begin{aligned} &\mathrm{Stab}(\mathfrak{s}) = \mathrm{Stab}(\omega) X_{\mathrm{nr}}(L/L^\sharp Z(G)), \\ &\mathrm{Stab}(\omega) = \{1, (w,\gamma)\}, \end{aligned}$$

and these groups act freely on $T_{\mathfrak{s}}$. It follows that $I_P^G(\omega)$ is irreducible and remains so upon restriction to $G^\sharp$. Writing $\mathfrak{s}^\sharp = [L^\sharp,\omega]_{G^\sharp}$, Theorems 4.13 and 4.15 provide Morita equivalences

$$\begin{aligned} &\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}} \sim_M \mathcal{O}(T_{\mathfrak{s}}) \rtimes \mathrm{Stab}(\mathfrak{s}) \sim_M \mathcal{O}(X_{\mathrm{nr}}(L^\sharp Z(G))) \rtimes \mathrm{Stab}(\omega), \\ &\mathcal{H}(G^\sharp)^{\mathfrak{s}^\sharp} \sim_M \mathcal{O}(X_{\mathrm{nr}}(L^\sharp)) \rtimes \mathrm{Stab}(\omega). \end{aligned}$$

Example 5.4 (Decomposition into 4 irreducibles upon restriction to $G^\sharp$)**.**
This and the next example are based on [10, §6.3]. Let ϕ be a Langlands parameter for $\mathrm{GL}_2(F)$ with image $\{ \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid a,b \in \{\pm 1\}\}$ and whose kernel contains a Frobenius element of the Weil group of F. The representation $\pi \in \mathrm{Irr}(\mathrm{GL}_2(F))$ associated to ϕ via the local Langlands correspondence has $X^{\mathrm{GL}_2(F)}(\pi)$ consisting of four ramified characters of $F^\times$ of order at most two, say $\{1,\gamma,\eta,\eta\gamma\}$. The cocycle κ_π is trivial, so by (16)

$$\mathrm{End}_{\mathrm{SL}_2(F)}(\pi) \cong \mathbb{C}\big[X^{\mathrm{GL}_2(F)}(\pi)\big] \cong \mathbb{C}^4,$$

and $\mathrm{Res}^{\mathrm{GL}_2(F)}_{\mathrm{SL}_2(F)}(\pi)$ consists of 4 inequivalent irreducible representations.
Next, let St be the Steinberg representation of $\mathrm{GL}_2(F)$ and consider

$$\omega = \pi \otimes \mathrm{St} \otimes \gamma\mathrm{St} \otimes \eta\mathrm{St} \otimes \gamma\eta\mathrm{St} \in \mathrm{Irr}(L), \tag{175}$$

where $L = \mathrm{GL}_2(F)^5$, a Levi subgroup of $G = \mathrm{GL}_{10}(F)$. In this setting

$$W_{\mathfrak{s}} = 1,\ X^L(\omega) = 1,\ X^G(I_P^G(\omega)) = X^{\mathrm{GL}_2(F)}(\pi).$$

Identifying $W(G,L)$ with S_5, we quickly deduce

$$\mathrm{Stab}(\omega) = \{1, ((23)(45),\gamma\eta), ((24)(35),\eta), ((25)(34),\gamma)\},$$

and $\mathfrak{R}^\sharp_{\mathfrak{s}} = W^\sharp_{\mathfrak{s}} \cong (\mathbb{Z}/2\mathbb{Z})^2$. However, the action of $\mathrm{Stab}(\omega)$ on $T_{\mathfrak{s}}$ does not involve translations, it is the same action as that of $\mathfrak{R}^\sharp_{\mathfrak{s}}$. The cocycle $\kappa_{I_P^G(\omega)}$ can be determined by looking carefully at the intertwining operators. Only in the first factor of L something interesting happens, in the other four factors the intertwining operators can be regarded as permutations. Hence the isomorphism $\mathrm{Stab}(\omega) \to X^{\mathrm{GL}_2(F)}(\pi)$ induces an equality $\kappa_{I_P^G(\omega)} = \kappa_\pi$. As we observed above, this cocycle is trivial, so by (16)

$$\mathrm{End}_{G^\sharp}(I_P^G(\omega)) \cong \mathbb{C}\big[\mathrm{Stab}(\omega)\big] \cong \mathbb{C}^4$$

and $\mathrm{Res}^G_{G^\sharp}(I_P^G(\omega))$ decomposes as a direct sum of 4 inequivalent irreducible representations. Theorems 4.13 and 4.15 tell us that there are Morita equivalences

$$\begin{aligned}
&\mathcal{H}(G^\sharp Z(G))^{\mathfrak s} \sim_M \mathcal{O}(T_{\mathfrak s})^{X_{\mathrm{nr}}(L/L^\sharp Z(G))} \rtimes \mathfrak{R}^\sharp_{\mathfrak s} = \mathcal{O}(X_{\mathrm{nr}}(L^\sharp Z(G))) \rtimes \mathfrak{R}^\sharp_{\mathfrak s},\\
&\mathcal{H}(G^\sharp)^{\mathfrak s} \sim_M \mathcal{O}(T_{\mathfrak s})^{X_{\mathrm{nr}}(L/L^\sharp)} \rtimes \mathfrak{R}^\sharp_{\mathfrak s} = \mathcal{O}(X_{\mathrm{nr}}(L^\sharp)) \rtimes \mathfrak{R}^\sharp_{\mathfrak s}.
\end{aligned}$$

Example 5.5 (non-trivial 2-cocycles)**.**

Let D be a central division algebra of dimension 4 over F and recall that D^1 denotes the group of elements of reduced norm 1 in $D^\times = \mathrm{GL}_1(D)$, which is also the maximal compact subgroup and the derived group of $D^\times$.
Take ϕ, π, γ, η as in the previous example and let $\tau \in \mathrm{Irr}(D^\times)$ be the image of π under the Jacquet–Langlands correspondence. Equivalently, τ has Langlands parameter ϕ. Then

$$X^{D^\times}(\tau) = X^{\mathrm{GL}_2(F)}(\pi) = \{1, \gamma, \eta, \gamma\eta\}.$$

As already observed in [1], the 2-cocycle κ_τ of $X^{D^\times}(\tau)$ is nontrivial. The group $X^{D^\times}(\tau)$ has one irreducible projective non-linear representation, of dimension two. Therefore

$$\mathrm{End}_{D^1}(\tau) \cong \mathbb{C}\big[X^{D^\times}(\tau), \kappa_\tau\big] \cong M_2(\mathbb{C})$$

and $\mathrm{Res}^{D^\times}_{D^1}(\tau) \cong \tau^\sharp \oplus \tau^\sharp$ with $\tau^\sharp$ irreducible.
Now we consider $G = \mathrm{GL}_5(D), L = \mathrm{GL}_1(D)^5$ and

$$\sigma = \tau \otimes 1 \otimes \gamma \otimes \eta \otimes \gamma\eta \in \mathrm{Irr}(L).$$

This representation is the image of (175) under the Jacquet–Langlands correspondence. It is clear that

$$X^L(\sigma) = 1,\ X^G(I_P^G(\sigma)) = X^{D^\times}(\tau) \text{ and } W_{\mathfrak s} = 1,$$

where $\mathfrak s = [L, \sigma]_G$. Just as in the previous example, we find

$$\begin{aligned}
&X^L(\mathfrak s) = X_{\mathrm{nr}}(L/L^\sharp Z(G)) \cong \mathbb{Z}/10\mathbb{Z},\\
&W^\sharp_{\mathfrak s} = \mathfrak{R}^\sharp_{\mathfrak s} \cong (\mathbb{Z}/2\mathbb{Z})^2 \subset W(G, L),\\
&\mathrm{Stab}(\sigma) = \{1, ((23)(45), \gamma\eta), ((24)(35), \eta), ((25)(34), \gamma)\},\\
&X^G(\mathfrak s) = \mathrm{Stab}(\sigma) X^L(\mathfrak s).
\end{aligned}$$

We refer to Subsection 2.2 for the definitions of these groups. The same reasoning as for κ_π shows that $\kappa_{I_P^G(\sigma)} = \kappa_\tau$ via the isomorphism $\mathrm{Stab}(\sigma) \to X^{D^\times}(\tau)$. Hence

$$\mathrm{End}^G_{G^\sharp}(I_P^G(\sigma)) \cong \mathbb{C}\big[\mathrm{Stab}(\sigma), \kappa_{I_P^G(\sigma)}\big] \cong M_2(\mathbb{C}),$$

and $\mathrm{Res}^G_{G^\sharp}(I_P^G(\sigma))$ is direct sum of two isomorphic irreducible $G^\sharp$-representations.
To analyse the Hecke algebras associated to $\mathfrak s$, we need to exhibit some types. A type for γ as a $D^\times$-representation is $(D^1, \lambda_\gamma = \gamma \circ \mathrm{Nrd})$. The same works for other characters of $D^\times$. We know that τ admits a type, and we may assume that it is of the form (D^1, λ_τ). It is automatically stable under $X^{D^\times}(\tau)$ and $\dim(\lambda_\tau) > 1$ because τ is not a character. Then

$$\big((D^1)^5, \lambda = \lambda_\tau \otimes \lambda_1 \otimes \lambda_\gamma \otimes \lambda_\eta \otimes \lambda_{\eta\gamma}\big)$$

is a type for $[L,\sigma]_L$. The underlying vector space V_λ can be identified with V_{λ_τ}. We note that $M = L$ and $T_{\mathfrak s} = X_{\mathrm{nr}}(L) \cong (\mathbb C^\times)^5$. Proposition 3.9 and Theorem 4.1 show that there is a Morita equivalence

$$\mathcal H(GZ(G))^{\mathfrak s} \sim_M \big(\mathcal O(T_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\lambda)\big)^{X_{\mathrm{nr}}(L/L^\sharp Z(G))} \rtimes \mathfrak R^\sharp_{\mathfrak s}.$$

The action of $\mathfrak R^\sharp_{\mathfrak s}$ on $\mathrm{End}_{\mathbb C}(V_\lambda)$ comes from a projective representation of $X^{D^\times}(\tau)$ on V_{λ_τ}. It does not lift a to linear representation because κ_τ is nontrivial. Therefore $\mathcal H(GZ(G))^{\mathfrak s}$ is not Morita equivalent with

$$\mathcal O(T_{\mathfrak s})^{X_{\mathrm{nr}}(L/L^\sharp Z(G))} \rtimes \mathfrak R^\sharp_{\mathfrak s} = \mathcal O(X_{\mathrm{nr}}(L^\sharp Z(G)) \rtimes \mathfrak R^\sharp_{\mathfrak s}.$$

Similarly the algebras $\mathcal H(G^\sharp)^{\mathfrak s}$ and

$$\big(\mathcal O(T_{\mathfrak s}) \otimes \mathrm{End}_{\mathbb C}(V_\lambda)\big)^{X_{\mathrm{nr}}(L/L^\sharp)} \rtimes \mathfrak R^\sharp_{\mathfrak s} = \big(\mathcal O(X_{\mathrm{nr}}(L^\sharp)) \otimes \mathrm{End}_{\mathbb C}(V_\lambda)\big) \rtimes \mathfrak R^\sharp_{\mathfrak s} \tag{176}$$

are Morita equivalent, but

$$\mathcal O(T_{\mathfrak s})^{X_{\mathrm{nr}}(L/L^\sharp)} \rtimes \mathfrak R^\sharp_{\mathfrak s} = \mathcal O(X_{\mathrm{nr}}(L^\sharp)) \rtimes \mathfrak R^\sharp_{\mathfrak s} \tag{177}$$

has a different module category. One can show that (176) and (177) are quite far apart, in the sense that they have different periodic cyclic homology.

Example 5.6 (Type does not see all $G^\sharp$-subrepresentations)**.**
Take $G = \mathrm{GL}_2(F)$ and let χ_- be the unique unramified character of order 2. There exists a supercuspidal $\omega \in \mathrm{Irr}(G)$ with $X_{\mathrm{nr}}(G,\omega) = \{1,\chi_-\}$. Then $\chi_- \in X^G(\omega)$ and

$$I(\omega,\chi_-) \in \mathrm{Hom}_G(\omega, \omega \otimes \chi_-).$$

This operator can be normalized so that its square is the identity on V_ω. Let G^1 be the subgroup of G generated by all compact subgroups. The $+1$-eigenspace and the -1-eigenspace of $I(\omega,\chi_-)$ are irreducible G^1-subrepresentations of $\mathrm{Res}^G_{G^1}(\omega)$, and these are conjugate via an element $a \in G \setminus G^1 Z(G)$. Any type for $[G,\omega]_G$ is based on a subgroup of G^1, so it sees only one of the two irreducible G^1-subrepresentations of ω.
This phenomenon forces us to introduce the group L/H_λ in Lemma 3.3 (here $G/G^1Z(G) \cong \{1,a\}$) and carry it with us through a large part of the paper.

Example 5.7 (Types conjugate in G but not in $G^1G^\sharp$)**.**
Consider a supercuspidal representation ω of $\mathrm{GL}_m(D)$ which contains a simple type (K,λ). Fix a uniformizer ϖ_D of D and denote the unit of $\mathrm{GL}_m(D)$ by 1_m. Assume that there exists $\gamma \in X^G(\mathfrak s)$ such that $\varpi_D 1_m$ normalizes K and

$$\varpi_D^{-1} 1_m \cdot \lambda \cong \lambda \otimes \gamma \not\cong \lambda.$$

Then λ and $\lambda \otimes \gamma$ are conjugate in G but not in $G^1G^\sharp$.
This can be constructed as follows. For simplicity we consider the case where $K = \mathrm{GL}_m(\mathfrak o_D)$ and λ has level zero. Then λ is inflated from a cuspidal representation σ of the finite group $\mathrm{GL}_m(k_D)$, where k_D denotes the residue field of D. On this group conjugation by ϖ_D has the same effect as some field automorphism of k_D/k_F. We assume that it is the Frobenius automorphism $x \mapsto x^q$, where $q = |k_F|$. Recall that k_D/k_F has degree d, so $|k_D| = q^d$.
We need a $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}_m(k_D))$ such that

$$\sigma \circ \mathrm{Frob}_{k_F} \cong \sigma \otimes \bar\gamma \not\cong \sigma, \tag{178}$$

where $\bar\gamma \in \mathrm{Irr}(\mathrm{GL}_m(k_D)/k_D^\times \mathrm{SL}_m(k_D))$ is induced by γ.
To find an example, we recall the classification of the characters of $\mathrm{GL}_m(k_D)$ by Green [15]. In his notation, every irreducible cuspidal character of $\mathrm{GL}_m(k_D)$ is of the form $(-1)^{m-1} I_m^k[1]$, where $k \in \mathbb{Z}/(q^{dm}-1)\mathbb{Z}$ is such that $k, kq^d, \ldots, kq^{d(m-1)}$ are m different elements of $\mathbb{Z}/(q^{dm}-1)\mathbb{Z}$.

Let us make the class function $I_m^k[1]$ a bit more explicit. [15, Theorems 12 and 13] entail that it is determined by its values on principal elements, that is, elements of $\mathrm{GL}_m(k_D)$ which do not belong to any proper parabolic subgroup. Let k_E be a field with q^{md} elements, which contains k_D. Suppose that $x \in k_E^\times$ is a generator and let $f_x \in \mathrm{GL}_m(k_D)$ be such that $\det(t - f_x)$ is the minimal polynomial of x over k_D. Then $f_x \in \mathrm{GL}_m(k_D)$ is principal, and every principal element is of this form. We identify $\mathrm{Irr}(k_E^\times)$ with $\mathbb{Z}/(q^{dm}-1)\mathbb{Z}$ by fixing a generator, a character $\theta : k_E^\times \to \mathbb{C}^\times$ of order $q^{dm}-1$. From [15, §3] one can see that $I_m^k[1](f_x) = \theta^k(x)$.

In this setting, the above condition on k becomes that $\theta^{kq^{ds}} \neq \theta^k$ for any divisor s of m with $1 \leq s < m$. Let us call such a character of $k_E^\times$ regular. The Galois group $\mathrm{Gal}(k_E/k_D) = \langle \mathrm{Frob}_{k_D} \rangle$ acts on $\mathrm{Irr}(k_E^\times)$ by $\theta^k \circ \mathrm{Frob}_{k_D} = \theta^{kq^d}$, and the regular characters are precisely those whose orbit contains $m = |\mathrm{Gal}(k_E/k_D)|$ elements. Now [15, Theorem 13] sets up a bijection

$$\begin{array}{ccc} \mathrm{Irr}(k_E^\times)_{\mathrm{reg}}/\mathrm{Gal}(k_E/k_D) & \to & \mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}_m(k_D)) \\ \theta^k & \mapsto & \sigma_k, \end{array} \tag{179}$$

determined by

$$\mathrm{tr}(\sigma_k(f_x)) = (-1)^{m-1}\theta^k(x) \text{ for every generator } x \text{ of } k_E^\times. \tag{180}$$

Let us describe the effects of tensoring with elements of $\mathrm{Irr}(G/G^\sharp Z(G))$ and of conjugation with powers of ϖ_D in these terms. As noted above, the conjugation action of ϖ_D on $\mathrm{GL}_m(D)$ is the same as entrywise application of $\mathrm{Frob}_{k_F} \in \mathrm{Gal}(k_D/k_F)$. With (180) we deduce

$$\varpi_D^{-1} \cdot \sigma_k = \sigma_k \circ \mathrm{Frob}_{k_F} = \sigma_{kq}.$$

This corresponds to the natural action of $\mathrm{Gal}(k_D/k_F)$ on $\mathrm{Irr}(k_E^\times)_{\mathrm{reg}}/\mathrm{Gal}(k_E/k_D)$.
Consider a $\gamma \in \mathrm{Irr}(G/G^\sharp Z(G))$ which is trivial on $\ker(\mathrm{GL}_m(\mathfrak{o}_D) \to \mathrm{GL}_m(k_D))$. It induces a character of $\mathrm{GL}_m(k_D)$ of the form

$$\bar\gamma = \gamma' \circ N_{k_D/k_F} \circ \det \quad \text{with} \quad \gamma' \in \mathrm{Irr}(k_F^\times).$$

We assume that $\gamma' = \theta|_{k_F^\times}$. By (180)

$$\begin{aligned} \bar\gamma(f_x) = \theta\big(N_{k_D/k_F}(N_{k_E/k_D}(x))\big) = \theta\big(N_{k_E/k_F}(x)\big) \\ = \theta\big(x^{(q^{dm}-1)/(q-1)}\big) = \theta^{(q^{dm}-1)/(q-1)}(x). \end{aligned}$$

Comparing with (178) we find that we want to arrange that

$$kq \equiv k + \frac{q^{dm}-1}{q-1} \mod q^{dm}-1, \text{ but } kq \notin \{k, kq^d, \ldots, kq^{d(m-1)}\} \mod q^{dm}-1.$$

For example, we can take $q = 3, d = 3$ and $m = 2$. Then $q^{dm} = 729$,
$(q^{dm}-1)/(q-1) = 364$ and suitable k's are 182 or $182q^d \equiv -110$.

Example 5.8 (Types conjugate in $G^1 G^\sharp$ but not in G^1)**.**

Take $G = \mathrm{GL}_{2m}(D)$, $L = \mathrm{GL}_m(D)^2$, $\omega = \omega_1 \otimes \omega_2$ with ω_1, K_1, λ_1 as in the previous example (but there without subscripts). Also assume that the supercuspidal $\omega_2 \in \mathrm{Irr}(\mathrm{GL}_m(D))$ contains a simple type (K_2, λ_2) such that

$$\varpi_D 1_m \cdot \lambda_2 \cong \lambda_2 \otimes \gamma \not\cong \lambda_2.$$

Then $(K, \lambda) = (K_1 \times K_2, \lambda_1 \otimes \lambda_2)$ is a type for $[L, \omega]_L$. It is conjugate to $(K, \lambda \otimes \gamma)$ by $(\varpi_D^{-1} 1_m, \varpi_D 1_m) \in G^\sharp \setminus G^\sharp \cap G^1$, but not by an element of G^1.

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