

# Hochschild homology of affine Hecke algebras

Maarten Solleveld

Radboud Universiteit Nijmegen  
Heyendaalseweg 135, 6525AJ Nijmegen, the Netherlands  
email: m.solleveld@science.ru.nl

## Abstract.

Let  $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$  be an affine Hecke algebra with complex, possibly unequal parameters  $q$ , which are not roots of unity. We compute the Hochschild and the cyclic homology of  $\mathcal{H}$ . It turns out that these are independent of  $q$  and that they admit an easy description in terms of the extended quotient of a torus by a Weyl group, both of which are canonically associated to the root datum  $\mathcal{R}$ .

For positive  $q$  we also prove that the representations of the family of algebras  $\mathcal{H}(\mathcal{R}, q^\epsilon)$ ,  $\epsilon \in \mathbb{C}$  come in families which depend analytically on  $\epsilon$ .

Analogous results are obtained for graded Hecke algebras and for Schwartz completions of affine Hecke algebras.

**Correction:** in 2021 some problems with the construction of families of representations surfaced, these are discussed on pages 16, 18, 19 and 22.

## 2010 Mathematics Subject Classification.

primary: 20C08, secondary: 18H15

## Contents

<b>Introduction</b>	<b>2</b>
<b>1 Preliminaries</b>	<b>5</b>
1.1 Affine Hecke algebras . . . . .	5
1.2 Graded Hecke algebras . . . . .	7
1.3 Parabolic subalgebras . . . . .	8
1.4 Lusztig's reduction theorems . . . . .	10
1.5 Schwartz algebras . . . . .	12
<b>2 Analytic families of representations</b>	<b>16</b>
2.1 Positive parameters . . . . .	16
2.2 Complex parameters . . . . .	21
<b>3 Hochschild homology</b>	<b>24</b>
3.1 Graded Hecke algebras . . . . .	24
3.2 Affine Hecke algebras . . . . .	29
3.3 Schwartz algebras . . . . .	32
3.4 Comparison of different parameters . . . . .	34
<b>References</b>	<b>38</b>

## Introduction

The representation theory of affine Hecke algebras has been studied extensively, for a large part motivated by the connection with reductive  $p$ -adic groups. By now this theory is in a very good state, thanks to work of Kazhdan–Lusztig, Barbasch–Moy, Delorme–Opdam and many others. Given that the classification of irreducible representations of affine Hecke algebras is more or less completed [KaLu, OpSo1, Sol3], it is natural to look at subtler properties like extensions, derived categories and homology [BaNi, Nis, OpSo2]. In this paper we will compute the Hochschild and cyclic homology of affine Hecke algebras with possibly unequal parameters. Related results are obtained for graded Hecke algebras and for Schwartz completions of affine Hecke algebras. This work can be regarded as a sequel to [Sol1], where the author determined the Hochschild homology of graded Hecke algebras (but not the structure as a module over the centre).

We discuss the main results in some detail. Let  $\mathcal{R} = (X, R, Y, R^\vee, \Delta)$  be a based root datum with finite Weyl group  $W = W(R)$ . Let  $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$  be the associated affine Hecke algebra over  $\mathbb{C}$ , with positive (possibly unequal) parameters  $q$ . There is also a Schwartz algebra  $\mathcal{S}(\mathcal{R}, q)$ , a topological completion of  $\mathcal{H}$  which is useful for the study of tempered  $\mathcal{H}$ -representations. We refer to Section 1 for proper definitions of these objects.

Write  $T = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$  and let  $\langle W \rangle$  be a collection of representatives for the conjugacy classes in  $W$ . The extended quotient of  $T$  by  $W$  is

$$\tilde{T}/W = \bigsqcup_{w \in \langle W \rangle} T^w/Z_W(w).$$

It can be used to parametrize the irreducible  $\mathcal{H}$ -representations [Sol3, Theorem 5.4.2], in agreement with a conjecture of Aubert, Baum and Plymen [ABP]. Then the irreducible  $\mathcal{S}(\mathcal{R}, q)$ -representations correspond to  $\tilde{T}_{un}/W \subset \tilde{T}/W$ , where  $\tilde{T}_{un} = \mathrm{Hom}_{\mathbb{Z}}(X, S^1)$ .

We provide a more precise version of the above parametrization, which at the same time extends to the case  $q \in \mathbb{C}^\times, |q| \neq 1$ . Recall that  $\mathcal{H}(\mathcal{R}, q)$  has a commutative subalgebra  $\mathcal{A} \cong \mathcal{O}(T)$ , such that  $\mathcal{A}^W \cong \mathcal{O}(T)^W$  is the centre of  $\mathcal{H}(\mathcal{R}, q)$ . For  $w \in \langle W \rangle$  let  $T_i^w/Z_w(w)$ ,  $1 \leq i \leq c(w)$  be the connected components of  $T^w/Z_W(w)$ . We consider the family of algebras  $\{\mathcal{H}(\mathcal{R}, q^\epsilon) \mid \epsilon \in \mathbb{C}\}$ .

**Theorem 1.** (See Theorem 2.6.)

There exist families of  $\mathcal{H}(\mathcal{R}, q^\epsilon)$ -representations

$$\{\pi(w, i, t, \epsilon) \mid w \in \langle W \rangle, 1 \leq i \leq c(w), t \in T_i^w, \epsilon \in \mathbb{C}\}$$

such that:

- (a) The representations are irreducible for generic  $t$  and  $\epsilon$ .
- (b) The vector space underlying  $\pi(w, i, t, \epsilon)$  depends only on  $w$  and  $i$ . The matrix coefficients of this representation are algebraic in  $t$  and complex analytic in  $\epsilon$ . (The latter makes sense because we can identify the vector spaces  $\mathcal{H}(\mathcal{R}, q^\epsilon)$  and  $\mathcal{H}(\mathcal{R}, q)$  in a canonical way.)

- (c) The central character of  $\pi(w, i, t, \epsilon)$  is of the form  $Wtc_{w,i}^\epsilon$ , where  $c_{w,i}$  is a homomorphism from  $X$  to the subgroup of  $\mathbb{R}_{>0}$  generated by the variables  $q_{\alpha^\vee}^{\pm 1/2}$  for all possible coroots  $\alpha^\vee$ .
- (d) For  $\epsilon \in \mathbb{C} \setminus \sqrt{-1}\mathbb{R}^\times$  the  $\mathcal{H}(\mathcal{R}, q^\epsilon)$ -representations  $\pi(w, i, t, \epsilon)$  and  $\pi(w', i', t', \epsilon)$  have the same trace if and only if  $w = w', i = i'$  and  $t$  and  $t'$  are in the same  $Z_W(w)$ -orbit.
- (e) For  $\epsilon \in \mathbb{C} \setminus \sqrt{-1}\mathbb{R}^\times$  the collection

$$\{\pi(w, i, t, \epsilon) \mid w \in \langle W \rangle, 1 \leq i \leq c(w), t \in T_i^w / Z_W(w)\}$$

is a  $\mathbb{Q}$ -basis of the Grothendieck group of finite dimensional  $\mathcal{H}(\mathcal{R}, q^\epsilon)$ -representations.

- (f) For  $\epsilon \in \mathbb{R}$  the collection

$$\{\pi(w, i, t, \epsilon) \mid w \in \langle W \rangle, 1 \leq i \leq c(w), t \in (T_i^w \cap T_{un}) / Z_W(w)\}$$

is a  $\mathbb{Q}$ -basis of the Grothendieck group of finite dimensional  $\mathcal{S}(\mathcal{R}, q^\epsilon)$ -representations.

By analogy with the case of a single parameter  $q$  [KaLu], the author expects that even more is true. Namely, in (b) the matrix coefficients should depend algebraically on the variables  $q_{\alpha^\vee}^{\pm 1/2}$ , while parts (d) and (e) should hold for all but finitely many  $\epsilon \in \mathbb{C}$ .

The families of representations from Theorem 1 are our main tool for the computation of the Hochschild homology. For fixed  $w, i, \epsilon$  the representations  $\{\pi(w, i, t, \epsilon) \mid t \in T_i^w\}$  yield an algebra homomorphism

$$\pi_{w,i,\epsilon} : \mathcal{H}(\mathcal{R}, q^\epsilon) \rightarrow \mathcal{O}(T_i^w) \otimes \text{End}(V_{w,i}),$$

where  $V_{w,i}$  is the finite dimensional vector space on which these representations are defined. Recall that by the Hochschild–Kostant–Rosenberg theorem

$$HH_*(\mathcal{O}(T_i^w) \otimes \text{End}(V_{w,i})) \cong \Omega^*(T_i^w),$$

the space of algebraic differential forms on the complex affine variety  $T_i^w$ .

**Theorem 2.** (See Theorems 3.4, 3.6 and 3.8.)

- (a) Let  $\epsilon \in \mathbb{C} \setminus \sqrt{-1}\mathbb{R}^\times$ . The maps  $\pi_{w,i,\epsilon}$  induce an isomorphism

$$HH_*(\mathcal{H}(\mathcal{R}, q^\epsilon)) \rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} \Omega^*(T_i^w)^{Z_W(w)} \cong \Omega^*(\tilde{T})^W.$$

The induced action of  $Z(\mathcal{H}(\mathcal{R}, q^\epsilon)) \cong \mathcal{O}(T)^W$  on  $\Omega^*(T_i^w)$  is the same as the action via the embedding

$$T_i^w \rightarrow T : t \mapsto c_{w,i}^\epsilon t,$$

where  $c_{w,i}$  is as in Theorem 1.c.

(b) Let  $\epsilon \in \mathbb{R}$ . Part (a) extends to an isomorphism of topological vector spaces

$$HH_*(\mathcal{S}(\mathcal{R}, q^\epsilon)) \rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} \Omega_{sm}^*(T_i^w \cap T_{un})^{Z_W(w)} \cong \Omega_{sm}^*(\tilde{T}_{un})^W,$$

where  $\Omega_{sm}^*$  stands for smooth differential forms.

Thus the Hochschild homology of  $\mathcal{H}(\mathcal{R}, q^\epsilon)$  is independent of the parameters  $q^\epsilon$  and can be expressed in terms of  $T$  and  $W$ . Only the action of the centre depends on  $q^\epsilon$ , but in a simple way. Another way to look at it is that  $HH_n(\mathcal{H}(\mathcal{R}, q^\epsilon))$  is the space of algebraic  $n$ -forms on the dual space of  $\mathcal{H}(\mathcal{R}, q^\epsilon)$ , regarded as a nonseparated variety with a stratification coming from Theorem 1.

All the above results have natural counterparts for graded Hecke algebras. These are considerably easier to prove and serve as intermediate steps towards Theorems 1 and 2. One can easily deduce from Theorem 2 what the (periodic) cyclic homology and the Hochschild cohomology of  $\mathcal{H}(\mathcal{R}, q^\epsilon)$  look like. The outcome is again that they do not depend on  $q^\epsilon$  and can be expressed with only  $T$  and  $W$ .

There is a second form of Hochschild cohomology,

$$H^*(\mathcal{H}, \mathcal{H}) = \text{Ext}_{\mathcal{H} \otimes \mathcal{H}^{op}}^*(\mathcal{H}, \mathcal{H}).$$

It would be quite interesting to compute this, since it is related to deformation theory and extensions of  $\mathcal{H}$ -bimodules. However, this theory is not dual to Hochschild homology, so Theorem 2 says little about it.

In view of Theorems 1 and 2 we might wonder how similar two affine Hecke algebras with the same root datum but different parameter functions are. From  $HH_0$  we see already that  $\mathcal{H}(\mathcal{R}, q)$  and  $\mathcal{H}(\mathcal{R}, q')$  can only be Morita equivalent if there is an automorphism of  $T/W$  that sends every subvariety  $c_{w,i}T_i^w/Z_W(w)$  to a subvariety  $c_{w',i'}T_{i'}^{w'}/Z_W(w')$ . It turns out that this condition is rather strong as soon as  $R$  contains nonperpendicular roots. Indeed, for  $\mathcal{R}$  of type  $\widetilde{A}_2$  it was shown in [Yan] that it is only fulfilled if  $q' = q$  or  $q' = q^{-1}$ .

**Question 3.** *Suppose that  $\mathcal{H}(\mathcal{R}, q)$  and  $\mathcal{H}(\mathcal{R}, q')$  are Morita equivalent or even isomorphic. What are the possibilities for  $(q, q')$ ?*

For  $\mathcal{R}$  of type  $\widetilde{A}_1$  it is easily seen that  $\mathcal{H}(\mathcal{R}, q) \cong \mathbb{C}[D_\infty]$  for all  $q \in \mathbb{C} \setminus \{-1\}$ , where  $D_\infty$  is the infinite dihedral group. But this root datum is exceptional, because it corresponds to the only connected Dynkin diagram for which the generators of the Coxeter group do not satisfy a braid relation. For irreducible  $\mathcal{R}$  of rank at least 2, the aforementioned result of [Yan] suggests that there are only very few positive answers to Question 3.

Let us briefly describe the organization of the paper. We work in somewhat larger generality than in the introduction, in the sense that we allow Hecke algebras extended by finite groups of automorphisms of the root datum. This enhances the applicability, as such algebras appear naturally in the representation theory of reductive  $p$ -adic groups.

The first section is meant to introduce the notation and to recall several relevant theorems. In Section 2 we study analytic families of representations of Hecke algebras, in increasing generality. This builds upon the author's previous work [Sol3] and leads to Theorem 1. The various homologies of Hecke algebras are computed in Section 3. We do it first for graded Hecke algebras, by hand so to say. Subsequently we transfer the results to affine Hecke algebras and their Schwartz completions, via some arguments involving localization at central characters.

We do not study the consequences for the Hochschild homology of reductive  $p$ -adic groups here, the author intends to do so in a forthcoming paper.

## 1 Preliminaries

### 1.1 Affine Hecke algebras

Let  $\mathfrak{a}$  be a finite dimensional real vector space and let  $\mathfrak{a}^*$  be its dual. Let  $Y \subset \mathfrak{a}$  be a lattice and  $X = \text{Hom}_{\mathbb{Z}}(Y, \mathbb{Z}) \subset \mathfrak{a}^*$  the dual lattice. Let

$$\mathcal{R} = (X, R, Y, R^\vee, \Delta).$$

be a based root datum. Thus  $R$  is a reduced root system in  $X$ ,  $R^\vee \subset Y$  is the dual root system and  $\Delta$  is a basis of  $R$ . Furthermore we are given a bijection  $R \rightarrow R^\vee$ ,  $\alpha \mapsto \alpha^\vee$  such that  $\langle \alpha, \alpha^\vee \rangle = 2$  and such that the corresponding reflections  $s_\alpha : X \rightarrow X$  (resp.  $s_\alpha^\vee : Y \rightarrow Y$ ) stabilize  $R$  (resp.  $R^\vee$ ). We do not assume that  $R$  spans  $\mathfrak{a}^*$ .

The reflections  $s_\alpha$  generate the Weyl group  $W = W(R)$  of  $R$ , and  $S_\Delta := \{s_\alpha \mid \alpha \in \Delta\}$  is the collection of simple reflections. We have the affine Weyl group  $W^{\text{aff}} = \mathbb{Z}R \rtimes W$  and the extended (affine) Weyl group  $W^e = X \rtimes W$ . Both can be considered as groups of affine transformations of  $\mathfrak{a}^*$ . We denote the translation corresponding to  $x \in X$  by  $t_x$ . As is well known,  $W^{\text{aff}}$  is a Coxeter group, and the basis of  $R$  gives rise to a set  $S^{\text{aff}}$  of simple (affine) reflections. More explicitly, let  $\Delta_M^\vee$  be the set of maximal elements of  $R^\vee$ , with respect to the dominance ordering coming from  $\Delta$ . Then

$$S^{\text{aff}} = S_\Delta \cup \{t_\alpha s_\alpha \mid \alpha^\vee \in \Delta_M^\vee\}.$$

We write

$$\begin{aligned} X^+ &:= \{x \in X \mid \langle x, \alpha^\vee \rangle \geq 0 \forall \alpha \in \Delta\}, \\ X^- &:= \{x \in X \mid \langle x, \alpha^\vee \rangle \leq 0 \forall \alpha \in \Delta\} = -X^+. \end{aligned}$$

It is easily seen that the centre of  $W^e$  is the lattice

$$Z(W^e) = X^+ \cap X^-.$$

The length function  $\ell$  of the Coxeter system  $(W^{\text{aff}}, S^{\text{aff}})$  extends naturally to  $W^e$ . The elements of length zero form a subgroup  $\Omega \subset W^e$  and  $W^e = W^{\text{aff}} \rtimes \Omega$ . With  $\mathcal{R}$  we also associate some other root systems. There is the non-reduced root system

$$R_{nr} := R \cup \{2\alpha \mid \alpha^\vee \in 2Y\}.$$

Obviously we put  $(2\alpha)^\vee = \alpha^\vee/2$ . Let  $R_l$  be the reduced root system of long roots in  $R_{nr}$ :

$$R_l := \{\alpha \in R_{nr} \mid \alpha^\vee \notin 2Y\}.$$

We introduce a complex parameter function for  $\mathcal{R}$  in two equivalent ways. Firstly, it is a map  $q : S^{\text{aff}} \rightarrow \mathbb{C}^\times$  such that  $q(s) = q(s')$  if  $s$  and  $s'$  are conjugate in  $W^e$ . This extends naturally to a map  $q : W^e \rightarrow \mathbb{C}^\times$  which is 1 on  $\Omega$  and satisfies  $q(ww') = q(w)q(w')$  if  $\ell(ww') = \ell(w) + \ell(w')$ . Secondly, a parameter function is a  $W$ -invariant map  $q : R_{nr}^\vee \rightarrow \mathbb{C}^\times$ . The relation between the two definitions is given by

$$\begin{aligned} q_{\alpha^\vee} &= q(s_\alpha) = q(t_\alpha s_\alpha) & \text{if } \alpha \in R \cap R_l, \\ q_{\alpha^\vee} &= q(t_\alpha s_\alpha) & \text{if } \alpha \in R \setminus R_l, \\ q_{\alpha^\vee/2} &= q(s_\alpha)q(t_\alpha s_\alpha)^{-1} & \text{if } \alpha \in R \setminus R_l. \end{aligned} \quad (1)$$

We speak of equal parameters if  $q(s) = q(s') \forall s, s' \in S^{\text{aff}}$  and of positive parameters if  $q(s) \in \mathbb{R}_{>0} \forall s \in S^{\text{aff}}$ .

We fix a square root  $q^{1/2} : S^{\text{aff}} \rightarrow \mathbb{C}^\times$ . The affine Hecke algebra  $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$  is the unique associative complex algebra with basis  $\{N_w \mid w \in W\}$  and multiplication rules

$$\begin{aligned} N_w N_v &= N_{wv} & \text{if } \ell(wv) = \ell(w) + \ell(v), \\ (N_s - q(s)^{1/2})(N_s + q(s)^{-1/2}) &= 0 & \text{if } s \in S^{\text{aff}}. \end{aligned} \quad (2)$$

In the literature one also finds this algebra defined in terms of the elements  $q(s)^{1/2}N_s$ , in which case the multiplication can be described without square roots. This explains why  $q^{1/2}$  does not appear in the notation  $\mathcal{H}(\mathcal{R}, q)$ .

Notice that  $N_w \mapsto N_{w^{-1}}$  extends to a  $\mathbb{C}$ -linear anti-automorphism of  $\mathcal{H}$ , so  $\mathcal{H}$  is isomorphic to its opposite algebra. The span of the  $N_w$  with  $w \in W$  is a finite dimensional Iwahori–Hecke algebra, which we denote by  $\mathcal{H}(W, q)$ .

Now we describe the Bernstein presentation of  $\mathcal{H}$ . For  $x \in X^+$  we put  $\theta_x := N_{t_x}$ . The corresponding semigroup morphism  $X^+ \rightarrow \mathcal{H}(\mathcal{R}, q)^\times$  extends to a group homomorphism

$$X \rightarrow \mathcal{H}(\mathcal{R}, q)^\times : x \mapsto \theta_x.$$

**Theorem 1.1.** (Bernstein presentation)

- (a) The sets  $\{N_w \theta_x \mid w \in W, x \in X\}$  and  $\{\theta_x N_w \mid w \in W, x \in X\}$  are bases of  $\mathcal{H}$ .
- (b) The subalgebra  $\mathcal{A} := \text{span}\{\theta_x \mid x \in X\}$  is isomorphic to  $\mathbb{C}[X]$ .
- (c) The centre of  $Z(\mathcal{H}(\mathcal{R}, q))$  of  $\mathcal{H}(\mathcal{R}, q)$  is  $\mathcal{A}^W$ , where we define the action of  $W$  on  $\mathcal{A}$  by  $w(\theta_x) = \theta_{wx}$ .
- (d) For  $f \in \mathcal{A}$  and  $\alpha \in \Delta$

$$f N_{s_\alpha} - N_{s_\alpha} s_\alpha(f) = q(s_\alpha)^{-1/2}(f - s_\alpha(f))(q(s_\alpha)c_\alpha - 1).$$

Here the  $c$ -functions are defined as

$$c_\alpha = \begin{cases} \frac{\theta_\alpha + q(s_\alpha)^{-1/2}q(t_\alpha s_\alpha)^{1/2}}{\theta_\alpha + 1} \frac{\theta_\alpha - q(s_\alpha)^{-1/2}q(t_\alpha s_\alpha)^{-1/2}}{\theta_\alpha - 1} & \alpha \in R \setminus R_l \\ (\theta_\alpha - q(s_\alpha)^{-1})(\theta_\alpha - 1)^{-1} & \alpha \in R \cap R_l. \end{cases}$$

*Proof.* These results are due to Bernstein, see [Lus2, §3].  $\square$

Consider the complex algebraic torus

$$T = \text{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times) \cong Y \otimes_{\mathbb{Z}} \mathbb{C}^\times,$$

so  $\mathcal{A} \cong \mathcal{O}(T)$  and  $Z(\mathcal{H}) = \mathcal{A}^W \cong \mathcal{O}(T/W)$ . From Theorem 1.1 we see that  $\mathcal{H}$  is of finite rank over its centre. Let  $\mathfrak{t} = \text{Lie}(T)$  and  $\mathfrak{t}^*$  be the complexifications of  $\mathfrak{a}$  and  $\mathfrak{a}^*$ . The direct sum  $\mathfrak{t} = \mathfrak{a} \oplus i\mathfrak{a}$  corresponds to the polar decomposition

$$T = T_{rs} \times T_{un} = \text{Hom}_{\mathbb{Z}}(X, \mathbb{R}_{>0}) \times \text{Hom}_{\mathbb{Z}}(X, S^1)$$

of  $T$  into a real split (or positive) part and a unitary part. The exponential map  $\exp : \mathfrak{t} \rightarrow T$  is bijective on the real parts, and we denote its inverse by  $\log : T_{rs} \rightarrow \mathfrak{a}$ .

An automorphism of the Dynkin diagram of the based root system  $(R, \Delta)$  is a bijection  $\gamma : \Delta \rightarrow \Delta$  such that

$$\langle \gamma(\alpha), \gamma(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle \quad \forall \alpha, \beta \in \Delta. \quad (3)$$

Such a  $\gamma$  naturally induces automorphisms of  $R, R^\vee, W$  and  $W^{\text{aff}}$ . It is easy to classify all diagram automorphisms of  $(R, \Delta)$ : they permute the irreducible components of  $R$  of a given type, and the diagram automorphisms of a connected Dynkin diagram can be seen immediately.

We will assume that the action of  $\gamma$  on  $W^{\text{aff}}$  has been extended in some way to  $W^e$ , and then we call it a diagram automorphism of  $\mathcal{R}$ . For example, this is the case if  $\gamma$  belongs to the Weyl group of some larger root system contained in  $X$ . We regard two diagram automorphisms as the same if and only if their actions on  $W^e$  coincide.

Let  $\Gamma$  be a finite group of diagram automorphisms of  $\mathcal{R}$  and assume that  $q_{\alpha^\vee} = q_{\gamma(\alpha^\vee)}$  for all  $\alpha \in R_{nr}$ . Then  $\Gamma$  acts on  $\mathcal{H}$  by algebra automorphisms  $\psi_\gamma$  that satisfy

$$\begin{aligned} \psi_\gamma(N_w) &= N_{\gamma(w)} & w \in W, \\ \psi_\gamma(\theta_x) &= \theta_{\gamma(x)} & x \in X. \end{aligned} \quad (4)$$

Hence one can form the crossed product algebra  $\Gamma \ltimes \mathcal{H} = \mathcal{H} \rtimes \Gamma$ , whose natural basis is indexed by the group  $(X \rtimes W) \rtimes \Gamma = X \rtimes (W \rtimes \Gamma)$ . It follows easily from (4) and Theorem 1.1.c that  $Z(\mathcal{H} \rtimes \Gamma) = \mathcal{A}^{W \rtimes \Gamma}$ . We say that the central character of an (irreducible)  $\mathcal{H} \rtimes \Gamma$ -representation is positive if it lies in  $T^{rs}/(W \rtimes \Gamma)$ .

## 1.2 Graded Hecke algebras

Graded Hecke algebras are also known as degenerate (affine) Hecke algebras. They were introduced by Lusztig in [Lus2]. We call

$$\tilde{\mathcal{R}} = (\mathfrak{a}^*, R, \mathfrak{a}, R^\vee, \Delta) \quad (5)$$

a degenerate root datum. We pick complex numbers  $k_\alpha$  for  $\alpha \in \Delta$ , such that  $k_\alpha = k_\beta$  if  $\alpha$  and  $\beta$  are in the same  $W$ -orbit. The graded Hecke algebra associated to these data is the complex vector space

$$\mathbb{H} = \mathbb{H}(\tilde{\mathcal{R}}, k) = S(\mathfrak{t}^*) \otimes \mathbb{C}[W],$$

with multiplication defined by the following rules:

- $\mathbb{C}[W]$  and  $S(\mathfrak{t}^*)$  are canonically embedded as subalgebras;
- for  $x \in \mathfrak{t}^*$  and  $s_\alpha \in S$  we have the cross relation

$$x \cdot s_\alpha - s_\alpha \cdot s_\alpha(x) = k_\alpha \langle x, \alpha^\vee \rangle. \quad (6)$$

Multiplication with any  $\epsilon \in \mathbb{C}^\times$  defines a bijection  $m_\epsilon : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ , which clearly extends to an algebra automorphism of  $S(\mathfrak{t}^*)$ . From the cross relation (6) we see that it extends even further, to an algebra isomorphism

$$m_\epsilon : \mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rightarrow \mathbb{H}(\tilde{\mathcal{R}}, k) \quad (7)$$

which is the identity on  $\mathbb{C}[W]$ . For  $\epsilon = 0$  this map is well-defined, but obviously not bijective.

Let  $\Gamma$  be a group of diagram automorphisms of  $\mathcal{R}$ , and assume that  $k_{\gamma(\alpha)} = k_\alpha$  for all  $\alpha \in R, \gamma \in \Gamma$ . Then  $\Gamma$  acts on  $\mathbb{H}$  by the algebra automorphisms

$$\begin{aligned} \psi_\gamma : \mathbb{H} &\rightarrow \mathbb{H}, \\ \psi_\gamma(x s_\alpha) &= \gamma(x) s_{\gamma(\alpha)} \quad x \in \mathfrak{t}^*, \alpha \in \Pi. \end{aligned} \quad (8)$$

By [Sol1, Proposition 5.1.a] the centre of the resulting crossed product algebra is

$$Z(\mathbb{H} \rtimes \Gamma) = S(\mathfrak{t}^*)^{W \rtimes \Gamma} = \mathcal{O}(\mathfrak{t}/(W \rtimes \Gamma)). \quad (9)$$

We say that the central character of an  $\mathbb{H} \rtimes \Gamma$ -representation is real if it lies in  $\mathfrak{a}/(W \rtimes \Gamma)$ .

### 1.3 Parabolic subalgebras

For a set of simple roots  $P \subset \Delta$  we introduce the notations

$$\begin{aligned} R_P &= \mathbb{Q}P \cap R & R_P^\vee &= \mathbb{Q}P^\vee \cap R^\vee, \\ \mathfrak{a}_P &= \mathbb{R}P^\vee & \mathfrak{a}^P &= (\mathfrak{a}_P^*)^\perp, \\ \mathfrak{a}_P^* &= \mathbb{R}P & \mathfrak{a}^{P*} &= (\mathfrak{a}_P)^\perp, \\ \mathfrak{t}_P &= \mathbb{C}P^\vee & \mathfrak{t}^P &= (\mathfrak{t}_P^*)^\perp, \\ \mathfrak{t}_P^* &= \mathbb{C}P & \mathfrak{t}^{P*} &= (\mathfrak{t}_P)^\perp, \\ X_P &= X/(X \cap (P^\vee)^\perp) & X^P &= X/(X \cap \mathbb{Q}P), \\ Y_P &= Y \cap \mathbb{Q}P^\vee & Y^P &= Y \cap P^\perp, \\ T_P &= \text{Hom}_{\mathbb{Z}}(X_P, \mathbb{C}^\times) & T^P &= \text{Hom}_{\mathbb{Z}}(X^P, \mathbb{C}^\times), \\ \mathcal{R}_P &= (X_P, R_P, Y_P, R_P^\vee, P) & \mathcal{R}^P &= (X, R_P, Y, R_P^\vee, P), \\ \tilde{\mathcal{R}}_P &= (\mathfrak{a}_P^*, R_P, \mathfrak{a}_P, R_P^\vee, P) & \tilde{\mathcal{R}}^P &= (\mathfrak{a}^*, R_P, \mathfrak{a}, R_P^\vee, P). \end{aligned} \quad (10)$$

We define parameter functions  $q_P$  and  $q^P$  on the root data  $\mathcal{R}_P$  and  $\mathcal{R}^P$ , as follows. Restrict  $q$  to a function on  $(R_P)_{nr}^\vee$  and determine the value on simple (affine) reflections in  $W(\mathcal{R}_P)$  and  $W(\mathcal{R}^P)$  by (1). Similarly the restriction of  $k$  to  $P$  is a



parameter function for the degenerate root data  $\tilde{\mathcal{R}}_P$  and  $\tilde{\mathcal{R}}^P$ , and we denote it by  $k_P$  or  $k^P$ . Now we can define the parabolic subalgebras

$$\begin{aligned}\mathcal{H}_P &= \mathcal{H}(\mathcal{R}_P, q_P) & \mathcal{H}^P &= \mathcal{H}(\mathcal{R}^P, q^P), \\ \mathbb{H}_P &= \mathbb{H}(\tilde{\mathcal{R}}_P, k_P) & \mathbb{H}^P &= \mathbb{H}(\tilde{\mathcal{R}}^P, k^P).\end{aligned}$$

We notice that  $\mathbb{H}^P = S(\mathfrak{t}^{P*}) \otimes \mathbb{H}_P$ , a tensor product of algebras. Despite our terminology  $\mathcal{H}^P$  and  $\mathcal{H}_P$  are not subalgebras of  $\mathcal{H}$ , but they are close. Namely,  $\mathcal{H}(\mathcal{R}^P, q^P)$  is isomorphic to the subalgebra of  $\mathcal{H}(\mathcal{R}, q)$  generated by  $\mathcal{A}$  and  $\mathcal{H}(W(R_P), q_P)$ .

We denote the image of  $x \in X$  in  $X_P$  by  $x_P$  and we let  $\mathcal{A}_P \subset \mathcal{H}_P$  be the commutative subalgebra spanned by  $\{\theta_{x_P} \mid x_P \in X_P\}$ . There is natural surjective quotient map

$$\mathcal{H}^P \rightarrow \mathcal{H}_P : \theta_x N_w \mapsto \theta_{x_P} N_w. \quad (11)$$

Suppose that  $\gamma \in \Gamma \times W$  satisfies  $\gamma(P) = Q \subseteq \Delta$ . Then there are algebra isomorphisms

$$\begin{aligned}\psi_\gamma : \mathcal{H}_P &\rightarrow \mathcal{H}_Q, & \theta_{x_P} N_w &\mapsto \theta_{\gamma(x_P)} N_{\gamma w \gamma^{-1}}, \\ \psi_\gamma : \mathcal{H}^P &\rightarrow \mathcal{H}^Q, & \theta_x N_w &\mapsto \theta_{\gamma x} N_{\gamma w \gamma^{-1}}, \\ \psi_\gamma : \mathbb{H}_P &\rightarrow \mathbb{H}_Q, & f_P w &\mapsto (f_P \circ \gamma^{-1}) \gamma w \gamma^{-1}, \\ \psi_\gamma : \mathbb{H}_P &\rightarrow \mathbb{H}_Q, & f w &\mapsto (f \circ \gamma^{-1}) \gamma w \gamma^{-1},\end{aligned} \quad (12)$$

where  $f_P \in \mathcal{O}(\mathfrak{t}_P)$  and  $f \in \mathcal{O}(\mathfrak{t})$ . Sometimes we will abbreviate  $W \rtimes \Gamma$  to  $W'$ . For example the group

$$W'_P := \{\gamma \in \Gamma \times W \mid \gamma(P) = P\} \quad (13)$$

acts on the algebras  $\mathcal{H}_P$  and  $\mathcal{H}^P$ . Although  $W'_\Delta = \Gamma$ , for proper subsets  $P \subsetneq \Delta$  the group  $W'_P$  need not be contained in  $\Gamma$ .

To avoid confusion we do not use the notation  $W_P$ . Instead the parabolic subgroup of  $W$  generated by  $\{s_\alpha \mid \alpha \in P\}$  will be denoted  $W(R_P)$ . Suppose that  $\gamma \in W'$  stabilizes either the root system  $R_P$ , the lattice  $\mathbb{Z}P$  or the vector space  $\mathbb{Q}P \subset \mathfrak{a}^*$ . Then  $\gamma(P)$  is a basis of  $R_P$ , so  $\gamma(P) = w(P)$  and  $w^{-1}\gamma \in W'_P$  for a unique  $w \in W(R_P)$ . Therefore

$$W'_{\mathbb{Z}P} := \{\gamma \in W' \mid \gamma(\mathbb{Z}P) = \mathbb{Z}P\} \text{ equals } W(R_P) \rtimes W'_P. \quad (14)$$

For  $t \in T^P$  and  $\lambda \in \mathfrak{t}^P$  we define an algebra automorphisms

$$\begin{aligned}\phi_t : \mathcal{H}^P &\rightarrow \mathcal{H}^P, & \phi_t(\theta_x N_w) &= t(x) \theta_x N_w & x \in X, w \in W, \\ \phi_\lambda : \mathbb{H}^P &\rightarrow \mathbb{H}^P, & \phi_\lambda(fh) &= f(\lambda) fh & f \in S(\mathfrak{t}^{P*}), h \in \mathbb{H}_P.\end{aligned} \quad (15)$$

For  $t \in K_P := T^P \cap T_P$  this descends to an algebra automorphism

$$\psi_t : \mathcal{H}_P \rightarrow \mathcal{H}_P, \quad \theta_{x_P} N_w \mapsto t(x_P) \theta_{x_P} N_w \quad t \in K_P. \quad (16)$$

We can regard any representation  $(\sigma, V_\sigma)$  of  $\mathcal{H}(\mathcal{R}_P, q_P)$  as a representation of  $\mathcal{H}^P = \mathcal{H}(\mathcal{R}^P, q^P)$  via the quotient map (11). Thus we can construct the  $\mathcal{H}$ -representation

$$\pi(P, \sigma, t) := \text{Ind}_{\mathcal{H}_P}^{\mathcal{H}}(\sigma \circ \phi_t).$$

Representations of this form are said to be parabolically induced. Similarly, for any  $\mathbb{H}_P$ -representation  $(\rho, V_\rho)$  and any  $\lambda \in \mathfrak{t}^P$  there is an  $\mathbb{H}^P$ -representation  $\rho \circ \phi_\lambda$ . The corresponding parabolically induced representation is

$$\pi(P, \rho, \lambda) := \text{Ind}_{\mathbb{H}^P}^{\mathbb{H}}(\rho \circ \phi_\lambda).$$

In case we include a group of diagram automorphisms  $\Gamma$ , we will also use the representations

$$\begin{aligned} \pi^\Gamma(P, \sigma, t) &:= \text{Ind}_{\mathcal{H}^P}^{\mathcal{H} \times \Gamma}(\sigma \circ \phi_t), \\ \pi^\Gamma(P, \rho, \lambda) &:= \text{Ind}_{\mathbb{H}^P}^{\mathbb{H} \times \Gamma}(\rho \circ \phi_\lambda). \end{aligned}$$

#### 1.4 Lusztig's reduction theorems

The study of irreducible representations of  $\mathcal{H} \times \Gamma$  is simplified by two reduction theorems, which are essentially due to Lusztig [Lus2]. The first one reduces to the case of modules whose central character is positive on the lattice  $\mathbb{Z}R_l$ . The second one relates these to modules of an associated graded Hecke algebra.

Given  $t \in T$  and  $\alpha \in R$ , [Lus2, Lemma 3.15] tells us that

$$s_\alpha(t) = t \text{ if and only if } \alpha(t) = \begin{cases} 1 & \text{if } \alpha^\vee \notin 2Y \\ \pm 1 & \text{if } \alpha^\vee \in 2Y. \end{cases} \quad (17)$$

We define  $R_t := \{\alpha \in R \mid s_\alpha(t) = t\}$ . The collection of long roots in  $R_{t, nr}$  is  $\{\beta \in R_l \mid \beta(t) = 1\}$ . Let  $F_t$  be the unique basis of  $R_t$  consisting of roots that are positive with respect to  $\Delta$ . We can define a parameter function  $q_t$  for the based root datum

$$\mathcal{R}_t := (X, R_t, Y, R_t^\vee, F_t)$$

via restriction from  $R_{nr}^\vee$  to  $R_{t, nr}^\vee$ . Furthermore we write

$$P(t) := \Delta \cap \mathbb{Q}R_t.$$

Then  $R_{P(t)}$  is a parabolic root subsystem of  $R$  that contains  $R_t$  as a subsystem of full rank. Let  $t = uc \in T_{un}T_{rs}$  be the polar decomposition of  $t \in T$ . We note that  $R_{uc} \subset R_u$ , that  $W'_{uc} \subset W'_u$  and that the lattice

$$\mathbb{Z}P(t) = \mathbb{Z}R \cap \mathbb{Q}R_u$$

can be strictly larger than  $\mathbb{Z}R_t$ . We will phrase the first reduction theorem such that it depends mainly on the unitary part  $u$  of  $t$ , it will decompose a representation in parts corresponding to the point of the orbit  $W'u$ .

For a finite set  $U \subset T/W$ , let  $Z_U(\mathcal{H}) \subset Z(\mathcal{H})$  be the ideal of functions vanishing at  $U$ . Let  $\widehat{Z(\mathcal{H})}_U$  be the formal completion of  $Z(\mathcal{H})$  with respect to the powers of the ideal  $Z_U(\mathcal{H})$  and define

$$\widehat{\mathcal{H}}_U = \widehat{Z(\mathcal{H})}_U \otimes_{Z(\mathcal{H})} \mathcal{H}. \quad (18)$$

Similarly, for  $t \in T$  let  $\widehat{\mathcal{A}}_t$  denote the formal completion of  $\mathcal{A} \cong \mathcal{O}(T)$  with respect to the powers of the ideal  $\{f \in \mathcal{O}(T) \mid f(t) = 0\}$ . Inside  $\widehat{\mathcal{H}}_U$  we have the formal completion of  $\mathcal{A}$  at  $U$ , which by the Chinese remainder theorem is isomorphic to

$$\widehat{\mathcal{A}}_U := \widehat{Z(\mathcal{H})}_U \otimes_{\mathcal{A}^W} \mathcal{A} \cong \bigoplus_{t \in U} \widehat{\mathcal{A}}_t. \quad (19)$$

In this notation we can rewrite

$$\widehat{Z(\mathcal{H})}_{Wt} \cong \widehat{\mathcal{A}}_{Wt}^W \cong \widehat{\mathcal{A}}_t^{Wt}.$$

Analogous statements hold for  $\mathcal{H} \rtimes \Gamma$ . Given a subset  $\varpi \subset W't$  we let  $1_\varpi \in \widehat{\mathcal{A}}_{W't}$  be the idempotent corresponding to  $\bigoplus_{s \in \varpi} \widehat{\mathcal{A}}_s$ .

**Theorem 1.2.** (First reduction theorem)

*There is a natural isomorphism of  $Z(\mathcal{H} \rtimes \Gamma)_{W'uc}$ -algebras*

$$\mathcal{H}(\mathcal{R}^{P(u)}, \widehat{q^{P(u)}})_{W'_{ZP(u)}uc} \rtimes W'_{P(u)} \cong 1_{W'_{ZP(u)}uc} (\widehat{\mathcal{H}}_{W'uc} \rtimes \Gamma) 1_{W'_{ZP(u)}uc}$$

*It can be extended (not naturally) to isomorphism of  $Z(\mathcal{H} \rtimes \Gamma)_{W'uc}$ -algebras*

$$\widehat{\mathcal{H}}_{W'uc} \rtimes \Gamma \cong M_{[W':W'_{ZP(u)}]} \left( 1_{W'_{ZP(u)}uc} (\widehat{\mathcal{H}}_{W'uc} \rtimes \Gamma) 1_{W'_{ZP(u)}uc} \right),$$

where  $M_n(A)$  denotes the algebra of  $n \times n$ -matrices with coefficients in an algebra  $A$ . In particular the algebras

$$\mathcal{H}(\mathcal{R}^{P(u)}, \widehat{q^{P(u)}})_{W'_{ZP(u)}uc} \rtimes W'_{P(u)} \quad \text{and} \quad \widehat{\mathcal{H}}_{W'uc} \rtimes \Gamma$$

are Morita equivalent.

*Proof.* This is a variation on [Lus2, Theorem 8.6]. Compared to Lusztig we substituted his  $R_{uc}$  by a larger root system, we replaced the subgroup  $Y \otimes \langle v_0 \rangle \subset T$  by  $T_{rs} = Y \otimes \mathbb{R}_{>0}$  and we included the automorphism group  $\Gamma$ . These changes are justified in the proof of [Sol3, Theorem 2.1.2].  $\square$

By (17) we have  $\alpha(u) = 1$  for all  $\alpha \in R_l \cap \mathbb{Q}R_u$ , so  $\alpha(t) = \alpha(u)\alpha(c) > 0$  for such roots. Hence Theorem 1.2 allows us to restrict our attention to  $\mathcal{H} \rtimes \Gamma$ -modules whose central character is positive on the sublattice  $\mathbb{Z}R_l \subseteq X$ .

By definition  $u$  is fixed by  $W'_u \supset W(R_{P(u)})$ , so the map

$$\exp_u : \mathfrak{t} \rightarrow T, \quad \lambda \mapsto u \exp(\lambda) \quad (20)$$

is  $W'_u$ -equivariant.

Analogous to (18), let  $Z_{W\lambda}(\mathbb{H}) \subset Z(\mathbb{H})$  be the maximal ideal of functions vanishing at  $W\lambda \in \mathfrak{t}/W$ . Let  $\widehat{Z(\mathbb{H})}_{W\lambda}$  be the formal completion of  $Z(\mathbb{H})$  with respect to  $Z_{W\lambda}(\mathbb{H})$  and define

$$\widehat{\mathbb{H}}_{W\lambda} := \widehat{Z(\mathbb{H})}_{W\lambda} \otimes_{Z(\mathbb{H})} \mathbb{H}. \quad (21)$$

The corresponding formal completion of  $S(\mathfrak{t}^*) \cong \mathcal{O}(\mathfrak{t})$  is

$$\widehat{S(\mathfrak{t}^*)}_{W\lambda} := \widehat{Z(\mathbb{H})}_{W\lambda} \otimes_{S(\mathfrak{t}^*)^W} S(\mathfrak{t}^*) \cong \bigoplus_{\mu \in W\lambda} \widehat{S(\mathfrak{t}^*)}_\mu.$$

The map (20) induces a  $W'_u$ -equivariant isomorphism

$$\widehat{\mathcal{A}}_{u \exp \lambda} \rightarrow \widehat{S(\mathfrak{t}^*)}_\lambda : f \mapsto f \circ \exp_u,$$

which restricts to an isomorphism

$$\Phi_{u, W'\lambda} : Z(\mathcal{H} \rtimes \Gamma)_{W'u \exp(\lambda)} \rightarrow Z(\mathbb{H} \rtimes \Gamma)_{W'\lambda}. \quad (22)$$

We define a parameter function  $k_u$  for the degenerate root datum  $\tilde{\mathcal{R}}_u$  by

$$2k_{u, \alpha} = \log q_{\alpha^\vee/2} + (1 + \alpha(u)) \log q_{\alpha^\vee} \quad \alpha \in R_u. \quad (23)$$

Let  $q^{\mathbb{Z}/2}$  be the subgroup of  $\mathbb{C}^\times$  generated by  $\{q_{\alpha^\vee}^{\pm 1/2} \mid \alpha^\vee \in R_{nr}^\vee\}$ .

**Theorem 1.3.** (Second reduction theorem)

Suppose that 1 is the only root of unity in  $q^{\mathbb{Z}/2}$ . Let  $u \in T_{un}^{W'}$  and let  $\lambda \in \mathfrak{t}$  be such that

$$\langle \alpha, \lambda \rangle, \langle \alpha, \lambda \rangle + k_{u, \alpha} \notin \pi i \mathbb{Z} \setminus \{0\} \quad \forall \alpha \in R. \quad (24)$$

(a) The map (22) extends to an algebra isomorphism

$$\Phi_{u, W'\lambda} : \mathcal{H}_{W'u \exp(\lambda)} \rtimes \Gamma \rightarrow \mathbb{H}(\tilde{\mathcal{R}}, k_u)_{W'\lambda} \rtimes \Gamma.$$

(b) The algebras  $\mathcal{H}_{W'u \exp(\lambda)} \rtimes \Gamma$  and  $\mathbb{H}(\tilde{\mathcal{R}}^{P(u)}, k^{P(u)})_{W'_{\mathbb{Z}P(u)}\lambda} \rtimes W'_{P(u)}$  are Morita equivalent.

*Proof.* For part (a) see [Lus2, Theorem 9.3]. Our conditions on  $q$  replace the assumption [Lus2, 9.1]. Part (b) follows from part (a) and Theorem 1.2.  $\square$

## 1.5 Schwartz algebras

An important tool to study  $\mathcal{H}$ -representations is restriction to the commutative subalgebra  $\mathcal{A} \cong \mathcal{O}(T)$ . We say that  $t \in T$  is a weight of  $(\pi, V)$  if there exists a  $v \in V \setminus \{0\}$  such that  $\pi(a)v = a(t)v$  for all  $a \in \mathcal{A}$ . Temperedness of  $\mathcal{H}$ -representations, which is defined via  $\mathcal{A}$ -weights, is analogous to temperedness of representations of reductive groups. Via the Langlands classification for affine Hecke algebras [Sol3, Section 2.2] these representations are essential in the classification of irreducible representations.

The antidual of  $\mathfrak{a}^{*+} := \{x \in \mathfrak{a}^* \mid \langle x, \alpha^\vee \rangle \geq 0 \forall \alpha \in \Delta\}$  is

$$\mathfrak{a}^- = \{\lambda \in \mathfrak{a} \mid \langle x, \lambda \rangle \leq 0 \forall x \in \mathfrak{a}^{*+}\} = \left\{ \sum_{\alpha \in \Delta} \lambda_\alpha \alpha^\vee \mid \lambda_\alpha \leq 0 \right\}. \quad (25)$$

The interior  $\mathfrak{a}^{--}$  of  $\mathfrak{a}^-$  equals  $\{\sum_{\alpha \in \Delta} \lambda_\alpha \alpha^\vee \mid \lambda_\alpha < 0\}$  if  $\Delta$  spans  $\mathfrak{a}^*$ , and is empty otherwise. We write  $T^- = \exp(\mathfrak{a}^-)$  and  $T^{--} = \exp(\mathfrak{a}^{--})$ .

Let  $t = |t| \cdot t|t|^{-1} \in T_{rs} \times T_{un}$  be the polar decomposition of  $t$ . A finite dimensional  $\mathcal{H}$ -representation is called tempered if  $|t| \in T^-$  for all its  $\mathcal{A}$ -weights  $t$ , and anti-tempered if  $|t|^{-1} \in T^-$  for all such  $t$ .

We say that an irreducible  $\mathcal{H}$ -representation belongs to the discrete series (or simply: is discrete series) if all its  $\mathcal{A}$ -weights lie in  $T^- T_{un}$ . In particular the discrete series is empty if  $\Delta$  does not span  $\mathfrak{a}^*$ . The discrete series is a starting point for the construction of irreducible representations: they can all be realized as a subrepresentation of the induction of a discrete series representation of a parabolic subalgebra of  $\mathcal{H}$ . The notions tempered and discrete series apply equally well to  $\mathcal{H} \rtimes \Gamma$ , since that algebra contains  $\mathcal{A}$  and the action of  $\Gamma$  on  $T$  preserves  $T^-$ .

This terminology also extends naturally to graded Hecke algebras, via the  $S(\mathfrak{t}^*)$ -weights of a representation. Thus we say that a finite dimensional  $\mathbb{H} \rtimes \Gamma$ -representation is tempered if all its  $S(\mathfrak{t}^*)$ -weights lie in  $\mathfrak{a}^- + i\mathfrak{a}$  and we say that it is discrete series if it is irreducible and all its  $S(\mathfrak{t}^*)$ -weights lie in  $\mathfrak{a}^{--} + i\mathfrak{a}$ . By construction Lusztig's two reduction theorems from Section 1.4 preserve the properties temperedness and discrete series.

**Proposition 1.4.** *Let  $P \subset \Delta$ .*

- (a) *Let  $\sigma$  be a finite dimensional  $\mathcal{H}_P \rtimes \Gamma_P$ -representation. For  $t \in T^P$  the  $\mathcal{H} \rtimes \Gamma$ -representation  $\text{Ind}_{\mathcal{H}_P \rtimes \Gamma_P}^{\mathcal{H} \rtimes \Gamma}(\sigma \circ \phi_t)$  is tempered if and only if  $t \in T_{un}^P$  and  $\sigma$  is tempered.*
- (b) *Let  $\rho$  be a finite dimensional  $\mathbb{H}_P \rtimes \Gamma_P$ -representation. For  $\lambda \in \mathfrak{t}^P$  the  $\mathbb{H} \rtimes \Gamma$ -representation  $\text{Ind}_{\mathbb{H}_P \rtimes \Gamma_P}^{\mathbb{H} \rtimes \Gamma}(\rho \circ \phi_\lambda)$  is tempered if and only if  $\lambda \in i\mathfrak{a}^P$  and  $\rho$  is tempered.*

*Proof.* For (a) see [Sol3, Lemma 3.1.1.b] and for (b) see [Sol2, Lemma 2.2].  $\square$

Now we will recall the construction of the Schwartz algebra  $\mathcal{S}$  of an affine Hecke algebra [DeOp], which we will use in Section 3.3. For this we assume that the parameter function  $q$  is positive, because otherwise there would not be a good link with  $C^*$ -algebras. As a topological vector space  $\mathcal{S}$  will consist of rapidly decreasing functions on  $W^e$ , with respect to a suitable length function  $\mathcal{N}$ . For example we can take a  $W$ -invariant norm on  $X \otimes_{\mathbb{Z}} \mathbb{R}$  and put  $\mathcal{N}(wt_x) = \|x\|$  for  $w \in W$  and  $x \in X$ . Then we can define, for  $n \in \mathbb{N}$ , the following norm on  $\mathcal{H}$ :

$$p_n \left( \sum_{w \in W^e} h_w N_w \right) = \sup_{w \in W^e} |h_w| (\mathcal{N}(w) + 1)^n.$$

The completion of  $\mathcal{H}$  with respect to these norms is the Schwartz algebra  $\mathcal{S} = \mathcal{S}(\mathcal{R}, q)$ . It is known from [Opd, Section 6.2] that it is a Fréchet algebra. Diagram automorphisms of  $\mathcal{R}$  induce automorphisms of  $\mathcal{S}$ , so the crossed product algebra  $\mathcal{S} \rtimes \Gamma$  is well-defined. By [Opd, Lemma 2.20] a finite dimensional  $\mathcal{H} \rtimes \Gamma$ -representation is tempered if and only if it extends continuously to an  $\mathcal{S} \rtimes \Gamma$ -representation.

Next we describe the Fourier transform for  $\mathcal{H}$  and  $\mathcal{S}$ . An induction datum for  $\mathcal{H}$  is a triple  $(P, \delta, t)$ , with  $P \subset \Delta$ ,  $t \in T^P$  and  $\delta$  a discrete series representation of  $\mathcal{H}_P$ .

A large part of the representation theory of  $\mathcal{H}$  is built on the parabolically induced representations

$$\pi(P, \delta, t) = \text{Ind}_{\mathcal{H}^P}^{\mathcal{H}}(\delta \circ \phi_t) \quad \text{and} \quad \pi^\Gamma(P, \delta, t) = \text{Ind}_{\mathcal{H}^P}^{\mathcal{H} \times \Gamma}(\delta \circ \phi_t).$$

Let  $\Xi$  be the space of all such induction data  $(P, \delta, t)$ , with  $\delta$  up to equivalence of  $\mathcal{H}_P$ -representations. It carries a natural structure of a complex affine variety with finitely many components of different dimensions. Furthermore  $\Xi$  has a compact submanifold

$$\Xi_{un} := \{(P, \delta, t) \in \Xi \mid t \in T_{un}^P\},$$

which is a disjoint union of finitely many compact tori.

Given a discrete series representation  $(\delta, V_\delta)$  of a parabolic subalgebra  $\mathcal{H}_P$  of  $\mathcal{H}$ , the  $\mathcal{H} \times \Gamma$ -representation  $\pi^\Gamma(P, \delta, t)$  can be realized on the vector space  $V_\delta^\Gamma := \mathbb{C}[\Gamma W^P] \otimes V_\delta$ , which does not depend on  $t \in T^P$ . Here  $W^P$  is a specified set of representatives for  $W/W(R_P)$ . Let  $\mathcal{V}_\Xi^\Gamma$  be the vector bundle over  $\Xi$  whose fiber at  $\xi = (P, \delta, t)$  is  $V_\delta^\Gamma$ , and let

$$\mathcal{O}(\Xi; \text{End}(\mathcal{V}_\Xi^\Gamma)) := \bigoplus_{P, \delta} \mathcal{O}(T^P) \otimes \text{End}_{\mathbb{C}}(V_\delta^\Gamma)$$

be the algebra of polynomial sections of the endomorphism bundle  $\text{End}(\mathcal{V}_\Xi^\Gamma)$ . The Fourier transform for  $\mathcal{H} \times \Gamma$  is

$$\begin{aligned} \mathcal{F} : \mathcal{H} \times \Gamma &\rightarrow \mathcal{O}(\Xi; \text{End}(\mathcal{V}_\Xi^\Gamma)), \\ \mathcal{F}(h)(\xi) &= \pi^\Gamma(\xi)(h). \end{aligned}$$

It extends to an algebra homomorphism

$$\mathcal{F} : \mathcal{S} \times \Gamma \rightarrow C^\infty(\Xi_{un}; \text{End}(\mathcal{V}_\Xi^\Gamma)) := \bigoplus_{P, \delta} C^\infty(T_{un}^P) \otimes \text{End}_{\mathbb{C}}(V_\delta^\Gamma), \quad (26)$$

defined by the same formula.

We will need a groupoid  $\mathcal{G}$  over the power set of  $\Delta$ , defined as follows. For  $P, Q \subset \Delta$  the collection of arrows from  $P$  to  $Q$  is

$$\mathcal{G}_{PQ} = \{(g, u) \in \Gamma \times W \times K_P \mid g(P) = Q\}.$$

Whenever it is defined, the multiplication in  $\mathcal{G}$  is

$$(g', u') \cdot (g, u) = (g'g, g^{-1}(u)u').$$

This groupoid acts from the left on  $\Xi$  by

$$(g, u)(P, \delta, t) = (g(P), \delta \circ \psi_u^{-1} \circ \psi_g^{-1}, g(ut)). \quad (27)$$

This is the projection of an action of  $\mathcal{G}$  on parabolically induced representations via intertwining operators. These operators provide an action of  $\mathcal{G}$  on  $C^\infty(\Xi_{un}; \text{End}(\mathcal{V}_\Xi^\Gamma))$ , see [Opd, Theorem 4.33] and [Sol3, (3.16)]. The Plancherel isomorphism for (extended) affine Hecke algebras with positive parameters reads:

**Theorem 1.5.** *The Fourier transform  $\mathcal{F}$  induces an isomorphism of Fréchet algebras*

$$\mathcal{S} \rtimes \Gamma \rightarrow C^\infty(\Xi_{un}; \text{End}(\mathcal{V}_{\Xi}^\Gamma))^{\mathcal{G}}.$$

*Proof.* See [DeOp, Theorem 5.3] and [Sol3, Theorem 3.2.2].  $\square$

From this isomorphism we see in particular that there are unique central idempotents  $e_{P,\delta} \in \mathcal{S} \rtimes \Gamma$  such that

$$\begin{aligned} e_{P,\delta} \mathcal{S} \rtimes \Gamma &\cong (C^\infty(T_{un}^P) \otimes \text{End}_{\mathbb{C}}(V_\delta^\Gamma))^{\mathcal{G}_{P,\delta}}, \\ \mathcal{G}_{P,\delta} &:= \{g \in \mathcal{G} \mid g(P, \delta, T_{un}^P) = (P, \delta, T_{un}^P)\}. \end{aligned} \quad (28)$$

For a suitable collection  $\mathcal{P}$  of pairs  $(P, \delta)$  we obtain decompositions

$$\begin{aligned} \mathcal{S} \rtimes \Gamma &= \bigoplus_{(P,\delta) \in \mathcal{P}} e_{P,\delta} \mathcal{S} \rtimes \Gamma, \\ Z(\mathcal{S} \rtimes \Gamma) &= \bigoplus_{(P,\delta) \in \mathcal{P}} C^\infty(T_{un}^P)^{\mathcal{G}_{P,\delta}}. \end{aligned} \quad (29)$$

Via (28) the subalgebra  $e_{P,\delta} \mathcal{H} \rtimes \Gamma$  of  $\mathcal{S} \rtimes \Gamma$  is isomorphic to a subalgebra of  $(\mathcal{O}(T^P) \otimes \text{End}_{\mathbb{C}}(V_\delta^\Gamma))^{\mathcal{G}_{P,\delta}}$ . We note that  $e_{P,\delta} Z(\mathcal{H} \rtimes \Gamma)$  is isomorphic to the restriction of  $Z(\mathcal{H} \rtimes \Gamma) \cong \mathcal{O}(T)^{W'}$  to the image of  $(P, \delta, T_{un}^P)$  in  $T/W'$ . We will see from Lemma 2.3 that it is just  $\mathcal{O}(T^P)^{\mathcal{G}_{P,\delta}}$ .

Clearly  $e_{P,\delta} \mathcal{H} \rtimes \Gamma$  is dense in  $e_{P,\delta} \mathcal{S} \rtimes \Gamma$ , so for any closed ideal  $I \subset e_{P,\delta} \mathcal{S} \rtimes \Gamma$  of finite codimension the canonical maps

$$e_{P,\delta} \mathcal{H} \rtimes \Gamma / (I \cap e_{P,\delta} \mathcal{H} \rtimes \Gamma) \rightarrow (e_{P,\delta} \mathcal{H} \rtimes \Gamma + I) / I \rightarrow e_{P,\delta} \mathcal{S} \rtimes \Gamma / I \quad (30)$$

are isomorphisms of  $e_{P,\delta} \mathcal{H} \rtimes \Gamma$ -modules.

**Lemma 1.6.** *The multiplication map*

$$\mu_{P,\delta} : e_{P,\delta} Z(\mathcal{S} \rtimes \Gamma) \otimes_{e_{P,\delta} Z(\mathcal{H} \rtimes \Gamma)} e_{P,\delta} \mathcal{H} \rtimes \Gamma \rightarrow e_{P,\delta} \mathcal{S} \rtimes \Gamma$$

*is an isomorphism of  $e_{P,\delta} Z(\mathcal{S} \rtimes \Gamma)$ -modules.*

*Proof.* Clearly  $\mu_{P,\delta}$  is  $e_{P,\delta} Z(\mathcal{S} \rtimes \Gamma)$ -linear. By Theorem 1.5  $e_{P,\delta} \mathcal{S} \rtimes \Gamma$  is of finite rank over  $e_{P,\delta} Z(\mathcal{S} \rtimes \Gamma)$  and we know that  $e_{P,\delta} \mathcal{H} \rtimes \Gamma$  is dense in  $e_{P,\delta} \mathcal{S} \rtimes \Gamma$ . Thus the image of  $\mu_{P,\delta}$  is closed and dense in  $e_{P,\delta} \mathcal{S} \rtimes \Gamma$ , which means that  $\mu_{P,\delta}$  is surjective.

Let  $I_\xi$  be the maximal ideal of  $e_{P,\delta} Z(\mathcal{S} \rtimes \Gamma) \cong C^\infty(T_{un}^P)^{\mathcal{G}_{P,\delta}}$  corresponding to  $\xi = (P, \delta, t)$ . Applying (30) to  $I_\xi^n e_{P,\delta} \mathcal{S} \rtimes \Gamma$  we see that  $\mu_{P,\delta}$  induces isomorphisms

$$e_{P,\delta} Z(\mathcal{S} \rtimes \Gamma) / I_\xi^n \otimes_{e_{P,\delta} Z(\mathcal{H} \rtimes \Gamma)} e_{P,\delta} \mathcal{H} \rtimes \Gamma \rightarrow e_{P,\delta} \mathcal{S} \rtimes \Gamma / I_\xi^n (e_{P,\delta} \mathcal{S} \rtimes \Gamma), \quad (31)$$

for all  $n \in \mathbb{Z}_{>0}$ . Consider any  $x \in \ker \mu_{P,\delta}$ . Its annihilator  $\text{Ann}(x)$  is a closed ideal of  $e_{P,\delta} Z(\mathcal{S} \rtimes \Gamma)$  and  $e_{P,\delta} Z(\mathcal{S} \rtimes \Gamma)x \cong e_{P,\delta} Z(\mathcal{S} \rtimes \Gamma) / \text{Ann}(x)$ . By (31)  $I_\xi^n / I_\xi^n \text{Ann}(x) = 0$  for all  $\xi \in (P, \delta, T_{un}^P)$ . Therefore  $\text{Ann}(x)$  is not contained in any of the closed maximal ideals  $I_\xi$ . Consequently  $\text{Ann}(x) = e_{P,\delta} Z(\mathcal{S} \rtimes \Gamma)$  and  $x = 0$ .  $\square$

## 2 Analytic families of representations

### 2.1 Positive parameters

In this subsection we assume that  $q$  is positive. We will define analytic families of representations and show that they can be used to construct the dual space of  $\mathcal{H}(\mathcal{R}, q)$ .

**Theorem 2.1.** *Let  $\xi, \xi' \in \Xi$ .*

- (a) *The  $\mathcal{H} \rtimes \Gamma$ -representations  $\pi^\Gamma(\xi)$  and  $\pi^\Gamma(\xi')$  have the same trace if and only if there exists a  $g \in \mathcal{G}$  such that  $g\xi = \xi'$ .*
- (b) *Suppose that  $\xi, \xi' \in \Xi_{un}$ . Then  $\pi^\Gamma(\xi)$  and  $\pi^\Gamma(\xi')$  are completely reducible, and they have a common irreducible subquotient if and only if there exists a  $g \in \mathcal{G}$  such that  $g\xi = \xi'$ . Moreover  $\pi^\Gamma(\xi) \cong \pi^\Gamma(\xi')$  in this situation.*

*Proof.* (a) [Sol3, Lemma 3.1.7] provides the "if"-part. By [Sol3, page 44] the "only if"-part can be reduced to certain positive induction data, to which [Sol3, Theorem 3.3.1] applies.

(b) See Corollary 3.1.3 and Theorem 3.3.1 of [Sol3]. We note that this is essentially a consequence of Theorem 1.5.  $\square$

Since  $\gamma(P) \subset \Delta$  for all  $\gamma \in \Gamma$  and  $P \subset \Delta$ , (27) describes an action of  $\Gamma$  on  $\Xi$ . Given an induction datum  $\xi = (P, \delta, t) \in \Xi$  we put

$$\Gamma_\xi = \{ \gamma \in \Gamma_P \mid \delta \circ \phi_t \cong \delta \circ \phi_t \circ \psi_\gamma^{-1} \text{ as } \mathcal{H}^P\text{-representations} \}.$$

Now we fix  $(P, \delta, u) \in \Xi_{un}$  and we let  $\sigma$  be an irreducible direct summand of  $\text{Ind}_{\mathcal{H}^P}^{\mathcal{H}^P \rtimes \Gamma_{(P, \delta, u)}}(\delta \circ \phi_u)$ . In this case we abbreviate  $\Gamma_{(P, \delta, u)} = \Gamma_\sigma$ . By Clifford theory the representations  $\text{Ind}_{\mathcal{H}^P}^{\mathcal{H}^P \rtimes \Gamma_\sigma}(\delta \circ \phi_u)$  and  $\text{Ind}_{\mathcal{H}^P \rtimes \Gamma_\sigma}^{\mathcal{H}^P \rtimes \Gamma_P}(\sigma)$  are completely reducible.

Let  $T^\sigma$  be the connected component of  $T^{W(R_P) \rtimes \Gamma_\sigma}$  that contains  $1 \in T$ . Notice that  $T^\sigma \subset T^P$  because  $T_P^{W(R_P)}$  is finite. We call

$$\{ \pi_\sigma(t) = \text{Ind}_{\mathcal{H}^P \rtimes \Gamma_\sigma}^{\mathcal{H} \rtimes \Gamma}(\sigma \circ \phi_t) \mid t \in T^\sigma \} \quad (32)$$

an analytic  $d$ -dimensional family of  $\mathcal{H} \rtimes \Gamma$ -representations, where  $d$  is the dimension of the complex algebraic variety  $T^\sigma$ . By Proposition 1.4 the representations  $\pi_\sigma(t)$  are tempered if and only if  $t \in T_{un}^\sigma := T^\sigma \cap T_{un}$ . We refer to  $\{ \pi_\sigma(t) \mid t \in T_{un}^\sigma \}$  as a tempered analytic family or an analytic family of  $\mathcal{S}$ -representations.

**Correction (2021):** To guarantee that the union of analytic families of representations spans  $G_{\mathbb{Q}}(\mathcal{H} \rtimes \Gamma)$ , we need to allow more general  $\sigma$ . Namely, we need all irreducible tempered elliptic representations of a parabolic subalgebra  $\mathcal{H}^P \rtimes \Gamma'_P$ , where  $\Gamma'_P$  is an arbitrary subgroup of  $\Gamma_P$ . Here elliptic means that the image in  $G_{\mathbb{Q}}(\mathcal{H}^P \rtimes \Gamma'_P)$  does not belong to the span of the representations induced from proper parabolic subalgebras of  $\mathcal{H}^P \rtimes \Gamma'_P$ . Such a  $\sigma$  is a direct summand of the  $\mathcal{H}^P \rtimes \Gamma'_P$ -representation associated to a triple  $(P, \delta, u) \in \Xi_{un}$ . With that, the below



arguments hold in the necessary generality.

Since there are only finitely many pairs  $(P, \delta)$  and since two  $u$ 's in the same  $T^\sigma$ -coset give rise to the same analytic family, there exist only finitely many analytic families of  $\mathcal{H} \rtimes \Gamma$ -representations. Recall from [OpSo1, Theorem 2.58] that the absolute value of any  $\mathcal{A}_P$ -weight of  $\delta$  is a monomial in the variables  $q(s_\alpha)^{\pm 1/2}$ . That is, it lies in

$$W(R_P)u_\delta Y_P \otimes_{\mathbb{Z}} q^{\mathbb{Z}/2} \quad (33)$$

where  $u_\delta \in T_{P,un}$  is the unitary part of such a weight. Hence all  $\mathcal{A}$ -weights of  $\pi_\sigma(t)$  lie in

$$W'tuu_\delta Y \otimes_{\mathbb{Z}} q^{\mathbb{Z}/2}. \quad (34)$$

We note that the  $\mathcal{H}^P \rtimes \Gamma_\sigma$ -representation  $\pi_\sigma(t)$  is only defined for  $t \in T^{W(R_P) \rtimes \Gamma_\sigma}$ , so it is impossible to extend (32) to a larger connected subset of  $T^P$ . As discussed in [Sol3] after (3.30), the representations  $\pi_\sigma(t)$  are irreducible for  $t$  in a Zariski-open dense subset of  $T^\sigma$ . Like in (28) the group

$$\mathcal{G}_\sigma := \{g \in \mathcal{G} \mid g(P) = P, \delta \circ \psi_g^{-1} \cong \delta, g \text{ stabilizes } \{\pi_\sigma(t) \mid t \in T_{un}^\sigma\}\} \quad (35)$$

acts on  $T^\sigma$ . (Here the advantage of considering only  $T_{un}^\sigma$  is that Theorem 2.1.b applies.) By Theorem 2.1.a  $\pi_\sigma(t)$  and  $\pi_\sigma(gt)$  have the same trace for all  $t \in T^\sigma$  and  $g \in \mathcal{G}_\sigma$ . We will see later that every representation  $\pi_\sigma(t')$  with  $t'$  in a different  $\mathcal{G}_\sigma$ -orbit has a different trace.

Let  $G(A)$  denote the Grothendieck group of finite dimensional  $A$ -representations, for suitable algebras or groups  $A$ . As explained in [Sol3, Section 3.4],

$$G_{\mathbb{Q}}(\mathcal{H} \rtimes \Gamma) := \mathbb{Q} \otimes_{\mathbb{Z}} G(\mathcal{H} \rtimes \Gamma)$$

can be built from analytic families of representations (called smooth families in that paper). By this we mean that we can choose a collection of analytic families  $\{\pi_{\sigma_i}(t) \mid t \in T^{\sigma_i}\}$  such that the set

$$\bigcup_i \{\pi_{\sigma_i}(t) \mid t \in T^{\sigma_i} / \mathcal{G}_{\sigma_i}\} \quad (36)$$

spans  $G_{\mathbb{Q}}(\mathcal{H} \rtimes \Gamma)$ . In [Sol3, Section 3.4] this is actually done with "Langlands constituents" of  $\pi_{\sigma_i}(t)$ . But by [Sol3, Lemma 2.2.6] the Langlands formalism is such that the other constituents of  $\pi_{\sigma_i}(t)$  are smaller in a certain sense. Hence the non-Langlands constituents are already accounted for by families of smaller dimension.

All this can be made much more concrete for the algebra

$$\mathcal{H}(\mathcal{R}, 1) \rtimes \Gamma = \mathbb{C}[X \rtimes (W \rtimes \Gamma)] = \mathcal{O}(T) \rtimes W'.$$

By classical results which go back to Frobenius and Clifford, its irreducible representations with  $\mathcal{O}(T)^{W'}$ -character  $W't$  are in bijection with the irreducible representations  $\sigma$  of the isotropy group  $W'_t$ , via

$$\sigma \mapsto \text{Ind}_{X \rtimes W'_t}^{X \rtimes W'}(\mathbb{C}_t \otimes \sigma). \quad (37)$$

With this one can easily determine the dual space of  $X \rtimes W'$ . Let

$$\tilde{T} = \{(w, t) \in T \times W' \mid w(t) = t\}.$$

Then  $W'$  acts on  $\tilde{T}$  by  $w \cdot (w', t) = (ww'w^{-1}, w(t))$  and  $\tilde{T}/W'$  is called the extended quotient of  $T$  by  $W'$ . Let  $\langle W' \rangle$  be a set of representatives for the conjugacy classes in  $W'$ . Then

$$\tilde{T}/W' \cong \bigsqcup_{w \in \langle W' \rangle} T^w/Z_{W'}(w), \quad (38)$$

where  $Z_{W'}(w)$  denotes the centralizer of  $w$  in  $W'$ . Let  $T_i^w/Z_{W'}(w), i = 1, \dots, c(w)$  be the connected components of  $T^w/Z_{W'}(w)$ , so that

$$\tilde{T}/W' \cong \bigsqcup_{w \in \langle W' \rangle} \bigsqcup_{1 \leq i \leq c(w)} T_i^w/Z_{W'}(w). \quad (39)$$

By the above there exists a continuous bijection from  $\tilde{T}/W'$  to the space of irreducible complex representations of  $X \rtimes W'$ . We vary a little and take the Grothendieck group of  $X \rtimes W'$  instead. Let  $C_w \subset W'$  be the cyclic group generated by  $w$  and let  $\rho_w$  be a  $C_w$ -representation. This yields an analytic family of  $X \rtimes W'$ -representations

$$\{\pi_{w,i}(t) := \text{Ind}_{X \rtimes C_w}^{X \rtimes W'}(\mathbb{C}_t \otimes \rho_w) \mid t \in T_i^w\}.$$

For generic  $t$ ,  $\pi_{w,i}(t)$  is irreducible if  $W'_t = C_w$ . Moreover  $\pi_{w,i}(t)$  arise by induction from an elliptic representation of  $X \rtimes (W(R_P) \rtimes \Gamma'_P)$ , where  $W(R_P) \rtimes \Gamma'_P$  is a parabolic subgroup of  $W \rtimes \Gamma$  minimally containing  $w$ .

By Artin's Theorem [Ser, Theorem 17] it is possible to choose the  $\rho_w$  such that  $G_{\mathbb{Q}}(W'_t)$  is spanned by  $\{\text{Ind}_{C_w}^{W'_t}(\rho_w) \mid w \in W'_t\}$ ,

**Correction (2021):** The next claim is false, it can fail for instance if  $W' = \mathbb{Z}/4\mathbb{Z}$ .

and a basis is obtained by taking only one  $w$  from every conjugacy class in  $W'_t$ . It follows that the representations

$$\{\pi_{w,i}(t) \mid w \in \langle W' \rangle, 1 \leq i \leq c(w), t \in T_i^w/Z_{W'}(w)\} \quad (40)$$

form a basis of  $G_{\mathbb{Q}}(X \rtimes W')$ . Notice that the space underlying (40) is exactly the extended quotient  $\tilde{T}/W'$ , since all the involved representations are different. Having described this Grothendieck group conveniently, we recall how it can be compared with the Grothendieck groups of associated affine Hecke algebras.

**Theorem 2.2.** [Sol3, Section 2.3]

*Let  $q$  be positive. There exists a natural  $\mathbb{Q}$ -linear bijection*

$$\zeta^{\vee} : G_{\mathbb{Q}}(\mathcal{H} \rtimes \Gamma) \rightarrow G_{\mathbb{Q}}(X \rtimes W')$$

*such that:*

- (1)  $\zeta^\vee$  restricts to a bijection between the corresponding Grothendieck groups of tempered representations;
- (2)  $\zeta^\vee$  commutes with parabolic induction, in the sense that for a subgroup  $\Gamma'_P \subset \Gamma_P$ , a tempered  $\mathcal{H}^P \rtimes \Gamma'_P$ -representation  $\pi$  and  $t \in T^{W(R_P) \rtimes \Gamma'_P}$  we have

$$\zeta^\vee \left( \text{Ind}_{\mathcal{H}^P \rtimes \Gamma'_P}^{\mathcal{H} \rtimes \Gamma} (\pi \circ \phi_t) \right) = \text{Ind}_{X \rtimes (W(R_P) \rtimes \Gamma'_P)}^{X \rtimes (W \rtimes \Gamma)} (\zeta^\vee(\pi) \circ \phi_t);$$

- (3) if  $u \in T_{un}$  and  $\pi$  is a virtual representation with central character in  $W'uT_{rs}$ , then so is  $\zeta^\vee(\pi)$ .

Let  $d \in \mathbb{N}$ . By property (2)  $\zeta^\vee$  sends analytic  $d$ -dimensional families of  $\mathcal{H} \rtimes \Gamma$ -representations to analytic  $d$ -dimensional families of  $X \rtimes W'$ -representations - except that the latter need not be generically irreducible. The image can not be part of a higher dimensional analytic family, for that would violate the bijectivity of  $\zeta^\vee$ . Since no  $d$ -dimensional variety is a finite union of varieties of dimension strictly smaller than  $d$ , and since there are only finitely many analytic families,  $\zeta^\vee$  restricts to a bijection between the subspaces (of the Grothendieck groups) spanned by analytic families of dimension at least  $d$ . Let us call these subspaces  $G_{\mathbb{Q}}^d(\mathcal{H} \rtimes \Gamma)$  and  $G_{\mathbb{Q}}^d(X \rtimes W')$ . Then  $\zeta^\vee$  induces a bijection

$$G_{\mathbb{Q}}^d(\mathcal{H} \rtimes \Gamma) / G_{\mathbb{Q}}^{d+1}(\mathcal{H} \rtimes \Gamma) \rightarrow G_{\mathbb{Q}}^d(X \rtimes W') / G_{\mathbb{Q}}^{d+1}(X \rtimes W'). \quad (41)$$

By (40) the right-hand side has a basis which is parametrized by the  $d$ -dimensional part of  $\tilde{T}/W'$ .

**Correction (2022):** Unfortunately the below construction of families of representations does not work in general, because of the problems with (40) and because an application of Theorem 2.2 could result in virtual  $\mathcal{H} \rtimes \Gamma$ -representations. Nevertheless a collection of families of representations satisfying Lemma 2.3 below can be found in most cases:

- When the parameters  $q$  are of “geometric type”, one can use families of standard representations from [AMS, Paragraph 3.3].
- For most positive-valued  $q$  one can use the technique with parameter deformations from [Sol4, proof of Lemma 6.4], to reduce to the previous case.
- We are not aware of any examples of graded Hecke algebras for which it is clear that they do not possess families of representations satisfying Lemma 2.3.

In all these cases, the remainder of the paper is valid without further changes.

We can use this to pick a suitable collection of analytic families of  $\mathcal{H} \rtimes \Gamma$ -representations. Fix a  $d$ -dimensional component of  $\tilde{T}/W'$  and let  $C$  be its image in  $T/W'$ . In the notation of (39) let  $\tilde{C}$  be the collection of  $(w, i)$  such that

$w \in \langle W' \rangle$ ,  $1 \leq i \leq c(w)$  and  $T_i^w/W' = C$ . By Theorem 2.2 and (41) there exist analytic families of  $\mathcal{H} \rtimes \Gamma$ -representations  $\{\pi_{\sigma_{w,i}}(t) \mid (w, i) \in \tilde{C}, t \in T^{\sigma_{w,i}}\}$  such that

$$\zeta^\vee(G_{\mathbb{Q}}^{d+1}(\mathcal{H} \rtimes \Gamma)) + \zeta^\vee(\{\pi_{\sigma_{w,i}}(t) \mid (w, i) \in \tilde{C}, t \in T^{\sigma_{w,i}}\})$$

contains a basis of

$$\text{span}\{\pi \in \text{Irr}(X \rtimes W') \mid c \text{ is the central character of } \pi\} \subset G_{\mathbb{Q}}(X \rtimes W'),$$

for all generic points  $c \in C$ . This procedure yields analytic  $d$ -dimensional families whose span in  $G_{\mathbb{Q}}(\mathcal{H} \rtimes \Gamma)$  is mapped bijectively to  $G_{\mathbb{Q}}^d(X \rtimes W')/G_{\mathbb{Q}}^{d+1}(X \rtimes W')$  by  $\zeta^\vee$ . By (35) we may take the parameters for  $\pi_{\sigma_{w,i}}$  in  $T^{\sigma_{w,i}}/\mathcal{G}_{\sigma_{w,i}}$ , when we consider these representations in  $G(\mathcal{H} \rtimes \Gamma)$ . Let us list some properties of

$$\{\pi_{\sigma_{w,i}}(t) \mid w \in \langle W' \rangle, 1 \leq i \leq c(w), t \in T^{\sigma_{w,i}}/\mathcal{G}_{\sigma_{w,i}}\}. \quad (42)$$

**Lemma 2.3.** (a) *The parameter space  $T^{\sigma_{w,i}}/\mathcal{G}_{\sigma_{w,i}}$  is isomorphic to  $T_i^w/Z_{W'}(w)$ , as orbifolds.*

(b)  *$\pi_{\sigma_{w,i}}(t)$  and  $\pi_{\sigma_{w,i}}(t')$  have the same trace if and only if  $t$  and  $t'$  lie in the same  $\mathcal{G}_{\sigma_{w,i}}$ -orbit.*

(c) *The family of representations (42) contains no repetitions and forms a basis of  $G_{\mathbb{Q}}(\mathcal{H} \rtimes \Gamma)$ .*

*Proof.* By the above construction and since different  $w \in W'$  act differently on  $T$ ,

$$\begin{aligned} & \zeta^\vee(G_{\mathbb{Q}}^{d+1}(\mathcal{H} \rtimes \Gamma)) + \zeta^\vee(\text{span}\{\pi_{\sigma_{w,i}}(t) \mid t \in T^{\sigma_{w,i}}\}) = \\ & \zeta^\vee(G_{\mathbb{Q}}^{d+1}(X \rtimes W')) + \zeta^\vee(\text{span}\{\pi_w(t) \mid t \in T_i^w/Z_{W'}(w)\}). \end{aligned}$$

Hence the parameter space  $T_i^w/Z_{W'}(w)$  is diffeomorphic to  $T^{\sigma_{w,i}}$  modulo the equivalence relation

$$t \sim t' \iff \pi_{\sigma_{w,i}}(t) = \pi_{\sigma_{w,i}}(t') \text{ in } G_{\mathbb{Q}}(\mathcal{H} \rtimes \Gamma).$$

By Theorem 2.1.b  $t \sim t'$  if and only if there exists a  $g \in \mathcal{G}$  with  $g(P, \delta, ut) = (P, \delta, ut')$ , where  $\pi_{\sigma_{w,i}}$  is a direct summand of  $\pi(P, \delta, u)$ . By definition any  $g \in \mathcal{G}_{\sigma_{w,i}}$  yields such equivalences for all  $t \in T^{\sigma_{w,i}}$ , so  $T^{\sigma_{w,i}}/\sim$  is a quotient of  $T^{\sigma_{w,i}}/\mathcal{G}_{\sigma_{w,i}}$ . On the other hand, elements  $h \in \mathcal{G}_{(P,\delta)} \setminus \mathcal{G}_{\sigma_{w,i}}$  can only produce such equivalences on lower dimensional subsets of  $T^{\sigma_{w,i}}$ . But any such relation which does not already come from  $\mathcal{G}_{\sigma_{w,i}}$  would destroy the orbifold structure of  $T^{\sigma_{w,i}}/\mathcal{G}_{\sigma_{w,i}}$ , so

$$T_i^w/Z_{W'}(w) \cong T^{\sigma_{w,i}}/\sim \cong T^{\sigma_{w,i}}/\mathcal{G}_{\sigma_{w,i}}.$$

This proves (a) and since we work in  $G_{\mathbb{Q}}(\mathcal{H} \rtimes \Gamma)$ , it also implies (b).

We noted that (40) contains no repetitions and is a basis of  $G_{\mathbb{Q}}(X \rtimes W')$ . Since the collection of analytic families

$$\{\zeta^\vee(\pi_{\sigma_{w,i}}(t)) \mid t \in w \in \langle W' \rangle, 1 \leq i \leq c(w), t \in T^{\sigma_{w,i}}/\mathcal{G}_{\sigma_{w,i}}\}$$

is parametrized by the same space  $\tilde{T}/W'$ , it also enjoys these properties. As  $\zeta^\vee$  is bijective, these two properties already hold for (42) in  $G_{\mathbb{Q}}(\mathcal{H} \rtimes \Gamma)$ .  $\square$

## 2.2 Complex parameters

Everything we discussed in the previous subsection has a natural analogue for graded Hecke algebras. The condition  $q$  positive translates to  $k$  real valued, but with the scaling isomorphisms (7) we can also reach complex parameters.

In the graded setting we have to replace  $T^P$  and  $T_{un}^P$  by the vector spaces  $\mathfrak{t}^P$  and  $i\mathfrak{a}^P$ . In the groupoid  $\mathcal{G}$  we have to omit the parts  $K_P$ , so we get a groupoid  $\mathcal{W}'$  with

$$\mathcal{W}'_{PQ} = \{g \in \Gamma \times W \mid g(P) = Q\} \quad P, Q \subset \Delta.$$

**Theorem 2.4.** *Suppose that  $k_\alpha \in \mathbb{R}$  for all  $\alpha \in R$ . There exists a natural  $\mathbb{Q}$ -linear bijection*

$$\zeta^\vee : G_{\mathbb{Q}}(\mathbb{H} \rtimes \Gamma) \rightarrow G_{\mathbb{Q}}(S(\mathfrak{t}^*) \rtimes W')$$

such that:

- (1)  $\zeta^\vee(\pi)$  is a tempered virtual  $S(\mathfrak{t}^*) \rtimes W'$ -representation if and only if  $\pi$  is a tempered virtual  $\mathbb{H} \rtimes \Gamma$ -representation.
- (2)  $\zeta^\vee$  commutes with parabolic induction, in the sense that for a subgroup  $\Gamma'_P \subset \Gamma_P$ , a tempered  $\mathbb{H}^P \rtimes \Gamma'_P$ -representation  $\pi$  and  $\lambda \in \mathfrak{t}^{W(R_P) \rtimes \Gamma'_P}$  we have

$$\zeta^\vee(\text{Ind}_{\mathbb{H}^P \rtimes \Gamma'_P}^{\mathbb{H} \rtimes \Gamma}(\pi \circ \phi_\lambda)) = \text{Ind}_{S(\mathfrak{t}^*) \rtimes (W(R_P) \rtimes \Gamma'_P)}^{S(\mathfrak{t}^*) \rtimes (W \rtimes \Gamma)}(\zeta^\vee(\pi) \circ \phi_\lambda);$$

- (3) if  $\lambda \in i\mathfrak{a}$  and  $\pi$  is a virtual representation with central character in  $W'\lambda + \mathfrak{a}$ , then so is  $\zeta^\vee(\pi)$ ;
- (4) if  $\pi$  is tempered and admits a real central character, then  $\zeta^\vee(\pi) = \pi \circ m_0$  with  $m_0$  as in (7).

*Proof.* For  $\mathbb{H} \rtimes \Gamma$ -representations which admit a central character whose component in  $i\mathfrak{a}$  is small, this follows from Theorems 2.2 and 1.3. The extra condition (4) is actually the starting point for the construction of  $\zeta^\vee$ , see [Sol3, Section 2.3]. We can generalize this to all  $\mathbb{H} \rtimes \Gamma$ -representations via the scaling isomorphisms

$$m_\epsilon : \mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma \rightarrow \mathbb{H}(\tilde{\mathcal{R}}, k) \rtimes \Gamma$$

for  $\epsilon > 0$ .  $\square$

An induction datum for  $\mathbb{H}$  is a triple  $(P, \delta, \lambda)$  such that  $P \subset \Delta$ ,  $\delta$  is a discrete series representation of  $\mathbb{H}_P$  and  $\lambda \in \mathfrak{t}^P$ . Suppose that  $\mu \in i\mathfrak{a}^P$  and that  $\rho$  is an irreducible direct summand of the representation  $\text{Ind}_{\mathbb{H}^P}^{\mathbb{H} \rtimes \Gamma(P, \delta, \mu)}(\delta \circ \phi_\mu)$  (which is completely reducible). We abbreviate  $\Gamma_\rho = \Gamma_{(P, \delta, \mu)}$  and  $\mathfrak{t}^\rho = (\mathfrak{t}^P)^{\Gamma_\rho}$ . We say that

$$\{\pi_\rho(\lambda) := \text{Ind}_{\mathbb{H}^P \rtimes \Gamma_\rho}^{\mathbb{H} \rtimes \Gamma}(\rho \circ \phi_\lambda) \mid \lambda \in \mathfrak{t}^\rho\} \quad (43)$$

is an analytic family of  $\mathbb{H} \rtimes \Gamma$ -representations. Notice that

$$\pm \mu \in \mathfrak{t}^\rho \quad (44)$$

and that the central character of  $\pi_\rho(-\mu)$  equals the central character of  $\pi^\Gamma(P, \delta, 0) = \text{Ind}_{\mathbb{H}^P}^{\mathbb{H} \rtimes \Gamma} \delta$ . Moreover the weights satisfy an integrality property with respect to

$$\mathbb{Z}k := \mathbb{Z}\{k_\alpha \mid \alpha \in R\}.$$

By [Opd, Theorem A.7] every  $S(\mathfrak{t}_P^*)$ -weight  $\nu$  of  $\delta$  is a residual point for  $(\tilde{\mathcal{R}}_P, k_P)$ . The classification of such distinguished points in [HeOp, Section 4] shows that  $\alpha(\nu) \in \mathbb{Z}k_P$  for all  $\alpha \in R_P$ . By [BaMo, Theorem 6.4] this property is preserved under induction, that is,  $\alpha(\nu) \in \mathbb{Z}k$  for every  $\alpha \in R$  and every  $S(\mathfrak{t}^*)$ -weight  $\nu$  of  $\text{Ind}_{\mathbb{H}^P}^{\mathbb{H} \rtimes \Gamma} \delta$ . We conclude that all  $S(\mathfrak{t}^*)$ -weights of  $\pi_\rho(\lambda)$  lie in

$$W'(\lambda + \mu) + \text{Hom}(\mathbb{Z}R, \mathbb{Z}k), \quad (45)$$

where we consider  $\text{Hom}(\mathbb{Z}R, \mathbb{Z}k)$  as a lattice in  $\mathfrak{a}_R \subset \mathfrak{t}$ .

To the analytic family  $\pi_\rho$  we associate the group

$$W'_\rho = \{g \in W' \mid g(P) = P, \delta \circ \psi_g^{-1} \cong \delta, g \text{ stabilizes } \{\pi_\rho(\lambda) \mid \lambda \in i\mathfrak{a}^\rho\}\}.$$

**Correction (2022):** The new comments on pages 16, 18 and 19 also apply to the below construction and to Corollary 2.5. If we can instead use one of the good cases listed on page 19, the remainder of the paper holds for those families of  $\mathbb{H} \rtimes \Gamma$ -representations. Otherwise, there is still a satisfactory substitute for Corollary 2.5 in terms of families of virtual  $\mathbb{H} \rtimes \Gamma$ -representations, worked out in [Sol5].

Our discussion of  $\mathbb{C}[X] \rtimes W' = \mathcal{O}(T) \rtimes W'$  before Theorem 2.2 applies just as well to  $\mathbb{H}(\tilde{\mathcal{R}}, 0) \rtimes \Gamma = S(\mathfrak{t}^*) \rtimes W'$ . The considerations that lead to Lemma 2.3 also show:

**Corollary 2.5.** *There exist analytic families of  $\mathbb{H} \rtimes \Gamma$ -representations  $\pi_{\rho_w}$  for  $w \in \langle W' \rangle$ , such that:*

- (a) *The parameter space  $\mathfrak{t}^{\rho_w} / \mathcal{W}'_{\rho_w}$  is isomorphic to  $\mathfrak{t}^w / Z_{W'}(w)$ , as orbifolds.*
- (b)  *$\pi_{\rho_w}(\lambda)$  and  $\pi_{\rho_w}(\lambda')$  have the same trace if and only if  $\lambda$  and  $\lambda'$  lie in the same  $\mathcal{W}'_{\rho_w}$ -orbit.*
- (c) *The family of representations*

$$\{\pi_{\rho_w}(\lambda) \mid w \in \langle W' \rangle, \lambda \in \mathfrak{t}^{\rho_w} / \mathcal{W}'_{\rho_w}\}$$

*contains no repetitions and forms a basis of  $G_{\mathbb{Q}}(\mathbb{H} \rtimes \Gamma)$ .*

The scaling isomorphisms  $m_\epsilon$  for  $\epsilon \in \mathbb{C}^\times$  from (7) provide an immediate generalization of Theorem 2.4 and Corollary 2.5 to graded Hecke algebras  $\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma$  such that  $k$  is real valued and  $\epsilon \in \mathbb{C}$ . To retain the formulation of Theorem 2.4 one must modify the notions tempered and discrete series, in the sense that everywhere in Section 1.5 one must use  $\epsilon \mathfrak{a}^-$  instead of  $\mathfrak{a}^-$ . Let us call this  $\epsilon$ -tempered and  $\epsilon$ -discrete series. We can also consider the family of algebras

$$\{\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma \mid \epsilon \in \mathbb{C}\}.$$

For every  $\epsilon \in \mathbb{C}$  the analytic families of  $\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma$ -representations

$$\{\pi_{m_\epsilon^*(\rho_w)}(\lambda) \mid w \in \langle W' \rangle, \lambda \in \mathfrak{t}^{\rho_w} / \mathcal{W}'_{\rho_w}\}$$

fulfill Corollary 2.5. We note that the matrix coefficients of  $\pi_{m_\epsilon^*(\rho_w)}(\lambda)$  depend algebraically on  $\epsilon$  and  $\lambda$ . More specifically, by (45) the  $S(\mathfrak{t}^*)$ -weights of  $\pi_{m_\epsilon^*(\rho_w)}(-\epsilon\mu)$  lie in  $\text{Hom}(\mathbb{Z}R, \mathbb{Z}k\epsilon) \subset \mathfrak{t}_R$  and depend linearly on  $\epsilon$ .

Next we generalize Section 2.1 to certain complex parameter functions  $q$ . As the analogue of a line  $\epsilon\mathbb{R}$  we use a one-parameter subgroup of  $\mathbb{C}^\times$ . We consider  $q^\epsilon$  for a positive parameter function  $q$  and  $\epsilon \in \mathbb{C}$ . Then (23) becomes

$$k_{u,\epsilon,\alpha} = (\log q_{\alpha^\vee/2} + (1 + \alpha(u)) \log q_{\alpha^\vee}) \epsilon / 2 \quad \alpha \in R_u.$$

Notice that  $\epsilon = 0$  corresponds to the algebra  $\mathcal{H}(\mathcal{R}, q^0) \rtimes \Gamma = \mathbb{C}[X] \rtimes W'$ , which we understand very well. On the other hand, for  $\epsilon \in i\mathbb{R} \setminus \{0\}$  the parameter  $q_{\alpha^\vee}$  can be a root of unity (unequal to 1), in which case the affine Hecke algebra  $\mathcal{H}(\mathcal{R}, q)$  can differ substantially from those with positive parameters. Therefore we assume that  $\Re(\epsilon) \neq 0$ .

**Theorem 2.6.** *Theorem 2.2 and Lemma 2.3 also hold for parameter functions  $q^\epsilon$  with  $q$  positive and  $\epsilon \in \mathbb{C} \setminus i\mathbb{R}$ . For the interpretation of Theorem 2.2 we have to replace  $\mathfrak{a}^-$  by  $\epsilon\mathfrak{a}^-$ ,  $T_{rs}$  by  $\exp(\epsilon\mathfrak{a})$  and tempered by  $\epsilon$ -tempered.*

*Moreover we can choose the required analytic families of  $\mathcal{H}(\mathcal{R}, q^\epsilon) \rtimes \Gamma$ -representations such that the matrix coefficients of the representations depend analytically on  $\epsilon$  and all  $\mathcal{A}$ -weights depend polynomially on the variables  $\{q_{\alpha^\vee}^{\pm\epsilon/2} \mid \alpha^\vee \in R_{nr}^\vee\}$ .*

*Proof.* Every  $t \in T$  can be written uniquely as  $t = uc$  with  $u \in T_{un}$  and  $c \in \exp(\epsilon\mathfrak{a})$ , since  $\Re(\epsilon) \neq 0$ . In this setting Theorem 1.2 remains valid. Moreover  $k_{u,\epsilon,\alpha} \in \epsilon\mathbb{R}$  and  $\langle \alpha, \lambda \rangle \in \epsilon\mathbb{R}$  for all  $\alpha \in R$  and  $\lambda \in \epsilon\mathfrak{a}$ , so  $\langle \alpha, \epsilon\mathfrak{a} \rangle$  and  $\langle \alpha, \epsilon\mathfrak{a} \rangle + k_{u,\epsilon,\alpha}$  have positive distance to  $\pi i\mathbb{Z} \setminus \{0\}$ . This ensures that Theorem 1.3 applies to all  $\lambda$  in a suitable tubular neighborhood of  $\epsilon\mathfrak{a}$  in  $\mathfrak{t}$ . Its part (b) tells us that  $\widehat{\mathcal{H}(\mathcal{R}, q^\epsilon)}_{W'u \exp(\lambda)} \rtimes \Gamma$  is Morita equivalent to  $\mathbb{H}(\widehat{\tilde{\mathcal{R}}^{P(u)}, \epsilon k^{P(u)}})_{W'_{\mathbb{Z}P(u)\lambda}} \rtimes W'_{P(u)}$ , while  $\mathbb{H}(\tilde{\mathcal{R}}^{P(u)}, \epsilon k^{P(u)})$  is isomorphic to a graded Hecke algebra with real parameter function  $k^{P(u)}$  by means of  $m_\epsilon$ . To  $\mathbb{H}(\tilde{\mathcal{R}}^{P(u)}, k^{P(u)})$  we can apply Theorem 2.4, which gives us Theorem 2.2 for  $\mathcal{H} \rtimes \Gamma$ -representations with central character in  $W'u \exp(\epsilon\mathfrak{a})$ . Combining these results for all  $u \in T_{un}$  yields the map

$$\zeta^\vee : G(\mathcal{H}(\mathcal{R}, q^\epsilon) \rtimes \Gamma) \rightarrow G(X \rtimes W').$$

It makes Theorem 2.2 true for  $q^\epsilon$ , because it does so on all sets  $u \exp(\epsilon\mathfrak{a})$ .

Let us construct the required analytic families of  $\mathcal{H}(\mathcal{R}, q^\epsilon) \rtimes \Gamma$ -representations. We start with (42) and let  $\sigma$  be any of the  $\sigma_{w,i}$ . With Theorem 1.3 we transfer the  $\mathcal{H}^P \rtimes \Gamma_\sigma$ -representation to a tempered representation  $\tilde{\sigma}$  of  $\mathbb{H}^{P(u)} \rtimes \tilde{\Gamma}$ , for a suitable subgroup  $\tilde{\Gamma} \subset W'$ . Taking for  $u$  the unitary part of an  $\mathcal{A}$ -weight of  $\sigma$ , we can achieve that  $\tilde{\sigma}$  has real central character. Then  $m_\epsilon^*(\tilde{\sigma})$  is an  $\epsilon$ -tempered representation of  $\mathbb{H}(\tilde{\mathcal{R}}^{P(u)}, \epsilon k^{P(u)}) \rtimes \tilde{\Gamma}$ . Now Theorem 1.3 in the opposite direction

yields an  $\epsilon$ -tempered representation of  $\mathcal{H}(\mathcal{R}^P, (q^P)^\epsilon) \rtimes \Gamma_\sigma$ , which we call  $m_\epsilon^*(\sigma)$ . We claim that the analytic families of  $\mathcal{H}(\mathcal{R}, q^\epsilon) \rtimes \Gamma$ -representations

$$\{\pi_{m_\epsilon^*(\sigma_{w,i})}(t) \mid w \in \langle W' \rangle, 1 \leq i \leq c(w), t \in T^{\sigma_{w,i}}/\mathcal{G}_{\sigma_{w,i}}\}$$

fulfill Lemma 2.3. By construction  $\zeta^\vee(\pi_{m_\epsilon^*(\sigma_{w,i})}(t))$  equals  $\zeta^\vee(\sigma_{w,i})(t)$ . In view of Lemma 2.3 and since Theorem 2.2 holds for  $q$  and  $q^\epsilon$ , this implies the claim.

In view of [Sol3, (4.6)],  $m_\epsilon^*(\sigma)$  is also the composition of  $\sigma$  with the algebra homomorphism  $\rho_\epsilon$  from [Sol3, Proposition 4.1.2]. Although in [Sol3] the author assumed that  $\epsilon \in [-1, 1]$ , the relevant calculations remain valid for  $\epsilon \in \mathbb{C} \setminus i\mathbb{R}$ . This shows that the matrix coefficients of  $m_\epsilon^*(\sigma)$  depend analytically on  $\epsilon$ . More precisely, for every  $v \in W^e$  the operator  $\pi_{m_\epsilon^*(\sigma_{w,i})}(t, N_v)$  depends in a complex analytic way on  $\epsilon$  and  $t$ .

By (34) all  $\mathcal{A}$ -weights of  $\pi_{\sigma_{w,i}}(t)$  are of the form  $\gamma(tuc)$  where  $\gamma \in W'$  and  $uc$  is a weight of  $\sigma_{w,i}$  with unitary part  $u \in T_{un}$  and absolute value  $c \in Y \otimes_{\mathbb{Z}} q^{\mathbb{Z}/2}$ . This in turn implies that all  $\mathcal{A}$ -weights of  $\pi_{m_\epsilon^*(\sigma_{w,i})}(t)$  are of the form

$$\gamma(tuc^\epsilon) \quad c^\epsilon \in q^{\mathbb{Z}/2}, \quad (46)$$

and in particular that (for fixed  $t$ ) they depend polynomially on  $\{q_{\alpha^\vee}^{\pm\epsilon/2} \mid \alpha^\vee \in R_{nr}^\vee\}$ .  $\square$

Since the whole situation depends in complex analytic fashion on  $\epsilon$ , it is natural to expect that Theorem 2.6 extends to all but finitely many  $\epsilon \in \mathbb{C}$ . Indeed this is known when  $q$  is an equal parameter function [KaLu], but the methods of this paper are insufficient to prove it.

### 3 Hochschild homology

#### 3.1 Graded Hecke algebras

In [Sol1, Theorem 3.4] the author showed that, for any parameter function  $k$ ,

$$HH_n(\mathbb{H} \rtimes \Gamma) \cong HH_n(S(\mathfrak{t}^*) \rtimes W') \quad (47)$$

as vector spaces. Here we will provide a more explicit version of (47) which highlights the  $Z(\mathbb{H} \rtimes \Gamma)$ -module structure of  $HH_*(\mathbb{H} \rtimes \Gamma)$  and is suitable for generalization to affine Hecke algebras.

In the proof we will use some rather deep results from [Sol3], which were established under the assumption that all the parameters  $k_\alpha$  are real. In view of the scaling isomorphisms (7) the only cases not covered are those where  $R$  contains nonperpendicular roots  $\alpha, \beta$  of different length, such that  $k_\alpha$  and  $k_\beta$  are  $\mathbb{R}$ -linearly independent in  $\mathbb{C}$ .

Let  $\langle W' \rangle$  be a set of representatives for the conjugacy classes in  $W' = W \rtimes \Gamma$ . For each  $w \in \langle W' \rangle$  we pick an analytic family of  $\mathbb{H}$ -representations  $\{\pi_{\rho_w}(\lambda) \mid \lambda \in t^{\rho_w}\}$  as in Corollary 2.5. This yields an algebra homomorphism

$$\tilde{\rho} : \mathbb{H} \rtimes \Gamma \rightarrow \bigoplus_{w \in \langle W' \rangle} \mathcal{O}(t^{\rho_w}) \otimes \text{End}_{\mathbb{C}}(V_{\rho_w}), \quad (48)$$



where  $V_{\rho_w}$  is the vector space underlying the representations  $\pi_{\rho_w}(\lambda)$ . Since these  $\mathbb{H} \rtimes \Gamma$ -representations are irreducible for generic  $\lambda$ ,  $Z(\mathbb{H} \rtimes \Gamma) \cong \mathcal{O}(\mathfrak{t})^{W'}$  acts on them by scalars. In other words,

$$\tilde{\rho}(Z(\mathbb{H} \rtimes \Gamma)) \subset \bigoplus_{w \in \langle W' \rangle} \mathcal{O}(\mathfrak{t}^{\rho_w}) = Z\left(\bigoplus_{w \in \langle W' \rangle} \mathcal{O}(\mathfrak{t}^{\rho_w}) \otimes \text{End}_{\mathbb{C}}(V_{\rho_w})\right). \quad (49)$$

Suppose that  $\rho_w$  comes from a discrete series representation of  $\mathbb{H}_P$ . Then its central character is an element of  $\mathfrak{t}_P/W(R_P) \rtimes \Gamma_\rho$ , and for any representative  $cc(\rho_w) \in \mathfrak{t}_P$  and any  $f \in Z(\mathbb{H} \rtimes \Gamma)$ :

$$\tilde{\rho}(f)(w, \lambda) = f(cc(\rho_w) + \lambda). \quad (50)$$

By Morita equivalence and by the Hochschild–Kostant–Rosenberg theorem

$$HH_*(\mathcal{O}(\mathfrak{t}^{\rho_w}) \otimes \text{End}_{\mathbb{C}}(V_{\rho_w})) \cong HH_*(\mathcal{O}(\mathfrak{t}^{\rho_w})) \cong \Omega^*(\mathfrak{t}^{\rho_w}), \quad (51)$$

where  $\Omega^n$  stands for algebraic  $n$ -forms. Recall that the Morita invariance of Hochschild homology is implemented by a generalized trace map [Lod, Section I.2]. For  $g \in \mathcal{G}_{\rho_w}$  the  $\mathbb{H} \rtimes \Gamma$ -representations  $\pi_{\rho_w}(\lambda)$  and  $\pi_{\rho_w}(g\lambda)$  have the same trace, so the image of  $HH_*(\rho)$  cannot distinguish the points  $(w, \lambda)$  and  $(w, g\lambda)$ . Thus from (48) and (51) we obtain a linear map

$$HH_*(\tilde{\rho}) = \bigoplus_{w \in \langle W' \rangle} HH_*(\tilde{\rho})_w : HH_*(\mathbb{H} \rtimes \Gamma) \rightarrow \bigoplus_{w \in \langle W' \rangle} \Omega^*(\mathfrak{t}^{\rho_w})^{\mathcal{G}_{\rho_w}}. \quad (52)$$

In view of (50) the action of  $Z(\mathbb{H} \rtimes \Gamma) \cong \mathcal{O}(\mathfrak{t})^{W'}$  on the right hand side becomes the natural evaluation if we replace  $\mathfrak{t}^{\rho_w}$  by  $cc(\rho_w) + \mathfrak{t}^{\rho_w}$ .

**Theorem 3.1.** *Suppose that there exists  $\epsilon \in \mathbb{C}^\times$  such that  $k_\alpha \in \epsilon\mathbb{R}$  for all  $\alpha \in R$ . The map*

$$HH_*(\tilde{\rho}) : HH_*(\mathbb{H} \rtimes \Gamma) \rightarrow \bigoplus_{w \in \langle W' \rangle} \Omega^*(cc(\rho_w) + \mathfrak{t}^{\rho_w})^{\mathcal{G}_{\rho_w}}$$

*is an isomorphism of  $Z(\mathbb{H} \rtimes \Gamma)$ -modules.*

*Proof.* The isomorphism  $\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rightarrow \mathbb{H}(\tilde{\mathcal{R}}, k)$  from (7) allows us to assume that  $k$  is real valued. Recall that the Hochschild complex for a unital algebra is  $C_n(A) = A^{\otimes n+1}$  with differential

$$b_n(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^n a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

As discussed in the proof of [Sol1, Theorem 3.4], we can choose representatives  $w$  for the conjugacy classes in  $W'$ , such that the following hold:

- $w = \gamma w_P$  with  $\gamma \in \Gamma$  and  $w_P \in W(R_P)$ ;
- $\mathfrak{a}^w \subset \mathfrak{a}^P \cap \mathfrak{a}^\gamma \cap \mathfrak{a}^{w_P}$ ;
- all elements of  $\mathcal{O}(\mathfrak{t}_w) \subset \mathcal{O}(\mathfrak{t}/\mathfrak{t}_P) \subset \mathbb{H}$  commute with  $w \in \mathbb{H} \rtimes \Gamma$ , where  $\mathfrak{t}_w := \mathfrak{t}/(w-1)\mathfrak{t} \cong \mathfrak{t}^w$ ;

- every element of  $HH_*(\mathbb{H} \rtimes \Gamma)$  can be represented by a Hochschild cycle in  $\bigoplus_{w \in \langle W' \rangle} wC_*(\mathcal{O}(\mathfrak{t}_w))$ ;
- the differential complex  $wC_*(\mathcal{O}(\mathfrak{t}_w))$  contributes precisely  $\Omega^*(\mathfrak{t}^w)^{Z_{W'}(w)}$  to  $HH_*(\mathbb{H} \rtimes \Gamma)$ , via the natural map

$$C_*(\mathcal{O}(\mathfrak{t}_w)) \rightarrow \Omega^*(\mathfrak{t}^w) : a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto a_0 da_1 \cdots da_n. \quad (53)$$

In the last point it also suffices to take a subspace (say  $w\Omega_w^*$ ) of the cycles in  $wC_*(\mathcal{O}(\mathfrak{t}_w))$  which is mapped bijectively to  $\Omega^*(\mathfrak{t}^w)^{Z_{W'}(w)}$  by (53). We intend to show that the map

$$\bigoplus_{w \in \langle W' \rangle} w\Omega_w^* \rightarrow \bigoplus_{w \in \langle W' \rangle} \Omega^*(\mathfrak{t}^w)^{\mathcal{G}_{\rho_w}} \quad (54)$$

induced by  $HH_*(\tilde{\rho})$  is bijective. Our strategy is first to show that this map is upper triangular with respect to some reflexive transitive order on  $\langle W' \rangle$  and then to apply the results of Section 2.2 to the elements of  $\langle W' \rangle$  that equivalent in this sense.

For the first point we study what happens to  $wC_*(\mathcal{O}(\mathfrak{t}_w))$  under  $HH_*(\tilde{\rho})_{w'}$ . We need to apply the generalized trace map

$$HH_*(\text{gtr}) : HH_*(\mathcal{O}(\mathfrak{t}^{\rho_{w'}}) \otimes \text{End}_{\mathbb{C}}(V_{\rho_{w'}})) \rightarrow HH_*(\mathcal{O}(\mathfrak{t}^{\rho_{w'}})),$$

which is defined on the level of Hochschild complexes by

$$\text{gtr}(m_0 a_0 \otimes m_1 a_1 \otimes \cdots \otimes m_n a_n) = \text{tr}(m_0 m_1 \cdots m_n) a_0 \otimes a_1 \otimes \cdots \otimes a_n$$

for endomorphisms  $m_i \in \text{End}_{\mathbb{C}}(V_{\rho_{w'}})$  and functions  $a_i \in \mathcal{O}(\mathfrak{t}^{\rho_{w'}})$ . Since  $wC_*(\mathcal{O}(\mathfrak{t}_w))$  is contained in the Hochschild complex of the commutative algebra

$$A_w = \mathbb{C}[C_w] \otimes \mathcal{O}(\mathfrak{t}_w)$$

generated by  $w$  and  $\mathcal{O}(\mathfrak{t}_w)$ , we can analyse  $HH_*(\tilde{\rho})_{w'}(wC_*(\mathcal{O}(\mathfrak{t}_w)))$  via a filtration of the representation  $\pi_{\rho_{w'}}(\lambda)$  by onedimensional  $A_w$ -modules. With respect to a corresponding basis of  $V_{\rho_{w'}}$ , all the matrices  $\pi_{\rho_{w'}}(\lambda)(A_w)$  are upper triangular. Combining that with the formula for the generalized trace map, one concludes that  $HH_*(\tilde{\rho})_{w'}(A_w)$  sees  $\pi_{\rho_{w'}}(\lambda)$  only via the diagonal parts of these matrices, that is, via the semisimplification of this  $A_w$ -representation.

Let us assume for the moment that  $\lambda \in \mathfrak{t}_w$  is generic. The semisimplification of the restriction of  $\pi_{\rho_{w'}}(\lambda)$  to  $\mathcal{O}(\mathfrak{t}_w)$  is a direct sum of onedimensional modules with characters  $s + g(\lambda)$ , where  $s$  runs through the  $\mathcal{O}(\mathfrak{t}_w)$ -characters of  $\rho_{w'}$  and  $g$  runs through  $\mathcal{W}'_{\rho_{w'}}$ . As  $\lambda$  is generic, this yields a decomposition into a direct sum of exactly  $|\mathcal{W}'_{\rho_{w'}}|$   $\mathcal{O}(\mathfrak{t}_w)$ -submodules, the summand parametrized by  $g \in \mathcal{W}'_{\rho_{w'}}$  having weights  $s + g(\lambda)$ . The intertwining operator associated to  $g$ , as in [Sol2, Section 8], provides a linear bijection between the summands corresponding to  $(\lambda, 1)$  and to  $(g^{-1}\lambda, g)$ .

From the above submodules of the semisimplification of  $\pi_{\rho_{w'}}(\lambda)$  we take the ones with the same  $\mathcal{O}(\mathfrak{t}_w)$ -character together, thus creating a decomposition into  $\mathcal{O}(\mathfrak{t}_w)$ -submodules  $M_i(\lambda)$  with pairwise different weights. The  $M_i(\lambda)$  are stabilized by  $w$ , because  $w$  commutes with  $\mathcal{O}(\mathfrak{t}_w)$ . As a representation of the finite group  $C_w$

generated by  $w$ ,  $M_i(\lambda)$  is independent of  $\lambda$  because  $\lambda$  can be varied continuously. Neither does it depend on  $i$ , since  $M_i(\lambda)$  is isomorphic to  $M_j(g\lambda)$  via the intertwining operator associated to a suitable  $g \in \mathcal{W}'_{\rho_{w'}}$ . We conclude that the semisimplification of  $\pi_{\rho_{w'}}(\lambda)$  as an  $A_w$ -module is of the form

$$V_w \otimes V_{\mathcal{O}}(\lambda) \tag{55}$$

for some  $C_w$ -representation  $V_w$  and some  $\mathcal{O}(t_w)$ -module  $V_{\mathcal{O}}(\lambda)$ . Although the above argument uses that  $\lambda$  is generic, the conclusion extends to all  $\lambda \in \mathfrak{t}_w$  because we are dealing with a continuous family of representations.

From (55) we see that for any  $f_i \in \mathfrak{t}_w$

$$HH_n(\tilde{\rho})_{w'}(wf_0 \otimes f_1 \cdots \otimes f_n) = \text{tr}(w, V_w) \dim(V_w)^{-1} HH_*(\tilde{\rho})_{w'}(f_0 \otimes f_1 \cdots \otimes f_n). \tag{56}$$

If  $(w-1)\mathfrak{t}$  is not contained in  $g(w'-1)\mathfrak{t}$  for any  $g \in W'$  and  $\lambda \in \mathfrak{t}_{w'}$  is generic, then  $w$  permutes the  $\mathcal{O}(t_w)$ -weight spaces of  $\pi_{\rho_{w'}}(\lambda)$  in a nontrivial way. Hence  $\text{tr}(w, V_w) = 0$  in such cases, and by continuity this extends to all  $\lambda \in \mathfrak{t}_w$ . For reference we state this as

$$HH_*(\tilde{\rho})_{w'}(wC_*(\mathcal{O}(t_w))) = 0 \quad \text{when} \quad (w-1)\mathfrak{t} \not\subset W'(w'-1)\mathfrak{t}_w. \tag{57}$$

Now pick  $\lambda \in i\mathfrak{a}^{w'}$  with  $\|\lambda\|$  so small that  $g(\lambda) = \lambda$  is equivalent to  $\mathfrak{t}^g \supset \mathfrak{t}^{w'}$  for  $g \in W'$ . By parts (3) and (4) of Theorem 2.4, among the  $\mathcal{O}(\mathfrak{t}) \rtimes W'$ -representations  $\zeta^\vee(\pi_{\rho_g}(\mu))$ , only those with  $\mu \in W'\lambda$  and  $W'\mathfrak{t}^{\rho_g} \supset \mathfrak{t}^{w'}$  have central character  $W'\lambda$ . For the duration of the proof we write  $g \geq w'$  to describe this situation. Similarly we write  $g > w'$  if  $g \geq w'$  but  $W'\mathfrak{t}^{\rho_g} \neq W'\mathfrak{t}^{w'}$ , and we write  $g \approx w'$  if  $W'\mathfrak{t}^{\rho_g} = W'\mathfrak{t}^{w'}$ . To simplify the situation somewhat, we may and will assume that the elements of  $\langle W' \rangle$  are chosen such that  $\mathfrak{t}^{\rho_g} \supset \mathfrak{t}^{w'}$  for all  $g \in \langle W' \rangle_{\geq w'}$ .

By (37) the number of irreducible  $\mathcal{O}(\mathfrak{t}) \rtimes W'$ -representations with central character  $W'\lambda$  is the number of conjugacy classes in the isotropy group  $W'_\lambda$ . Together with the bijectivity of  $\zeta^\vee$  this shows that every conjugacy class of  $W'_\lambda$  is contained in a unique conjugacy class of  $W'$ , and that all elements of  $\langle W' \rangle_{\geq w'}$  are obtained in this way. Given  $z \in Z_{W'}(w')$ , we have  $z\lambda \in \mathfrak{t}^{w'} \subset \mathfrak{t}^{\rho_g}$  for all  $g \in \langle W' \rangle_{\geq w'}$ . By Lemma 2.3.c and Corollary 2.5 the representations

$$\{\zeta^\vee(\pi_{\rho_g}(z\lambda)) : g \in \langle W' \rangle_{\geq w'}\}$$

form a basis of the subspace of  $G_{\mathbb{Q}}(\mathbb{H} \times \Gamma)$  corresponding to the central character  $W'\lambda$ . By parts (a) and (b) of Lemma 2.3 this means that  $\pi_{\rho_g}(z\lambda)$  and  $\pi_{\rho_g}(\lambda)$  are  $\mathcal{W}'_{\rho_g}$ -associate, for all  $g \in \langle W' \rangle_{\geq w'}$ . For similar reasons other generic  $\lambda' \in \mathfrak{t}^{w'}$  yield representations with other traces. For  $g \approx w'$  this allows us to conclude that the orbifolds  $\mathfrak{t}^{\rho_g}/\mathcal{W}'_{\rho_g}$  and  $\mathfrak{t}^{w'}/Z_{W'}(w')$  have the same generic parts, and hence are isomorphic.

Recall from Theorem 2.4 that  $\zeta^\vee$  does not change the  $W'$ -type of representations and induces a bijection between Grothendieck groups. Therefore the matrix

$$(\text{tr}(w, \pi_{\rho_g}(\lambda)))_{w, g \in \langle W' \rangle_{\geq w'}}$$

is invertible. We note that this matrix does not depend on  $\lambda \in \mathfrak{t}^w$ . As discussed before (57), all entries with  $w > g$  are 0, so the submatrix  $(\text{tr}(w, \pi_{\rho_g}(\lambda)))_{w, g \in \langle W' \rangle_{\approx w'}}$  is also invertible. Together with Corollary 2.5 and (56) this implies that the map

$$\bigoplus_{w \in \langle W' \rangle_{\approx w'}} w \mathcal{O}(\mathfrak{t}^w)^{Z_{W'}(w)} \rightarrow \bigoplus_{g \in \langle W' \rangle_{\approx w'}} \mathcal{O}(\mathfrak{t}^{\rho_g})^{\mathcal{W}'_{\rho_g}} \quad (58)$$

induced by  $HH_0(\tilde{\rho})$  becomes a bijection upon specializing at any  $\lambda \in \mathfrak{t}^{w'}$ . As shown above, the underlying orbifolds on both sides are  $\langle W' \rangle_{\approx w'} \times (\mathfrak{t}^w)^{Z_{W'}(w)}$ , so by (56) we can regard (58) as an invertible complex matrix of size  $|\langle W' \rangle_{\approx w'}|$  tensored with the unique isomorphisms of orbifolds  $\mathfrak{t}^{\rho_g} / \mathcal{W}'_{\rho_g} \rightarrow \mathfrak{t}^w / Z_{W'}(w')$  that respect the projections to  $\mathfrak{t} / W'$ . This description and (56) show that the natural extension of (58) to differential forms,

$$\bigoplus_{w \in \langle W' \rangle_{\approx w'}} w \Omega^*(\mathfrak{t}^w)^{Z_{W'}(w)} \rightarrow \bigoplus_{g \in \langle W' \rangle_{\approx w'}} \Omega^*(\mathfrak{t}^{\rho_g})^{\mathcal{W}'_{\rho_g}}, \quad (59)$$

is also bijective. In view of (53) the map (59) is induced by  $HH_*(\tilde{\rho})$ . Together with (57) this proves that (54) is bijective, as required.  $\square$

In fact the above proof establishes a slightly more precise statement: the map of differential complexes

$$\text{gtr} \circ C_*(\tilde{\sigma}) : C_*(\mathbb{H} \rtimes \Gamma) \rightarrow \bigoplus_{w \in \langle W' \rangle} C_*(\mathfrak{t}^{\rho_w})^{\mathcal{W}'_{\rho_w}} \quad (60)$$

is a quasi-isomorphism.

Theorem 3.1 has a direct analogue for the (periodic) cyclic homology of graded Hecke algebras:

**Corollary 3.2.** *Let  $k$  be as in Theorem 3.1. The algebra homomorphism (48) induces isomorphisms*

$$\begin{aligned} HC_*(\tilde{\rho}) : HC_*(\mathbb{H} \rtimes \Gamma) &\rightarrow \bigoplus_{w \in \langle W' \rangle} HC_*(\mathcal{O}(\mathfrak{t}^{\rho_w}))^{\mathcal{W}'_{\rho_w}}, \\ HP_{ev/odd}(\tilde{\rho}) : HP_{ev/odd}(\mathbb{H} \rtimes \Gamma) &\rightarrow \bigoplus_{w \in \langle W' \rangle} H_{DR}^{ev/odd}(\mathfrak{t}^{\rho_w})^{\mathcal{W}'_{\rho_w}} \cong H_{DR}^{ev/odd}(\tilde{\mathfrak{t}})^{W'}. \end{aligned}$$

We remark that the (periodic) cyclic homology of smooth commutative algebras like  $\mathcal{O}(\mathfrak{t}^{\rho_w})$  is well-known from Hochschild–Kostant–Rosenberg theorem, see for example [Lod, Theorem 3.4.12].

*Proof.* Recall that the relation between the Hochschild complex  $C_*(A)$  and the cyclic bicomplex  $CC_{**}(A)$  of an algebra  $A$  gives rise to Connes' periodicity exact sequence:

$$\cdots \rightarrow HH_n(A) \rightarrow HC_n(A) \rightarrow HC_{n-2}(A) \rightarrow HH_{n-1}(A) \rightarrow \cdots \quad (61)$$

In particular this holds for  $A = \mathbb{H} \rtimes \Gamma$ . The algebra homomorphism  $\tilde{\rho}$  and the generalized trace map send (61) to the corresponding long exact sequence for the cyclic bicomplex  $\bigoplus_{w \in \langle W' \rangle} CC_{**}(\mathcal{O}(\mathfrak{t}^{\rho_w}))^{\mathcal{W}'_{\rho_w}}$ . Theorem 3.1 says that this map induces an

isomorphism on Hochschild homology. Since  $HC_0 = HH_0$ , it follows with induction to  $n$  from (61) and the five lemma that  $\tilde{\rho}$  induces the asserted isomorphism on  $HC_*$ . Since  $HH_n(\mathbb{H} \rtimes \Gamma) = 0$  for  $n > \dim_{\mathbb{C}} \mathfrak{t}$ , it follows that  $HC_{n+2}(\mathbb{H} \rtimes \Gamma) \cong HC_n(\mathbb{H} \rtimes \Gamma)$  for such  $n$ . By [Lod, Proposition 5.1.9] this implies that  $HP_n(\mathbb{H} \rtimes \Gamma) \cong HC_n(\mathbb{H} \rtimes \Gamma)$  for all  $n > \dim_{\mathbb{C}} \mathfrak{t}$ . The same holds for the (periodic) cyclic homology of  $\mathcal{O}(\mathfrak{t}^{\rho_w})$ , so  $\tilde{\rho}$  also induces an isomorphism

$$HP_{\text{ev/odd}}(\mathbb{H} \rtimes \Gamma) \rightarrow \bigoplus_{w \in \langle W' \rangle} HP_{\text{ev/odd}}(\mathcal{O}(\mathfrak{t}^{\rho_w}))^{\mathcal{W}_{\rho_w}} \cong \bigoplus_{w \in \langle W' \rangle} H_{DR}^{\text{ev/odd}}(\mathfrak{t}^{\rho_w})^{\mathcal{W}_{\rho_w}}. \quad (62)$$

By Corollary 2.5 and (38)

$$\bigsqcup_{w \in \langle W' \rangle} \mathfrak{t}^{\rho_w} / \mathcal{W}_{\rho_w} \cong \bigsqcup_{w \in \langle W' \rangle} \mathfrak{t}^w / Z_{W'}(w) \cong \tilde{\mathfrak{t}} / W'$$

as orbifolds. Therefore the right hand side of (62) is isomorphic to  $H_{DR}^{\text{ev/odd}}(\tilde{\mathfrak{t}})^{W'}$ .  $\square$

### 3.2 Affine Hecke algebras

We will transfer the description of the Hochschild homology of graded Hecke algebras, as given in the previous section, to affine Hecke algebras. The connection is provided by Lusztig's reduction theorems. Since these involve formal completions of algebras, we spend a few words on the modifications which are necessary for homological algebra in that context.

Given an algebra  $A$  and a central ideal  $I$  of the multiplier algebra of  $A$ , we endow

$$\widehat{A} := \varprojlim_{\infty \leftarrow n} A / I^n A$$

with the  $I$ -adic topology. Then it is natural to use completed tensor products and relative homological algebra, as in [KNS, Section 2]. The resulting Hochschild homology is denoted  $HH_*^{\text{top}}(\widehat{A})$  and it is a module over  $\widehat{Z(A)}$ .

Let  $V$  be any complex affine variety. Recall that a finite type  $\mathcal{O}(V)$ -algebra is an algebra  $A$  together with a homomorphism from  $\mathcal{O}(V)$  to the centre of the multiplier algebra of  $A$ , which makes  $A$  into a  $\mathcal{O}(V)$ -module of finite rank.

**Theorem 3.3.** [KNS, Theorem 3]

*Let  $A$  be a finite type  $\mathcal{O}(V)$ -algebra and let  $I$  be an ideal of  $\mathcal{O}(V)$ . The inclusion  $A \rightarrow \widehat{A}$  induces an isomorphism*

$$HH_*(A) \otimes_{\mathcal{O}(V)} \widehat{\mathcal{O}(V)} \rightarrow HH_*^{\text{top}}(\widehat{A}).$$

Because we intend to use the results from Section 2.2, we fix  $\epsilon \in \mathbb{C} \setminus i\mathbb{R}$  and we assume that  $q_{\alpha^\vee} \in \exp(\epsilon\mathbb{R})$  for every  $\alpha^\vee \in \mathbb{R}_{nr}^\vee$ . We pick analytic families of  $\mathcal{H} \rtimes \Gamma$ -representations as in (42) and Theorem 2.6. These give rise to an algebra homomorphism

$$\tilde{\sigma} : \mathcal{H} \rtimes \Gamma \rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} \mathcal{O}(T^{\sigma_{w,i}}) \otimes \text{End}_{\mathbb{C}}(V_{\sigma_{w,i}}). \quad (63)$$

This induces a map  $C_*(\tilde{\sigma})$  between the corresponding Hochschild complexes, which we can compose with the generalized trace map to reach the direct sum of the Hochschild complexes  $C_*(\mathcal{O}(T^{\sigma_{w,i}}))$ . Since the image cannot distinguish representations with the same trace, we obtained a morphism of differential complexes

$$\text{gtr} \circ C_*(\tilde{\sigma}) : C_*(\mathcal{H} \rtimes \Gamma) \rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} C_*(\mathcal{O}(T^{\sigma_{w,i}}))^{\mathcal{G}_{\sigma_{w,i}}} \quad (64)$$

Just as for graded Hecke algebras the induced map on Hochschild homology lands in a space of invariant differential forms:

$$HH_*(\tilde{\sigma}) : HH_*(\mathcal{H} \rtimes \Gamma) \rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} \Omega^*(cc(\sigma_{w,i})T^{\sigma_{w,i}})^{\mathcal{G}_{\sigma_{w,i}}}. \quad (65)$$

Here  $cc(\sigma_{w,i})$  is an  $\mathcal{A}$ -weight of the irreducible  $\mathcal{H}^P \rtimes \Gamma_{\sigma_{w,i}}$ -representation  $\sigma_{w,i}$ , which serves mainly to make the module structure over the centre clear. An element  $f \in Z(\mathcal{H} \rtimes \Gamma) \cong \mathcal{O}(T)^{W'}$  acts on the right hand side of (65) via evaluation on  $cc(\sigma_{w,i})T^{\sigma_{w,i}} \subset T$ . This construction makes (65)  $Z(\mathcal{H} \rtimes \Gamma)$ -linear.

**Theorem 3.4.** *Suppose that there exists  $\epsilon \in \mathbb{C} \setminus i\mathbb{R}$  such that  $q_{\alpha^\vee} \in \exp(\epsilon\mathbb{R})$  for all  $\alpha^\vee \in R_{nr}^\vee$ . The map (64) is a quasi-isomorphism and (65) is an isomorphism of  $Z(\mathcal{H} \rtimes \Gamma)$ -modules.*

*Proof.* We only have to deal with the second claim, since it encompasses the first. Pick any  $u \in T_{un}$  and  $\lambda \in \epsilon\mathfrak{a}$ . By Theorem 1.3 the algebras  $\mathcal{H} \rtimes \Gamma$  and  $\mathbb{H}(\tilde{\mathcal{R}}^{P(u)}, k^{P(u)}) \rtimes W'_{P(u)}$  become isomorphic upon formally completing at  $W'u \exp(\lambda)$ , respectively at  $W'_{\mathbb{Z}P(u)}\lambda$ . This enables us to transform the  $\pi_{\sigma_{w,i}}(t)$  for which

$$cc(\sigma_{w,i})T^{\sigma_{w,i}} \cap W'u \exp(\epsilon\mathfrak{a}) \neq \emptyset$$

into analytic families of  $\mathbb{H}(\tilde{\mathcal{R}}^{P(u)}, k^{P(u)}) \rtimes W'_{P(u)}$ -representations

$$\{\pi_{\rho_w}(\lambda) : w \in \langle W'_u \rangle, \lambda \in \mathfrak{t}^{\rho_w}\},$$

like we did in the proof of Theorem 2.6. By Theorem 2.6 the formal completions

$$\bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} \widehat{\mathcal{O}(cc(\sigma_{w,i})T^{\sigma_{w,i}})^{\mathcal{G}_{\sigma_{w,i}}}}_{W'u \exp(\lambda)} \otimes \text{End}_{\mathbb{C}}(V_{\sigma_{w,i}})$$

and

$$\bigoplus_{w \in \langle W'_u \rangle} \widehat{\mathcal{O}(cc(\rho_w) + \mathfrak{t}^{\rho_w})^{\mathcal{W}_{\rho_w}}}_{W'_{\mathbb{Z}P(u)}\lambda} \otimes \text{End}_{\mathbb{C}}(V_{\rho_w})$$

are Morita equivalent. All this makes it possible to fit (65) in a commutative diagram

$$\begin{array}{ccc}
HH_*(\mathcal{H} \rtimes \Gamma) & \rightarrow & \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} \Omega^*(cc(\sigma_{w,i})T^{\sigma_{w,i}})^{\mathcal{G}_{\sigma_{w,i}}} \\
\downarrow & & \downarrow \\
HH_*(\widehat{\mathcal{H}}_{W'u \exp(\lambda)} \rtimes \Gamma) & \rightarrow & \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} \Omega^*(cc(\sigma_{w,i})T^{\sigma_{w,i}})^{\mathcal{G}_{\sigma_{w,i}}} \widehat{W'u \exp(\lambda)} \\
\downarrow & & \downarrow \\
HH_*(\mathbb{H}(\widehat{\mathcal{R}}^{P(u)}, k^{P(u)})_{W'_{ZP(u)\lambda}} \rtimes W'_{P(u)}) & \rightarrow & \bigoplus_{w \in \langle W'_u \rangle} \Omega^*(cc(\rho_w) + \mathfrak{t}^{\rho_w})^{\mathcal{W}_{\rho_w}} W'_{ZP(u)\lambda}
\end{array}$$

By Theorems 3.1 and 3.3 the lower horizontal map is an isomorphism of  $Z(\mathbb{H}(\widehat{\mathcal{R}}^{P(u)}, k^{P(u)})_{W'_{ZP(u)\lambda}} \rtimes W'_{P(u)})$ -modules. By the above constructions the same goes for the two lower vertical maps, and hence for the middle horizontal map as well. Thus the diagram and Theorem 3.3 show that (65) is an homomorphism of  $Z(\mathcal{H} \rtimes \Gamma)$ -modules which becomes an isomorphism upon completing with respect to an arbitrary maximal ideal of this commutative algebra.

Now consider any  $x \in \ker HH_*(\tilde{\sigma})$ . Then

$$Z(\mathcal{H} \rtimes \Gamma)x \cong Z(\mathcal{H} \rtimes \Gamma)/J_x$$

for some ideal  $J_x \subset Z(\mathcal{H} \rtimes \Gamma)$ . The above implies that  $Z(\widehat{\mathcal{H} \rtimes \Gamma})/J_x = 0$  for every maximal ideal of  $Z(\mathcal{H} \rtimes \Gamma)$ , which means that  $J_x$  is not contained in any maximal ideal. Hence  $J_x = Z(\mathcal{H} \rtimes \Gamma)$  and  $x = 0$ . A similar argument shows that  $\text{coker } HH_*(\tilde{\sigma}) = 0$ , so (65) is an isomorphism.  $\square$

**Corollary 3.5.** *The algebra homomorphism (65) induces isomorphisms*

$$\begin{aligned}
HC_*(\tilde{\sigma}) : \quad HC_*(\mathcal{H} \rtimes \Gamma) & \rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} HC_*(\mathcal{O}(T^{\sigma_{w,i}}))^{\mathcal{G}_{\sigma_{w,i}}}, \\
HP_{ev/odd}(\tilde{\sigma}) : \quad HP_{ev/odd}(\mathcal{H} \rtimes \Gamma) & \rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} H_{DR}^{ev/odd}(T^{\sigma_{w,i}})^{\mathcal{G}_{\sigma_{w,i}}} \cong H_{DR}^{ev/odd}(\tilde{T})^{W'}.
\end{aligned}$$

*Proof.* This follows in the same way from Theorem 3.4 as Corollary 3.2 follows from Theorem 3.1.  $\square$

We note that the periodic cyclic homology of  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$  was already known from [Sol3, Section 5.2] for positive  $q$ . However, there it was obtained in a rather indirect way, via the topological  $K$ -theory of a  $C^*$ -completion of  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ . With Corollary 3.5 we can understand  $HP_*(\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma)$  more explicitly and for more general parameter functions  $q$ .

Maybe one can improve on Theorem 3.4 by finding suitable maps  $\phi_{w,i} : \mathcal{H} \rtimes \Gamma \rightarrow A_{w,i}$ , where  $A_{w,i}$  is an algebra with dual space  $T_i^w/Z_{W'}(w)$  and  $HH_*(A_{w,i}) \cong \Omega^*(T_i^w)^{Z_{W'}(w)}$ . Yet this is difficult in the present setup, since the representations  $\pi_{\sigma_{w,i}}(t)$  and  $\pi_{\sigma_{w,i}}(gt)$  with  $g \in \mathcal{G}_{\sigma_{w,i}}$  need not be isomorphic.

For equal parameter functions  $q$ , Lusztig achieved something similar with an "asymptotic Hecke algebra" [Lus1]. This was used in [BaNi] to compute the periodic

cyclic homology of  $\mathcal{H}(\mathcal{R}, q)$ . It is not known whether this strategy can be employed for Hochschild homology or for unequal parameter functions.

### 3.3 Schwartz algebras

In this section we will compute the Hochschild homology of the Schwartz completion  $\mathcal{S} \rtimes \Gamma$  of an (extended) affine Hecke algebra  $\mathcal{H} \rtimes \Gamma$  with positive parameters  $q$ . It turns out that there is a clear relation between  $HH_*(\mathcal{S} \rtimes \Gamma)$  and  $HH_*(\mathcal{H} \rtimes \Gamma)$ , similar to the relation between  $C^\infty(S^1)$  and  $\mathcal{O}(\mathbb{C}^\times) \cong \mathbb{C}[z, z^{-1}]$ .

To get the correct answer we must take the topology of  $\mathcal{S} \rtimes \Gamma$  into account. The best way to do so is with complete bornological algebras [Mey] and bornological tensor products, which we will denote by  $\widehat{\otimes}$ . Thus  $HH_*(\mathcal{S} \rtimes \Gamma)$  is the homology of the differential complex  $((\mathcal{S} \rtimes \Gamma)^{\widehat{\otimes} n+1}, b_n)$ . It is isomorphic to

$$\mathrm{Tot}_*^{\mathcal{S} \rtimes \Gamma \widehat{\otimes} (\mathcal{S} \rtimes \Gamma)^{op}}(\mathcal{S} \rtimes \Gamma, \mathcal{S} \rtimes \Gamma),$$

computed in the category of complete bornological  $\mathcal{S} \rtimes \Gamma$ -bimodules.

We will use the notation from (42). By Theorem 1.5 the evaluation map

$$\mathcal{H} \rtimes \Gamma \rightarrow \mathcal{O}(T^{\sigma_{w,i}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma_{w,i}})$$

extends to a homomorphism of Fréchet algebras

$$\mathcal{S} \rtimes \Gamma \rightarrow C^\infty(T_{un}^{\sigma_{w,i}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma_{w,i}}).$$

Applying this to all  $w$  and  $i$  yields a homomorphism

$$\tilde{\sigma}_{un} : \mathcal{S} \rtimes \Gamma \rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} C^\infty(T_{un}^{\sigma_{w,i}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma_{w,i}}) \quad (66)$$

which extends (63). Recall that by the topological version of the Hochschild–Kostant–Rosenberg Theorem

$$HH_*(C^\infty(T_{un})) \cong \Omega_{sm}^*(T_{un}),$$

the algebra of smooth complex valued differential forms on the real manifold  $T_{un}$ . Hence (66), like  $\tilde{\sigma}$ , induces a  $\mathbb{C}$ -linear map

$$HH_*(\tilde{\sigma}_{un}) : HH_*(\mathcal{S} \rtimes \Gamma) \rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} \Omega_{sm}^*(T_{un}^{\sigma_{w,i}})^{\mathcal{G}_{\sigma_{w,i}}}. \quad (67)$$

It becomes  $Z(\mathcal{S} \rtimes \Gamma)$ -linear if we write  $cc(\sigma_{w,i})T_{un}^{\sigma_{w,i}}$  instead of  $T_{un}^{\sigma_{w,i}}$  and interpret the  $Z(\mathcal{S} \rtimes \Gamma)$ -action accordingly.

**Theorem 3.6.** *Let  $q$  be positive. The map (67) is an isomorphism of  $Z(\mathcal{S} \rtimes \Gamma)$ -modules and it extends Theorem 3.4.*



*Proof.* Since  $e_{P,\delta} \in \mathcal{S} \rtimes \Gamma$  is a central idempotent,

$$HH_n(e_{P,\delta}\mathcal{S} \rtimes \Gamma) = H_n((e_{P,\delta}\mathcal{S} \rtimes \Gamma)^{\widehat{\otimes}(*+1)}, b_*) \cong H_n(e_{P,\delta}\mathcal{S} \rtimes \Gamma \widehat{\otimes} \mathcal{S} \rtimes \Gamma)^{\widehat{\otimes}*}, b_*). \quad (68)$$

For  $\xi = (P, \delta, t) \in \Xi_{un}$  let  $\mathcal{F}_\xi$  be the ring of formal power series on  $(P, \delta, T_{un}^P)$  centred at  $\xi$ . By [OpSo2, Theorem 3.5] the functor

$$\mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \widehat{\otimes}_{Z(\mathcal{S} \rtimes \Gamma)} = \mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \widehat{\otimes}_{e_{P,\delta}Z(\mathcal{S} \rtimes \Gamma)}$$

is exact on a large class of  $Z(\mathcal{S} \rtimes \Gamma)$ -modules, which contains all modules that we use here. Applying this to the right-hand side of (68) we obtain

$$\mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \widehat{\otimes}_{Z(\mathcal{S} \rtimes \Gamma)} HH_n(e_{P,\delta}\mathcal{S} \rtimes \Gamma) \cong H_n(\mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \widehat{\otimes}_{Z(\mathcal{S} \rtimes \Gamma)} e_{P,\delta}\mathcal{S} \rtimes \Gamma \widehat{\otimes} \mathcal{S} \rtimes \Gamma)^{\widehat{\otimes}*}, b_*). \quad (69)$$

It follows from a result of Borel [Tou, Remarque IV.3.5] that the Taylor series map

$$e_{P,\delta}Z(\mathcal{S} \rtimes \Gamma) \cong C^\infty(T_{un}^P)^{\mathcal{G}_{P,\delta}} \rightarrow \mathcal{F}_\xi^{\mathcal{G}_{P,\delta}}$$

is surjective, and (by definition) its kernel  $I_\xi^\infty$  consists of all functions that are flat at  $\xi$ . Clearly the  $Z(\mathcal{S} \rtimes \Gamma)$ -action on (69) factors through  $\mathcal{F}_\xi^{\mathcal{G}_{P,\delta}}$ , so  $I_\xi^\infty$  annihilates (69). Together with Lemma 1.6 this shows that (69) is isomorphic to

$$\begin{aligned} H_n((\mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \widehat{\otimes}_{Z(\mathcal{S} \rtimes \Gamma)} e_{P,\delta}\mathcal{S} \rtimes \Gamma)^{\widehat{\otimes}(*+1)}, b_*) &\cong H_n((\mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \otimes_{e_{P,\delta}Z(\mathcal{H} \rtimes \Gamma)} e_{P,\delta}\mathcal{H} \rtimes \Gamma)^{\widehat{\otimes}(*+1)}, b_*) \\ &= HH_n(\mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \otimes_{e_{P,\delta}Z(\mathcal{H} \rtimes \Gamma)} e_{P,\delta}\mathcal{H} \rtimes \Gamma). \end{aligned} \quad (70)$$

Let  $pr : \Xi \rightarrow T/W'$  be the projection that sends  $\xi$  to the  $Z(\mathcal{H} \rtimes \Gamma)$ -character of the representation  $\pi^\Gamma(\xi)$ . With (29) and Lemma 2.3.c we can associate to every tempered analytic family of  $\mathcal{H} \rtimes \Gamma$ -representations a unique  $(P, \delta) \in \mathcal{P}$ . In the notation of (42) we write  $\sigma_{w,i} \prec (P, \delta)$  if  $\pi_{\sigma_{w,i}}(t)(e_{P,\delta}) \neq 0$  for any  $t \in T_{un}^P$ .

If we want to apply the algebra homomorphism  $\tilde{\sigma}$  (or  $\tilde{\sigma}_{un}$ , that comes down to the same thing) to  $\mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \otimes_{e_{P,\delta}Z(\mathcal{H} \rtimes \Gamma)} e_{P,\delta}\mathcal{H} \rtimes \Gamma$ , we must replace the target by

$$\begin{aligned} \mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \otimes_{e_{P,\delta}Z(\mathcal{H} \rtimes \Gamma)} \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} \mathcal{O}(T^{\sigma_{w,i}}) \otimes \text{End}_{\mathbb{C}}(V_{\sigma_{w,i}}) \\ = \bigoplus_{\sigma_{w,i} \prec (P,\delta)} \widehat{\mathcal{O}(T^{\sigma_{w,i}})}_{pr(\xi)} \otimes \text{End}_{\mathbb{C}}(V_{\sigma_{w,i}}). \end{aligned}$$

Let us denote the resulting algebra homomorphism by  $\tilde{\sigma}_\xi$ . It induces a morphism  $C_*(\tilde{\sigma}_\xi)$  on Hochschild complexes, whose composition with the generalized trace map  $\text{gtr}$  lands in a sum of  $\mathcal{G}_{\sigma_{w,i}}$ -invariant subcomplexes:

$$\text{gtr} \circ C_*(\tilde{\sigma}_\xi) : C_*(\mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \otimes_{e_{P,\delta}Z(\mathcal{H} \rtimes \Gamma)} e_{P,\delta}\mathcal{H} \rtimes \Gamma) \rightarrow \bigoplus_{\sigma_{w,i} \prec (P,\delta)} C_*(\widehat{\mathcal{O}(T^{\sigma_{w,i}})}_{pr(\xi)})^{\mathcal{G}_{\sigma_{w,i}}}. \quad (71)$$

More precisely, the image of  $\text{gtr} \circ C_*(\tilde{\sigma}_\xi)$  is the same as the formal completion of

$$\bigoplus_{\sigma_{w,i} \prec (P,\delta)} \text{gtr} \circ C_*(\tilde{\sigma})_{w,i}(C_*(\mathcal{H} \rtimes \Gamma))$$

at  $\xi$ . Since  $\text{gtr} \circ C_*(\tilde{\sigma})$  is a quasi-isomorphism (by Theorem 3.4), and since

$$\mathcal{H} \rtimes \Gamma \mapsto \mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \otimes_{e_{P,\delta}Z(\mathcal{H} \rtimes \Gamma)} e_{P,\delta}\mathcal{H} \rtimes \Gamma$$

just has the effect of formally completing at  $\text{pr}(\xi)$  and forgetting all analytic families with  $\sigma_{w,i} \not\prec (P, \delta)$ , (71) is also a quasi-isomorphism. Thus  $\tilde{\sigma}_\xi$  induces

$$HH_n(\mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \otimes_{e_{P,\delta}Z(\mathcal{H} \rtimes \Gamma)} e_{P,\delta}\mathcal{H} \rtimes \Gamma) \cong \bigoplus_{\sigma_{w,i} \prec (P,\delta)} \Omega^*(\widehat{T^{\sigma_{w,i}}})_{\text{pr}(\xi)}^{\mathcal{G}_{\sigma_{w,i}}}.$$

By (70), (69) and (68) this means that, for all  $\xi = (P, \delta, t) \in \Xi_{un}$ ,  $\tilde{\sigma}$  induces an isomorphism

$$\mathcal{F}_\xi^{\mathcal{G}_{P,\delta}} \otimes_{e_{P,\delta}Z(\mathcal{S} \rtimes \Gamma)} HH_n(e_{P,\delta}\mathcal{S} \rtimes \Gamma) \cong \bigoplus_{\sigma_{w,i} \prec (P,\delta)} \Omega^*(\widehat{T^{\sigma_{w,i}}})_{\text{pr}(\xi)}^{\mathcal{G}_{\sigma_{w,i}}}.$$

In other words, the homomorphism of  $Z(\mathcal{S} \rtimes \Gamma)$ -modules (67) becomes an isomorphism upon formally completing with respect to any closed maximal ideal of  $Z(\mathcal{S} \rtimes \Gamma)$ . Now the same argument as in the proof of Lemma 1.6 shows that (67) is itself an isomorphism.  $\square$

The following consequence of Theorem 3.6 can be proved in the same way as Corollaries 3.2 and 3.5.

**Corollary 3.7.** *The algebra homomorphism  $\tilde{\sigma}_{un}$  induces isomorphisms*

$$\begin{aligned} HC_*(\tilde{\sigma}_{un}) : HC_*(\mathcal{S} \rtimes \Gamma) &\rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} HC_*(C^\infty(T_{un}^{\sigma_{w,i}}))^{\mathcal{G}_{\sigma_{w,i}}}, \\ HP_{ev/odd}(\tilde{\sigma}) : HP_{ev/odd}(\mathcal{S} \rtimes \Gamma) &\rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} H_{DR}^{ev/odd}(T_{un}^{\sigma_{w,i}})^{\mathcal{G}_{\sigma_{w,i}}} \cong H_{DR}^{ev/odd}(\tilde{T}_{un})^{W'}. \end{aligned}$$

### 3.4 Comparison of different parameters

Let  $k$  be real valued and let  $q$  be positive. We will investigate what happens to the homology of the algebras  $\mathbb{H}(\tilde{\mathcal{R}}, k) \rtimes \Gamma$ ,  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$  and  $\mathcal{S}(\mathcal{R}, q) \rtimes \Gamma$  when we replace  $k$  by  $\epsilon k$  ( $\epsilon \in \mathbb{C}^\times$ ) and  $q$  by  $q^\epsilon$  ( $\epsilon \in \mathbb{C} \setminus i\mathbb{R}$ ). For the Schwartz algebras we must of course take  $\epsilon$  real.

Recall the analytic families of representations  $\pi_{\sigma_{w,i}}$  from Lemma 2.3 and  $\pi_{\rho_w}$  from Corollary 2.5. As discussed in the proof Theorem 2.6, adjusting the parameter function by  $\epsilon$  corresponds to applying  $m_\epsilon^*$  to  $\sigma_{w,i}$  and  $\rho_w$ . Let

$$\begin{aligned} \mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma &\rightarrow \mathcal{O}(\mathfrak{t}^{\rho_w}) \otimes \text{End}_{\mathbb{C}}(V_{\rho_w}^\Gamma), \\ \mathcal{H}(\mathcal{R}, q^\epsilon) \rtimes \Gamma &\rightarrow \mathcal{O}(T^{\sigma_{w,i}}) \otimes \text{End}_{\mathbb{C}}(V_{\sigma_{w,i}}^\Gamma), \\ \mathcal{S}(\mathcal{R}, q) \rtimes \Gamma &\rightarrow C^\infty(T_{un}^{\sigma_{w,i}}) \otimes \text{End}_{\mathbb{C}}(V_{\sigma_{w,i}}^\Gamma) \end{aligned}$$

be the corresponding algebra homomorphisms. By Theorems 3.1, 3.4 and 3.6 these induce isomorphisms

$$\begin{aligned} HH_*(\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma) &\rightarrow \bigoplus_{w \in \langle W' \rangle} \Omega^*(\mathfrak{t}^{\rho_w})^{\mathcal{G}_{\rho_w}} \cong \bigoplus_{w \in \langle W' \rangle} \Omega(\mathfrak{t}^w)^{Z_{W'}(w)}, \\ HH_*(\mathcal{H}(\mathcal{R}, q^\epsilon) \rtimes \Gamma) &\rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} \Omega^*(T^{\sigma_{w,i}})^{\mathcal{G}_{\sigma_{w,i}}} \cong \bigoplus_{w \in \langle W' \rangle} \Omega^*(T^w)^{Z_{W'}(w)}, \\ HH_*(\mathcal{S}(\mathcal{R}, q^\epsilon) \rtimes \Gamma) &\rightarrow \bigoplus_{w \in \langle W' \rangle} \bigoplus_{i=1}^{c(w)} \Omega_{sm}^*(T_{un}^{\sigma_{w,i}})^{\mathcal{G}_{\sigma_{w,i}}} \cong \bigoplus_{w \in \langle W' \rangle} \Omega_{sm}^*(T_{un}^w)^{Z_{W'}(w)}. \end{aligned}$$

The action of  $Z(\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma) \cong \mathcal{O}(\mathfrak{t})^{W'}$  on  $\Omega^*(\mathfrak{t}^{\rho_w})$  is via the embedding

$$\mathfrak{t}^{\rho_w} \rightarrow \mathfrak{t} : \lambda \mapsto cc(m_\epsilon^*(\rho_w)) + \lambda = \epsilon cc(\rho_w) + \lambda.$$

The translation factor  $cc(m_\epsilon^*(\rho_w)) \in \mathfrak{t}$  is a representative for the central character of  $m_\epsilon^*(\rho_w)$ . Similarly, the actions of  $Z(\mathcal{H}(\mathcal{R}, q^\epsilon) \rtimes \Gamma) \cong \mathcal{O}(T)^{W'}$  on  $\Omega^*(T^w)$  and on  $\Omega_{sm}^*(T_{un}^w)$  go via the embedding

$$T^{\sigma_{w,i}} \rightarrow T : t \mapsto cc(m_\epsilon^*(\sigma_{w,i}))t.$$

Here  $cc(m_\epsilon^*(\sigma_{w,i})) \in T_{un}$  is a representative for the central character of  $\sigma_{w,i}$ , and by (46) it lies in  $Y \otimes S^1 q^{\epsilon\mathbb{Z}/2}$ .

Now we will generalize these results to the case  $\epsilon = 0$ . Recall the maps

$$\zeta^\vee : G(\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma) \rightarrow G(\mathcal{S}(\mathfrak{t}^*) \rtimes \Gamma)$$

from Theorem 2.4 and

$$\zeta^\vee : G(\mathcal{H}(\mathcal{R}, q^\epsilon) \rtimes \Gamma) \rightarrow G(\mathbb{C}[X] \rtimes W')$$

from Theorem 2.6. These maps send analytic families to other analytic families, because they commute with parabolic induction.

**Theorem 3.8.** (a) *Let  $k$  be real valued and let  $\epsilon \in \mathbb{C}^\times$ . There exists a unique isomorphism*

$$HH_*(\zeta^\vee) : HH_*(\mathcal{S}(\mathfrak{t}^*) \rtimes W') \rightarrow HH_*(\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma)$$

such that

$$HH_*(\pi_\rho) \circ HH_*(\zeta^\vee) = HH_*(\zeta^\vee(\pi_\rho))$$

for all analytic families of  $\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma$ -representations  $\{\pi_\rho(\lambda) \mid \lambda \in \mathfrak{t}^\rho\}$ .

(b) *Let  $q$  be positive and let  $\epsilon \in \mathbb{C} \setminus i\mathbb{R}$ . There exists a unique isomorphism*

$$HH_*(\zeta^\vee) : HH_*(\mathbb{C}[X] \rtimes W') \rightarrow HH_*(\mathcal{H}(\mathcal{R}, q^\epsilon) \rtimes \Gamma)$$

such that

$$HH_*(\pi_\sigma) \circ HH_*(\zeta^\vee) = HH_*(\zeta^\vee(\pi_\sigma))$$

for all analytic families of  $\mathcal{H}(\mathcal{R}, q^\epsilon) \rtimes \Gamma$ -representations  $\{\pi_\sigma(t) \mid t \in T^\sigma\}$ .

(c) Let  $q$  be positive. There exists a unique isomorphism

$$HH_*(\zeta^\vee) : HH_*(\mathcal{S}(X) \rtimes W') \rightarrow HH_*(\mathcal{S}(\mathcal{R}, q) \rtimes \Gamma)$$

such that

$$HH_*(\pi_\sigma) \circ HH_*(\zeta^\vee) = HH_*(\zeta^\vee(\pi_\sigma))$$

for all analytic families of  $\mathcal{S}(\mathcal{R}, q) \rtimes \Gamma$ -representations  $\{\pi_\sigma(t) \mid t \in T_{un}^\sigma\}$ .

In all three settings the analogous statements for cyclic and periodic cyclic homology are also valid.

*Proof.* We will only prove (a), the other parts follow in the same way. The construction of the  $\pi_{\rho_w}$  and the proof of Lemma 2.3 show that the analytic families

$$\{\zeta^\vee(\pi_{\rho_w}(\lambda)) \mid w \in \langle W' \rangle, \lambda \in \mathfrak{t}^{\rho_w}\} \quad (72)$$

fulfill Corollary 2.5 for  $\mathbb{H}(\tilde{\mathcal{R}}, 0) \rtimes \Gamma = S(\mathfrak{t}^*) \rtimes W'$ . We remark that the representations (72) can be reducible, even for generic  $\lambda$ . But since this does not affect the arguments in Section 3.1, we may ignore it. In particular Theorem 3.1 holds for  $S(\mathfrak{t}^*) \rtimes W'$  with (72). Now the condition of the theorem enforces that  $HH_*(\zeta^\vee)$  is the composite isomorphism

$$HH_*(S(\mathfrak{t}^*) \rtimes W') \longrightarrow \bigoplus_{w \in \langle W' \rangle} \Omega^*(\mathfrak{t}^{\rho_w}) \mathcal{W}_{\rho_w}' \xrightarrow{HH_*(\tilde{\rho})^{-1}} HH_*(\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma).$$

For all  $w \in \langle W' \rangle$  this map satisfies

$$HH_*(\pi_{\rho_w}) \circ HH_*(\zeta^\vee) = HH_*(\zeta^\vee(\pi_{\rho_w})).$$

Thus it remains to check compatibility with an arbitrary analytic family of  $\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma$ -representations  $\{\pi_\rho(\lambda) \mid \lambda \in \mathfrak{t}^\rho\}$ . Let  $\lambda$  be a generic point of  $\mathfrak{t}^\rho$ . By Corollary 2.5.c we can find  $n_\rho \in \mathbb{N}, n_w \in \mathbb{Z}$  and  $\mu_{w,\lambda} \in \mathfrak{t}^{\rho_w}$  such that

$$n_\rho \pi_\rho(\lambda) = \sum_{w \in \langle W' \rangle} n_w \pi_{\rho_w}(\mu_{w,\lambda}) \quad \text{in } G(\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma). \quad (73)$$

The genericity of  $\lambda$  implies  $n_w = 0$  unless  $W' \mathfrak{t}^{\rho_w} \supset W' \mathfrak{t}^\rho$ . Comparing the central characters also shows that there exist  $\tilde{w} \in W'$  with

$$\tilde{\lambda} + cc(\pi_\rho(0)) = \mu_{w,\lambda} + cc(\pi_{\rho_w}(0)),$$

for suitable representatives  $cc(\pi)$  of the central character of the representations  $\pi$  under consideration. Hence we may take

$$\mu_{w,\lambda} = \tilde{w}\lambda + cc(\pi_\rho(0)) - cc(\pi_{\rho_w}(0)) =: \tilde{\lambda} + \mu_w.$$

Since  $\lambda$  was generic, with this choice of  $\mu_{w,\lambda}$  (73) becomes valid for all generic  $\lambda \in \mathfrak{t}^\rho$ , which then by continuity extends to the whole of  $\mathfrak{t}^\rho$ . Next we bring the terms  $n_w < 0$  to the other side and obtain

$$n_\rho \pi_\rho(\lambda) + \sum_{w \in \langle W' \rangle_-} -n_w \pi_{\rho_w}(\tilde{\lambda} + \mu_w) = \sum_{w \in \langle W' \rangle_+} n_w \pi_{\rho_w}(\tilde{\lambda} + \mu_w).$$

This is an equality in  $G(\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma)$ , but the representations on the both sides need not be isomorphic. Since Hochschild homology does not distinguish direct sums from nontrivial extensions [Lod, Theorem 1.2.15], we may nevertheless treat it as an equivalence of  $\mathbb{H}(\tilde{\mathcal{R}}, k) \rtimes \Gamma$ -representations. Consequently

$$n_\rho HH_*(\pi_\rho) + \sum_{w \in \langle W' \rangle_-} -n_w HH_*(\pi_{\rho_w})|_{\mu_w + \tilde{w}t^\rho} = \sum_{w \in \langle W' \rangle_+} n_w HH_*(\pi_{\rho_w})|_{\mu_w + \tilde{w}t^\rho}$$

as maps

$$HH_*(\mathbb{H}(\tilde{\mathcal{R}}, \epsilon k) \rtimes \Gamma) \rightarrow HH_*(\mathcal{O}(t^\rho)) \cong \Omega^*(t^\rho).$$

Now the aforementioned compatibility of  $HH_*(\zeta^\vee)$  with the  $\pi_{\rho_w}$  and the additivity of  $\zeta^\vee$  show that

$$HH_*(\pi_\rho) \circ HH_*(\zeta^\vee) = HH_*(\zeta^\vee(\pi_\rho)),$$

as required. The analogous statements for (periodic) cyclic homology can be proved in the same way, using Corollary 3.2.  $\square$

In [Sol3, Theorem 4.4.2] the author constructed a homomorphism of Fréchet algebras

$$\zeta_0 : \mathcal{S}(X) \rtimes W' \rightarrow \mathcal{S}(\mathcal{R}, q) \rtimes \Gamma.$$

It is related to Theorem 2.2 by [Sol3, Corollary 4.4.3]:  $\zeta^\vee(\pi) \cong \pi \circ \zeta_0$  for all irreducible  $\mathcal{S}(\mathcal{R}, q) \rtimes \Gamma$ -representations. Furthermore it was shown in [Sol3, Theorems 5.1.4 and 5.2.1] that  $\zeta_0$  induces isomorphisms on topological  $K$ -theory and on periodic cyclic homology. With Theorem 3.8 we can improve on this:

**Proposition 3.9.** *Let  $q$  be positive.*

- (a)  $HH_*(\zeta_0) = HH_*(\zeta^\vee) : HH_*(\mathcal{S}(X) \rtimes W') \rightarrow HH_*(\mathcal{S}(\mathcal{R}, q) \rtimes \Gamma)$ .
- (b) *There exists a natural commutative diagram*

$$\begin{array}{ccc} HH_*(\mathbb{C}[X] \rtimes W') & \rightarrow & HH_*(\mathcal{S}(X) \rtimes W') \\ \downarrow HH_*(\zeta^\vee) & & \downarrow HH_*(\zeta_0) \\ HH_*(\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma) & \rightarrow & HH_*(\mathcal{S}(\mathcal{R}, q) \rtimes \Gamma) \end{array}$$

*in which the horizontal maps are induced by inclusions of algebras and the vertical maps are isomorphisms.*

- (c) *The analogues of (a) and (b) for cyclic and periodic cyclic homology are also valid.*

*Proof.* By [Sol3, Lemma 4.2.3 and Theorem 4.4.2.e]

$$\pi_\sigma(t) \circ \zeta_0 = \zeta^\vee(\pi_\sigma(t))$$

for every tempered analytic family of  $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$ -representations  $\{\pi_\sigma(t) \mid t \in T_{un}^\sigma\}$ . Thus (a) follows from the unicity part of Theorem 3.8.c. That and Theorem 3.8 directly imply (b) and (c).  $\square$

## References

- [ABP] A.-M. Aubert, P.F. Baum, R.J. Plymen, “The Hecke algebra of a reductive  $p$ -adic group: a view from noncommutative geometry”, pp. 1–34 in: *Noncommutative geometry and number theory*, Aspects of Mathematics **E37**, Vieweg Verlag, 2006
- [BaMo] D. Barbasch, A. Moy, “Reduction to real infinitesimal character in affine Hecke algebras”, *J. Amer. Math. Soc.* **6.3** (1993), 611–635
- [BaNi] P.F. Baum, V. Nistor, “Periodic cyclic homology of Iwahori-Hecke algebras”, *K-Theory* **27.4** (2002), 329–357
- [DeOp] P. Delorme, E.M. Opdam, “The Schwartz algebra of an affine Hecke algebra”, *J. reine angew. Math.* **625** (2008), 59–114
- [HeOp] G.J. Heckman, E.M. Opdam, “Yang’s system of particles and Hecke algebras”, *Ann. of Math.* **145.1** (1997), 139–173
- [KaLu] D. Kazhdan, G. Lusztig, “Proof of the Deligne-Langlands conjecture for Hecke algebras”, *Invent. Math.* **87** (1987), 153–215
- [KNS] D. Kazhdan, V. Nistor, P. Schneider, “Hochschild and cyclic homology of finite type algebras”, *Sel. Math. New Ser.* **4.2** (1998), 321–359
- [Lod] J.-L. Loday, *Cyclic homology 2nd ed.*, Mathematischen Wissenschaften **301** Springer Verlag, 1997
- [Lus1] G. Lusztig, “Cells in affine Weyl groups II”, *J. Algebra* **109** (1987), 536–548
- [Lus2] G. Lusztig, “Affine Hecke algebras and their graded version”, *J. Amer. Math. Soc.* **2.3** (1989), 599–635
- [Mey] R. Meyer, “Embeddings of derived categories of bornological modules”, arXiv:math.FA/0410596, 2004
- [Nis] V. Nistor, “A non-commutative geometry approach to the representation theory of reductive  $p$ -adic groups: Homology of Hecke algebras, a survey and some new results”, pp. 301–323 in: *Noncommutative geometry and number theory*, Aspects of Mathematics **E37**, Vieweg Verlag, 2006
- [Opd] E.M. Opdam, “On the spectral decomposition of affine Hecke algebras”, *J. Inst. Math. Jussieu* **3.4** (2004), 531–648
- [OpSo1] E.M. Opdam, M. Solleveld, “Discrete series characters for affine Hecke algebras and their formal dimensions”, *Acta Mathematica* **105** (2010), 105–187
- [OpSo2] E.M. Opdam, M. Solleveld, “Extensions of tempered representations”, preprint, arXiv:1105.3802
- [Ser] J.-P. Serre, *Linear representations of finite groups*, Springer Verlag, 1977

- [Sol1] M. Solleveld, “Homology of graded Hecke algebras” *J. Algebra* **323** (2010), 1622–1648
- [Sol2] M. Solleveld, “Parabolically induced representations of graded Hecke algebras”, *Algebras and Representation Theory* **15.2** (2012), 233–271
- [Sol3] M. Solleveld, “On the classification of irreducible representations of affine Hecke algebras with unequal parameters”, *Represent. Theory* **16** (2012), 1–87
- [Tou] J-C. Tougeron, *Idéaux de Fonctions Différentiables*, Springer Verlag, 1972
- [Yan] Rong Yan, “Isomorphisms between affine Hecke algebras of type  $\widetilde{A}_2$ ”, *J. Algebra* **324.5** (2010), 984–999
- [AMS] A.-M. Aubert, A. Moussaoui, M. Solleveld, “Affine Hecke algebras for Langlands parameters”, arXiv:1701.03593, 2019
- [Sol4] M. Solleveld, “Affine Hecke algebras and their representations”, *Indagationes Mathematica* **32.5** (2021), 1005–1082
- [Sol5] M. Solleveld, “Hochschild homology of twisted crossed products and twisted graded Hecke algebras”, arXiv:2201.11186, 2022