A LOCAL LANGLANDS CORRESPONDENCE FOR UNIPOTENT REPRESENTATIONS

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Abstract. Let G be a connected reductive group over a non-archimedean local field K, and assume that G splits over an unramified extension of K. We establish a local Langlands correspondence for irreducible unipotent representations of G. It comes as a bijection between the set of such representations (up to isomorphism) and the collection of enhanced L-parameters for G, which are trivial on the inertia subgroup of the Weil group of K. We show that this correspondence has many of the expected properties, for instance with respect to central characters, tempered representations, the discrete series, cuspidality and parabolic induction.

The core of our strategy is the investigation of affine Hecke algebras on both sides of the local Langlands correspondence. When a Bernstein component of G-representations is matched with a Bernstein component of enhanced L-parameters, we prove a comparison theorem for the two associated affine Hecke algebras.

This generalizes work of Lusztig in the case of adjoint K-groups.

Contents

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Introduction

Let K be a non-archimedean local field and let $\mathcal{G}$ be a connected reductive K-group. We consider smooth, complex representations of the group $G = \mathcal{G}(K)$. An irreducible smooth G-representation π is called unipotent if there exists a parahoric subgroup $P_{\mathfrak{f}} \subset G$ and an irreducible $P_{\mathfrak{f}}$-representation σ, which is inflated from a cuspidal unipotent representation of the finite reductive quotient of $P_{\mathfrak{f}}$, such that $\pi|_{P_{\mathfrak{f}}}$ contains σ. These notions behave best when $\mathcal{G}$ splits over an unramified extension of K, so that assume that in the introduction (and in most of the paper).

We will exhibit a local Langlands correspondence (LLC) for all irreducible unipotent representations of such reductive p-adic groups. This generalizes results of Lusztig [Lus4, Lus5] for simple adjoint K-groups.

Let us make the statement more precise, referring to Section 2 for the details. We denote the set of isomorphism classes of irreducible unipotent G-representations by $\mathrm{Irr}_{\mathrm{unip}}(G)$ and we consider Langlands parameters

$$\phi : \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G = G^\vee \rtimes \mathbf{W}_K .$$

As component group of ϕ we take the group $\mathcal{S}_\phi$ from [Art, HiSa, ABPS], a central extension of the component group of the centralizer of ϕ in $G^\vee$. An enhancement of ϕ is an irreducible representation ρ of $\mathcal{S}_\phi$, and there is a G-relevance condition for such enhancements. We let $\Phi_e(G)$ be the set of $G^\vee$-association classes of G-relevant enhanced L-parameters $\mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$.

Let $\mathbf{I}_K$ be the inertia subgroup of the Weil group $\mathbf{W}_K$. An enhanced L-parameter (ϕ, ρ) is called unramified if $\phi(w) = (1, w)$ for all $w \in \mathbf{I}_K$. If we consider $\phi|_{\mathbf{W}_K}$ as a 1-cocycle of $\mathbf{W}_K$ with values in $G^\vee$, then unramified means trivial on $\mathbf{I}_K$. We denote the resulting subset of $\Phi_e(G)$ by $\Phi_{\mathrm{nr},e}(G)$. Let $X_{\mathrm{wr}}(G)$ be the group of characters of G that are "weakly unramified", i.e. trivial on all parahoric subgroups of G.

Theorem 1. (see Section 5)
There exists a bijection

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{unip}}(G) & \longrightarrow & \Phi_{\mathrm{nr},e}(G) \\ \pi & \mapsto & (\phi_\pi, \rho_\pi) \\ \pi(\phi,\rho) & \leftarrow\!\shortmid & (\phi, \rho) \end{array}$$

with the following properties.

(a) Compatibility with direct products of reductive K-groups.
(b) Equivariance with respect to the canonical actions of $X_{\mathrm{wr}}(G)$.
(c) The central character of π equals the character of $Z(G)$ determined by ϕ_π.
(d) π is tempered if and only if ϕ_π is bounded.
(e) π is essentially square-integrable if and only if ϕ_π is discrete.
(f) π is supercuspidal if and only if (ϕ_π, ρ_π) is cuspidal.
(g) The analogous bijections for the Levi subgroups of G and the cuspidal support maps form a commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{unip}}(G) & \longrightarrow & \Phi_{\mathrm{nr},e}(G) \\ \downarrow & & \downarrow \\ \bigsqcup_L \mathrm{Irr}_{\mathrm{cusp,unip}}(L)/(N_G(L)/L) & \longrightarrow & \bigsqcup_L \Phi_{\mathrm{nr,cusp}}(L)/(N_G(L)/L) \end{array} .$$

Here L runs over a collection of representatives for the conjugacy classes of Levi subgroups of G. See Section 2 for explanation of the notation in the diagram.

(h) *Suppose that $P = LU$ is a parabolic subgroup of G and that $(\phi, \rho^L) \in \Phi_{\mathrm{nr},e}(L)$ is bounded. Then the normalized parabolically induced representation $I_P^G \pi(\phi, \rho^L)$ is a direct sum of representations $\pi(\phi, \rho)$, with multiplicities $[\rho^L : \rho]_{\mathcal{S}_\phi^L}$.*

(i) *Compatibility with the Langlands classification for representations of reductive groups and the Langlands classification for enhanced L-parameters.*

Since there are so many properties, one may wonder to what extent the LLC is characterized by them. First we note that ρ_π is certainly not uniquely determined by π alone. Namely, in many cases one can twist the LLC by a character of $\mathcal{S}_{\phi_\pi}$ and retain all the above properties.

The obvious next question is: do the above conditions determine the map

$$\mathrm{Irr}_{\mathrm{unip}}(G) \to \Phi_{\mathrm{nr}}(G) : \pi \mapsto \phi_\pi$$

uniquely? Again the answer is no, for (sometimes) one can still adjust the map by the action of a weakly unramified character of G. Then one may enquire whether the map $\pi \mapsto \phi_\pi$ is canonical in a weaker sense, up to twists by elements of $X_{\mathrm{wr}}(G)$. That is the case, and it is worked out in [FOS2, Theorem 2.1].

Now we provide an overview of the setup and the general strategy of the paper. Foremostly, everything runs via affine Hecke algebras. Usually an affine Hecke algebra is associated to one Bernstein component in $\mathrm{Irr}(G)$. To get them into play on the Galois side of the LLC, one first needs a good notion of a Bernstein component there. That was achieved in [AMS1], by means of a cuspidal support map for enhanced L-parameters. (To this end the enhancements are essential, without them one cannot even define cuspidality of L-parameters.) In [AMS1] the cuspidal support of an enhanced L-parameter for G is given as the $G^\vee$-association class of a cuspidal L-parameter for a G-relevant Levi subgroup of ${}^L G$. For Theorem 1.(g) and for later comparison, we need to translate this to a cuspidal L-parameter for a Levi subgroup of G, unique up to G-conjugation. That is the purpose of the next result (which we actually prove in greater generality).

Proposition 2. (see Corollary 1.3)
There exists a canonical bijection between:

- *the set of G-conjugacy classes of Levi subgroups of G;*
- *the set of $G^\vee$-conjugacy classes of G-relevant Levi subgroups of ${}^L G$.*

In Section 2 we show how one can associate, to every Bernstein component $\Phi_e(G)^{\mathfrak{s}^\vee}$ of enhanced L-parameters for G, an affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$. Here the array of complex parameters $\vec{v}$ can be chosen freely. This relies entirely on [AMS3]. The crucial properties of this algebra are:

- the irreducible representations of $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ are canonically parametrized by $\Phi_e(G)^{\mathfrak{s}^\vee}$ (at least when the parameters are chosen in $\mathbb{R}_{>0}$);
- the maximal commutative subalgebra of $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ (coming from the Bernstein presentation) is the ring of regular functions on a complex torus $T_{\mathfrak{s}^\vee}$. When the cuspidal support of $\Phi_e(G)^{\mathfrak{s}^\vee}$ is $\Phi_e(G)^{\mathfrak{s}_L^\vee}$ (for some Levi subgroup L of G), $T_{\mathfrak{s}^\vee}$ is in bijection with $\Phi_e(G)^{\mathfrak{s}_L^\vee}$.

Only after that we really turn to unipotent G-representations. From work of Morris and Lusztig [Mor1, Mor3, Lus4] it is known that every Bernstein block $\mathrm{Rep}(G)_{\mathfrak{s}}$ of smooth unipotent G-representations admits a type, and that it is equivalent to the module category of an affine Hecke algebra. In the introduction, we will denote that algebra simply by $\mathcal{H}_\mathfrak{s}$. In Section 3 we work out the Bernstein presentation of $\mathcal{H}_\mathfrak{s}$, that is, we make the underlying torus and Weyl group explicit in terms of $\mathfrak{s}$.

Armed with a good understanding of the affine Hecke algebras on both sides of the LLC, we set out to compare them. Here we make use of a local Langlands correspondence for supercuspidal unipotent representations, which was established in [FOS1]. Together with Proposition 2 that gives rise to:

Proposition 3. (see Proposition 4.2)
There exists a bijection, induced by a LLC for supercuspidal unipotent representations, between:

- *the set $\mathfrak{Be}(G)_{\mathrm{unip}}$ of Bernstein components in $\mathrm{Irr}(G)$ consisting of unipotent representations;*
- *the set $\mathfrak{Be}^\vee(G)_{\mathrm{nr}}$ of Bernstein components in $\Phi_e(G)$ consisting of unramified enhanced L-parameters.*

Roughly speaking, every affine Hecke algebra is determined by a complex torus, a finite Weyl group and array of q-parameters. When $\mathfrak{s} \in \mathfrak{Be}(G)_{\mathrm{unip}}$ corresponds to $\mathfrak{s}^\vee \in \mathfrak{Be}^\vee(G)_{\mathrm{nr}}$ via Proposition 3, we deduce from [FOS1] that the two associated Hecke algebras $\mathcal{H}_\mathfrak{s}$ and $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ have isomorphic Weyl groups, and that the underlying tori are isomorphic via the LLC on the cuspidal level. By reduction to the case of adjoint groups, which was settled in [Lus4, Lus5], we prove that the parameters of these two affine Hecke algebras match. That leads to:

Theorem 4. (see Theorem 4.4)
When $\mathfrak{s}$ corresponds to $\mathfrak{s}^\vee$ via Proposition 3 and all the parameters in $\vec{v}$ are equal to $q_K^{1/2}$, the algebra $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ is canonically isomorphic to $\mathcal{H}_\mathfrak{s}$.

In combination with the aforementioned properties of the involved affine Hecke algebras, Theorem 4 provides the bijection in Theorem 1. Most of the further properties mentioned in our main theorem follow rather quickly from earlier work on such algebras [AMS2, AMS3, Sol2].

A few properties which can be expected of a local Langlands correspondence remain open in Theorem 1. Comparing with Borel's list of desiderata in [Bor, §10], one notes that we have shown all of them, except for the functoriality with respect to homomorphisms of reductive groups with commutative kernel and commutative cokernel. This holds in a sense which is more precise and stronger than the formulation in [Bor], see [Sol3, §7].

In [FOS1, §16] the HII conjectures (about formal degrees and adjoint γ-factors) were established for supercuspidal unipotent representations, and in [Opd] a weaker version was shown for all unipotent representations. While this paper was under review, Feng, Opdam and the author proved the full HII conjectures for all unipotent representations [FOS2].

A rather ambitious issue is the stability of the L-packets constructed in this paper. Given a bounded $\phi \in \Phi_{\mathrm{nr}}(G)$, is there a linear combination of the members of the L-packet $\Pi_\phi(G)$ whose trace gives a stable distribution on G? Conjecturally there is a unique (up to scalars) such distribution, namely $\sum_{\pi \in \Pi_\phi(G)} \dim(\rho_\pi) \mathrm{tr}_\pi$. Further investigations are needed to verify this.

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1. Langlands dual groups and Levi subgroups

For more background on the material in this section, cf. [Bor, §1–3] and [SiZi, §2]. Let K be field with an algebraic closure $\overline{K}$ and a separable closure $K_s \subset \overline{K}$. Let Γ_K be a dense subgroup of the Galois group of K_s/K, for example $\mathrm{Gal}(K_s/K)$ or, when K is local and nonarchimedean, the Weil group of K.

Let $\mathcal{G}$ be a connected reductive K-group. Let $\mathcal{T}$ be a maximal torus of $\mathcal{G}$, and let $\Phi(\mathcal{G},\mathcal{T})$ be the associated root system. We also fix a Borel subgroup $\mathcal{B}$ of $\mathcal{G}$ containing $\mathcal{T}$, which determines a basis Δ of $\Phi(\mathcal{G},\mathcal{T})$. For every $\gamma \in \Gamma_K$ there exists a $g_\gamma \in \mathcal{G}(K_s)$ such that

$$g_\gamma \gamma(\mathcal{T}) g_\gamma^{-1} = \mathcal{T} \quad \text{and} \quad g_\gamma \gamma(\mathcal{B}) g_\gamma^{-1} = \mathcal{B}.$$

One defines an action of Γ_K on $\mathcal{T}$ by

$$\mu_{\mathcal{B}}(\gamma)(t) = \mathrm{Ad}(g_\gamma) \circ \gamma(t). \tag{1}$$

This also determines an action $\mu_{\mathcal{B}}$ of Γ_K on $\Phi(\mathcal{G},\mathcal{T})$, which stabilizes Δ.

Let $\Phi(\mathcal{G},\mathcal{T})^\vee$ be the dual root system of $\Phi(\mathcal{G},\mathcal{T})$, contained in the cocharacter lattice $X_*(\mathcal{T})$. The based root datum of $\mathcal{G}$ is

$$\big(X^*(\mathcal{T}), \Phi(\mathcal{G},\mathcal{T}), X_*(\mathcal{T}), \Phi(\mathcal{G},\mathcal{T})^\vee, \Delta \big).$$

Let $\mathcal{S}$ be a maximal K-split torus in $\mathcal{G}$. Then $Z_{\mathcal{G}}(\mathcal{S})$ is a minimal K-Levi subgroup of $\mathcal{G}$. By [Spr, Theorem 13.3.6.(i)] applied to $Z_{\mathcal{G}}(\mathcal{S})$, we can choose $\mathcal{T}$ so that it is defined over K and contains $\mathcal{S}$. Let

$$\Delta_0 := \{ \alpha \in \Delta : \mathcal{S} \subset \ker \alpha \}$$

be the set of simple roots of $(Z_{\mathcal{G}}(\mathcal{S}),\mathcal{T})$. Recall from [Spr, Lemma 15.3.1] that the root system $\Phi(\mathcal{G},\mathcal{S})$ is the image of $\Phi(\mathcal{G},\mathcal{T})$ in $X^*(\mathcal{S})$, without 0. The set of simple roots of $(\mathcal{G},\mathcal{S})$ can be identified with $(\Delta \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$.

Let $N_{\mathcal{G}}(\mathcal{S},\mathcal{T})$ be the intersection of the normalizers of $\mathcal{S}$ and $\mathcal{T}$ in $\mathcal{G}$. The Weyl group of $(\mathcal{G},\mathcal{S})$ can be expressed in various ways:

$$\begin{aligned} W(\mathcal{G},\mathcal{S}) &= N_{\mathcal{G}}(\mathcal{S})/Z_{\mathcal{G}}(\mathcal{S}) \cong N_{\mathcal{G}(K)}(\mathcal{S}(K))/Z_{\mathcal{G}(K)}(\mathcal{S}(K)) \\ &\cong N_{\mathcal{G}}(\mathcal{S},\mathcal{T})/N_{Z_{\mathcal{G}}(\mathcal{S})}(\mathcal{T}) = \big(N_{\mathcal{G}}(\mathcal{S},\mathcal{T})/\mathcal{T} \big) \big/ \big(N_{Z_{\mathcal{G}}(\mathcal{S})}(\mathcal{T})/\mathcal{T} \big) \\ &\cong \mathrm{Stab}_{W(\mathcal{G},\mathcal{T})}(\mathcal{S})/W(Z_{\mathcal{G}}(\mathcal{S}),\mathcal{T}). \end{aligned} \tag{2}$$

Let $\mathcal{P}_{\Delta_0} = Z_{\mathcal{G}}(\mathcal{S})\mathcal{B}$ the minimal parabolic K-subgroup of $\mathcal{G}$ associated to Δ_0. It is well-known [Spr, Theorem 15.4.6] that the following sets are canonically in bijection:

- $\mathcal{G}(K)$-conjugacy classes of parabolic K-subgroups of $\mathcal{G}$;
- standard (i.e. containing $\mathcal{P}_{\Delta_0}$) parabolic K-subgroups of $\mathcal{G}$;
- subsets of $(\Delta \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$;
- $\mu_{\mathcal{B}}(\Gamma_K)$-stable subsets of Δ containing Δ_0.

By [Spr, Lemma 15.4.5] every $\mu_{\mathcal{B}}(\Gamma_K)$-stable subset $I \subset \Delta$ containing Δ_0 gives rise to a standard Levi K-subgroup $\mathcal{L}_I$ of $\mathcal{G}$, namely the group generated by $Z_{\mathcal{G}}(\mathcal{S})$ and the root subgroups for roots in $\mathbb{Z}I \cap \Phi(\mathcal{G},\mathcal{T})$.

By definition conjugacy classes of Levi K-subgroups of $\mathcal{G}$ are in bijection with association classes of parabolic K-subgroups of $\mathcal{G}$. The next lemma gives the concrete classification of these classes. It has been known for considerable time, for instance from [Cas, Proposition 1.3.4]. For a complete proof and generalizations we refer to [Sol4, Lemma 1].

Lemma 1.1. (*a*) *Every Levi K-subgroup of $\mathcal{G}$ is $\mathcal{G}(K)$-conjugate to a standard Levi K-subgroup of $\mathcal{G}$.*
(*b*) *For two standard Levi K-subgroups $\mathcal{L}_I$ and $\mathcal{L}_J$ the following are equivalent:*
(i) $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(K)$-conjugate;
(ii) $(I \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$ and $(J \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$ are $W(\mathcal{G},\mathcal{S})$-associate.

Let $\mathcal{G}^\vee$ be the split reductive group with based root datum

$$\big(X_*(\mathcal{T}), \Phi(\mathcal{G},\mathcal{T})^\vee, X^*(\mathcal{T}), \Phi(\mathcal{G},\mathcal{T}), \Delta^\vee\big).$$

Then $G^\vee = \mathcal{G}^\vee(\mathbb{C})$ is the complex dual group of $\mathcal{G}$. Via the choice of a pinning, the action $\mu_{\mathcal{B}}$ of Γ_K on the root datum of $\mathcal{G}$, from (1), determines an action of Γ_K of $G^\vee$. That action stabilizes the torus $T^\vee = X^*(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{C}^\times$ and the Borel subgroup $B^\vee$ determined by $T^\vee$ and $\Delta^\vee$. The version of the Langlands dual group of $\mathcal{G}(K)$ based on Γ_K is ${}^L G := G^\vee \rtimes \Gamma_K$.

Every subset $I \subset \Delta$ corresponds to a unique subset $I^\vee \subset \Delta^\vee$, and as such gives rise to a standard parabolic subgroup $P_I^\vee \subset G^\vee$ and a standard Levi subgroup $L_I^\vee$. Following [Bor, AMS1], we define a L-parabolic subgroup $\tilde{P}^\vee$ of ${}^L G$ to be the normalizer of a parabolic subgroup $P^\vee \subset G^\vee$ for which the canonical map $N_{G^\vee \rtimes \Gamma_K}(P^\vee) \to \Gamma_K$ is surjective. As $\Gamma_K \subset \mathrm{Gal}(K_s/K)$ is totally disconnected, $(\tilde{P}^\vee)^\circ = P^\vee$.

Let $T_L^\vee \subset G^\vee$ be a torus such that $Z_{G^\vee \rtimes \Gamma_K}(T_L^\vee) \to \Gamma_K$ is surjective. Then we call $Z_{G^\vee \rtimes \Gamma_K}(T_L^\vee)$ a Levi L-subgroup of ${}^L G$. Notice that $Z_{G^\vee \rtimes \Gamma_K}(T_L^\vee)^\circ = Z_{G^\vee}(T_L^\vee)$ is a Levi subgroup of $G^\vee$.

Special cases include $P_I^\vee \rtimes \Gamma_K$ and $L_I^\vee \rtimes \Gamma_K$, where $P_I^\vee$ (resp. $L_I^\vee$) is a standard Levi subgroup of $G^\vee$ such that I is Γ_K-stable. We call these standard L-parabolic (resp. L-Levi) subgroups of ${}^L G$.

We say that a L-parabolic (resp. L-Levi) subgroup $\tilde{H}^\vee \subset {}^L G$ is $\mathcal{G}(K)$-relevant if the $G^\vee$-conjugacy class of $(\tilde{H}^\vee)^\circ \subset G^\vee$ corresponds to a conjugacy class of parabolic (resp. Levi) K-subgroups of $\mathcal{G}$. As observed in [Bor, §3], for Γ_K-stable $I \subset \Delta$:

(3) the groups $P_I^\vee \rtimes \Gamma_K$ and $L_I^\vee \rtimes \Gamma_K$ are $\mathcal{G}(K)$-relevant if and only if $\Delta_0 \subset I$.

Moreover the correspondence

$$\mathcal{P}_I \longleftrightarrow P_I^\vee \rtimes \Gamma_K \tag{4}$$

provides a bijection between the set of $\mathcal{G}(K)$-conjugacy classes of parabolic K-subgroups of $\mathcal{G}$ and the set of $G^\vee$-conjugacy classes of $\mathcal{G}(K)$-relevant L-parabolic subgroups of ${}^L G$ [Bor, §3]. Similarly, there is a bijective correspondence between the set of standard Levi K-subgroups of $\mathcal{G}$ and the set of standard $\mathcal{G}(K)$-relevant L-Levi subgroups of ${}^L G$:

$$\mathcal{L}_I \longleftrightarrow L_I^\vee \rtimes \Gamma_K. \tag{5}$$

The actions of Γ_K on $\Phi(\mathcal{G},\mathcal{T})$ and on $\Phi(\mathcal{G},\mathcal{T})^\vee = \Phi(G^\vee, T^\vee)$ induce Γ_K-actions on the associated Weyl groups. The Γ_K-equivariant isomorphism

$$W(\mathcal{G},\mathcal{T}) \cong W(G^\vee, T^\vee)$$

can be modified to a version for $\mathcal{S}$. Namely, it was shown in [ABPS, Proposition 3.1 and (43)] that there are canonical isomorphisms

$$W(\mathcal{G},\mathcal{S}) \to \mathrm{Stab}_{W(G^\vee,T^\vee)^{\Gamma_K}}(\mathbb{Z}\Delta_0^\vee)/W(L^\vee_{\Delta_0},T^\vee)^{\Gamma_K} \to N_{G^\vee}(L^\vee_{\Delta_0} \rtimes \Gamma_K)/L^\vee_{\Delta_0}. \tag{6}$$

As $W(G^\vee,T^\vee)$ acts naturally on $X_*(\mathcal{T}) = X^*(T^\vee)$, $\mathrm{Stab}_{W(G^\vee,T^\vee)^{\Gamma_K}}(\mathbb{Z}\Delta_0^\vee)$ acts on $X_*(\mathcal{T})/\mathbb{Z}\Delta_0^\vee$. This descends to a natural action of

$$\mathrm{Stab}_{W(G^\vee,T^\vee)^{\Gamma_K}}(\mathbb{Z}\Delta_0^\vee)/W(L^\vee_{\Delta_0},T^\vee)^{\Gamma_K} \quad \text{on} \quad X_*(\mathcal{T})/\mathbb{Z}\Delta_0^\vee,$$

which stabilizes the image of $\Phi(\mathcal{G},\mathcal{T})^\vee$ in $X_*(\mathcal{T})/\mathbb{Z}\Delta_0^\vee$. As observed in [SiZi, Proposition 2.5.4], the correspondences (4) and (5) are $W(\mathcal{G},\mathcal{S})$-equivariant, with respect to (6).

Lemma 1.2. (*a*) *Every $\mathcal{G}(K)$-relevant L-Levi subgroup of ${}^LG$ is $G^\vee$-conjugate to a $\mathcal{G}(K)$-relevant standard L-Levi subgroup of ${}^LG$.*

(*b*) *Let $I^\vee, J^\vee \subset \Delta^\vee$ be Γ_K-stable subsets containing $\Delta_0^\vee$. The two $\mathcal{G}(K)$-relevant standard L-Levi subgroups $L_I^\vee \rtimes \Gamma_K$ and $L_J^\vee \rtimes \Gamma_K$ are $G^\vee$-conjugate if and only if there exists a $w^\vee \in \mathrm{Stab}_{W(G^\vee,T^\vee)^{\Gamma_K}}(\mathbb{Z}\Delta_0^\vee)/W(L^\vee_{\Delta_0},T^\vee)^{\Gamma_K}$ with $w^\vee(I^\vee \setminus \Delta_0^\vee) = J^\vee \setminus \Delta_0^\vee$.*

Proof. (a) By [AMS1, Lemma 6.2] every L-Levi subgroup of ${}^LG$ is $G^\vee$-conjugate to a standard L-Levi subgroup. By definition $G^\vee$-conjugacy preserves $\mathcal{G}(K)$-relevance.
(b) Suppose that a $w^\vee$ with the indicated properties exists. Let $\tilde{w} \in N_{G^\vee}(T^\vee)$ be a lift of $w^\vee$. Then $\tilde{w}(L_I^\vee \rtimes \Gamma_K)\tilde{w}^{-1}$ contains $L^\vee_{\Delta_0} \rtimes \Gamma_K$ and the roots of

$$\big(\tilde{w}(L_I^\vee \rtimes \Gamma_K)\tilde{w}^{-1}\big)^\circ = \tilde{w}L_I^\vee\tilde{w}^{-1}$$

with respect to $T^\vee$ are

$$w(\Phi(L_I^\vee,T^\vee)) = w(\mathbb{Z}I^\vee \cap \Phi(G^\vee,T^\vee)) = \mathbb{Z}J^\vee \cap \Phi(G^\vee,T^\vee) = \Phi(L_J^\vee,T^\vee).$$

Hence $\tilde{w}(L_I^\vee \rtimes \Gamma_K)\tilde{w}^{-1} = L_J^\vee$.

Conversely, suppose that $g(L_I^\vee \rtimes \Gamma_K)g^{-1} = L_J^\vee \rtimes \Gamma_K$ for some $g \in G^\vee$. Then

$$L_J^\vee = (L_J^\vee \rtimes \Gamma_K)^\circ = \big(g(L_I^\vee \rtimes \Gamma_K)g^{-1}\big)^\circ = gL_I^\vee g^{-1}.$$

By the conjugacy of maximal tori in connected linear algebraic groups, there exists a $l_1 \in L_J^\vee$ such that $l_1 g \in N_{G^\vee}(T^\vee)$. Then

$$(l_1 g)L_I^\vee \rtimes \Gamma_K(l_1 g)^{-1} = l_1(L_J^\vee \rtimes \Gamma_K)l_1^{-1} = L_J^\vee \rtimes \Gamma_K.$$

Now $(l_1 g)(L_I^\vee \cap B^\vee)(l_1 g)^{-1}$ is a Borel subgroup of $L_J^\vee$ containing $T^\vee$. By the conjugacy of Borel subgroups and maximal tori in $L_J^\vee$, there exists $l_2 \in N_{L_J^\vee}(T^\vee)$ such that

$$l_2 l_1 g(L_I^\vee \cap B^\vee)g^{-1}l_1^{-1}l_2^{-1} = L_J^\vee \cap B^\vee.$$

Then conjugation by $l_2 l_1 g$ sends the set Δ_0 of simple roots for $L^\vee_{\Delta_0}$ to the set of simple roots for $L_J^\vee \cap B^\vee$. Hence

$$l_2 l_1 g(L^\vee_{\Delta_0} \rtimes \Gamma_K)g^{-1}l_1^{-1}l_2^{-1} \;\subset\; L_J^\vee \rtimes \Gamma_K$$

is a standard L-Levi subgroup of ${}^LG$. It is conjugate to $L^\vee_{\Delta_0} \rtimes \Gamma_K$, so $\mathcal{G}(K)$-relevant and minimal for that property. As $L^\vee_{\Delta_0} \rtimes \Gamma_K$ is the unique standard minimal $\mathcal{G}(K)$-relevant L-Levi subgroup of ${}^LG$, it must be normalized by $l_2 l_1 g$. Thus $l_2 l_1 g \in N_{G^\vee}(L^\vee_{\Delta_0} \rtimes \Gamma_K, T^\vee)$ sends $I^\vee$ to $J^\vee$. By (6) there exists a $w^\vee \in \mathrm{Stab}_{W(G^\vee,T^\vee)^{\Gamma_K}}(\mathbb{Z}\Delta_0^\vee)$ mapping to $l_2 l_1 g L^\vee_{\Delta_0}$, and then $w^\vee(I^\vee \setminus \Delta_0^\vee) = J^\vee \setminus \Delta_0^\vee$. $\square$

By (6) the set of orbits of $W(\mathcal{G},\mathcal{S})$ on $(\Delta\setminus\Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$ is canonically in bijection with the set of orbits of $\mathrm{Stab}_{W(G^\vee,T^\vee)^{\Gamma_K}}(\mathbb{Z}\Delta_0^\vee)/W(L^\vee_{\Delta_0},T^\vee)^{\Gamma_K}$ on $(\Delta^\vee\setminus\Delta_0^\vee)/\Gamma_K$. This and Lemmas 1.1 and 1.2 yield the version of (4) for Levi subgroups that we were after:

Corollary 1.3. *The assignment $\mathcal{L}_I\mapsto L_I^\vee\rtimes\Gamma_K$ from (5) provides a bijection between the set of $\mathcal{G}(K)$-conjugacy classes of Levi K-subgroups of $\mathcal{G}$ and the set of $G^\vee$-conjugacy classes of $\mathcal{G}(K)$-relevant L-Levi subgroups of ${}^LG$.*

2. Hecke algebras for Langlands parameters

From now on K is a non-archimedean local field with ring of integers $\mathfrak{o}_K$ and a uniformizer ϖ_K. Let $k=\mathfrak{o}_K/\varpi_K\mathfrak{o}_K$ be its residue field, of cardinality q_K. Let $\mathbf{W}_K\subset\mathrm{Gal}(K_s/K)$ be the Weil group of K and let Frob be a geometric Frobenius element. Let $\mathbf{I}_K\subset\mathbf{W}_K$ be the inertia subgroup, so that $\mathbf{W}_K/\mathbf{I}_K\cong\mathbb{Z}$ is generated by Frob.

We let $\mathcal{G}$ and its subgroups be as in Section 1. We write $G=\mathcal{G}(K)$ and similarly for other K-groups. Recall that a Langlands parameter for G is a homomorphism

$$\phi:\mathbf{W}_K\times\mathrm{SL}_2(\mathbb{C})\to{}^LG=G^\vee\rtimes\mathbf{W}_K,$$

with some extra requirements. In particular $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ has to be algebraic, $\phi(\mathbf{W}_K)$ must consist of semisimple elements and ϕ must respect the projections to $\mathbf{W}_K$.

We say that a L-parameter ϕ for G is

- discrete if there does not exist any proper L-Levi subgroup of ${}^LG$ containing the image of ϕ;
- bounded if $\phi(\mathrm{Frob})=(s,\mathrm{Frob})$ with s in a bounded subgroup of $G^\vee$;
- unramified if $\phi(w)=(1,w)$ for all $w\in\mathbf{I}_K$.

Let ${G^\vee}_{\mathrm{ad}},{G^\vee}_{\mathrm{der}}$ and ${G^\vee}_{\mathrm{sc}}$ be, respectively, the adjoint group of $G^\vee$, the derived group of $G^\vee$ and the simply connected cover of ${G^\vee}_{\mathrm{der}}$. Let $\mathcal{G}^*$ be the unique K-quasi-split inner form of $\mathcal{G}$. We consider $\mathcal{G}$ as an inner twist of $\mathcal{G}^*$, so it is (implicitly) endowed with a K_s-isomorphism $\psi:\mathcal{G}\to\mathcal{G}^*$. Then ψ defines an element $[\psi]$ of the Galois cohomology group $H^1(K,\mathcal{G}_{\mathrm{ad}})$. Via the Kottwitz isomorphism (see [Kot, Theorem 6.4] and [Tha, Theorem 2.1]), $[\psi]$ becomes a character $\zeta_{\mathcal{G}}$ of $Z({G^\vee}_{\mathrm{sc}})^{\mathbf{W}_K}$, which we use to label $\mathcal{G}$ as inner twist of $\mathcal{G}^*$. We also choose an extension ζ of $\zeta_{\mathcal{G}}$ to $Z({G^\vee}_{\mathrm{sc}})$.

Both ${G^\vee}_{\mathrm{ad}}$ and ${G^\vee}_{\mathrm{sc}}$ act on $G^\vee$ by conjugation. Since

$$Z_{G^\vee}(\mathrm{im}\ \phi)\cap Z(G^\vee)=Z(G^\vee)^{\mathbf{W}_K},$$

we can regard $Z_{G^\vee}(\mathrm{im}\ \phi)/Z(G^\vee)^{\mathbf{W}_K}$ as a subgroup of ${G^\vee}_{\mathrm{ad}}$. Let $Z^1_{{G^\vee}_{\mathrm{sc}}}(\mathrm{im}\ \phi)$ be its inverse image in ${G^\vee}_{\mathrm{sc}}$. It contains $Z({G^\vee}_{\mathrm{sc}})$ and consists of the $g\in{G^\vee}_{\mathrm{sc}}$ for which $g\phi g^{-1}$ equals ϕ times a 1-coboundary of $\mathbf{W}_K$ in $Z({G^\vee}_{\mathrm{der}})$ [AMS1, (84)]. Using the surjectivity of $Z({G^\vee}_{\mathrm{sc}})\to Z({G^\vee}_{\mathrm{der}})$ we obtain

$$Z^1_{{G^\vee}_{\mathrm{sc}}}(\mathrm{im}\ \phi)=Z_{{G^\vee}_{\mathrm{sc}}}(\mathrm{im}\ \phi)Z({G^\vee}_{\mathrm{sc}})\cong\frac{Z_{{G^\vee}_{\mathrm{sc}}}(\mathrm{im}\ \phi)\times Z({G^\vee}_{\mathrm{sc}})}{Z_{{G^\vee}_{\mathrm{sc}}}(\mathrm{im}\ \phi)\cap Z({G^\vee}_{\mathrm{sc}})}. \tag{7}$$

Here the group

$$\begin{aligned}Z({G^\vee}_{\mathrm{sc}})^+&=Z_{{G^\vee}_{\mathrm{sc}}}(\mathrm{im}\ \phi)\cap Z({G^\vee}_{\mathrm{sc}})\\&=\{z\in Z({G^\vee}_{\mathrm{sc}}):\mathbf{W}_K\text{ fixes the image of }z\text{ in }Z(G^\vee)\}\end{aligned}$$

is embedded in the numerator in (7) via $z \mapsto (z, z^{-1})$. An appropriate component group for ϕ is

$$\mathcal{S}_\phi := \pi_0\big(Z^1_{G^\vee{}_{\mathrm{sc}}}(\mathrm{im}\ \phi)\big).$$

An enhancement of ϕ is an irreducible representation ρ of $\mathcal{S}_\phi$. Via the canonical map $Z(G^\vee{}_{\mathrm{sc}}) \to Z(\mathcal{S}_\phi)$, ρ determines a character ζ_ρ of $Z(G^\vee{}_{\mathrm{sc}})$. We say that an enhanced L-parameter (ϕ, ρ) is relevant for G if $\zeta_\rho = \zeta$. From (7) we see that the set of G-relevant enhancements of ρ is naturally in bijection with the set of irreducible representations of

$$\pi_0\big(Z_{G^\vee{}_{\mathrm{sc}}}(\mathrm{im}\ \phi)\big) \text{ whose } Z(G^\vee{}_{\mathrm{sc}})^+\text{-character is } \zeta\big|_{Z(G^\vee{}_{\mathrm{sc}})^+}. \tag{8}$$

This shows in particular that if ζ' is another extension of $\zeta_{\mathcal{G}}$ to $Z(G^\vee{}_{\mathrm{sc}})$, which agrees with ζ on $Z(G^\vee{}_{\mathrm{sc}})^+$, then there is a canonical bijection between the set of G-relevant enhancements of ϕ with respect to ζ and the corresponding set with respect to ζ'.

Relevance can also be reformulated with G-relevance of ϕ in terms of Levi subgroups [HiSa, Lemma 9.1]. To be precise, in view of (3) there exists an enhancement ρ such that (ϕ, ρ) is G-relevant if and only if every L-Levi subgroup of ${}^L G$ containing the image of ϕ is G-relevant. The group $G^\vee$ acts naturally on the collection of G-relevant enhanced L-parameters, by

$$g \cdot (\phi, \rho) = (g \phi g^{-1}, \rho \circ \mathrm{Ad}(g)^{-1}).$$

We denote the set of $G^\vee$-equivalence classes of G-relevant (resp. enhanced) L-parameters by $\Phi(G)$, resp. $\Phi_e(G)$. A local Langlands correspondence for G (in its modern interpretation) should be a bijection between $\Phi_e(G)$ and the set of irreducible smooth G-representations, with several nice properties.

Let $H^1(\mathbf{W}_K, Z(G^\vee))$ be the first Galois cohomology group of $\mathbf{W}_K$ with values in $Z(G^\vee)$. It acts on $\Phi(G)$ by

$$(z\phi)(w, x) = z'(w)\phi(w, x) \qquad \phi \in \Phi(G), w \in \mathbf{W}_K, x \in \mathrm{SL}_2(\mathbb{C}), \tag{9}$$

where $z' : \mathbf{W}_K \to Z(G^\vee)$ represents $z \in H^1(\mathbf{W}_K, Z(G^\vee))$. This extends to an action of $H^1(\mathbf{W}_K, Z(G^\vee))$ on $\Phi_e(G)$, which does nothing to the enhancements.

A character of G is called weakly unramified if it is trivial on the kernel of the Kottwitz homomorphism, or equivalently is trivial on all parahoric subgroups of G. The group $X_{\mathrm{wr}}(G)$ of weakly unramified characters $G \to \mathbb{C}^\times$ is naturally isomorphic to an object coming from ${}^L G$:

$$X_{\mathrm{wr}}(G) \cong \big(Z(G^\vee)^{\mathbf{I}_K}\big)_{\mathrm{Frob}} \quad \subset H^1(\mathbf{W}_K, Z(G^\vee)), \tag{10}$$

see [Hai, §3.3.1]. The identity component of $X_{\mathrm{wr}}(G)$ is the group $X_{\mathrm{nr}}(G)$ of unramified characters $G \to \mathbb{C}^\times$. Via (10) and (9), $X_{\mathrm{wr}}(G)$ acts naturally on $\Phi_e(G)$, while it acts on $\mathrm{Rep}(G)$ by tensoring.

Let us focus on cuspidality for enhanced L-parameters [AMS1, §6]. Consider

$$G^\vee_\phi := Z^1_{G^\vee{}_{\mathrm{sc}}}(\phi|_{\mathbf{W}_K}),$$

a possibly disconnected complex reductive group. Then $u_\phi := \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)$ is a unipotent element of $(G^\vee_\phi)^\circ$ and $\mathcal{S}_\phi \cong \pi_0(Z_{G^\vee_\phi}(u_\phi))$. We say that $(\phi, \rho) \in \Phi_e(G)$ is cuspidal if ϕ is discrete and (u_ϕ, ρ) is a cuspidal pair for $G^\vee_\phi$. The latter means that (u_ϕ, ρ) determines a $G^\vee_\phi$-equivariant cuspidal local system on the $G^\vee_\phi$-conjugacy class of u_ϕ. Notice that a L-parameter alone does not contain enough information to detect cuspidality, for that we really need an enhancement. Therefore we will often say "cuspidal L-parameter" for an enhanced L-parameter which is cuspidal.

The set of $G^\vee$-equivalence classes of G-relevant cuspidal L-parameters is denoted $\Phi_{\mathrm{cusp}}(G)$. It is conjectured that under the LLC $\Phi_{\mathrm{cusp}}(G)$ corresponds to the set of supercuspidal irreducible smooth G-representations.

The cuspidal support of any $(\phi,\rho) \in \Phi_e(G)$ is defined in [AMS1, §7]. It is unique up to $G^\vee$-conjugacy and consists of a G-relevant L-Levi subgroup ${}^L L$ of ${}^L G$ and a cuspidal L-parameter $(\phi_v, q\epsilon)$ for ${}^L L$. By Corollary 1.3 this ${}^L L$ corresponds to a unique (up to G-conjugation) Levi K-subgroup $\mathcal{L}$ of $\mathcal{G}$. This allows us to express the aforementioned cuspidal support map as

$$\mathbf{Sc}(\phi,\rho) = (\mathcal{L}(K), \phi_v, q\epsilon), \quad \text{where } (\phi_v, q\epsilon) \in \Phi_{\mathrm{cusp}}(\mathcal{L}(K)). \tag{11}$$

It is conjectured that under the LLC this map should correspond to Bernstein's cuspidal support map for irreducible smooth G-representations.

Sometimes we will be a little sloppy and write that $L = \mathcal{L}(K)$ is a Levi subgroup of G. Via (9) the group of unramified characters $X_{\mathrm{nr}}(L)$ acts on $\Phi_e(L)$ and on $\Phi_{\mathrm{cusp}}(L)$. A cuspidal Bernstein component of $\Phi_e(L)$ is a set of the form

$$\Phi_e(L)^{\mathfrak{s}_L^\vee} := X_{\mathrm{nr}}(L) \cdot (\phi_L, \rho_L) \text{ for some } (\phi_L, \rho_L) \in \Phi_{\mathrm{cusp}}(L).$$

The group $G^\vee$ acts on the set of cuspidal Bernstein components for arbitrary Levi subgroups of G. The $G^\vee$-action is just by conjugation, but to formulate it precisely, more general L-Levi subgroups of ${}^L G$ are necessary. We prefer to keep those out of the notations, since we do not need them to get all classes up to equivalence. With that convention, we can define an inertial equivalence class for $\Phi_e(G)$ as

$$\mathfrak{s} \text{ is the } G^\vee\text{-orbit of } (L, X_{\mathrm{nr}}(L) \cdot (\phi_L, \rho_L)), \text{ where } (\phi_L, \rho_L) \in \Phi_{\mathrm{cusp}}(L).$$

The underlying inertial equivalence class for $\Phi_e(L)$ is $\mathfrak{s}_L^\vee = (L, X_{\mathrm{nr}}(L) \cdot (\phi_L, \rho_L))$. Here it is not necessary to take the $L^\vee$-orbit, for $(\phi_L, \rho_L) \in \Phi_e(L)$ is fixed by $L^\vee$-conjugation.

We denote the set of inertial equivalence classes for $\Phi_e(G)$ by $\mathfrak{Be}^\vee(G)$. Every $\mathfrak{s}^\vee \in \mathfrak{Be}^\vee(G)$ gives rise to a Bernstein component in $\Phi^e(G)$ [AMS1, §8], namely

$$\Phi_e(G)^{\mathfrak{s}^\vee} = \{(\phi,\rho) \in \Phi_e(G) : \mathbf{Sc}(\phi,\rho) \in \mathfrak{s}^\vee\}. \tag{12}$$

The set of such Bernstein components is also parametrized by $\mathfrak{Be}^\vee(G)$, and forms a partition of $\Phi_e(G)$.

Notice that $\Phi_e(L)^{\mathfrak{s}_L^\vee} \cong \mathfrak{s}_L^\vee$ has a canonical topology, coming from the transitive action of $X_{\mathrm{nr}}(L)$. More precisely, let $X_{\mathrm{nr}}(L, \phi_L)$ be the stabilizer in $X_{\mathrm{nr}}(L)$ of ϕ_L. Then the complex torus

$$T_{\mathfrak{s}_L^\vee} := X_{\mathrm{nr}}(L) / X_{\mathrm{nr}}(L, \phi_L)$$

acts simply transitively on $\mathfrak{s}_L^\vee$. This endows $\mathfrak{s}_L^\vee$ with the structure of an affine variety. (There is no canonical group structure on $\mathfrak{s}_L^\vee$ though, for that one still needs to choose a basepoint.)

To $\mathfrak{s}^\vee$ we associate a finite group $W_{\mathfrak{s}^\vee}$, in many cases a Weyl group. For that, we choose $\mathfrak{s}_L^\vee = (L, X_{\mathrm{nr}}(L) \cdot (\phi_L, \rho_L))$ representing $\mathfrak{s}^\vee$ (up to isomorphism, the below does not depend on this choice). We define $W_{\mathfrak{s}^\vee}$ as the stabilizer of $\mathfrak{s}_L^\vee$ in $N_{G^\vee}(L^\vee \rtimes \mathbf{W}_K)/L^\vee$. In this setting we write $T_{\mathfrak{s}^\vee}$ for $T_{\mathfrak{s}_L^\vee}$. Thus $W_{\mathfrak{s}^\vee}$ acts on $\mathfrak{s}_L^\vee$ by algebraic automorphisms and on $T_{\mathfrak{s}^\vee}$ by group automorphisms (but the bijection $T_{\mathfrak{s}^\vee} \to \mathfrak{s}_L^\vee$ need not be $W_{\mathfrak{s}^\vee}$-equivariant).

Next we quickly review the construction of an affine Hecke algebra from a Bernstein component of enhanced Langlands parameters. We fix a basepoint ϕ_L for $\mathfrak{s}_L^\vee$ as in [AMS3, Proposition 3.9.b], and use that to identify $\mathfrak{s}_L^\vee$ with $T_{\mathfrak{s}_L^\vee}$. Consider the possibly disconnected reductive group

$$G^\vee_{\phi_L} = Z^1_{G^\vee{}_{\mathrm{sc}}}(\phi_L|\mathbf{W}_K).$$

Let $L_c^\vee$ be the Levi subgroup of $G^\vee{}_{\mathrm{sc}}$ determined by $L^\vee$. There is a natural homomorphism

$$Z(L_c^\vee)^{\mathbf{W}_K,\circ} \to X_{\mathrm{nr}}(L) \to T_{\mathfrak{s}_L^\vee} \tag{13}$$

with finite kernel [AMS3, Lemma 3.7]. Using that and [AMS3, Lemma 3.12], $\Phi(G^\circ_{\phi_L}, Z(L_c^\vee)^{\mathbf{W}_K,\circ})$ gives rise to a reduced root system $\Phi_{\mathfrak{s}^\vee}$ in $X^*(T_{\mathfrak{s}^\vee})$. The coroot system $\Phi^\vee_{\mathfrak{s}^\vee}$ is contained in $X_*(T_{\mathfrak{s}^\vee})$. That gives a root datum $\mathcal{R}_{\mathfrak{s}^\vee}$, whose basis can still be chosen arbitrarily.

The construction of label functions λ and λ^* for $\mathcal{R}_{\mathfrak{s}^\vee}$ consists of several steps. The numbers $\lambda(\alpha), \lambda^*(\alpha) \in \mathbb{Z}_{\geq 0}$ will be defined for all roots $\alpha \in \Phi_{\mathfrak{s}^\vee}$. First, we pick $t \in (Z(L_c^\vee)^{\mathbf{I}_K})^\circ_{\mathrm{Frob}}$ such that the reflection s_α fixes $t\phi_L(\mathrm{Frob})$. Then $q\alpha$ lies in $\Phi\big((G^\vee_{t\phi_L})^\circ, Z(L_c^\vee)^{\mathbf{W}_K,\circ}\big)$ for a well-chosen $q \in \mathbb{Q}_{>0}$. In [AMS3, Proposition 3.14], $\lambda(\alpha)$ and $\lambda^*(\alpha)$ are deduced from the labels $c(q\alpha), c(2q\alpha)$ for a graded Hecke algebra [AMS3, §1] associated to

$$(G^\vee_{t\phi_L})^\circ = Z_{G^\vee{}_{\mathrm{sc}}}(t\phi_L(\mathbf{W}_K))^\circ, Z(L_c^\vee)^{\mathbf{W}_K,\circ}, u_{\phi_L} \text{ and } \rho_L. \tag{14}$$

We need only few $t \in (Z(L_c^\vee)^{\mathbf{I}_K})^\circ_{\mathrm{Frob}}$ we need to determine all labels: for each $\alpha \in \Phi_{\mathfrak{s}^\vee}$ just one with $\alpha(t) = 1$, and sometimes one with $\alpha(t) = -1$. The aforementioned integers $c(q\alpha), c(2q\alpha)$ were defined in [Lus2, Propositions 2.8, 2.10 and 2.12], in terms of the adjoint action of $\log(u_{\phi_L})$ on

$$\mathrm{Lie}(G^\vee_{t\phi_L})^\circ = \mathrm{Lie}\big(Z_{G^\vee{}_{\mathrm{sc}}}(t\phi_L(\mathbf{W}_K))\big).$$

Finally, we choose an array $\vec{v}$ of nonzero complex numbers, one v_j for every irreducible component of $\Phi_{\mathfrak{s}^\vee}$. To these data one can attach an affine Hecke algebra $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{v})$, as in [AMS3, §3.3].

The group $W_{\mathfrak{s}^\vee}$ acts on $\Phi_{\mathfrak{s}^\vee}$ and contains the Weyl group $W^\circ_{\mathfrak{s}^\vee}$ of that root system. It admits a semidirect factorization

$$W_{\mathfrak{s}^\vee} = W^\circ_{\mathfrak{s}^\vee} \rtimes \mathfrak{R}_{\mathfrak{s}^\vee},$$

where $\mathfrak{R}_{\mathfrak{s}^\vee}$ is the stabilizer of a chosen basis of $\Phi_{\mathfrak{s}^\vee}$.

Using the above identification of $T_{\mathfrak{s}^\vee}$ with $\mathfrak{s}_L^\vee$, we can reinterpret $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{v})$ as an algebra $\mathcal{H}(\mathfrak{s}_L^\vee, W^\circ_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{v})$ whose underlying vector space is $\mathcal{O}(\mathfrak{s}_L^\vee) \otimes \mathbb{C}[W^\circ_{\mathfrak{s}^\vee}]$. The group $\mathfrak{R}_{\mathfrak{s}^\vee}$ acts naturally on the based root datum $\mathcal{R}_{\mathfrak{s}^\vee}$, and hence on $\mathcal{H}(\mathfrak{s}_L^\vee, W^\circ_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{v})$ by algebra automorphisms [AMS3, Proposition 3.15.a]. From [AMS3, Proposition 3.15.b] we get a 2-cocycle $\natural : \mathfrak{R}^2_{\mathfrak{s}^\vee} \to \mathbb{C}^\times$ and a twisted group algebra $\mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \natural]$. Now we can define the twisted affine Hecke algebra

$$\mathcal{H}(\mathfrak{s}^\vee, \vec{v}) := \mathcal{H}(\mathfrak{s}_L^\vee, W^\circ_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{v}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \natural]. \tag{15}$$

Up to isomorphism it depends only on $\mathfrak{s}^\vee$ and $\vec{v}$ [AMS3, Lemma 3.16].

The multiplication relations in $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ are based on the Bernstein presentation of affine Hecke algebras, let us make them explicit. The vector space $\mathbb{C}[W^\circ_{\mathfrak{s}^\vee}] \subset \mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ is the Iwahori–Hecke algebra $\mathcal{H}(W^\circ_{\mathfrak{s}^\vee}, \vec{v}^{\,2\lambda})$, where $\vec{v}^{\,\lambda}(\alpha) = v_j^{\lambda(\alpha)}$ for the entry v_j of

$\vec{v}$ specified by α. The conjugation action of $\mathfrak{R}_{\mathfrak{s}^\vee}$ on $W^\circ_{\mathfrak{s}^\vee}$ induces an action on $\mathcal{H}(W^\circ_{\mathfrak{s}^\vee}, \vec{v}^{2\lambda})$.

The vector space $\mathcal{O}(\mathfrak{s}^\vee_L)$ is embedded in $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ as a maximal commutative subalgebra. The multiplication map

$$\mathcal{O}(\mathfrak{s}^\vee_L) \otimes_{\mathbb{C}} \big(\mathcal{H}(W^\circ_{\mathfrak{s}^\vee}, \vec{v}^{2\lambda}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \natural] \big) \longrightarrow \mathcal{H}(\mathfrak{s}^\vee, \vec{v}) \tag{16}$$

is a linear bijection. The group $W_{\mathfrak{s}^\vee}$ acts on $\mathcal{O}(\mathfrak{s}^\vee_L)$ via its action of $\mathfrak{s}^\vee_L$, and every root $\alpha \in \Phi_{\mathfrak{s}^\vee} \subset X^*(T_{\mathfrak{s}^\vee})$ determines an element $\theta_\alpha \in \mathcal{O}(\mathfrak{s}^\vee_L)^\times$, which does not depend on the choice of the basepoint ϕ_L of $\mathfrak{s}^\vee_L$ by [AMS3, Proposition 3.9]. For $f \in \mathcal{O}(\mathfrak{s}^\vee_L)$ and a simple reflection $s_\alpha \in W^\circ_{\mathfrak{s}^\vee}$ the following version of the Bernstein–Lusztig–Zelevinsky relations holds:

$$f N_{s_\alpha} - N_{s_\alpha} f = \big((v_j^{\lambda(\alpha)} - v_j^{-\lambda(\alpha)}) + \theta_{-\alpha} (v_j^{\lambda^*(\alpha)} - v_j^{-\lambda^*(\alpha)}) \big) (f - s_\alpha \cdot f)/(1 - \theta_{-\alpha}^2). \tag{17}$$

Thus $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ depends on the following objects: $\mathfrak{s}^\vee_L, W_{\mathfrak{s}^\vee}$ and the simple reflections therein, the label functions λ, λ^*, the parameters $\vec{v}$ and the functions $\theta_\alpha : \mathfrak{s}^\vee_L \to \mathbb{C}^\times$ for reduced roots $\alpha \in \Phi_{\mathfrak{s}^\vee}$. When $W_{\mathfrak{s}^\vee} \neq W^\circ_{\mathfrak{s}^\vee}$, we also need the 2-cocycle $\natural$ on $\mathfrak{R}_{\mathfrak{s}^\vee}$.

As in [Lus3, §3], the above relations entail that the centre of $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ is $\mathcal{O}(\mathfrak{s}^\vee_L)^{W_{\mathfrak{s}^\vee}}$. In other words, the space of central characters for $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$-representations is $\mathfrak{s}^\vee_L / W_{\mathfrak{s}^\vee}$.

We note that when $\mathfrak{s}^\vee$ is cuspidal,

$$\mathcal{H}(\mathfrak{s}^\vee, \vec{v}) = \mathcal{O}(\mathfrak{s}^\vee) \tag{18}$$

and every element of $\mathfrak{s}^\vee$ determines a character of $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$.

The main reason for introducing $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ is the next result. (See [AMS3, Definition 2.6] for the definition of tempered and essentially discrete series representations.)

Theorem 2.1. [AMS3, Theorem 3.18]
Let $\mathfrak{s}^\vee$ be an inertial equivalence class for $\Phi_e(G)$ and assume that the parameters $\vec{v}$ lie in $\mathbb{R}_{>1}$. Then there exists a canonical bijection

$$\begin{array}{ccc} \Phi_e(G)^{\mathfrak{s}^\vee} & \to & \mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, \vec{v})) \\ (\phi, \rho) & \mapsto & \bar{M}(\phi, \rho, \vec{v}) \end{array}$$

with the following properties.

- *$\bar{M}(\phi, \rho, \vec{v})$ is tempered if and only if ϕ is bounded.*
- *ϕ is discrete if and only if $\bar{M}(\phi, \rho, \vec{v})$ is essentially discrete series and the rank of $\Phi_{\mathfrak{s}^\vee}$ equals $\dim_{\mathbb{C}}(T_{\mathfrak{s}^\vee}/X_{\mathrm{nr}}(G))$.*
- *The central character of $\bar{M}(\phi, \rho, \vec{v})$ is the product of $\phi(\mathrm{Frob})$ and a term depending only on $\vec{v}$ and a cocharacter associated to u_ϕ.*
- *Suppose that $\mathbf{Sc}(\phi, \rho) = (L, \chi_L \phi_L, \rho_L)$, where $\chi_L \in X_{\mathrm{nr}}(L)$. Then $\bar{M}(\phi, \rho, \vec{v})$ is a constituent of $\mathrm{ind}_{\mathcal{H}(\mathfrak{s}^\vee_L, \vec{v})}^{\mathcal{H}(\mathfrak{s}^\vee, \vec{v})} (L, \chi_L \phi_L, \rho_L)$.*

The irreducible module $\overline{M}(\phi, \rho, \vec{v})$ in Theorem 2.1 is a quotient of a "standard module" $\overline{E}(\phi, \rho, \vec{v})$, also studied in [AMS3, Theorem 3.18]. By [AMS3, Lemma 3.19.a] such a standard module is (under a weak additional condition) a direct summand of a module obtained by induction from a standard module associated to a discrete enhanced L-parameter for a Levi subgroup of G.

The action of $H^1(\mathbf{W}_K, Z(G^\vee))$ on $\Phi_e(G)$ commutes with that of its subgroup $X_{\mathrm{nr}}(G)$, so it induces an action on $\mathfrak{Be}^\vee(G)$. For $z \in H^1(\mathbf{W}_K, Z(G^\vee))$ we write that as $\mathfrak{s}^\vee \mapsto z\mathfrak{s}^\vee$. Since $z\phi_L$ differs from ϕ_L only by central elements (of $\mathcal{G}^\vee$), important data used to construct $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ are the same for $z\mathfrak{s}^\vee$:

$$T_{z\mathfrak{s}^\vee} = T_{\mathfrak{s}^\vee},\ W_{z\mathfrak{s}^\vee} = W_{\mathfrak{s}^\vee} \text{ and } \Phi_{z\mathfrak{s}^\vee} = \Phi_{\mathfrak{s}^\vee}.$$

Furthermore the objects $\lambda, \lambda^*, \natural$ for $\mathfrak{s}^\vee$ and $z\mathfrak{s}^\vee$ can be identified, and the action of z gives a bijection $\mathfrak{s}_L^\vee \to z\mathfrak{s}_L^\vee$. Thus z canonically determines an algebra isomorphism

$$\begin{array}{llcll} \mathcal{H}(z): & \mathcal{H}(\mathfrak{s}^\vee, \vec{v}) & \to & \mathcal{H}(z\mathfrak{s}^\vee, \vec{v}) & \\ & fN_w & \mapsto & (f \circ z^{-1})N_w & \qquad f \in \mathcal{O}(\mathfrak{s}_L^\vee), w \in W_{\mathfrak{s}^\vee}. \end{array} \tag{19}$$

This defines a group action of $H^1(\mathbf{W}_K, Z(G^\vee))$ on the algebra $\bigoplus_{\mathfrak{s}^\vee \in \mathfrak{S}^\vee} \mathcal{H}(\mathfrak{s}^\vee, \vec{v})$, where $\mathfrak{S}^\vee$ is a union of $H^1(\mathbf{W}_K, Z(G^\vee))$-orbits in $\mathfrak{Be}^\vee(G)$.

Composition with $\mathcal{H}(z)^{-1}$ gives a functor between module categories:

$$z\otimes : \mathrm{Mod}(\mathcal{H}(\mathfrak{s}^\vee, \vec{v})) \to \mathrm{Mod}(\mathcal{H}(z\mathfrak{s}^\vee, \vec{v})).$$

Lemma 2.2. (*a*) *The bijections from Theorem 2.1 are $H^1(\mathbf{W}_K, Z(G^\vee))$-equivariant:*

$$\bar{M}(z\phi, \rho, \vec{v}) = z \otimes \bar{M}(\phi, \rho, \vec{v}) \qquad (\phi, \rho) \in \Phi_e(G)^{\mathfrak{s}^\vee}, z \in H^1(\mathbf{W}_K, Z(G^\vee)).$$

(*b*) *The same holds for the standard modules from* [AMS3, Theorem 3.18]*:*

$$\bar{E}(z\phi, \rho, \vec{v}) = z \otimes \bar{E}(\phi, \rho, \vec{v}) \qquad (\phi, \rho) \in \Phi_e(G)^{\mathfrak{s}^\vee}, z \in H^1(\mathbf{W}_K, Z(G^\vee)).$$

(*c*) *Suppose that ϕ is bounded and that $z \in H^1(\mathbf{W}_K, Z(G^\vee))$. Then*

$$\bar{M}(z\phi, \rho, \vec{v}) = \bar{E}(z\phi, \rho, \vec{v}).$$

Proof. (a) For $z \in X_{\mathrm{nr}}(G) \cong (Z(G^\vee)^{\mathbf{I}_K})^\circ_{\mathbf{W}_K}$ this was shown in [AMS3, Theorem 3.18.e]. For general z, Theorem 2.1 and the definition of $\mathcal{H}(z)^{-1}$ show that $\bar{M}(z\phi, \rho, \vec{v})$ and $z \otimes \bar{M}(\phi, \rho, \vec{v})$ have the same central character (an element of $z\mathfrak{s}_L^\vee / W_{\mathfrak{s}^\vee}$). Then the complete analogy between the construction of $\bar{M}(z\phi, \rho, \vec{v})$ and of $\bar{M}(\phi, \rho, \vec{v})$ in [AMS3] entails that $\bar{M}(z\phi, \rho, \vec{v}) = z \otimes \bar{M}(\phi, \rho, \vec{v})$.
(b) This can be shown in the same way as (a).
(c) For $z = 1$ this is [AMS3, Theorem 3.18.f]. Apply parts (a) and (b) to that. $\square$

Let us investigate the compatibility of Theorem 2.1 with suitable versions of the Langlands classification. The Langlands classification for (extended) affine Hecke algebras [Sol1, Corollary 2.2.5] says, roughly, that every irreducible module of $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ can be obtained from an irreducible tempered module of a parabolic subalgebra, by first twisting with a strictly positive character, then parabolic induction and subsequently taking the unique irreducible quotient.

Let $\phi \in \Phi(G)$ be arbitrary. The Langlands classification for L-parameters [SiZi, Theorem 4.6] says that there exists a parabolic subgroup P of G with Levi factor M, such that $\mathrm{im}(\phi) \subset {}^L M$ and ϕ can be written as $z\phi_b$ with $\phi_b \in \Phi(M)$ bounded and $z \in X_{\mathrm{nr}}(M)$ strictly positive with respect to P. Furthermore P is unique up to G-conjugation, and this provides a bijection between L-parameters for G and such triples (P, ϕ_b, z) considered up to G-conjugacy.

Let ζ be the character of $Z(G^\vee{}_{\mathrm{sc}})$ determined by ρ, an extension of the character $\zeta_{\mathcal{G}} \in \mathrm{Irr}(Z(G^\vee{}_{\mathrm{sc}})^{\mathbf{W}_F})$ associated to $\mathcal{G}(F)$ by Kottwitz. Let $\zeta^{\mathcal{M}} \in \mathrm{Irr}(Z(M^\vee{}_{\mathrm{sc}}))$ be derived from ζ as in [AMS1, Lemma 7.4]. Let $p_\zeta \in \mathbb{C}[\mathcal{S}_\phi]$ and $p_{\zeta^{\mathcal{M}}} \in \mathbb{C}[\mathcal{S}_\phi^{\mathcal{M}}]$ be the central idempotents associated to these characters. By [AMS1, Theorem 7.10.b] there are natural isomorphisms

$$p_{\zeta^{\mathcal{M}}}\mathbb{C}[\mathcal{S}^{\mathcal{M}}_{\phi_b}] = p_{\zeta^{\mathcal{M}}}\mathbb{C}[\mathcal{S}^{\mathcal{M}}_{z\phi_b}] \to p_\zeta \mathbb{C}[\mathcal{S}_\phi]. \tag{20}$$

Hence ϕ and ϕ_b admit the same relevant enhancements.

Proposition 2.3. *Let $(\phi,\rho) \in \Phi_e(G)$, let P, M be as above and assume that the parameters $\vec{v}$ lie in $\mathbb{R}_{>1}$.*

(a) $\bar{E}(\phi,\rho,\vec{v}) \cong \mathcal{H}(\mathfrak{s}^\vee,\vec{v}) \underset{\mathcal{H}(\mathfrak{s}_M^\vee,\vec{v})}{\otimes} \bar{E}^{\mathcal{M}}(\phi,\rho,\vec{v})$.

(b) $\bar{M}(\phi,\rho,\vec{v})$ is the unique irreducible quotient of $\mathcal{H}(\mathfrak{s}^\vee,\vec{v}) \underset{\mathcal{H}(\mathfrak{s}_M^\vee,\vec{v})}{\otimes} \bar{M}^{\mathcal{M}}(\phi,\rho,\vec{v}) \cong \bar{E}(\phi,\rho,\vec{v})$.

(c) The $\mathcal{H}(\mathfrak{s}_M^\vee,\vec{v})$-module $\bar{M}^{\mathcal{M}}(\phi,\rho,\vec{v})$ is a twist of a tempered module by a character which is strictly positive with respect to P.

Proof. (a) In view of (20), the statement is an instance of [AMS3, Lemma 3.19.a]. But to apply that lemma we need to check its condition

$$\epsilon_{u_\phi,j}(z,\vec{v}) \neq 0, \tag{21}$$

where $\phi = z\phi_b$ with $z \in X_{\mathrm{nr}}(M) \cong \big(Z(M^\vee)^{\mathbf{I}_K}\big)^\circ_{\mathrm{Frob}}$ as above. Write $r_j = \log(v_j) \in \mathbb{R}_{>0}$. Via the definitions in [AMS3, pages 29 and 12] and using the notations from [AMS3, §1], (21) boils down to

$$\det\Big(\mathrm{ad}\big(\mathrm{d}\phi\left(\begin{smallmatrix} r_j & 0 \\ 0 & -r_j\end{smallmatrix}\right)-\log(z)\big)-2r_j\Big) \neq 0. \tag{22}$$

Here we take the determinant of an endomorphism of a vector space defined in terms of $P, M, \log(u_\phi)$ and a semisimple factor $G^\circ_{\phi_b,j}$ of G_{ϕ_b}. This brings us to the setting of modules for a graded Hecke algebra $\mathbb{H}^{\mathcal{M}}(r_j)$ with one parameter $r_j > 0$, associated a Levi subgroup of the complex group $G_{\phi_b,j}$. Using that $-\log(v_j)$ is strictly negative with respect to the parabolic subgroup of $G_{\phi_b,j}$ determined by P, it is not hard to verify (22) – this was done in [AMS2, Lemma A.2].
(b) By part (a) and Lemma 2.2.c

$$\bar{E}(\phi,\rho,\vec{v}) \cong \mathcal{H}(\mathfrak{s}^\vee,\vec{v}) \underset{\mathcal{H}(\mathfrak{s}_M^\vee,\vec{v})}{\otimes} \bar{E}^{\mathcal{M}}(\phi,\rho,\vec{v}) = \bar{E}(\phi,\rho,\vec{v}) \cong \mathcal{H}(\mathfrak{s}^\vee,\vec{v}) \underset{\mathcal{H}(\mathfrak{s}_M^\vee,\vec{v})}{\otimes} \bar{M}^{\mathcal{M}}(\phi,\rho,\vec{v}).$$

Via the construction of $\bar{E}(\phi,\rho,\vec{v})$ in [AMS3, §3] and [AMS3, Lemma 1.3 and Theorem 1.4], we can reduce the statement about quotients to modules over a graded Hecke algebra $\mathbb{H}(r_j)$ with one parameter r_j, associated to the complex group $G_{\phi_b,j}$. Since $r_j = \log(v_j) \neq 0$, [AMS2, Theorem 3.20.a] applies, and says that $\bar{E}(\phi,\rho,\vec{v})$ has a unique irreducible quotient, namely $\bar{M}(\phi,\rho,\vec{v})$.
(c) By Lemma 2.2.a

$$\bar{M}^{\mathcal{M}}(\phi,\rho,\vec{v}) = z \otimes \bar{M}^{\mathcal{M}}(\phi,\rho,\vec{v}),$$

and by Theorem 2.1 $\bar{M}^{\mathcal{M}}(\phi,\rho,\vec{v})$ is tempered. From (19) we see that $z\otimes$ is a twist by a character which on P-positive elements of $X^*(T_{\mathfrak{s}_L^\vee})$ takes values in $\mathbb{R}_{>1}$. Here P-positive refers to those elements of $X^*(T_{\mathfrak{s}_L^\vee})$ which lie in the interior of the positive cone associated to the root system $\Phi^{\mathcal{M}}_{\mathfrak{s}^\vee}$ (constructed similarly as $\Phi_{\mathfrak{s}^\vee}$) with the simple roots determined by P. $\square$

Suppose that K'/K is a finite extension inside the separable closure K_s. Suppose also that $\mathcal{G}'$ is a connected reductive K'-group and that $\mathcal{G} = \mathrm{Res}_{K'/K}(\mathcal{G}')$, the Weil restriction of $\mathcal{G}'$. Then $\mathcal{G}(K) = \mathcal{G}'(K')$ and

$$\mathcal{G}^\vee = \mathrm{ind}_{\mathbf{W}_{K'}}^{\mathbf{W}_K}\big(\mathcal{G}'^\vee\big) \quad \text{as } \mathbf{W}_K\text{-modules}. \tag{23}$$

According to [Bor, Proposition 8.4], Shapiro's lemma yields a natural bijection between L-parameters for the K-group $\mathcal{G}$ and for the K'-group $\mathcal{G}'$. By [FOS1, Lemma A.3] it extends naturally to a bijection

$$\Phi_e(\mathcal{G}(K)) \to \Phi_e(\mathcal{G}'(K')), \tag{24}$$

which preserves cuspidality. For any Levi K-subgroup $\mathcal{L}$ of $\mathcal{G}$ there is a unique Levi K'-subgroup $\mathcal{L}'$ of $\mathcal{G}'$ with $\mathcal{L} = \mathrm{Res}_{K'/K}\mathcal{L}'$. By [ABPS, Proposition 3.1] there are natural isomorphisms

$$N_{G'^\vee}(L'^\vee \rtimes \mathbf{W}_{K'})/L'^\vee \cong W(\mathcal{G}',\mathcal{L}') \cong W(\mathcal{G},\mathcal{L}) \cong N_{G^\vee}(L^\vee \rtimes \mathbf{W}_K)/L^\vee. \tag{25}$$

Applying (24) to all Levi subgroups of $\mathcal{G}(K)$ and invoking (25), we obtain a bijection

$$\mathfrak{Be}^\vee(\mathcal{G}(K)) \to \mathfrak{Be}^\vee(\mathcal{G}'(K')) : \mathfrak{s}^\vee \mapsto \mathfrak{s}'^\vee. \tag{26}$$

Let $f_{K'/K}$ be the residue degree of K'/K and denote the array with entries $v_j^{f_{K'/K}}$ by $\vec{v}^{f_{K'/K}}$.

Lemma 2.4. *The algebra $\mathcal{H}(\mathfrak{s}^\vee,\vec{v})$ from* (15) *is invariant under Weil restriction of reductive groups. More precisely, in the above situation there is a natural isomorphism $\mathcal{H}(\mathfrak{s}^\vee,\vec{v}) \cong \mathcal{H}(\mathfrak{s}'^\vee,\vec{v}^{f_{K'/K}})$.*

Proof. Let $(t\phi_L,\rho_L) \in \Phi_{\mathrm{cusp}}(\mathcal{L}(K))$ be as in (14). Via (24) we can regard it also as a cuspidal L-parameter for a (unique) Levi subgroup $\mathcal{L}'(K') \subset \mathcal{G}'(K')$. From (23) we see that $G^\vee{}_{\mathrm{sc}} = \mathrm{ind}_{\mathbf{W}_{K'}}^{\mathbf{W}_K}(G'^\vee{}_{\mathrm{sc}})$ and

$$G'^\vee_{t\phi_L} = Z^1_{G'^\vee{}_{\mathrm{sc}}}\big(t\phi_L|_{\mathbf{W}_{K'}}\big) = Z^1_{G^\vee{}_{\mathrm{sc}}}\big(t\phi_L|_{\mathbf{W}_K}\big) = G^\vee_{t\phi_L}.$$

Almost the entire construction of $\mathcal{H}(\mathfrak{s}^\vee,\vec{v})$ in [AMS3, §3.3], as recalled above, takes place in groups $G^\vee_{t\phi_L}$ for some $t \in (Z(L_c^\vee)^{\mathbf{I}_K})^\circ_{\mathrm{Frob}}$. In particular every $\alpha \in \Phi_{\mathfrak{s}^\vee}$ corresponds to a unique $\alpha' \in \Phi_{\mathfrak{s}'^\vee}$. Only the label functions λ and λ^* depend on $Z^1_{G^\vee{}_{\mathrm{sc}}}(\phi_L|_{\mathbf{I}_K})$ with the action of $\phi_L(\mathrm{Frob})$, and that is subtly different for K and K'.

Let K'' be the maximal unramified subextension of K'/K, and use analogous notations as for K', but now also with double primes. The extension K'/K'' is totally ramified, so $[\mathbf{I}_{K''} : \mathbf{I}_{K'}] = [K' : K'']$ and a Frobenius element in $\mathbf{W}_{K'}$ is also one in $\mathbf{W}_{K''}$. Then the construction of the labels in [AMS3, §3.2] is literally the same for $\mathcal{G}'(K')$ as for $\mathcal{G}''(K'')$, so $\lambda(\alpha') = \lambda(\alpha'')$ in suggestive notation. We end up with

$$\mathcal{H}(\mathfrak{s}'^\vee,\vec{v}) \cong \mathcal{H}(\mathfrak{s}''^\vee,\vec{v}). \tag{27}$$

Now we consider the unramified extension K''/K. Thus $\mathbf{I}_{K''} = \mathbf{I}_K$, $[K'' : K] = f_{K'/K}$ and $\mathrm{Frob}_K^{[K'':K]}$ is a Frobenius element in $\mathbf{W}_{K''}$. The only differences in the constructions for $\mathcal{G}''(K'')$ and $\mathcal{G}(K)$ lie in the integers m_β figuring in [AMS3, Lemma 3.12 and Proposition 3.13]. Here $\beta \in X^*(T)$ with $T = Z(L_c^\vee)^{\mathbf{W}_K,\circ}$ and $m_\beta\beta$ will be one of our roots in $\Phi_{\mathfrak{s}^\vee}$. By definition m_β is the minimal positive integer such that $\ker(m_\beta\beta)$ contains the finite group

$$T_{\phi_L} = T \cap (1 - \phi_L(\mathrm{Frob}_K))(L_c^\vee)^{\mathbf{I}_K}.$$

Whereas T is the same for $\mathcal{G}''(K'')$ and for $\mathcal{G}(K)$, the group $(L_c^\vee)^{\mathbf{I}_K}$ is a direct product of $[K'' : K]$ copies of $(L_c''^\vee)^{\mathbf{I}_K}$, permuted transitively by $\phi_L(\mathrm{Frob}_K)$. Then [AMS3, Definition 3.11 and Lemma 3.14] say that $\lambda(\alpha) = [K'' : K]\lambda(\alpha'')$, whenever $\alpha \in \Phi_{\mathfrak{s}^\vee}$ and $\alpha'' \in R_{\mathfrak{s}''^\vee}$ correspond (and similarly for λ^*).

From this and (27) we obtain $\lambda(\alpha) = f_{K'/K}\lambda(\alpha')$. We conclude that, although $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ and $\mathcal{H}(\mathfrak{s}'^\vee, \vec{v}')$ are not necessarily isomorphic, they become so if we specialize $\vec{v}'$ to $\vec{v}^{f_{K'/K}}$. $\square$

3. Hecke algebras for unipotent representations

We preserve the setup from the previous sections. Since we will discuss unipotent representations, it is convenient to require (for the remainder of the paper) that $\mathcal{G}$ splits over an unramified extension of K. A large part of this section will be based on [Lus4, Mor3]. Although Lusztig works with split simple adjoint K-groups in [Lus4], most of the first section of that paper holds just as well for our $\mathcal{G}$.

3.1. Buildings, facets and associated groups.

We denote the enlarged Bruhat–Tits building of G by $\mathcal{B}(\mathcal{G}, K)$. It is the Cartesian product of the semisimple Bruhat–Tits building $\mathcal{BT}(\mathcal{G}, K)$ and the vector space $X_*(Z(G)^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$. Whereas $Z(G)$ acts trivially on $\mathcal{BT}(\mathcal{G}, K)$, only its maximal compact subgroup $Z(G)_{\mathrm{cpt}}$ fixes $\mathcal{B}(\mathcal{G}, K)$ pointwise.

Let $\mathbb{A} = X_*(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ be the apartment of $\mathcal{B}(\mathcal{G}, K)$ associated to $\mathcal{S}$. The walls of $\mathbb{A}$ determine an affine root system Σ, which canonically projects onto the finite root system $\Phi = \Phi(\mathcal{G}, \mathcal{S}) = \Phi(G, S)$. Since $\mathcal{G}$ splits over F_{nr}, 0 is a special vertex of $\mathbb{A}$. Let C_0 be the unique chamber in the positive Weyl chamber in $\mathbb{A}$ (determined by Δ), whose closure contains 0.

Let Δ_{aff} be the set of simple affine roots in Σ determined by C_0. It contains Δ and one additional affine reflection for every simple factor of G which is not a torus and not anisotropic. The associated set of simple affine reflections S_{aff} generates an affine Weyl group W_{aff}. The standard Iwahori subgroup of G is P_{C_0} and the Iwahori–Weyl group of (G, S) is

$$W := N_G(S)/(N_G(S) \cap P_{C_0}) \cong Z_G(S)/(Z_G(S) \cap P_{C_0}) \rtimes W(G, S). \tag{28}$$

We note that it acts on

$$\mathbb{A} = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} = X_*(Z_G(S)) \otimes_{\mathbb{Z}} \mathbb{R} \cong Z_G(S)/Z_G(S)_{\mathrm{cpt}} \otimes_{\mathbb{Z}} \mathbb{R}$$

with $W(G, S)$ acting linearly and $Z_G(S)/(Z_G(S) \cap P_{C_0})$ by translations. The kernel of this action is the finite subgroup $Z_G(S)_{\mathrm{cpt}}/Z_{P_{C_0}}(S)$.

Furthermore W contains W_{aff} as the subgroup supported on the kernel of the Kottwitz homomorphism for G. The group $\Omega := \{w \in W : w(C_0) = C_0\}$ forms a complement to W_{aff}:

$$W = W_{\mathrm{aff}} \rtimes \Omega. \tag{29}$$

In particular $\Omega \cong W/W_{\mathrm{aff}}$, which is isomorphic to the image of the Kottwitz homomorphism for G, a subquotient of $\mathrm{Irr}(Z(G^\vee))$. This shows that Ω is abelian.

Every facet $\mathfrak{f}$ of $\mathcal{B}(\mathcal{G}, K)$ is the Cartesian product of $X_*(Z(G)^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$ and a facet in $\mathcal{BT}(\mathcal{G}, K)$. Let $P_{\mathfrak{f}} \subset G$ be the parahoric subgroup associated to $\mathfrak{f}$, and let $U_{\mathfrak{f}}$ be its pro-unipotent radical. Then $\overline{P}_{\mathfrak{f}} = P_{\mathfrak{f}}/U_{\mathfrak{f}}$ can be regarded as the k-points of a connected reductive group. More precisely, Bruhat and Tits [BrTi] constructed an $\mathfrak{o}_K$-model $\mathcal{G}_{\mathfrak{f}}^\circ$ of $\mathcal{G}$, such that $P_{\mathfrak{f}} = \mathcal{G}_{\mathfrak{f}}^\circ(\mathfrak{o}_K)$. Then $\overline{P}_{\mathfrak{f}}$ is the maximal reductive quotient of $\mathcal{G}_{\mathfrak{f}}^\circ(\mathfrak{o}_K/\varpi_K\mathfrak{o}_K)$. Let $\hat{P}_{\mathfrak{f}}$ be the pointwise stabilizer of $\mathfrak{f}$ in G. It contains $P_{\mathfrak{f}}$ with finite index, and $\hat{P}_{\mathfrak{f}}/U_{\mathfrak{f}}$ is the group of k-rational points of a (possibly disconnected) reductive group. As $P_\mathfrak{f}$ is a characteristic subgroup of $\hat{P}_\mathfrak{f}$, these two have the same normalizer in G.

Since G acts transitively on the collection of chambers of $\mathcal{B}(\mathcal{G}, K)$, we may assume without loss of generality that $\mathfrak{f}$ is contained in the closure of C_0. Let $\Sigma_\mathfrak{f}$ be the set of affine roots that vanish on $\mathfrak{f}$ and let $J := \Delta_{\mathrm{aff}} \cap \Sigma_\mathfrak{f}$ be its subset of simple affine roots. The associated set of (affine) reflections $\{s_j : j \in J\}$ generates a finite Weyl group W_J, which can be identified with the Weyl group of the k-group $\overline{P}_\mathfrak{f}$ (with respect to the torus $\mathcal{S}(k)$).

Let $\Phi_\mathfrak{f}^c$ be the set of roots for (G, S) that are constant on $\mathfrak{f}$, a parabolic root subsystem of $\Phi(G, S)$. Let $\mathcal{L}_\mathfrak{f}$ be the Levi K-subgroup of $\mathcal{G}$ determined by $\mathcal{S}$ and $\Phi_\mathfrak{f}^c$. By [Mor3, Theorem 2.1]

$$P_{L,\mathfrak{f}} := P_\mathfrak{f} \cap L_\mathfrak{f}$$

is a maximal parahoric subgroup of $L_\mathfrak{f}$ and

$$\begin{array}{rcccc} (30) \qquad & \hat{P}_{L,\mathfrak{f}} / P_{L,\mathfrak{f}} & = & (\hat{P}_\mathfrak{f} \cap L_\mathfrak{f}) / (P_\mathfrak{f} \cap L_\mathfrak{f}) & \cong & \hat{P}_\mathfrak{f} / P_\mathfrak{f}, \\ & \overline{P_{L,\mathfrak{f}}} & = & P_{L,\mathfrak{f}} / U_{L,\mathfrak{f}} & \cong & \overline{P_\mathfrak{f}}. \end{array}$$

Let $\Phi_\mathfrak{f}$ be the image of $\Sigma_\mathfrak{f}$ in $\Phi(G, S)$. Its closure $(\mathbb{Q}\Phi_\mathfrak{f}) \cap \Phi(G, S)$ equals $\Phi_\mathfrak{f}^c$. Although $\Phi_\mathfrak{f}^c$ and $\Phi_\mathfrak{f}$ have the same rank, it is quite possible that they have different Weyl groups. We write

$$\Omega_\mathfrak{f} = \{\omega \in \Omega : \omega(\mathfrak{f}) = \mathfrak{f}\} = \{\omega \in \Omega : P_\mathfrak{f} \omega P_\mathfrak{f} \subset N_G(P_\mathfrak{f})\} \cong N_G(P_\mathfrak{f}) / P_\mathfrak{f}.$$

Since Ω is abelian (see the lines following (29)), so is $\Omega_\mathfrak{f}$.

Next we analyse a group that underlies a relevant Hecke algebra:

$$W(J, \sigma) := N_W(W_J) / W_J \cong \{w \in W : w(J) = J\}, \tag{31}$$

This does not depend on $\sigma \in \mathrm{Irr}(\overline{P_\mathfrak{f}})$, which will only be introduced later. We include σ in the notation to comply with [Mor3] (in the cases where σ is cuspidal and unipotent). It is easy to see that the right hand side of (31) contains $\Omega_\mathfrak{f}$. When $P_\mathfrak{f}$ is a maximal parahoric subgroup of G, $W(J, \sigma)$ coincides with $\Omega_\mathfrak{f}$.

Otherwise G has at least one simple factor such that $\Delta_{\mathrm{aff}} \setminus J$ contains two elements belonging to that factor. Let $\Delta_{\mathfrak{f},\mathrm{aff}} \subset \Delta_{\mathrm{aff}} \setminus J$ be the collection of all roots belonging to such simple factors of G. For $\alpha_i \in \Delta_{\mathfrak{f},\mathrm{aff}}$, the reflections in the roots $J \cup \{\alpha_i\}$ generate a finite Weyl group (contained in W). Let w_i° be its longest element, then

$$w_i^\circ(J) \subset w_i^\circ(J \cup \{\alpha_i\}) = -J \cup \{-\alpha_i\}.$$

We call J self-opposed [Mor2, §4.1] if

$$w_i^\circ(J) = -J \quad \text{for all } \alpha_i \in \Delta_{\mathfrak{f},\mathrm{aff}}. \tag{32}$$

From now we assume that J is self-opposed. Then [Mor1, Lemma 2.4] says that every $\alpha_i \in \Delta_{\mathfrak{f},\mathrm{aff}}$ gives rise to a unique order two element $s_i \in W$ with $s_i(J) = J$. (Usually s_i is not the reflection with respect to α_i.) According to [Mor1, Proposition 2.12] the set

$$S_{\mathfrak{f},\mathrm{aff}} := \{s_i : \alpha_i \in \Delta_{\mathfrak{f},\mathrm{aff}}\}$$

generates an affine Weyl group $W_{\mathrm{aff}}(J, \sigma)$ in $W(J, \sigma)$, and

$$W(J, \sigma) = W_{\mathrm{aff}}(J, \sigma) \rtimes \Omega_\mathfrak{f}. \tag{33}$$

An advantage of this decomposition is that, under a central isogeny $\mathcal{G} \to \mathcal{G}'$, $W_{\mathrm{aff}}(J, \sigma)$ is preserved while $\Omega_\mathfrak{f}$ embeds in its version for $\mathcal{G}'(K)$.

The Coxeter group $W_{\mathrm{aff}}(J,\sigma)$ is the direct product of irreducible affine Weyl groups, one for every simple factor of G to which at least two elements of $\Delta_{\mathrm{aff}} \setminus J$ belong. Hence it can be written as

$$W_{\mathrm{aff}}(J,\sigma) = X(J) \rtimes W^\circ(J,\sigma), \tag{34}$$

where $X(J)$ is a lattice in $Z_G(S)/Z_G(S)_{\mathrm{cpt}}$ and $W^\circ(J,\sigma)$ is a finite Weyl group. The subgroup $X(J) \subset W_{\mathrm{aff}}(J,\sigma)$ is canonically defined, namely as the set of elements whose conjugacy class is finite.

The set of simple reflections $S_{\mathfrak{f}} = \{s_i : \alpha_i \in \Delta_{\mathfrak{f}}\}$ for $W^\circ(J,\sigma)$ is a subset of $S_{\mathfrak{f},\mathrm{aff}}$, and $\Delta_{\mathfrak{f},\mathrm{aff}}$ is the affine extension of $\Delta_{\mathfrak{f}}$. We note that

$$\begin{aligned} |S_{\mathfrak{f}}| \;&=\; |\Delta_{\mathfrak{f}}| \;=\; \mathrm{rk}(X(J)) \;=\; \dim(\mathfrak{f} \cap \mathcal{BT}(\mathcal{G},K)) \\ &=\; \mathrm{rk}_K(\mathcal{L}_{\mathfrak{f}}/Z(\mathcal{G})) \;=\; \mathrm{rk}\,\Phi(\mathcal{G}, Z(\mathcal{L}_{\mathfrak{f}})). \end{aligned} \tag{35}$$

The set $\Delta_{\mathfrak{f}} \cup J$ determines a unique vertex $x_{\mathfrak{f}}$ of $\bar{\mathfrak{f}} \cap \mathcal{BT}(\mathcal{G},K)$, and

$$W^\circ(J,\sigma) = \{w \in W_{\mathrm{aff}}(J,\sigma) : w(x_{\mathfrak{f}}) = x_{\mathfrak{f}}\}. \tag{36}$$

Let $\mathbb{A}_m \subset \mathbb{A}$ be the product of the standard apartments of those simple factors of G for which $\Delta_{\mathfrak{f}}$ has just one element. Let $\mathbb{A}_{\mathfrak{f}} \subset \mathbb{A}$ be the product of the standard apartments of the remaining simple factors of G (those for which $\Delta_{\mathfrak{f}}$ has more than one element or which are isotropic tori). We have a W-stable decomposition

$$\mathbb{A} = \mathbb{A}_m \times \mathbb{A}_{\mathfrak{f}}. \tag{37}$$

The group $\Omega_{\mathfrak{f}}$ acts by conjugation on the normal subgroup $W_{\mathrm{aff}}(J,\sigma)$ of $W(J,\sigma)$. This action stabilizes $S_{\mathfrak{f},\mathrm{aff}}$ setwise, and the pointwise stabilizer $\Omega_{\mathfrak{f}}^1$ of $S_{\mathfrak{f},\mathrm{aff}}$ consists of those $\omega \in \Omega_{\mathfrak{f}}$ which fix the image of $\mathfrak{f}$ in $\mathcal{BT}(\mathcal{G},K)$ pointwise. Since $\Omega_{\mathfrak{f}}$ is abelian and the centre of $W_{\mathrm{aff}}(J,\sigma)$ is trivial (as for every affine Weyl group),

$$Z(W(J,\sigma)) = \Omega_{\mathfrak{f}}^1. \tag{38}$$

Let $\Omega_{\mathfrak{f},\mathrm{tor}}$ be the pointwise stabilizer of $\mathfrak{f}$ in $\Omega_{\mathfrak{f}}$, a central subgroup of $W(J,\sigma)$. Since W acts on $\mathbb{A}$ with finite stabilizers, $\Omega_{\mathfrak{f},\mathrm{tor}}$ is finite. We note that

$$\hat{P}_{\mathfrak{f}}/P_{\mathfrak{f}} = \Omega_{\mathfrak{f},\mathrm{tor}}. \tag{39}$$

Proposition 3.1. *Assume that $\overline{P_{\mathfrak{f}}}$ has a cuspidal unipotent representation σ. There exists a lattice $X_{\mathfrak{f}}$ in $X_*(Z_G(S)) \cap \mathbb{A}_{\mathfrak{f}}$ such that*

$$W(J,\sigma)/\Omega_{\mathfrak{f},\mathrm{tor}} = W_{\mathrm{aff}}(J,\sigma) \rtimes \Omega_{\mathfrak{f}}/\Omega_{\mathfrak{f},\mathrm{tor}} \cong W^\circ(J,\sigma) \ltimes X_{\mathfrak{f}}.$$

Proof. Consider the action of any $\omega \in \Omega_{\mathfrak{f}}$ on $\mathbb{A}_{\mathfrak{f}}$. It can be written as $\omega^\circ \omega_t$, where $\omega^\circ \in W(G,S) \ltimes \mathbb{A}_{\mathfrak{f}}$ fixes $\{x_{\mathfrak{f}}\} \times X_*(Z(G)^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$ pointwise and $\omega_t \in \mathbb{A}_{\mathfrak{f}}$ is a translation.

We claim that $\omega^\circ \in W^\circ(J,\sigma)$. Since G acts on $X_*(Z(G)^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$ by translations and $W^\circ(J,\sigma)$ acts trivially on that space, it suffices to prove this claim under the assumption that $\mathcal{G}$ is semisimple. Then the group W is naturally a subgroup of the analogous group for the adjoint quotient of $\mathcal{G}$, and these two semisimple groups have the same $W_{\mathrm{aff}}(J,\sigma)$. The adjoint group of $\mathcal{G}$ is a direct product of simple adjoint K-groups, so we may even assume that $\mathcal{G}$ is simple and adjoint.

As we assumed the existence of a cuspidal unipotent $\sigma \in \mathrm{Irr}(\overline{P_{\mathfrak{f}}})$, we are now exactly in the setting of [Lus4, §1.20], and $\Omega_{\mathfrak{f},\mathrm{tor}}$ becomes the group denoted $\bar{\Omega}_1^u$ over there. Then $W(J,\sigma)/\Omega_{\mathfrak{f},\mathrm{tor}}$ underlies the "arithmetic" affine Hecke algebra in [Lus4, §1.18], with finite Weyl group $W^\circ(J,\sigma)$. By classification Lusztig showed in [Lus4, Theorem 6.3] and [Lus5, Theorem 10.11] that this algebra is isomorphic to a "geometric" affine Hecke algebra. By [Lus4, §4.1 and §5.12] and [Lus5, §8.2] such algebras admit a presentation in terms of root data, which means that every element of $W(J,\sigma)/\Omega_{\mathfrak{f},\mathrm{tor}}$ can be written as the product of a translation and an element of $W^\circ(J,\sigma)$. In particular $\omega^\circ \in W^\circ(J,\sigma)$, proving our claim.

The above argument also shows that ω° and ω_t depend only on the image of ω in $\Omega_{\mathfrak{f}}/\Omega_{\mathfrak{f},\mathrm{tor}}$. Put

$$X_{\mathfrak{f}} := X(J)\langle \omega_t : \omega \in \Omega_{\mathfrak{f}}/\Omega_{\mathfrak{f},\mathrm{tor}}\rangle. \tag{40}$$

Every element of $X_{\mathfrak{f}}$ gives the action of an element of $W(J,\sigma)$ on $\mathbb{A}_{\mathfrak{f}}$, so $X_{\mathfrak{f}}$ embeds naturally in $Z_G(S)/Z_G(S)_{\mathrm{cpt}} \cap \mathbb{A}_{\mathfrak{f}}$. As a subgroup of a lattice, it is itself a lattice.

From the action of (34) on $\mathbb{A}_{\mathfrak{f}}$ and from $\omega \mapsto \omega^\circ\omega_t$ we get a surjective group homomorphism

$$W(J,\sigma)/\Omega_{\mathfrak{f},\mathrm{tor}} = W_{\mathrm{aff}}(J,\sigma) \rtimes \Omega_{\mathfrak{f}}/\Omega_{\mathfrak{f},\mathrm{tor}} \longrightarrow W^\circ(J,\sigma) \ltimes X_{\mathfrak{f}}. \tag{41}$$

Suppose that its kernel is nontrivial, say it contains $w\omega$ with $w \in W_{\mathrm{aff}}(J,\sigma)$ and $\omega \in \Omega_{\mathfrak{f}}/\Omega_{\mathfrak{f},\mathrm{tor}}$. The homomorphism (41) is injective on $W_{\mathrm{aff}}(J,\sigma)$, for that group acts trivially on the factor $\mathbb{A}_m$ from (37). So $\omega \neq 1$ and the action of ω on $\mathbb{A}_{\mathfrak{f}}$ agrees with that of w^{-1}. By the definition of $\Omega_{\mathfrak{f}}$, ω stabilizes $\bar{\mathfrak{f}}$, whereas the affine Weyl group $W_{\mathrm{aff}}(J,\sigma)$ acts simply transitively on a chamber complex with fundamental chamber $\bar{\mathfrak{f}}$. Therefore w must be trivial, and ω lies in the kernel of (41). Then ω acts trivially on $\mathbb{A}_{\mathfrak{f}}$, so it only acts on the factor $\mathbb{A}_m$ of $\mathbb{A}$. As $\Omega_{\mathfrak{f}}$ stabilizes $\mathfrak{f}$, this means that ω fixes $\mathfrak{f}$ pointwise. Thus $\omega \in \Omega_{\mathfrak{f},\mathrm{tor}}$, and (41) is injective. $\square$

By Proposition 3.1 the centre of $W^\circ(J,\sigma) \ltimes X_{\mathfrak{f}}$ is the free abelian group $X_{\mathfrak{f}}^{W^\circ(J,\sigma)}$. From that and (38) it follows that $\Omega_{\mathfrak{f},\mathrm{tor}}$ is precisely the torsion subgroup of $Z(W(J,\sigma))$, which justifies our notation.

3.2. **Bernstein components and types.**

Let $\mathrm{Rep}(G)$ be the category of smooth representations of G on complex vector spaces, and let $\mathrm{Irr}(G)$ be the set of (isomorphism classes of) irreducible objects in $\mathrm{Rep}(G)$. We denote the subset of supercuspidal irreducible representations by $\mathrm{Irr}_{\mathrm{cusp}}(G)$. Recall from [BeDe] that every $\pi \in \mathrm{Irr}(G)$ has a cuspidal support $\mathbf{Sc}(\pi)$, which is unique up to G-conjugation and consists of a Levi subgroup of G and a supercuspidal irreducible representation thereof.

For a Levi subgroup $L \subset G$ and $\pi_L \in \mathrm{Irr}_{\mathrm{cusp}}(L)$ we write $[L,\pi_L]_L = \mathfrak{s}_L$ and

$$\mathrm{Irr}(L)_{\mathfrak{s}_L} = X_{\mathrm{nr}}(L)\pi_L = \{\chi \otimes \pi_L : \chi \in X_{\mathrm{nr}}(L)\},$$

an inertial class of supercuspidal L-representations. The G-orbit $[L,\pi_L]_G = \mathfrak{s}$ of $\mathfrak{s}_L$ is by definition an inertial equivalence class for G. Notice that the group $N_G(L)$ acts naturally on $\mathrm{Irr}_{\mathrm{cusp}}(L)$, with L acting trivially. To $\mathfrak{s}$ (and the choice of a representative $\mathfrak{s}_L$) one associates a finite group $W_{\mathfrak{s}}$, the stabilizer of $\mathfrak{s}_L$ in $N_G(L)/L$.

We denote the collection of inertial equivalence classes for G by $\mathfrak{Be}(G)$. Every $\mathfrak{s} \in \mathfrak{Be}(G)$ determines a Bernstein component of $\mathrm{Irr}(G)$:

$$\mathrm{Irr}(G)_{\mathfrak{s}} := \{\pi \in \mathrm{Irr}(G) : \mathbf{Sc}(\pi) \in \mathfrak{s}\}.$$

The associated Bernstein block $\mathrm{Rep}(G)_{\mathfrak{s}}$ is a direct factor of $\mathrm{Rep}(G)$. The theory of the Bernstein centre [BeDe] tells us that the centre of $\mathrm{Rep}(G)_{\mathfrak{s}}$ is $\mathcal{O}(\mathrm{Irr}(L)_{\mathfrak{s}_L}/W_{\mathfrak{s}})$.

As before we pick a facet $\mathfrak{f}$ in the closure of C_0. We assume that $\overline{P}_{\mathfrak{f}}$ has a cuspidal unipotent representation σ. This is a rather strong condition on the facet $\mathfrak{f}$, which by [Mor2, §4.4] implies that J is self-opposed in the sense of (32). The inflation of

σ to $P_{\mathfrak{f}}$ will also be denoted σ, and its underlying vector space V_σ. It was shown in [MoPr, §6] and [Mor3, Theorem 4.8] that $(P_{\mathfrak{f}}, \sigma)$ is a type for G. This has the following consequences [BuKu, Theorem 4.3]:

- Let $\mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)}$ be the category of smooth G-representations that are generated by their σ-isotypical vectors. Then $\mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)}$ is a direct factor of $\mathrm{Rep}(G)$, a direct sum of finitely Bernstein blocks.
- Let $\mathcal{H}(G, P_{\mathfrak{f}}, \sigma)$ be the G-endomorphism algebra of the module $\mathrm{ind}_{P_{\mathfrak{f}}}^G(\sigma, V_\sigma)$. Then

$$\begin{array}{ccc} \mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)} & \to & \mathrm{Mod}(\mathcal{H}(G, P_{\mathfrak{f}}, \sigma)) \\ V & \mapsto & \mathrm{Hom}_{P_{\mathfrak{f}}}(V_\sigma, V) \end{array} \tag{42}$$

  is an equivalence of categories.

If $(P_{\mathfrak{f}'}, \sigma')$ are data of the same kind as $(P_{\mathfrak{f}}, \sigma)$, then by [MoPr, Theorem 5.2] the two associated subcategories of $\mathrm{Rep}(G)$ are either disjoint or equal. Moreover, by [Lus4, 1.6.b]

$$\begin{array}{ll} \mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)} = \mathrm{Rep}(G)_{(P_{\mathfrak{f}'},\sigma')} & \text{if and only if} \\ & \text{there exists a } g \in G \text{ such that } P_{\mathfrak{f}'} = g P_{\mathfrak{f}} g^{-1} \text{ and } \sigma' \cong g \cdot \sigma. \end{array} \tag{43}$$

The category of unipotent G-representations is defined as the full subcategory of $\mathrm{Rep}(G)$ generated by the $\mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)}$ as above. By (43)

$$\mathrm{Rep}_{\mathrm{unip}}(G) = \prod\nolimits_{(P_{\mathfrak{f}},\sigma)/G\text{-conjugation}} \mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)}.$$

We note that this is finite product, because there are only finitely many G-orbits of facets in $\mathcal{B}(G, K)$. We want to make the structure of $\mathcal{H}(G, P_{\mathfrak{f}}, \sigma)$ more explicit. This will involve extending σ to a representation of $\hat{P}_{\mathfrak{f}}$ and analysing the Hecke algebra for that type. Up to twists by $X_{\mathrm{nr}}(L_{\mathfrak{f}})$, there are only finitely many ways to extend $\sigma|_{P_{L,\mathfrak{f}}}$ to a representation of $N_{L_{\mathfrak{f}}}(P_{L,\mathfrak{f}})$, say $\hat{\sigma}_i$ $(i = 1, \ldots, e_\sigma)$. Then

$$\mathfrak{s}_i = [L_{\mathfrak{f}}, \mathrm{ind}_{N_{L_{\mathfrak{f}}}(P_{L,\mathfrak{f}})}^{L_{\mathfrak{f}}}(\hat{\sigma}_i)]_G$$

is an inertial equivalence class for $\mathrm{Irr}(G)$ and by [Mor3, Theorem 4.3]:

$$\begin{array}{ccc} \mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)} & = & \prod_{i=1}^{e_\sigma} \mathrm{Rep}(G)_{\mathfrak{s}_i}, \\ \mathrm{Irr}(G)_{(P_{\mathfrak{f}},\sigma)} & = & \bigsqcup_{i=1}^{e_\sigma} \mathrm{Irr}(G)_{\mathfrak{s}_i}. \end{array} \tag{44}$$

By [BuKu, §2.6], $\mathcal{H}(G, P_{\mathfrak{f}}, \sigma)$ is naturally isomorphic to

$$\big\{ f \in C_c(G, \mathrm{End}_{\mathbb{C}}(V_\sigma^*)) : f(p_1 g p_2) = \sigma^\vee(p_1) f(g) \sigma^\vee(p_2) \; \forall g \in G, p_1, p_2 \in P_{\mathfrak{f}} \big\},$$

where $(\sigma^\vee, V_\sigma^*)$ denotes the contragredient of σ. According to [Mor3, §3.1] and [Lus4, 1.18] the support of $\mathcal{H}(G, P_{\mathfrak{f}}, \sigma)$ in G is

$$P_{\mathfrak{f}} N_W(W_J) P_{\mathfrak{f}} \; \subset \; P_{\mathfrak{f}} N_G(L_{\mathfrak{f}}) P_{\mathfrak{f}}.$$

To make sense of the left hand side, look at (28) and use that $N_G(S) \cap P_{C_0} = Z_G(S) \cap P_{\mathfrak{f}}$ is contained in $P_{\mathfrak{f}}$. Furthermore the group $W(J, \sigma)$ indexes a $\mathbb{C}$-basis $\{T_w\}$ of $\mathcal{H}(G, P_{\mathfrak{f}}, \sigma)$, such that T_w has support $P_{\mathfrak{f}} w P_{\mathfrak{f}}$. This is a little easier than in [Mor3] – the crucial difference is that the cuspidal unipotent representation σ of $P_{\mathfrak{f}}$ can be extended to a representation of $N_G(P_{\mathfrak{f}})$ [FOS1, Lemma 15.7], so the entire group $N_W(W_J)/W_J$ stabilizes σ.

The subgroup $N_G(P_\mathfrak{f})$ supports a subalgebra of $\mathcal{H}(G, P_\mathfrak{f}, \sigma)$, which by [Lus4, §1.19] is isomorphic to the group algebra

$$\mathbb{C}[\Omega_\mathfrak{f}] = \mathbb{C}[N_G(P_\mathfrak{f})/P_\mathfrak{f}]. \tag{45}$$

The construction of the isomorphism involves the choice of an extension of σ to a representation of $N_G(P_\mathfrak{f})$. For $X_*(Z(G)^\circ) \subset \Omega_\mathfrak{f}$ there exists a canonical extension. Namely, embed it in $N_G(P_\mathfrak{f})$ via evaluation of cocharacters $\lambda : K^\times \to Z(G)^\circ$ at the chosen uniformizer ϖ_K and then define $\sigma(\lambda(\varpi_K)) = 1$. As far as we know, the extension of σ from $X_*(Z(G)^\circ)P_\mathfrak{f}$ to $N_G(P_\mathfrak{f})$ is not canonical.

When $P_\mathfrak{f}$ is a maximal parahoric subgroup of G, (45) coincides with $\mathcal{H}(G, P_\mathfrak{f}, \sigma)$. The subalgebra $\mathcal{H}_{\mathrm{aff}}(G, P_\mathfrak{f}, \sigma)$ spanned by $\{T_w : w \in W_{\mathrm{aff}}(J, \sigma)\}$ (i.e. supported on $P_\mathfrak{f} W_{\mathrm{aff}}(J, \sigma) P_\mathfrak{f}$) is isomorphic to the Iwahori–Hecke algebra of the Coxeter system $(W_{\mathrm{aff}}(J, \sigma), S_{\mathfrak{f},\mathrm{aff}})$ with parameters as in [Lus4, §1.18]. Together with (45) that gives a description as an extended affine Hecke algebra:

$$\mathcal{H}(G, P_\mathfrak{f}, \sigma) = \mathcal{H}_{\mathrm{aff}}(G, P_\mathfrak{f}, \sigma) \rtimes \Omega_\mathfrak{f}. \tag{46}$$

The generators T_s with $s \in S_{\mathfrak{f},\mathrm{aff}}$ are uniquely determined by their support and by the quadratic relations that they satisfy. In view of the braid relations in Coxeter groups, all the T_w with $w \in W_{\mathrm{aff}}(J, \sigma)$ are canonical. With (33) we deduce that the only arbitrary choices for the realization of (46) come from (45).

According to [Lus4, §1–2], $\mathcal{H}_{\mathrm{aff}}(G, P_\mathfrak{f}, \sigma)$ is isomorphic to an affine Hecke algebra determined by:

- The lattice $X(J)$ and its dual $X(J)^\vee$.
- The root system $R_\mathfrak{f}$ in $X(J)$ from [Lus4, §2.22] (denoted $\mathcal{R}$ over there). It has a basis indexed by $\Delta_\mathfrak{f}$ and has Weyl group $W^\circ(J, \sigma)$.
- The dual root system $R_\mathfrak{f}^\vee$ from [Lus4, §2.22] (denoted $\mathcal{R}'$ over there).
- The set of affine reflections $S_{\mathfrak{f},\mathrm{aff}} = \{s_i : i \in \Delta_{\mathrm{aff}} \setminus J\}$ with parameter function $q_K^{\mathcal{N}}$ as in [Lus4, §1.18].

For a character ψ of $\Omega_{\mathfrak{f},\mathrm{tor}}$, let $e_\psi \in \mathbb{C}[\Omega_\mathfrak{f}]$ be the associated idempotent. We can decompose (46) as in [Lus4, §1.20]:

$$\mathcal{H}(G, P_\mathfrak{f}, \sigma) = \bigoplus\nolimits_{\psi \in \mathrm{Irr}(\Omega_{\mathfrak{f},\mathrm{tor}})} \mathcal{H}_{\mathrm{aff}}(G, P_\mathfrak{f}, \sigma) \rtimes e_\psi \mathbb{C}[\Omega_\mathfrak{f}]. \tag{47}$$

By (39) ψ can also be regarded as a character of $\hat{P}_\mathfrak{f}/P_\mathfrak{f}$.

Let $\hat\sigma$ be an extension of σ to an irreducible representation of $\hat{P}_\mathfrak{f}$, as in [Lus4, §1.16]. We may assume that it comes from the extension of σ used in [Lus4, §1.19] to construct an isomorphism with (45). The other extensions of σ to $\hat{P}_\mathfrak{f}$ are $\hat\sigma \otimes \psi$ with $\psi \in \mathrm{Irr}(\hat{P}_\mathfrak{f}/P_\mathfrak{f}) = \mathrm{Irr}(\Omega_{\mathfrak{f},\mathrm{tor}})$. By (43) different extensions of σ to $\hat{P}_\mathfrak{f}$ cannot be conjugate by elements of G. Comparing with (44) and taking (30) into account, we see that there is a unique i such that

$$(\hat\sigma \otimes \psi)\big|_{\hat{P}_{L,\mathfrak{f}}} = \hat\sigma_i\big|_{\hat{P}_{L,\mathfrak{f}}},$$

In this situation we henceforth write

$$\mathfrak{s}_\psi = \mathfrak{s}_i = [L_\mathfrak{f}, \mathrm{ind}_{N_{L_\mathfrak{f}}(P_{L,\mathfrak{f}})}^{L_\mathfrak{f}} (\hat\sigma_i)]_G,$$

and we denote the underlying inertial equivalence class for $L_\mathfrak{f}$ by $\mathfrak{s}_{L,\psi}$.

Theorem 3.2. *(a) $(\hat{P}_\mathfrak{f}, \hat\sigma \otimes \psi)$ is a type for the single Bernstein block* $\mathrm{Rep}(G)_{\mathfrak{s}_\psi}$.

(*b*) *There is an equivalence of categories*

$$\begin{array}{ccc} \mathrm{Rep}(G)_{\mathfrak{s}_\psi} & \to & \mathrm{Mod}(\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma} \otimes \psi)) \\ V & \mapsto & \mathrm{Hom}_{\hat{P}_{\mathfrak{f}}}(\hat{\sigma} \otimes \psi, V) \end{array}.$$

(*c*) *Part (b) restricts to an equivalence between the respective subcategories of tempered representations.*

(*d*) *Suppose that $V \in \mathrm{Rep}(G)_{\mathfrak{s}_\psi}$ has finite length. Then V is essentially square-integrable if and only if $\mathrm{Hom}_{\hat{P}_{\mathfrak{f}}}(\hat{\sigma} \otimes \psi, V)$ is an essentially discrete series $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma} \otimes \psi)$-module.*

Proof. (a) is a special case of [Mor3, Theorem 4.7].
(b) is a consequence of part (a) and [BuKu, Theorem 4.3].
(c) For finite length tempered representations see [DeOp, Theorem 10.1]. For general tempered representations see [Sol2, Theorem 3.12 and p.42].
(d) In (35) we saw that $\mathrm{rk}(R_{\mathfrak{f}}) = \mathrm{rk}(\Phi(G, L_{\mathfrak{f}}))$. It follows that under the isomorphism of $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma)$ with an affine Hecke algebra, as described after (46), $R_{\mathfrak{f}}$ corresponds to a full rank root subsystem of $\Phi(G, L_{\mathfrak{f}})$. This allows us to apply [Sol2, Theorem 3.9.b], which proves the claim. $\square$

3.3. **Relations with the adjoint case.**

We will relate the objects from Paragraph 3.2 for $\mathcal{G}$ with those for its adjoint group $\mathcal{G}_{\mathrm{ad}} = \mathcal{G}/Z(\mathcal{G})$. It is well-known that the enlarged Bruhat–Tits building depends only on $\mathcal{G}$ up to isogeny, and that the semisimple Bruhat–Tits buildings of $(\mathcal{G}, K)$ and $(\mathcal{G}_{\mathrm{ad}}, K)$ can be identified. Since parahoric subgroups correspond (bijectively) to facets of the semisimple Bruhat–Tits building, $\mathcal{G}(K)$ and $\mathcal{G}_{\mathrm{ad}}(K)$ have the same set of parahoric subgroups. (In $(\mathcal{G}_{\mathrm{ad}}(K)$ more of them can be conjugate, though.)

The $\mathfrak{o}_K$-group $\mathcal{G}_{\mathfrak{f}}^\circ$ is isogenous to the direct product of $\mathcal{G}_{\mathrm{ad},\mathfrak{f}}^\circ$ and an $\mathfrak{o}_K$-torus. Hence $\overline{P}_{\mathfrak{f}}$ is isogenous (as k-group) to the direct product of $\overline{P}_{\mathfrak{f},\mathrm{ad}}$ and a k-torus. The collection of (cuspidal) unipotent representations of a connected reductive k-group $\mathcal{H}(k)$ only depends on $\mathcal{H}$ up to isogeny [Lus1, Proposition 3.15] and a k-torus has just one irreducible unipotent representation, namely the trivial representation. Therefore we may identify the collections of cuspidal unipotent representations of $\overline{P}_{\mathfrak{f}}$ and $\overline{P}_{\mathfrak{f},\mathrm{ad}}$. The same goes for the collections of cuspidal unipotent representations of $P_{\mathfrak{f}}$ and $P_{\mathfrak{f},\mathrm{ad}}$. We will denote the cuspidal unipotent representation of $P_{\mathfrak{f},\mathrm{ad}}$ corresponding to $\sigma \in \mathrm{Irr}(P_{\mathfrak{f}})$ by σ_{ad}.

Lemma 3.3. *The following objects are the same for $(G, P_{\mathfrak{f}}, \sigma)$ and for $(G_{\mathrm{ad}}, P_{\mathfrak{f},\mathrm{ad}}, \sigma_{\mathrm{ad}})$: $S_{\mathfrak{f},\mathrm{aff}}, W_{\mathrm{aff}}(J, \sigma), \Delta_{\mathfrak{f}}, W^\circ(J, \sigma), R_{\mathfrak{f}}, R_{\mathfrak{f}}^\vee$ and $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma)$.*

Proof. The set Δ_{aff} depends only on $\mathcal{BT}(\mathcal{G}, K) = \mathcal{BT}(\mathcal{G}_{\mathrm{ad}}, K)$ and J is determined by the facet $\mathfrak{f}$, so the claim holds for $\Delta_{\mathrm{aff}} \setminus J$ and for $S_{\mathfrak{f},\mathrm{aff}}$. Hence also for the Coxeter group $W_{\mathrm{aff}}(J, \sigma)$ with generators $S_{\mathfrak{f},\mathrm{aff}}$. We can choose the simple roots $\Delta_{\mathfrak{f}}$ and the simple reflections $S_{\mathfrak{f}} \subset S_{\mathfrak{f},\mathrm{aff}}$ in the same way for G and for G_{ad}, so the Weyl group $W^\circ(J, \sigma)$ generated by $S_{\mathfrak{f}}$ does not change under passage to the adjoint case.

The construction of $R_{\mathfrak{f}}$ and $R_{\mathfrak{f}}^\vee$ in [Lus4, §2] depends only on $(W_{\mathrm{aff}}(J, \sigma), S_{\mathfrak{f},\mathrm{aff}})$, so it is the same for $(G, P_{\mathfrak{f}})$ and for $(G_{\mathrm{ad}}, P_{\mathfrak{f},\mathrm{ad}})$. The parameters $q_i = q_K^{\mathcal{N}(s_i)}$ for $S_{\mathfrak{f},\mathrm{aff}}$ used in $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma)$ are defined in [Lus4, §1.18] and [Mor1, §6.9 and §7.1]. For the parameter of $s_\alpha = s_i$, consider the standard parahoric subgroup $P_{J \cup \{i\}}$ of G determined by $J \cup \{i\}$. It contains $P_{\mathfrak{f}} = P_J$, and $\mathrm{ind}_{P_J}^{P_{J \cup \{i\}}}(\sigma)$ is a direct sum of two irreducible representations, say σ_1 and σ_2. Write $\dim(\sigma_j) = q_K^{n_j}$ with $n_j \in \mathbb{Z}_{\geq 0}$, then the parameter of s_i is

$$q_i = q_K^{|n_1 - n_2|}.$$

It follows from [Lus1, Proposition 2.6] that the class of unipotent representations of connected reductive groups over finite fields is closed under parabolic induction. In particular $\mathrm{ind}_{P_J}^{P_{J\cup\{i\}}}(\sigma)$ is again unipotent, while we already knew from [Lus1, Proposition 3.15] that it is independent of isogenies of the involved group. More explicitly, via the map $P_{J\cup\{i\}} \to P_{J\cup\{i\},\mathrm{ad}}$ this representation is isomorphic to $\mathrm{ind}_{P_{J,\mathrm{ad}}}^{P_{J\cup\{i\},\mathrm{ad}}}(\sigma_{\mathrm{ad}})$. (Of course the isomorphism can also be seen more elementarily.) It follows that pull-back from $P_{J\cup\{i\},\mathrm{ad}}$ to $P_{J\cup\{i\}}$ also defines isomorphisms $\sigma_{\mathrm{ad},j} \cong \sigma_j$. Comparing their dimensions, we find that $q_{\mathrm{ad},i} = q_i$.

The Iwahori–Hecke algebra $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma)$ depends only on $W_{\mathrm{aff}}(J,\sigma)$, S_{aff} and the parameters q_i for $i \in \Delta_{\mathrm{aff}} \setminus J$, so it is naturally isomorphic to $\mathcal{H}_{\mathrm{aff}}(G_{\mathrm{ad}}, P_{\mathfrak{f},\mathrm{ad}}, \sigma_{\mathrm{ad}})$. $\square$

From (43) and Theorem 3.2 we get an equivalence of categories

$$\begin{array}{ccc} \mathrm{Rep}_{\mathrm{unip}}(G) & \to & \mathrm{Mod}\Big(\bigoplus_{(\mathfrak{f},\hat\sigma\otimes\psi)/G} \mathcal{H}(G,\hat P_{\mathfrak{f}},\hat\sigma\otimes\psi)\Big) \\ V & \mapsto & \prod_{(\mathfrak{f},\hat\sigma\otimes\psi)/G} \mathrm{Hom}_{\hat P_{\mathfrak{f}}}(\hat\sigma\otimes\psi, V) \end{array} . \tag{48}$$

By [Lus4, §1.20] the Hecke algebra of $(\hat P_{\mathfrak{f}}, \hat\sigma\otimes\psi)$ can be written as

$$\mathcal{H}(G,\hat P_{\mathfrak{f}},\hat\sigma\otimes\psi) = \mathcal{H}_{\mathrm{aff}}(G,P_{\mathfrak{f}},\sigma) \rtimes e_\psi \mathbb{C}[\Omega_{\mathfrak{f}}] \cong \mathcal{H}_{\mathrm{aff}}(G,P_{\mathfrak{f}},\sigma) \rtimes \Omega_{\mathfrak{f}}/\Omega_{\mathfrak{f},\mathrm{tor}}. \tag{49}$$

Lemma 3.4. *Let $X_{\mathfrak{f}}^\vee$ be the lattice dual to $X_{\mathfrak{f}}$ (from Proposition 3.1).*
(*a*) *$(X_{\mathfrak{f}}, R_{\mathfrak{f}}, X_{\mathfrak{f}}^\vee, R_{\mathfrak{f}}^\vee)$ is a root datum.*
(*b*) *The algebra (49) is isomorphic to the affine Hecke algebra determined by the above root datum and the parameter function $q_K^{\mathcal{N}}$ from* [Lus4, §1.18].

Proof. (a) We already know from the Bernstein presentation of $\mathcal{H}_{\mathrm{aff}}(G,P_{\mathfrak{f}},\sigma)$ that it is a root datum with $X(J)$ (resp. $X(J)^\vee$) instead of $X_{\mathfrak{f}}$ (resp. $X_{\mathfrak{f}}^\vee$), and from Lemma 3.3 that $R_{\mathfrak{f}}, R_{\mathfrak{f}}^\vee$ are insensitive to central isogenies of G. It only remains to check that the root systems are contained in the given lattices.

From the definition (40) we see that $X_{\mathfrak{f}}$ contains $X(J)$, so it also contains $R_{\mathfrak{f}}$. For the adjoint group G_{ad}, the Bernstein presentation of (49) from [Lus4, §1.20] shows that $X_{\mathfrak{f},\mathrm{ad}}$ is contained in the weight lattice $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R_{\mathfrak{f}}^\vee, \mathbb{Z})$. In general

$$X_{\mathfrak{f}} \subset X_{\mathfrak{f},\mathrm{ad}} \times X_*(Z(G)^\circ) \otimes_{\mathbb{Z}} \mathbb{R},$$

by (33) and because Ω and $\Omega_{\mathfrak{f}}$ have that property. Hence

$$X_{\mathfrak{f}} \subset \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R_{\mathfrak{f}}^\vee, \mathbb{Z}) \times X_*(Z(G)^\circ) \otimes_{\mathbb{Z}} \mathbb{R},$$

which implies that $\mathbb{Z}R_{\mathfrak{f}}^\vee \subset X_{\mathfrak{f}}^\vee$.
(b) Start with the root datum from part (a) and the associated affine Hecke algebra in the Bernstein presentation. Translate that to an Iwahori–Matsumoto presentation with [Lus3, §3]. In view of the aforementioned description of $\mathcal{H}_{\mathrm{aff}}(G,P_{\mathfrak{f}},\sigma)$ and the isomorphism

$$W_{\mathrm{aff}}(J,\sigma) \rtimes \Omega_{\mathfrak{f}}/\Omega_{\mathfrak{f},\mathrm{tor}} \cong W^\circ(J,\sigma) \ltimes X_{\mathfrak{f}}$$

from Proposition 3.1, that yields exactly the algebra (49). $\square$

We note that the presentation of (49) in Lemma 3.4 is almost the same as the Bernstein presentation of $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma)$, the only difference lies in the lattices ($X(J)$ vs. $X_{\mathfrak{f}}$). That enables us to formulate a precise comparison between the affine Hecke algebras

$$\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma} \otimes \psi) \quad \text{and} \quad \mathcal{H}(G_{\mathrm{ad}}, \hat{P}_{\mathfrak{f},\mathrm{ad}}, \hat{\sigma}_{\mathrm{ad}} \otimes \psi_{\mathrm{ad}}), \tag{50}$$

for any choice of ψ_{ad}. From Lemmas 3.4 and 3.3 we see that the inputs for their Bernstein presentations are the based root data

$$(X_{\mathfrak{f}}, R_{\mathfrak{f}}, X_{\mathfrak{f}}^\vee, R_{\mathfrak{f}}^\vee, \Delta_{\mathfrak{f}}) \quad \text{and} \quad (X_{\mathfrak{f},\mathrm{ad}}, R_{\mathfrak{f}}, X_{\mathfrak{f},\mathrm{ad}}^\vee, R_{\mathfrak{f}}^\vee, \Delta_{\mathfrak{f}}) \tag{51}$$

and the parameter function $q_K^{\mathcal{N}}$ (for both). The only difference between the two sides comes from the underlying lattices: $X_{\mathfrak{f}}$ is usually not equal to $X_{\mathfrak{f},\mathrm{ad}}$.

3.4. **Relations with the cuspidal case.**

We want to relate the torus $\mathrm{Irr}(X_{\mathfrak{f}})$ and the Weyl group $W^\circ(J, \sigma)$ associated to $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma} \otimes \psi)$ with the torus $\mathrm{Irr}(L_{\mathfrak{f}})_{\mathfrak{s}_{L,\psi}}$ and the finite group $W_{\mathfrak{s}_\psi}$ associated by Bernstein to $\mathrm{Rep}(G)_{\mathfrak{s}_i}$. Let $X_{\mathrm{nr}}(L_{\mathfrak{f}}, \sigma)$ be the stabilizer of $\mathrm{ind}_{N_{L_{\mathfrak{f}}}(P_{L,\mathfrak{f}})}^{L_{\mathfrak{f}}}(\hat{\sigma}_i) \in \mathrm{Irr}_{\mathrm{cusp}}(L_{\mathfrak{f}})$ in $X_{\mathrm{nr}}(L_{\mathfrak{f}})$, with respect to the action of tensoring by unramified characters. Then $\big(\hat{P}_{L,\mathfrak{f}}, (\hat{\sigma} \otimes \psi)|_{\hat{P}_{Q,\mathfrak{f}}}\big)$ is a type for the supercuspidal Bernstein component

$$\mathrm{Rep}(L_{\mathfrak{f}})_{\mathfrak{s}_{L,\psi}} \quad \text{with} \quad \mathfrak{s}_{L,\psi} = \big[L_{\mathfrak{f}}, \mathrm{ind}_{N_{L_{\mathfrak{f}}}(P_{L,\mathfrak{f}})}^{L_{\mathfrak{f}}}(\hat{\sigma}_i)\big]_G. \tag{52}$$

Moreover, by [Lat, Theorem 6.2] $\big(\hat{P}_{L,\mathfrak{f}}, (\hat{\sigma} \otimes \psi)|_{\hat{P}_{Q,\mathfrak{f}}}\big)$ is the unique "archetype" for (52). This means that whenever $(K_{L_{\mathfrak{f}}}, \lambda)$ is a type for (52), there exists $l \in L_{\mathfrak{f}}$ such that $K_{L_{\mathfrak{f}}} \subset l\hat{P}_{L,\mathfrak{f}}l^{-1}$ and $\mathrm{ind}_{K_{L_{\mathfrak{f}}}}^{l\hat{P}_{L,\mathfrak{f}}l^{-1}} \lambda \cong l \cdot (\hat{\sigma} \otimes \psi)|_{\hat{P}_{Q,\mathfrak{f}}}$.

Theorem 3.5. (*a*) *Let Q be a Levi subgroup of G containing $L_{\mathfrak{f}}$ and write $\hat{P}_{Q,\mathfrak{f}} = \hat{P}_{\mathfrak{f}} \cap Q$. Then $(\hat{P}_{\mathfrak{f}}, \hat{\sigma} \otimes \psi)$ is a cover of $\big(\hat{P}_{Q,\mathfrak{f}}, (\hat{\sigma} \otimes \psi)|_{\hat{P}_{Q,\mathfrak{f}}}\big)$.*

(*b*) *There exists an algebra isomorphism $\mathcal{H}\big(L_{\mathfrak{f}}, \hat{P}_{L,\mathfrak{f}}, (\hat{\sigma} \otimes \psi)|_{\hat{P}_{L,\mathfrak{f}}}\big) \cong \mathbb{C}[X_{\mathfrak{f}}]$ which preserves supports. It is canonical up to twists by $X_{\mathrm{wr}}(G/Z(G)_s)$, where $Z(\mathcal{G})_s$ denotes the maximal K-split torus in $Z(\mathcal{G})$.*

(*c*) *There are homeomorphisms of complex tori*

$$X_{\mathrm{nr}}(L_{\mathfrak{f}})/X_{\mathrm{nr}}(L_{\mathfrak{f}}, \sigma) \to \mathrm{Irr}(L_{\mathfrak{f}})_{\mathfrak{s}_{L,\psi}} \to \mathrm{Irr}(X_{\mathfrak{f}}),$$

such that the composed map (between the outer terms) is a natural group homomorphism.

Proof. (a) For $Q = L_{\mathfrak{f}}$ this is [Mor3, Corollary 3.10]. Via the transitivity of covers [BuKu, Proposition 8.5] that implies it for other Q.

(b) We apply the earlier results from this section with $L_{\mathfrak{f}}$ instead of G. For all considerations in $L_{\mathfrak{f}}$ we must replace $\mathfrak{f}$ by its affine span in $\mathbb{A}$, which is a minimal facet in $\mathcal{B}(\mathcal{L}_{\mathfrak{f}}, K)$. Also $P_{L,\mathfrak{f}} = P_{\mathfrak{f}} \cap L_{\mathfrak{f}}$ is a maximal parahoric subgroup of $L_{\mathfrak{f}}$. Hence $W_{L,\mathrm{aff}}(J, \sigma|_{P_{L,\mathfrak{f}}}) = 1$ and

$$\mathcal{H}\big(L_{\mathfrak{f}}, \hat{P}_{L,\mathfrak{f}}, (\hat{\sigma} \otimes \psi)|_{\hat{P}_{L,\mathfrak{f}}}\big) \cong e_\psi \mathbb{C}[\Omega_{L,\mathfrak{f}}] \cong \mathbb{C}[\Omega_{L,\mathfrak{f}}/\Omega_{L,\mathfrak{f},\mathrm{tor}}].$$

We still need to identify the subgroup $\Omega_{L,\mathfrak{f}}$ of $W(J, \sigma)$. It consists of all elements of $W(J, \sigma)$ that have a representative in $N_{L_{\mathfrak{f}}}(S)$. By (30) and (39), $\Omega_{L,\mathfrak{f}}$ contains the group $\Omega_{\mathfrak{f},\mathrm{tor}} \cong \hat{P}_{L,\mathfrak{f}}/P_{L,\mathfrak{f}}$, namely as $\Omega_{L,\mathfrak{f},\mathrm{tor}}$, (the group $\Omega_{\mathfrak{f},\mathrm{tor}}$ with respect to $L_\mathfrak{f}$). From Proposition 3.1 we see that $X_\mathfrak{f} \subset X_*(Z_G(S))$ can be represented in $Z_G(S) = Z_{L_\mathfrak{f}}(S)$, so $X_\mathfrak{f} \subset \Omega_{L,\mathfrak{f}}/\Omega_{\mathfrak{f},\mathrm{tor}}$. Proposition 3.1 (for $L_\mathfrak{f}$) shows that $\Omega_{L,\mathfrak{f}}/\Omega_{\mathfrak{f},\mathrm{tor}}$ acts by translations on $\mathbb{A}_\mathfrak{f}$ (which we now view as the affine building of $Z(L_\mathfrak{f})^\circ$). By Proposition 3.1 the only elements of $W(J,\sigma)/\Omega_{\mathfrak{f},\mathrm{tor}}$ which act on $\mathbb{A}_\mathfrak{f}$ by translations, are those of $X_\mathfrak{f}$. Hence $\Omega_{L,\mathfrak{f}}/\Omega_{\mathfrak{f},\mathrm{tor}} = X_\mathfrak{f}$.

We remark that from the above, (33), (34) and Proposition 3.1 one can see that $\Omega_{L,\mathfrak{f}} \cap W_{\mathrm{aff}}(J,\sigma) = X(J)$.

From [Lus4, §1.18] and the explanation around (46) we know that the entire ambiguity comes from choosing an extension of the $P_\mathfrak{f}$-representation σ to $N_G(P_\mathfrak{f})$. By the remarks following (45), this can be done canonically on $X_*(Z(G)^\circ)P_\mathfrak{f}$. Taking that into account, all choices differ by a character of $\Omega_\mathfrak{f} \cong N_G(P_\mathfrak{f})/P_\mathfrak{f}$ which is trivial on $X_*(Z(G)^\circ) = X_*(Z(G)_s)$. That can be regarded as a character of $\Omega_\mathfrak{f}/X_*(Z(G)_s)$, a subgroup of $\Omega_{G/Z(G)_s}$. As the target group $\mathbb{C}^\times$ is divisible, every character of $\Omega_\mathfrak{f}/X_*(Z(G)_s)$ can be extended to $\Omega_{G/Z(G)_s}$. Thus the difference between any two choices of the algebra isomorphisms under consideration can obtained by tensoring with an element of $\mathrm{Irr}(\Omega_{G/Z(G)_s}) = X_{\mathrm{wr}}(G/Z(G)_s)$.

(c) The equivalence of categories from Theorem 3.2.b for $L_\mathfrak{f}$ restricts to a bijection

$$\mathrm{Irr}(L_\mathfrak{f})_{\mathfrak{s}_{L,\psi}} \to \mathrm{Irr}(\mathcal{H}(L_\mathfrak{f}, \hat{P}_{L,\mathfrak{f}}, (\hat\sigma \otimes \psi)|_{\hat{P}_{L,\mathfrak{f}}})) : V \mapsto \mathrm{Hom}_{\hat{P}_{L,\mathfrak{f}}}(\hat\sigma \otimes \psi, V). \tag{53}$$

The left hand side is

$$\big\{\chi \otimes \mathrm{ind}_{N_{L_\mathfrak{f}}(P_{L,\mathfrak{f}})}^{L_\mathfrak{f}}(\hat\sigma_i) : \chi \in X_{\mathrm{nr}}(L_\mathfrak{f})\big\},$$

which by construction admits a simply transitive action of $X_{\mathrm{nr}}(L_\mathfrak{f})/X_{\mathrm{nr}}(L_\mathfrak{f},\sigma)$. By part (a) the right hand side of (53) can be identified with $\mathrm{Irr}(\mathbb{C}[X_\mathfrak{f}]) = \mathrm{Irr}(X_\mathfrak{f})$. Let $i(\sigma)_\psi$ be the unique unramified twist of $\mathrm{ind}_{N_{L_\mathfrak{f}}(P_{L,\mathfrak{f}})}^{L_\mathfrak{f}}(\hat\sigma_i)$ which under (53) maps to the trivial representation of $X_\mathfrak{f}$. We take $\chi \mapsto \chi \otimes i(\sigma)_\psi$ as map $X_{\mathrm{nr}}(L_\mathfrak{f})/X_{\mathrm{nr}}(L_\mathfrak{f},\sigma) \to \mathrm{Irr}(L_\mathfrak{f})_{\mathfrak{s}_{L,\psi}}$. The canonical maps

$$X_\mathfrak{f} \hookrightarrow Z_G(S)/Z_G(S)_{\mathrm{cpt}} \leftarrow Z_G(S) \hookrightarrow L_\mathfrak{f} \tag{54}$$

induce a natural group homomorphism

$$X_{\mathrm{nr}}(L_\mathfrak{f}) \to \mathrm{Irr}(X_\mathfrak{f}) : \chi \mapsto \chi|_{X_\mathfrak{f}}. \tag{55}$$

Then Theorem 3.2.b and (53) send $\chi \otimes i(\sigma)_\psi$ to $\chi|_{X_\mathfrak{f}}$. As (53) is bijective, (55) induces a group isomorphism $X_{\mathrm{nr}}(L_\mathfrak{f})/X_{\mathrm{nr}}(L_\mathfrak{f},\sigma) \to \mathrm{Irr}(X_\mathfrak{f})$. $\square$

Lemma 3.6. (*a*) *Let QU_Q be a standard parabolic subgroup of G whose Levi factor Q contains $L_\mathfrak{f}$. There exists a natural embedding*

$$\mathcal{H}(Q, \hat{P}_{Q,\mathfrak{f}}, \hat\sigma \otimes \psi|_{\hat{P}_{Q,\mathfrak{f}}}) \to \mathcal{H}(G, \hat{P}_\mathfrak{f}, \hat\sigma \otimes \psi),$$

which identifies $\mathcal{H}(Q, \hat{P}_{Q,\mathfrak{f}}, \hat\sigma \otimes \psi|_{\hat{P}_{Q,\mathfrak{f}}})$ with a parabolic subalgebra of the affine Hecke algebra $\mathcal{H}(G, \hat{P}_\mathfrak{f}, \hat\sigma \otimes \psi)$.

(*b*) *Let $i(\sigma)_\psi \in \mathrm{Irr}(L_\mathfrak{f})_{\mathfrak{s}_{L,\psi}}$ be the unique element that maps to* $\mathrm{triv}_{X_\mathfrak{f}}$ *under* (53). *Then $i(\sigma)_\psi$ is fixed by $W_{\mathfrak{s}_\psi}$.*

(*c*) *The canonical map $W \to W(G,S)$ induces an isomorphism $W^\circ(J,\sigma) \to W_{\mathfrak{s}_\psi}$, and this makes the homeomorphisms in Theorem 3.5.c $W_{\mathfrak{s}_\psi}$-equivariant.*

Proof. (a) In view of Theorem 3.5.a, we may use the properties of covers, in particular [BuKu, Corollary 8.4]. The version of [BuKu, Corollary 8.4] with normalized parabolic induction [Sol2, Lemma 4.1] says that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Rep}(G)_{\mathfrak{s}_\psi} & \longrightarrow & \mathrm{Mod}\big(\mathcal{H}(G,\hat{P}_{\mathfrak{f}},\hat\sigma\otimes\psi)\big) \\ \uparrow I_{QU_Q}^G & & \uparrow \mathrm{ind}_{\mathcal{H}(Q,\hat{P}_{Q,\mathfrak{f}},\hat\sigma\otimes\psi|_{\hat{P}_{Q,\mathfrak{f}}})}^{\mathcal{H}(G,\hat{P}_{\mathfrak{f}},\hat\sigma\otimes\psi)} \\ \mathrm{Rep}(Q)_{\mathfrak{s}_{Q,\psi}} & \longrightarrow & \mathrm{Mod}\big(\mathcal{H}(Q,\hat{P}_{Q,\mathfrak{f}},\hat\sigma\otimes\psi|_{\hat{P}_{Q,\mathfrak{f}}})\big) \end{array} . \tag{56}$$

Here the embedding

$$\mathcal{H}(Q,\hat{P}_{Q,\mathfrak{f}},\hat\sigma\otimes\psi|_{\hat{P}_{Q,\mathfrak{f}}}) \to \mathcal{H}(G,\hat{P}_{\mathfrak{f}},\hat\sigma\otimes\psi) \tag{57}$$

comes from [BuKu, §7–8] and depends on the parabolic subgroup QU_Q of G. By Theorem 3.5.b we may take $\mathbb{C}[X_\mathfrak{f}]$ as the Hecke algebra for $(\hat{P}_{L,\mathfrak{f}},(\hat\sigma\otimes\psi)|_{\hat{P}_{L,\mathfrak{f}}})$. By naturality (57) is the identity on the canonical images of $\mathbb{C}[X_\mathfrak{f}]$. As (57) preserves supports, it sends every generator T_s with $s\in S_{Q,\mathfrak{f}}$ on the left side to T_s on the right side. Thus, in terms of the Bernstein presentation (57) is just a standard embedding of a parabolic subalgebra of an affine Hecke algebra.
(b) It is known from [Lus3, §3] that

$$Z\big(\mathcal{H}(G,\hat{P}_{\mathfrak{f}},\hat\sigma\otimes\psi)\big) \cong \mathbb{C}[X_\mathfrak{f}]^{W^\circ(J,\sigma)} = \mathcal{O}\big(\mathrm{Irr}(X_\mathfrak{f})/W^\circ(J,\sigma)\big).$$

On the other hand, by [BeDe, Théorème 2.13], the centre of the category $\mathrm{Rep}(G)_{\mathfrak{s}_\psi}$ is $\mathcal{O}(\mathrm{Irr}(L_\mathfrak{f})_{\mathfrak{s}_{L,\psi}}/W_{\mathfrak{s}_\psi})$. From (56) we see in particular that Theorem 3.2.b sends

$$I_{L_\mathfrak{f}U_{L_\mathfrak{f}}}^G(\chi\otimes i(\sigma)_\psi) \quad \text{to} \quad \mathrm{ind}_{\mathbb{C}[X_\mathfrak{f}]}^{\mathcal{H}(G,\hat{P}_\mathfrak{f},\hat\sigma\otimes\psi)}(\chi|_{X_\mathfrak{f}}). \tag{58}$$

On both sides of Theorem 3.2.b the centres of the categories can be detected by their actions on parabolically induced representations as in (58). Thus the bijection $\mathrm{Irr}(L_\mathfrak{f})_{\mathfrak{s}_{L,\psi}}\to\mathrm{Irr}(X_\mathfrak{f})$ from Theorem 3.5.c induces a bijection

$$\mathrm{Irr}(L_\mathfrak{f})_{\mathfrak{s}_{L,\psi}}/W_{\mathfrak{s}_\psi} \to \mathrm{Irr}(X_\mathfrak{f})/W^\circ(J,\sigma). \tag{59}$$

As both $W_{\mathfrak{s}_\psi}$ and $W^\circ(J,\sigma)$ are finite groups acting faithfully by automorphisms of complex affine varieties, the subspaces of these tori on which the isotropy groups are trivial form Zariski-open dense subvarieties. For $w\in W_{\mathfrak{s}_\psi}$ and $t\in\mathrm{Irr}(L_\mathfrak{f})_{\mathfrak{s}_{L,\psi}}$ with image $t'\in\mathrm{Irr}(X_\mathfrak{f})$ such that both have trivial stabilizer, the condition that $w(t)$ maps to $w'(t')$ completely determines w'. That yields a unique group isomorphism

$$W_{\mathfrak{s}_\psi} \to W^\circ(J,\sigma) \tag{60}$$

such that $\mathrm{Irr}(L_\mathfrak{f})_{\mathfrak{s}_{L,\psi}}\to\mathrm{Irr}(X_\mathfrak{f})$ becomes $W_{\mathfrak{s}_\psi}$-equivariant. Clearly the trivial representation of $X_\mathfrak{f}$ is fixed by $W^\circ(J,\sigma)$, and hence $i(\sigma)_\psi$ is fixed by $W_{\mathfrak{s}_\psi}$.
(c) By (60) the bijection

$$X_{\mathrm{nr}}(L_\mathfrak{f})/X_{\mathrm{nr}}(L_\mathfrak{f},\sigma)\to\mathrm{Irr}(L_\mathfrak{f})_{\mathfrak{s}_{L,\psi}} : \chi\mapsto\chi\otimes i(\sigma)_\psi$$

from Theorem 3.5.c is also $W_{\mathfrak{s}_\psi}$-equivariant. Consequently the group isomorphism

$$X_{\mathrm{nr}}(L_\mathfrak{f})/X_{\mathrm{nr}}(L_\mathfrak{f},\sigma)\to\mathrm{Irr}(X_\mathfrak{f}) \tag{61}$$

induced by (54) is $W_{\mathfrak{s}_\psi}$-equivariant. The group $W^\circ(J,\sigma)$ acts naturally on $X_\mathfrak{f}$, the action is induced by conjugation in $N_G(S)$. Conjugation also yields actions of

$N_G(S)$ and $W^\circ(J,\sigma)$ on $L_{\mathfrak f}$ and on $X_{\mathrm{nr}}(L_{\mathfrak f})$. Conjugation by $Z_G(S)$ does not change (unramified) characters, so the action of $W^\circ(J,\sigma)$ factors through

$$W^\circ(J,\sigma) \to W \to W(G,S). \tag{62}$$

Thus (61) is equivariant for the canonical actions of $W^\circ(J,\sigma)$, and is equivariant with respect to the above isomorphism $W^\circ(J,\sigma) \cong W_{\mathfrak{s}_\psi}$. By the uniqueness of that isomorphism, it must agree with the map induced by (62). $\square$

4. Comparison of Hecke algebras

Let $\mathcal{G}$ be a connected reductive K-group which splits over an unramified extension of K. We denote the set of ($G^\vee$-equivalence classes of) unramified L-parameters for G by $\Phi_{\mathrm{nr}}(G)$. We indicate the set of unipotent representations in $\mathrm{Irr}(G)$ (or $\mathrm{Rep}(G)$ etc.) by a subscript "unip".

We formulate the main result of [FOS1], and then derive some useful consequences. Recall that the HII conjectures [HII] compare the formal degree of a square-integrable modulo centre G-representation with (the specialization at $s = 0$ of) the adjoint γ-factor of its L-parameter.

Theorem 4.1. *There exists a bijective map*

$$\mathrm{Irr}_{\mathrm{cusp,unip}}(G) \to \Phi_{\mathrm{nr,cusp}}(G) : \pi \mapsto (\lambda_\pi, \rho_\pi)$$

with the following properties:

(i) Equivariance with respect to the natural actions of $X_{\mathrm{wr}}(G)$.
(ii) Compatibility with almost direct products of reductive groups.
(iii) Equivariance with respect to $\mathbf{W}_K$-automorphisms of the absolute root datum of $\mathcal{G}$.
(iv) The map $\pi \mapsto \lambda_\pi$ makes the HII conjectures true for $\mathrm{Irr}_{\mathrm{cusp,unip}}(G)$.
(v) Let $Z(\mathcal{G})_s$ be the maximal central K-split torus in $\mathcal{G}$. The map $\pi \mapsto \lambda_\pi$ is determined by the above properties (i), (ii) and (iv), up to twisting by weakly unramified characters of $G/Z(G)_s = (\mathcal{G}/Z(\mathcal{G})_s)(K)$.

Proof. The theorem is a combination of [FOS1, Theorems 2 and 3]. We take this opportunity to point out and fix some omissions in the proof of part (iii) in [FOS1]. It is explained in [FOS1, pages 1142–1143] how $\mathbf{W}_K$-automorphisms of the absolute root datum of $\mathcal{G}$ act on Langlands parameters and on representations of inner forms of $\mathcal{G}$.

In [FOS1] a few such automorphisms of G were overlooked, in the cases where $\mathcal{G}$ is simple, adjoint, an outer form of a split group and not quasi-split. Let K_{nr} denote the unramified closure of K, so that $G = \mathcal{G}(K) = \mathcal{G}(K_{\mathrm{nr}})^F$. Here the action F of the Frobenius element is the composition of

- the field automorphism Frob of K_{nr}/K, applied to matrix coefficients,
- an automorphism τ of the absolute root datum of $\mathcal{G}$,
- conjugation by an element $\omega \in \mathcal{G}_{\mathrm{ad}}(K_{\mathrm{nr}})$.

As G is an outer form and not quasi-split, τ and ω are nontrivial. In this situation $\mathrm{Ad}(\omega)\tau$ is an F-automorphism of $\mathcal{G}$, and on $G = \mathcal{G}(K)$ it reduces to the field automorphism Frob^{-1}.

In the cases under consideration $G^\vee$ is simply connected and $Z(G^\vee)^{\mathbf{W}_K} \neq Z(G^\vee)$. The automorphism for Langlands parameters associated to $\mathrm{Ad}(\omega)\tau$ is induced by

τ and stabilizes $\Phi(G)$, but it does not fix the character ζ of $Z(G^\vee)$. On the other hand, τ does fix

$$\zeta_{\mathcal{G}} \in \mathrm{Irr}(Z(G^\vee)^{\mathbf{W}_F}) = \mathrm{Irr}(Z(G^\vee)^+).$$

By (8) and the subsequent comment, that is just as good as fixing ζ, and τ canonically induces a permutation of $\Phi_e(G)$.

The omitted automorphisms of G should occur on [FOS1, pages 1149–1150] and in the cases $J = {}^2(D_t A_s D_t)$ and $J = {}^2 A_s$ on pages 1158–1159 of [FOS1]. With the above knowledge, the analysis on [FOS1, pages 1149–1150] can be extended to the missing automorphism by following the arguments for the corresponding quasi-split unitary groups. The omissions on [FOS1, pages 1158–1159] can be fixed by arguments completely analogous to the case $J = {}^2 A_s$ on [FOS1, page 1157]. $\square$

Let $\mathfrak{Be}(G)_{\mathrm{unip}}$ be the subset of $\mathfrak{Be}(G)$ obtained from $\mathrm{Irr}_{\mathrm{unip}}(G)$ by taking inertial equivalence classes, and similarly let $\mathfrak{Be}^\vee(G)_{\mathrm{nr}}$ be the subset of $\mathfrak{Be}^\vee(G)$ obtained from $\Phi_{\mathrm{nr},e}(G)$.

Proposition 4.2. (*a*) *Theorem 4.1 induces a bijection*

$$\mathfrak{Be}(G)_{\mathrm{unip}} \to \mathfrak{Be}^\vee(G)_{\mathrm{nr}} : \mathfrak{s} \mapsto \mathfrak{s}^\vee.$$

If $\mathfrak{s}$ can be represented by a cuspidal inertial class for a Levi subgroup L of G, then so can $\mathfrak{s}^\vee$, and conversely.

(*b*) *Suppose that $\mathfrak{s} = [L, \pi_L]_G$ for some $\pi_L \in \mathrm{Irr}_{\mathrm{cusp,unip}}(L)$. There is a natural isomorphism $W_{\mathfrak{s}} \cong W_{\mathfrak{s}^\vee}$, and it makes the bijection $\mathrm{Irr}(L)_{\mathfrak{s}_L} \to \Phi_e(L)^{\mathfrak{s}_L^\vee}$ $W_{\mathfrak{s}}$-equivariant.*

Proof. (a) By Corollary 1.3 it suffices to show this for inertial equivalence classes based on objects for a Levi subgroup L of G. By property (i), the bijection in Theorem 4.1 induces a bijection

$$\mathrm{Irr}_{\mathrm{cusp,unip}}(G)/X_{\mathrm{nr}}(G) \to \Phi_{\mathrm{nr,cusp}}(G)/X_{\mathrm{nr}}(G).$$

Applying this to L, we obtain a bijection between the cuspidal inertial classes for $\mathrm{Irr}_{\mathrm{unip}}(L)$ and $\Phi_{\mathrm{nr},e}(L)$, say

$$\mathfrak{s}_L \longleftrightarrow \mathfrak{s}_L^\vee. \tag{63}$$

We remark that in view of property (v) of Theorem 4.1, (63) is canonical up to twisting by $X_{\mathrm{wr}}(L/Z(L)_s)/X_{\mathrm{nr}}(L/Z(L)_s)$. Two such classes $\mathfrak{s}_L, \mathfrak{s}'_L$ become the same in $\mathfrak{Be}(G)$ if and only if they are conjugate by an element of $N_G(L)$. As L acts trivially here, this is equivalent to $\mathfrak{s}_L$ and $\mathfrak{s}'_L$ being in the same orbit under $N_G(L)/L$. The action of $N_G(L)/L$ on $\mathrm{Irr}(L)$ comes from its action (by $\mathbf{W}_K$-equivariant automorphisms) on the absolute root datum of $\mathcal{L}$.

Similarly [AMS1, (117)], the two classes $\mathfrak{s}_L^\vee, \mathfrak{s}'^\vee_L$ become the same in $\mathfrak{Be}^\vee(G)$ if and only if they are conjugate by an element of $N_{G^\vee}(L^\vee \rtimes \mathbf{W}_K)$, or equivalently by an element of $N_{G^\vee}(L^\vee \rtimes \mathbf{W}_K)/L^\vee$. This action of $N_{G^\vee}(L^\vee \rtimes \mathbf{W}_K)/L^\vee$ is determined by its action on the absolute root datum of $L^\vee$ (or equivalently that of $\mathcal{L}$).

By [ABPS, Proposition 3.1] there is a natural isomorphism

$$N_G(L)/L \cong N_{G^\vee}(L^\vee \rtimes \mathbf{W}_K)/L^\vee. \tag{64}$$

Its construction entails that both sides act in the same way on the absolute root datum of $\mathcal{L}$. Now property (iii) of Theorem 4.1 for L says that $\pi_L \mapsto (\phi_{\pi_L}, \rho_{\pi_L})$ is equivariant for the indicated actions of (64).

(b) By definition $W_{\mathfrak{s}}$ is the stabilizer of $\mathfrak{s}_L \cong \mathrm{Irr}(L)_{\mathfrak{s}_L}$ in $N_G(L)/L$, and $W_{\mathfrak{s}^\vee}$ is the stabilizer of $\mathfrak{s}_L^\vee \cong \Phi_e(L)^{\mathfrak{s}_L^\vee}$ in $N_{G^\vee}(L^\vee \rtimes \mathbf{W}_K)/L^\vee$. By the above $N_G(L)/L$-equivariance of (63), the isomorphism (64) restricts to $W_{\mathfrak{s}^\vee} \cong W_{\mathfrak{s}}$. In particular the bijection $\mathrm{Irr}(L)_{\mathfrak{s}_L} \to \Phi_e(L)^{\mathfrak{s}_L}$ from Theorem 4.1 becomes equivariant for $W_{\mathfrak{s}}$. $\square$

Let us compare the Hecke algebras for L-parameters to those in the adjoint case. Replacing $\mathcal{G}$ by $\mathcal{G}_{\mathrm{ad}}$ means that $G^\vee$ is replaced by $G^\vee{}_{\mathrm{sc}}$, the simply connected cover of the derived group of $G^\vee$. Let $\mathcal{L} \subset \mathcal{G}$ be a Levi K-subgroup and write $\mathcal{L}_c = \mathcal{L}/Z(\mathcal{G}) \subset \mathcal{G}_{\mathrm{ad}}$, so that $L_c^\vee \subset G^\vee{}_{\mathrm{sc}}$ is the Levi subgroup determined by $L \subset G$.

Let $(\phi_L, \rho_L) \in \Phi_{\mathrm{nr,cusp}}(L)$. Since $\phi_L(\mathrm{Frob})$ determines $\phi_L|_{\mathbf{W}_F}$ completely, it is easy to lift ϕ_L to a L-parameter $\phi_{L,\mathrm{ad}}$ for L_c: we only have to pick a lift of $\phi_L(\mathrm{Frob})$ in $L_c^\vee \rtimes \mathbf{W}_K$. Then

$$Z_{L^\vee{}_{\mathrm{sc}}}(\phi_{L,\mathrm{ad}})^\circ = Z_{L^\vee{}_{\mathrm{sc}}}(\phi_L)^\circ$$

(in fact these groups are trivial because ϕ_L is discrete) and

$$Z^1_{L^\vee{}_{\mathrm{sc}}}(\phi_{L,\mathrm{ad}}) \subset Z^1_{L^\vee{}_{\mathrm{sc}}}(\phi_L). \tag{65}$$

Hence $\mathcal{S}_{\phi_{L,\mathrm{ad}}}$ is naturally embedded in $\mathcal{S}_{\phi_L}$. Let $\rho_{L,\mathrm{ad}}$ be an irreducible representation of $\mathcal{S}_{\phi_{L,\mathrm{ad}}}$ appearing in $\rho\big|_{\mathcal{S}_{\phi_{L,\mathrm{ad}}}}$. The conditions for ρ_L to be cuspidal and L-relevant depend only on

$$(L^\vee_{\phi_{L,\mathrm{ad}}})^\circ = Z_{L^\vee{}_{\mathrm{sc}}}(\phi_{L,\mathrm{ad}}(\mathbf{W}_K))^\circ = Z_{L^\vee{}_{\mathrm{sc}}}(\phi_L(\mathbf{W}_K))^\circ = (L^\vee_{\phi_L})^\circ, \tag{66}$$

so $(\phi_{L,\mathrm{ad}}, \rho_{L,\mathrm{ad}}) \in \Phi_{\mathrm{nr,cusp}}(L_c)$.

Let $\pi_L \in \mathrm{Irr}_{\mathrm{cusp,unip}}(L)$ and $\pi_{L,\mathrm{ad}} \in \mathrm{Irr}_{\mathrm{cusp,unip}}(L_c)$ be the representations associated to, respectively, (ϕ_L, ρ_L) and $(\phi_{L,\mathrm{ad}}, \rho_{L,\mathrm{ad}})$ by Theorem 4.1. The constructions in [FOS1, §14–15] entail that, up to a twist by a weakly unramified character, π_L is contained in the pullback of $\pi_{L,\mathrm{ad}}$ along $q : L \to L_c$.

Let $\mathfrak{s}^\vee_{L,\mathrm{ad}}$ be the inertial class for $\Phi_e(L_c)$ containing $(\phi_{L,\mathrm{ad}}, \rho_{L,\mathrm{ad}})$, and let $\mathfrak{s}^\vee_{\mathrm{ad}}$ be the resulting inertial equivalence class for $\Phi_e(G)$. We note that the canonical homomorphism $q^\vee : L_c^\vee \to L^\vee$ induces maps

$$\mathfrak{s}^\vee_{L,\mathrm{ad}} \to \mathfrak{s}^\vee_L : (\phi'_{L,\mathrm{ad}}, \rho_{L,\mathrm{ad}}) \mapsto ({}^L q \circ \phi'_{L,\mathrm{ad}}, \rho_L) \tag{67}$$

and $\mathfrak{s}^\vee_{\mathrm{ad}} \to \mathfrak{s}^\vee$. These maps depend on the choice of $\rho_{L,\mathrm{ad}}$ and ρ_L, but given $\mathfrak{s}^\vee_{L,\mathrm{ad}}$ and $\mathfrak{s}^\vee_L$, they are canonical.

Let $\mathfrak{s} \in \mathfrak{Be}(G)$ and $\mathfrak{s}_{\mathrm{ad}} \in \mathfrak{Be}(G_{\mathrm{ad}})$ be the inertial equivalence classes obtained from $\mathfrak{s}^\vee$ and $\mathfrak{s}^\vee_{\mathrm{ad}}$ via Proposition 4.2.

Lemma 4.3. *The following objects are the same for $G^\vee$ and for $G^\vee{}_{\mathrm{sc}}$, up to natural isomorphisms: $W_{\mathfrak{s}^\vee}, \Phi_{\mathfrak{s}^\vee}, \lambda, \lambda^*$.*

For any $\alpha \in \Phi_{\mathfrak{s}^\vee}$ the function $\theta_{\alpha,\mathrm{ad}} \in \mathcal{O}(\mathfrak{s}^\vee_{L,\mathrm{ad}})$ is the composition of $\theta_\alpha \in \mathcal{O}(\mathfrak{s}^\vee_L)$ with the canonical map $\mathfrak{s}^\vee_{L,\mathrm{ad}} \to \mathfrak{s}^\vee_L$.

Proof. The canonical maps $G^\vee{}_{\mathrm{sc}} \to G^\vee, G \to G_{\mathrm{ad}}$ and (64) combine to a commutative diagram

$$\begin{array}{ccc} N_{G^\vee{}_{\mathrm{sc}}}(L_c^\vee \rtimes \mathbf{W}_K)/L_c^\vee & \cong & N_{G_{\mathrm{ad}}}(L_c)/L_c \\ \downarrow & & \downarrow \\ N_{G^\vee}(L^\vee \rtimes \mathbf{W}_K)/L^\vee & \cong & N_G(L)/L \end{array} \tag{68}$$

It is easy to see that the vertical maps in (68) are isomorphisms. The groups $W_{\mathfrak{s}^\vee}, W_{\mathfrak{s}^\vee_{\mathrm{ad}}}, W_{\mathfrak{s}}$ and $W_{\mathfrak{s}_{\mathrm{ad}}}$ are contained in the corners of this diagram, as the subgroups stabilizing, respectively, $\mathfrak{s}^\vee, \mathfrak{s}^\vee_{\mathrm{ad}}, \mathfrak{s}$ and $\mathfrak{s}_{\mathrm{ad}}$. In Proposition 4.2.b we showed that the rows in (68) restrict to isomorphisms $W_{\mathfrak{s}^\vee_{\mathrm{ad}}} \cong W_{\mathfrak{s}_{\mathrm{ad}}}$ and $W_{\mathfrak{s}^\vee} \cong W_{\mathfrak{s}}$. By Lemmas 3.3 and 3.6.b the right column of (68) restricts to an isomorphism $W_{\mathfrak{s}} \to W_{\mathfrak{s}_{\mathrm{ad}}}$. By the commutativity of the diagram, the left column restricts to an isomorphism $W_{\mathfrak{s}^\vee} \to W_{\mathfrak{s}^\vee_{\mathrm{ad}}}$.

By (66) and [AMS3, Definition 3.11 and Lemma 3.12], both $\Phi_{\mathfrak{s}^\vee}$ and $\Phi_{\mathfrak{s}^\vee_{\mathrm{ad}}}$ come from the same root system $\Phi\big((G^\vee_{\phi_L})^\circ, Z(L_c^\vee)^{\mathbf{W}_F,\circ}\big)$. This implies that the canonical map

$$T_{\mathfrak{s}^\vee_{\mathrm{ad}}} \to T_{\mathfrak{s}^\vee} \text{ provides a bijection } \Phi_{\mathfrak{s}^\vee_{\mathrm{ad}}} \to \Phi_{\mathfrak{s}^\vee}. \tag{69}$$

Recall that all these roots evaluate to 1 on the basepoints of $\mathfrak{s}_L^\vee$ and $\mathfrak{s}^\vee_{L,\mathrm{ad}}$ [AMS3, Proposition 3.9.b]. Hence the functions θ_α and $\theta_{\alpha,\mathrm{ad}}$ they determine on, respectively, $\mathfrak{s}_L^\vee$ and $\mathfrak{s}^\vee_{L,\mathrm{ad}}$, are related by composition with the canonical map from (67).

As described after (14), the label functions λ and λ^* depend only on objects living in $Z_{G^\vee{}_{\mathrm{sc}}}(t\phi_L(\mathbf{W}_K))^\circ$, for a few $t \in (Z(L^\vee)^{\mathbf{I}_K})^\circ_{\mathrm{Frob}}$. From the proof of [AMS3, Lemma 3.14] one sees that if $t_{\mathrm{ad}}\phi_{L,\mathrm{ad}} \in \Phi_{\mathrm{nr}}(L_c)$ is suitable to compute $\lambda_{\mathrm{ad}}(\alpha)$ and $\lambda^*_{\mathrm{ad}}(\alpha)$, then its image $t\phi_L$ in $\Phi_{\mathrm{nr}}(L)$ is suitable to compute $\lambda(\alpha)$ and $\lambda^*(\alpha)$. Using these t's, we see that λ, λ^* and $\lambda_{\mathrm{ad}}, \lambda^*_{\mathrm{ad}}$ are given by the same formulas, namely those in the proof of [AMS3, Lemma 3.14]. Hence $\lambda_{\mathrm{ad}} = \lambda$ and $\lambda^*_{\mathrm{ad}} = \lambda^*$, with the canonical bijection (69) in mind. $\square$

From Lemma 4.3 and the discussion around (15) and (16) it is clear that the affine Hecke algebras $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ and $\mathcal{H}(\mathfrak{s}^\vee_{\mathrm{ad}}, \vec{v})$ have almost the same presentation. That and the similar comparisons from Paragraph 3.4 will enable us to pass from the adjoint to the general case of the next result.

Theorem 4.4. *Let $(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma} \otimes \psi)$ and $\mathfrak{s}_\psi \in \mathfrak{Be}(G)$ be as in Theorem 3.2, and let $\mathfrak{s}^\vee_\psi \in \mathfrak{Be}^\vee(G)$ be the image of $\mathfrak{s}^\vee_\psi$ under Proposition 4.2.a. Theorem 4.1 and Proposition 4.2.b induce a unique algebra isomorphism*

$$\mathcal{H}(\mathfrak{s}^\vee_\psi, \vec{v}) \to \mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma} \otimes \psi).$$

It comes from a canonical isomorphism between the underlying based root data. The requirement that (under the correspondence between the roots on both sides) $\vec{v}^\lambda$ and $\vec{v}^{\lambda^*}$ must agree with the parameter function $q_K^{\mathcal{N}}$ for $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma)$, forces the choice $v_j = q_K^{1/2} = |k|^{1/2}$ for every entry v_j of $\vec{v}$.

Proof. We start with the p-adic side. Theorem 3.5 entails that there is a unique algebra isomorphism

$$\mathcal{H}(L_{\mathfrak{f}}, \hat{P}_{L,\mathfrak{f}}, (\hat{\sigma} \otimes \psi)|_{\hat{P}_{L,\mathfrak{f}}}) \cong \mathcal{O}(\mathfrak{s}_{L,\psi}) \tag{70}$$

such that Theorem 3.2.b for $\mathrm{Irr}(L_{\mathfrak{f}})_{\mathfrak{s}_{L,\psi}}$ just sends π_L to the character of $\mathcal{O}(\mathfrak{s}_{L,\psi})$ given by evaluation at (L, π_L). By Proposition 3.1, Theorem 3.5.b, (49) and (70) the multiplication map

$$\mathcal{O}(\mathfrak{s}_{L,\psi}) \otimes \mathcal{H}(W_{\mathfrak{s}_\psi}, q_K^{\mathcal{N}}) \to \mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma} \otimes \psi) \tag{71}$$

is an isomorphism of vector spaces. To compare these algebras with their analogues on the Galois side, we first look at a special case.

Lemma 4.5. *Theorem 4.4 holds if $\mathcal{G}$ is absolutely simple and adjoint.*

Proof. We are in the setting of [Lus4, Lus5]. Our affine Hecke algebra $\mathcal{H}(\mathfrak{s}_\psi^\vee, \vec{v})$ can be identified with $\mathbf{H}(G, G_J, \mathcal{C}, \mathcal{F})$ (from [Lus4, §5.17], when $\mathcal{G}$ is an inner form of a K-split group) or more generally with $\mathbf{H}(G\theta, G_J, \mathcal{C}, \mathcal{F})$ from [Lus5, §8.2].

To match Lusztig's notations with ours, we must take $G = G^\vee_{\phi_L}, G_J = L_c^\vee, \mathcal{C}$ the adjoint orbit of $\log(u_{\phi_L})$ and $\mathcal{F}$ the cuspidal local system on $\mathcal{C}$ determined by ρ_L. Then the construction of $\mathcal{H}(\mathfrak{s}_\psi^\vee, \vec{v})$ in [AMS1, AMS2, AMS3] boils down to the relevant parts of [Lus4, Lus5]. (In fact this was a starting point of the work of Aubert–Moussaoui–Solleveld.)

In [Lus4, Theorem 6.3] and [Lus5, Theorem 10.11] Lusztig exhibited, in particular, a matching between Bernstein components for $\mathrm{Irr}_{\mathrm{unip}}(G)$ and for $\Phi_{\mathrm{nr},e}(G)$. The bijection in Theorem 4.1 comes from [FeOp] and agrees with Lusztig's parametrization of supercuspidal unipotent representations [FOS1, Theorem 1]. Hence Lusztig's matching of Bernstein components is the same as in Proposition 4.2.a.

As explained in the proofs of [Lus4, Theorem 6.3] and [Lus5, Theorem 10.11], this matching is such that the corresponding affine Hecke algebras on both sides have the same Iwahori–Matsumoto presentation. We can reformulate that by saying that $\mathcal{H}(G, \hat{P}_\mathfrak{f}, \hat{\sigma} \otimes \psi)$ and $\mathbf{H}(G\theta, G_J, \mathcal{C}, \mathcal{F}) = \mathcal{H}(\mathfrak{s}_\psi^\vee, \vec{v})$ have the same Bernstein presentation. In particular the root data

$$(X_\mathfrak{f}, R_\mathfrak{f}, X_\mathfrak{f}^\vee, R_\mathfrak{f}^\vee) \quad \text{and} \quad \mathcal{R}_{\mathfrak{s}^\vee} = \big(X^*(T_{\mathfrak{s}_\psi^\vee}), \Phi_{\mathfrak{s}_\psi^\vee}, X_*(T_{\mathfrak{s}_\psi^\vee}), \Phi^\vee_{\mathfrak{s}_\psi^\vee}\big) \tag{72}$$

are isomorphic. This and Proposition 4.2.b imply

$$W_{\mathfrak{s}_\psi^\vee} \cong W_{\mathfrak{s}_\psi} \cong W(R_\mathfrak{f}) \cong W(\Phi_{\mathfrak{s}_\psi^\vee}). \tag{73}$$

Consequently $\mathfrak{R}_{\mathfrak{s}_\psi^\vee} \cong W_{\mathfrak{s}_\psi^\vee} / W(\Phi_{\mathfrak{s}_\psi^\vee})$ is trivial.

The isomorphism of root data (73) is induced by the $W_{\mathfrak{s}_\psi}$-equivariant bijection $\mathfrak{s}_{L,\psi} \to \mathfrak{s}_{L,\psi}^\vee$ from Theorem 4.1. (The choices of basepoints are not needed for this, since an adjustment of a basepoint only multiplies a (co)character by a complex number, and that still allows us to detect the same maps between (co)character lattices.) Although Theorem 4.1 is only canonical up to twists by weakly unramified characters, the isomorphism (72) is entirely canonical, for weakly unramified twists also just multiply (co)characters by nonzero scalars. (Such weakly unramified twists may move things to another Bernstein component, but the Hecke algebra for the new one is canonically identified with the original Hecke algebra.)

The Bernstein presentation (16) now entails that the multiplication map

$$\mathcal{O}(\mathfrak{s}_{L,\psi}^\vee) \otimes_\mathbb{C} \mathcal{H}(W_{\mathfrak{s}_\psi^\vee}, \vec{v}^{\,2\lambda}) \to \mathcal{H}(\mathfrak{s}_\psi^\vee, \vec{v}) \tag{74}$$

is a linear bijection. Theorem 4.1 induces an algebra isomorphism

$$\mathcal{O}(\mathfrak{s}_{L,\psi}^\vee) \to \mathcal{O}(\mathfrak{s}_{L,\psi}), \tag{75}$$

while Proposition 4.2.b gives rise to a linear bijection

$$\mathcal{H}(W_{\mathfrak{s}_\psi}, q_K^\mathcal{N}) \to \mathcal{H}(W_{\mathfrak{s}_\psi^\vee}, \vec{v}^{\,2\lambda}), \tag{76}$$

which sends a basis element T_w to a basis element $T_{w^\vee}$. The maps (75) and (76) will combine to an isomorphism between affine Hecke algebras, once we make the remaining choices appropriately.

The set of simple roots $\Delta_{\mathfrak{f}}$ of $R_{\mathfrak{f}}$ determines a (unique) basis $\Delta_{\mathfrak{s}^\vee_\psi}$ of $\Phi_{\mathfrak{s}^\vee_\psi}$ such that (72) becomes an isomorphism of based root data. As $W_{\mathfrak{s}^\vee_\psi} = W(\Phi_{\mathfrak{s}^\vee_\psi})$, we still had complete freedom to choose a basis for $\mathcal{R}_{\mathfrak{s}^\vee}$ in (15).

Since $\mathcal{G}$ is simple, so is $G^\vee_{\phi_L}$, and $\Phi_{\mathfrak{s}^\vee_\psi}$ is irreducible [Lus2, §2.13]. Hence the array of parameters $\vec{v}$ reduces to a single complex number v, and we may write $\mathcal{H}(\mathfrak{s}^\vee_\psi, v)$ for $\mathcal{H}(\mathfrak{s}^\vee_\psi, \vec{v})$. Lusztig showed in [Lus4, Lus5] that for $v = |k|^{1/2} = q_K^{1/2}$ the isomorphisms (72), (75) and (76) combine to an algebra isomorphism

$$\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma} \otimes \psi) \cong \mathcal{H}(\mathfrak{s}^\vee_\psi, q_K^{1/2}). \tag{77}$$

Notice that on the left hand side we have the parameters $q_K^{\mathcal{N}(\alpha)}$ for $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma}\otimes\psi)$ in the Iwahori–Matsumoto presentation, as given in [Lus4, §1.18], whereas on the right hand side we have the parameters $q_K^{\lambda(\alpha)/2}, q_K^{\lambda^*(\alpha)/2}$ for $\mathcal{H}(\mathfrak{s}^\vee_\psi, q_K^{1/2})$ in the Bernstein presentation. Transforming one presentation into the other, as in [Lus4, §5.12], yields the required relations between the parameters on both sides. $\square$

Continuation of the proof of Theorem 4.4

By (73) and Lemma 4.3, $W_{\mathfrak{s}^\vee_\psi} = W(\Phi_{\mathfrak{s}^\vee_\psi})$ and $\mathfrak{R}_{\mathfrak{s}^\vee_\psi} = \{1\}$. From that and (16) we see that the Bernstein presentation for $\mathcal{H}(\mathfrak{s}^\vee_\psi, \vec{v})$ expressed by (74) is valid in the current generality. Similarly the Bernstein presentation for $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma} \otimes \psi)$ is expressed by (71). Thus the Hecke algebras that we are interested in can be described as the vector spaces

$$\mathcal{O}(\mathfrak{s}_{L,\psi}) \otimes_{\mathbb{C}} \mathcal{H}(W_{\mathfrak{s}_\psi}, q_K^{\mathcal{N}}) \qquad \text{and} \qquad \mathcal{O}(\mathfrak{s}^\vee_{L,\psi}) \otimes_{\mathbb{C}} \mathcal{H}(W_{\mathfrak{s}^\vee_\psi}, \vec{v}^{\,2\lambda}), \tag{78}$$

together with the Bernstein–Lusztig–Zelevinsky relations from (17). Proposition 4.2 gives a natural isomorphism $W_{\mathfrak{s}_\psi} \cong W_{\mathfrak{s}^\vee_\psi}$, while the $W_{\mathfrak{s}_\psi}$-equivariant bijection $\mathfrak{s}_{L,\psi} \to \mathfrak{s}^\vee_{L,\psi}$ from Theorem 4.1 yields an algebra isomorphism $\mathcal{O}(\mathfrak{s}^\vee_{L,\psi}) \to \mathcal{O}(\mathfrak{s}_{L,\psi})$.

Together these provide a linear bijection between the two sides of (78). If the q-parameters match under this bijection, then the braid relations in the finite dimensional Iwahori–Hecke algebras and the Bernstein–Lusztig–Zelevinsky relations on both sides agree, and our bijection is an algebra isomorphism. When we replace $\mathcal{G}$ by its adjoint quotient, Lemmas 3.3 and 4.3 show that the only changes in (78) are that $\mathfrak{s}_{L,\psi}$ and $\mathfrak{s}^\vee_{L,\psi}$ are replaced by their versions for $\mathcal{G}_{\mathrm{ad}}$. Hence the setting with $\mathcal{G}_{\mathrm{ad}}$ is equally good to compare the q-parameters on both sides of (78).

Therefore we may assume that $\mathcal{G}$ is adjoint. Such a $\mathcal{G}$ is a direct product of simple, adjoint K-groups, and all objects under consideration factor accordingly. Thus, we may even assume that $\mathcal{G}$ is a simple adjoint K-group.

Then it is the restriction of scalars of an absolutely (i.e. over $\overline{K}$) simple K'-group $\mathcal{G}'$, for a finite unramified extension K'/K. On the p-adic side the identification $\mathcal{G}(K) = \mathcal{G}'(K')$ does not change the Hecke algebra of the type. We should, however, note that the parameter function $q_K^{\mathcal{N}}$ for $\mathcal{H}_{\mathrm{aff}}(G, P_{\mathfrak{f}}, \sigma)$ is now computed as $q_{K'}^{\mathcal{N}'}$, where $q_{K'}$ is the cardinality of the residue field of K'.

On the Galois side Lemma 2.4 says that $\mathcal{H}(\mathfrak{s}^\vee_\psi, \vec{v})$ is invariant under the Weil restriction $\mathcal{G} = \mathrm{Res}_{K'/K}\mathcal{G}'$. The parameters $v_j^{\lambda(\alpha)} = q_K^{\lambda(\alpha)/2}$ are then computed as $v_j'^{\,\lambda(\alpha')} = q_{K'}^{\lambda(\alpha')/2}$. Thus we reduced the verification of Theorem 4.4 to Lemma 4.5 for the absolutely simple adjoint group $\mathcal{G}'(K')$. $\square$

5. A local Langlands correspondence

Recall that $\mathcal{G}$ is a connected reductive K-group, which splits over an unramified extension of K. As in Section 2, we consider $\mathcal{G}$ as an inner twist of a quasi-split K-group. Let $\mathfrak{s}_\psi$ be a unipotent inertial equivalence class for $\mathrm{Irr}(G)$, as in Section 3. It is associated to a parahoric subgroup $P_\mathfrak{f}$, a cuspidal unipotent representation σ of $P_\mathfrak{f}$ and an extension $\hat\sigma \otimes \psi$ of σ to $\hat P_\mathfrak{f}$. Moreover $\mathfrak{s}_\psi$ comes from a cuspidal inertial class $\mathfrak{s}_{L,\psi}$ for a Levi subgroup $L = L_\mathfrak{f}$ of $G = \mathcal{G}(K)$. By Proposition 4.2.a $\mathfrak{s}_\psi$ gives rise to an inertial equivalence class $\mathfrak{s}_\psi^\vee$ of enhanced Langlands parameters for G.

Theorems 3.2.b, 4.4 and 2.1 yield bijections

$$\mathrm{Irr}(G)_{\mathfrak{s}_\psi} \longrightarrow \mathrm{Irr}\big(\mathcal{H}(G, \hat P_\mathfrak{f}, \hat\sigma \otimes \psi)\big) \longrightarrow \mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}_\psi^\vee, \vec{v})\big) \longrightarrow \Phi_e(G)^{\mathfrak{s}_\psi^\vee}. \tag{79}$$

By Theorem 3.5.b the first map in (79) is canonical up to twists by $X_{\mathrm{wr}}(G/Z(G)_s)$. From Theorem 4.4 and Theorem 4.1.v we see that the second map is canonical up to twisting by $X_{\mathrm{wr}}(L/Z(L)_s)$. By Theorem 2.1 the third map in (79) is entirely canonical.

However, there are (potentially) many choices for a $\mathfrak{s}_\psi$-type, and that may influence (79) as well. On the supercuspidal level, i.e. for $\mathrm{Rep}(L_\mathfrak{f})_{\mathfrak{s}_{L,\psi}})$, the type is essentially unique, see [Lat] and the discussion after (52). Beyond that different facets may support a $\mathfrak{s}_\psi$-type, and it seems difficult to analyse that.

Theorem 5.1. *The maps in* (79) *combine to a bijection*

$$\begin{array}{ccc} \mathrm{Irr}(G)_{\mathrm{unip}} & \longrightarrow & \Phi_{\mathrm{nr},e}(G) \\ \pi & \mapsto & (\phi_\pi, \rho_\pi) \end{array}.$$

Proof. Recall from [BeDe] and (12) that

$$\mathrm{Irr}(G)_{\mathrm{unip}} = \bigsqcup_{\mathfrak{s} \in \mathfrak{Be}(G)_{\mathrm{unip}}} \mathrm{Irr}(G)_\mathfrak{s} \quad \text{and} \quad \Phi_{\mathrm{nr},e}(G) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{Be}^\vee(G)_{\mathrm{nr}}} \Phi_e(G)^{\mathfrak{s}^\vee}.$$

In Proposition 4.2.a we found a bijection $\mathfrak{Be}(G)_{\mathrm{unip}} \leftrightarrow \mathfrak{Be}^\vee(G)_{\mathrm{nr}}$. Combine this with (79). $\square$

We check that the bijection in Theorem 5.1 satisfies many properties which are expected for a local Langlands correspondence.

Lemma 5.2. *Theorem 5.1 is compatible with direct products of reductive K-groups.*

Proof. Suppose that $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ as K-groups. Then all involved objects for $\mathcal{G}$ factorize naturally are direct products of the analogous objects for $\mathcal{G}_1$ and $\mathcal{G}_2$, for example

$$\Phi(G_1 \times G_2) = H^1(\mathbf{W}_K, G_1^\vee \times G_2^\vee) \cong H^1(\mathbf{W}_K, G_1^\vee) \times H^1(\mathbf{W}_K, G_2^\vee) = \Phi(G_1) \times \Phi(G_2).$$

Our constructions preserve these factorizations, that is implicit in all arguments. In particular $\pi_1 \otimes \pi_2 \in \mathrm{Irr}_{\mathrm{unip}}(G_1 \times G_2)$ is mapped to $(\phi_{\pi_1} \times \phi_{\pi_2}, \rho_{\pi_1} \otimes \rho_{\pi_2}) \in \Phi_{\mathrm{nr},e}(G_1 \times G_2)$. $\square$

Lemma 5.3. *The bijection in Theorem 5.1 is equivariant with respect to the natural actions of $X_{\mathrm{wr}}(G)$.*

Proof. First we reformulate (79) in a $X_{\mathrm{wr}}(G)$-stable setting. By (47) and (49)

$$\mathcal{H}(G, P_\mathfrak{f}, \sigma) = \bigoplus\nolimits_{\psi \in \mathrm{Irr}(\Omega_{\mathfrak{f},tor})} \mathcal{H}(G, \hat P_\mathfrak{f}, \hat\sigma \otimes \psi).$$

Hence the bijection from Theorem 5.1 can also be expressed as

$$
\begin{aligned}
(80)\quad \bigsqcup\nolimits_{\psi \in \mathrm{Irr}(\Omega_{\mathfrak{f},\mathrm{tor}})} \mathrm{Irr}(G)_{\mathfrak{s}_\psi} = \mathrm{Irr}(G)_{(P_{\mathfrak{f}},\sigma)} \to \mathrm{Irr}(\mathcal{H}(G,P_{\mathfrak{f}},\sigma)) \to \\
\mathrm{Irr}\Big(\bigoplus_{\psi \in \mathrm{Irr}(\Omega_{\mathfrak{f},\mathrm{tor}})} \mathcal{H}(\mathfrak{s}_\psi, \vec{v}) \Big) \to \bigsqcup_{\psi \in \mathrm{Irr}(\Omega_{\mathfrak{f},\mathrm{tor}})} \Phi_{nr,e}(G)^{\mathfrak{s}_\psi^\vee}.
\end{aligned}
$$

It is clear from (42) that the first arrow in (80) is $X_{\mathrm{wr}}(G)$-equivariant. The algebra isomorphism

$$
(81)\qquad \mathcal{H}(G,P_{\mathfrak{f}},\sigma) \to \bigoplus\nolimits_{\psi \in \mathrm{Irr}(\Omega_{\mathfrak{f},\mathrm{tor}})} \mathcal{H}(\mathfrak{s}_\psi, \vec{v})
$$

underlying the second arrow in (80) consists of two parts. Firstly (76) (which comes from the comparison of Weyl groups in Proposition 4.2.b) and secondly the bijection

$$
(82)\qquad \bigoplus\nolimits_{\psi \in \mathrm{Irr}(\Omega_{\mathfrak{f},\mathrm{tor}})} \mathcal{O}(\mathfrak{s}^\vee_{L,\psi}) \to \bigoplus\nolimits_{\psi \in \mathrm{Irr}(\Omega_{\mathfrak{f},tor})} \mathcal{O}(\mathfrak{s}_{L,\psi})
$$

induced by Theorem 4.1. By Theorem 4.1.i, (82) is $X_{\mathrm{wr}}(G)$-equivariant, while (76) is not affected by weakly unramified characters. Therefore (81) and the second arrow in (80) are $X_{\mathrm{wr}}(G)$-equivariant. By Lemma 2.2 the third arrow in (80) is $X_{\mathrm{wr}}(G)$-equivariant. $\square$

Lemma 5.4. *In Theorem 5.1 π is supercuspidal if and only if (ϕ_π, ρ_π) is cuspidal. In that setting Theorem 5.1 agrees with the bijection from 4.1 and* [FOS1].

Proof. The first part follows directly from Proposition 4.2.a.

In the cuspidal case the third map in (79) reduces to $\mathrm{Irr}(\mathcal{O}(\mathfrak{s}_\psi^\vee)) \leftrightarrow \mathfrak{s}_\psi^\vee$, see (18). By (70) the first map in (79) becomes the canonical bijection $\mathrm{Irr}(G)_{\mathfrak{s}_\psi} \leftrightarrow \mathrm{Irr}(\mathcal{O}(\mathfrak{s}_\psi))$ and (75) says the second map in (79) is induced by $\mathfrak{s}_\psi^\vee \leftrightarrow \mathfrak{s}_\psi$ from Proposition 4.2.a. Hence (79) and Theorem 5.1 boil down to Theorem 4.1. $\square$

Let $\mathfrak{Lev}(G)$ be a set of representatives for the Levi subgroups of G modulo G-conjugacy. Recall from Corollary 1.3 that the same set represents the $G^\vee$-conjugacy classes of G-relevant L-Levi subgroups of ${}^LG$.

Lemma 5.5. *The cuspidal support maps and Theorem 5.1 make a commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Irr}_{\mathrm{unip}}(G) & \longrightarrow & \Phi_{\mathrm{nr},e}(G) \\
\downarrow \mathbf{Sc} & & \downarrow \mathbf{Sc} \\
\bigsqcup\limits_{L \in \mathfrak{Lev}(G)} \mathrm{Irr}_{\mathrm{cusp,unip}}(L)/N_G(L) & \longrightarrow & \bigsqcup\limits_{L \in \mathfrak{Lev}(G)} \Phi_{\mathrm{nr,cusp}}(L)/N_{G^\vee}(L^\vee \rtimes \mathbf{W}_K)
\end{array}.
$$

Proof. In the bottom line the actions of $N_G(L)$ and $N_{G^\vee}(L^\vee \rtimes \mathbf{W}_K)$ factor through the finite group

$$
N_G(L)/L \cong N_{G^\vee}(L^\vee \rtimes \mathbf{W}_K)/L^\vee.
$$

In the proof of Proposition 4.2.a we saw that Theorem 4.1 is equivariant for the actions of this group. Now Lemma 5.4 says that the bottom line of the diagram is well-defined.

Suppose that (up to $G^\vee$-conjugacy) $\mathbf{Sc}(\phi_\pi, \rho_\pi) = (\chi_L \phi_L, \rho_L)$, with $\chi_L \in X_{\mathrm{nr}}(L)$. By Theorem 2.1 $\bar{M}(\phi, \rho, \vec{v})$ is a constituent of $\mathrm{ind}_{\mathcal{H}(\mathfrak{s}_L^\vee, \vec{v})}^{\mathcal{H}(\mathfrak{s}^\vee, \vec{v})} (L, \chi_L \phi_L, \rho_L)$. Here $\mathcal{H}(\mathfrak{s}_L^\vee, \vec{v})$ is embedded in $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ as $\mathcal{O}(\mathfrak{s}_L^\vee)$. By (75) this corresponds to the subalgebra

$$
\mathcal{O}(\mathfrak{s}_L) \cong \mathcal{H}(L, \hat{P}_{\mathfrak{f}} \cap L, (\hat{\sigma} \otimes \psi)|_{\hat{P}_{\mathfrak{f}} \cap L}) \quad \text{of} \quad \mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma} \otimes \psi).
$$

Hence $\mathrm{Hom}_{\hat P_{\mathfrak f}}(\hat\sigma \otimes \psi, \pi)$ is a constituent of $\mathrm{ind}_{\mathcal H(L,\hat P_{\mathfrak f}\cap L,(\hat\sigma\otimes\psi)|_{\hat P_{\mathfrak f}\cap L})}^{\mathcal H(G,\hat P_{\mathfrak f},\hat\sigma\otimes\psi)}(\chi)$, where $\chi \in \mathrm{Irr}_{\mathrm{cusp,unip}}(L)$ is the image of $(L, \chi_L \phi_L, \rho_L)$ under Theorem 4.1. By (56) π is a constituent of $I_{LN}^G(\chi)$, for a suitable parabolic subgroup $LN \subset G$ with Levi factor N. As χ is supercuspidal, this means that (L, χ) is the cuspidal support of π (up to G-conjugacy, as always). □

Lemma 5.6. *In Theorem 5.1 $\pi \in \mathrm{Irr}_{\mathrm{unip}}(G)$ is tempered if and only if $\phi_\pi \in \Phi_{\mathrm{nr}}(G)$ is bounded.*

Proof. Recall that Theorem 5.1 was built from (79). By Theorem 3.2.c the first map of (79) (and its inverse), preserve temperedness. By Theorem 4.4 the same holds for the second map in (79). By Theorem 2.1 the third map in (79) turns tempered representations into bounded (enhanced) L-parameters and conversely. □

Lemma 5.7. *In Theorem 5.1 $\pi \in \mathrm{Irr}_{\mathrm{unip}}(G)$ is essentially square-integrable if and only if $\phi_\pi \in \Phi_{\mathrm{nr}}(G)$ is discrete.*

Proof. Consider the chain of maps (79). By Theorem 3.2.d the first of those maps sends essentially square-integrable representations to essentially discrete series modules. The second map preserves the essentially discrete series property, because it comes from an isomorphism between all the structure defining these affine Hecke algebras (Theorem 4.4).

For $(\phi_L, \rho_L) \in \Phi_{\mathrm{nr},e}(L)$, [AMS3, Proposition 3.9.b] says that

$$R\big(G^\circ_{\phi_1}, Z(L_c^\vee)^{\mathbf W_K,\circ}\big)_{\mathrm{red}} = R\big(Z_{G^\vee_{\mathrm{sc}}}(\phi_L|_{\mathbf I_K}), Z(L_c^\vee)^{\mathbf W_K,\circ}\big)_{\mathrm{red}} = R\big(G^\vee_{\mathrm{sc}}, Z(L_c^\vee)^{\mathbf W_K,\circ}\big)_{\mathrm{red}}.$$

As $G^\vee_{\mathrm{sc}}$ is semisimple, the rank of this root system is $\dim_{\mathbb C}(Z(L_c^\vee)^{\mathbf W_K,\circ})$, which by [AMS3, Lemma 3.7] equals

$$\dim_{\mathbb C}\big(Z(L^\vee)^{\mathbf W_K,\circ}/X_{\mathrm{nr}}(G)\big) = \dim_{\mathbb C}(T_{\mathfrak s^\vee}/X_{\mathrm{nr}}(G)).$$

Now we can apply Theorem 2.1.e, which says that the first map in (79) sends essentially discrete series modules to discrete enhanced L-parameters.

These arguments also work for the inverses of the maps in (79). □

Recall from [Lan, p. 20–23] and [Bor, §10.1] that every $\phi \in \Phi(G)$ determines a character χ_ϕ of $Z(G)$. For the construction, one first embeds $\mathcal G$ in a connected reductive K-group $\overline{\mathcal G}$ with $\mathcal G_{\mathrm{der}} = \overline{\mathcal G}_{\mathrm{der}}$, such that $Z(\overline{\mathcal G})$ is connected. Then one lifts ϕ to a L-parameter $\overline\phi$ for $\overline G = \overline{\mathcal G}(K)$. The natural projection ${}^L\overline G \to {}^L Z(\overline G)$ produces an L-parameter $\overline\phi_z$ for $Z(\overline G) = Z(\overline{\mathcal G})(K)$, and via the local Langlands correspondence for tori $\overline\phi_z$ determines a character $\chi_{\overline\phi}$ of $Z(\overline G)$. Then χ_ϕ is given by restricting $\chi_{\overline\phi}$ to $Z(G)$. Langlands [Lan, p. 23] checked that χ_ϕ does not depend on the choices made above.

Lemma 5.8. *In Theorem 5.1 the central character of π equals the character χ_{ϕ_π} of $Z(G)$ determined by ϕ_π. This character is unramified, that is, trivial on every compact subgroup of $Z(G)$.*

Proof. The construction in [Lan, p. 20–21] can be executed such that $\overline{\mathcal G}$, like $\mathcal G$, splits over an unramified extension of K. As ϕ_π is unramified, it can be lifted to a $\overline\phi \in \Phi_{\mathrm{nr}}(\overline G)$. Then $\overline\phi_z \in \Phi(Z(\overline G))$ is also unramified. Since $Z(\overline{\mathcal G})$ is connected and splits over an unramified extension of K:

$$\Phi_{\mathrm{nr}}(Z(\overline G)) = \big((Z(\overline{\mathcal G})^\vee)^{\mathbf I_K}\big)_{\mathrm{Frob}} = \big(Z(\overline{\mathcal G})^\vee\big)_{\mathrm{Frob}} = \big(Z(\overline{\mathcal G})^\vee\big)^\circ_{\mathrm{Frob}} = X_{\mathrm{nr}}(Z(\overline G)).$$

Hence $\chi_{\overline{\phi}} \in \mathrm{Irr}(Z(\overline{G}))$ is unramified. Then its restriction χ_ϕ factors through

$$Z(G)/Z(G)_{\mathrm{cpt}} \cong X_*(Z(\mathcal{G})^\circ(K)). \tag{83}$$

The cuspidal unipotent representation $\bar\sigma$ of $\overline{P_{\mathfrak{f}}}$ is trivial on the centre of $\overline{P_{\mathfrak{f}}}$. Since $\mathcal{G}$ splits over an unramified extension, every parahoric subgroup of G contains the maximal compact subgroup $Z(G)_{\mathrm{cpt}}$ of $Z(G)$. In particular $Z(G)_{\mathrm{cpt}} \subset P_{\mathfrak{f}}$ projects to a central subgroup of $\overline{P_{\mathfrak{f}}}$. Hence the kernel of $\sigma \in \mathrm{Irr}(P_{\mathfrak{f}})$ contains $Z(G)_{\mathrm{cpt}}$. As $\pi \in \mathrm{Rep}(G)_{(P_{\mathfrak{f}},\sigma)}$, this implies $Z(G)_{\mathrm{cpt}} \subset \ker(\pi)$. In other words, the central character χ_π of π factors also through (83).

The lattice of K-rational cocharacters of $Z(\mathcal{G})^\circ(K)$ can be identified with the cocharacter lattice of the maximal K-split subtorus $Z(\mathcal{G})_s$ of $Z(\mathcal{G})^\circ$. Thus χ_π and χ_ϕ are determined by their restrictions to $Z(G)_s = Z(\mathcal{G})_s(K)$, which both are unramified characters.

This means that, to prove the lemma, it suffices to compare the characters of $Z(G)_s$ determined by π and by λ_π. The latter admits a more direct description than χ_ϕ. Namely, the inclusion $Z(\mathcal{G})_s \to \mathcal{G}$ has a dual surjection $\mathcal{G}^\vee \to (Z(\mathcal{G})_s)^\vee$. The image $\phi_s \in \Phi(Z(G)_s)$ of ϕ_π determines a character χ_{ϕ_s}, which equals the restriction of χ_{ϕ_π} of $Z(G)_s$.

Now we reduce to the cuspidal case. It is clear that π and $\mathbf{Sc}(\pi)$ have the same $Z(G)$-character. Let us write

$$\mathbf{Sc}(\phi_\pi, \rho_\pi) = (L, \phi_L, \rho_L) \quad \text{with} \quad (\phi_L, \rho_L) \in \Phi_{\mathrm{cusp}}(L).$$

From [AMS1, Lemma 7.6 and Definition 7.7] we see that $\phi_L|_{\mathbf{I}_K} = \phi_\pi|_{\mathbf{I}_K}$ and $\phi_L(\mathrm{Frob}) = \phi_\pi(\mathrm{Frob})t$, where $t \in G^\vee_{\mathrm{der}}$ commutes with the image of ϕ_π. As

$$G^\vee_{\mathrm{der}} \subset \ker\big(G^\vee \to Z(\mathcal{G})_s^\vee(\mathbb{C})\big),$$

ϕ_L and ϕ_π have the same image ϕ_s in $\Phi(Z(G)_s)$. So if we replace (ϕ_π, ρ_π) by its cuspidal support (L, ϕ_L, ρ_L), we do not change the $Z(G)_s$-character χ_{ϕ_s}. Although $\mathbf{Sc}(\phi_\pi, \rho_\pi)$ is determined only up to $G^\vee$-conjugacy, we may pick any representative for it, because conjugation by elements of $G^\vee$ does not affect ϕ_s. In view of this and Lemma 5.5 we may assume that π is supercuspidal.

Now, as explained after (15.5) in [FOS1], π can be written as $\pi' \otimes \chi$ with $\pi' \in \mathrm{Irr}_{\mathrm{unip}}(G/Z(G)_s)$ and $\chi \in X_{\mathrm{nr}}(G)$. Clearly $\chi_\pi|_{Z(G)_s} = \chi|_{Z(G)_s}$. The construction in [FOS1, (15.6) and (15.10)] says that $(\phi_\pi, \rho_\pi) = (\phi_{\pi'}\hat\chi, \rho_\pi)$, where $\hat\chi \in Z(\mathcal{G}^\vee)^\circ_{\mathrm{Frob}}$ is the image of χ. We see that ϕ_s equals the L-parameter of $\chi|_{Z(G)_s}$ and hence

$$\chi_\phi|_{Z(G)_s} = \chi_{\phi_s} = \chi|_{Z(G)_s} = \chi_\pi|_{Z(G)_s}.$$

As discussed above, this implies that $\chi_\phi = \chi_\pi$ on $Z(G)$. $\square$

Let $\mathcal{Q}\mathcal{U}_Q$ be a parabolic K-subgroup of $\mathcal{G}$, with unipotent radical $\mathcal{U}_Q$ and Levi factor $\mathcal{Q}$. Suppose that $\phi \in \Phi(G)$ factors via ${}^L Q$. Then we can compare representations of G and Q associated to enhancements of ϕ, via normalized parabolic induction. Let p_ζ and p_{ζ^Q} be as in (20). By [AMS1, Theorem 7.10.a] there is a natural injection

$$p_{\zeta^Q}\mathbb{C}[\mathcal{S}_\phi^Q] \to p_\zeta \mathbb{C}[\mathcal{S}_\phi].$$

This enables us to retract G-relevant enhancements of ϕ to representations of $\mathcal{S}_\phi^Q$.

Lemma 5.9. *Let $\phi \in \Phi_{\mathrm{nr}}(Q)$.*

(a) *Suppose that the function* $\epsilon_{u_\phi,j}(\phi(\mathrm{Frob})\phi_b(\mathrm{Frob})^{-1},\vec{v})$ *from* (21) *and* (22) *is nonzero. Let* $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$ *be G-relevant and let* $\rho^Q \in \mathrm{Irr}(\mathcal{S}_\phi^Q)$ *be Q-relevant.*

Then the multiplicity of $\pi(\phi,\rho) \in \mathrm{Irr}_{\mathrm{unip}}(G)$ *as a constituent of* $I_{QU_Q}^G \pi(\phi,\rho^Q)$ *is* $[\rho^Q : \rho]_{\mathcal{S}_\phi^Q}$. *It already appears that many times as a quotient.*

(b) *Let* $(\phi,\rho^Q) \in \Phi_{\mathrm{nr},e}(Q)$ *be bounded. Then*

$$I_{QU_Q}^G \pi(\phi,\rho^Q) \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\mathcal{S}_\phi^Q}(\rho^Q,\rho) \otimes \pi(\phi,\rho),$$

where the sum runs over all $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$ *with* $\mathbf{Sc}(\phi,\rho) = \mathbf{Sc}(\phi,\rho^Q)$.

Proof. (a) By [AMS3, Lemma 3.19.b] this holds for the modules $\bar{M}(\phi,\rho,\vec{v})$ and $\mathrm{ind}_{\mathcal{H}(\mathfrak{s}_{Q,\psi}^\vee,\vec{v})}^{\mathcal{H}(\mathfrak{s}_\psi^\vee,\vec{v})} \bar{M}(\phi,\rho^Q,\vec{v})$. Theorem 4.4 transfers that to a statement about modules for $\mathcal{H}(G,\hat{P}_{\mathfrak{f}},\hat{\sigma}\otimes\psi)$. By Theorem 3.2.b and (56) it becomes the desired statement about elements of $\mathrm{Rep}(G)_{\mathfrak{s}_\psi}$.
(b) By Lemma 2.2.c $\bar{M}(\phi,\rho,\vec{v})$ and $\bar{M}^Q(\phi,\rho^Q,\vec{v})$ are equal to the standard modules with the same parameters. Knowing that, [AMS3, Lemma 3.19.a] gives the desired statement for $\mathcal{H}(\mathfrak{s}_\psi^\vee,\vec{v})$-modules. As in the proof of part (a), that can be transferred to elements of $\mathrm{Rep}(G)_{\mathfrak{s}_\psi}$. □

Finally we work out the compatibility of Theorem 5.1 with the Langlands classification for representations of reductive p-adic groups [Kon, Ren]. We briefly recall the statement.

For every $\pi \in \mathrm{Irr}(G)$ there exists a triple (P,τ,ν), unique up to G-conjugation, such that:

- P is a parabolic subgroup of G;
- $\tau \in \mathrm{Irr}(P/U_P)$ is tempered, where U_P denotes the unipotent radical of P;
- the unramified character $\nu : P/U_P \to \mathbb{R}_{>0}$ is strictly positive with respect to P;
- π is the unique irreducible quotient of $I_P^G(\tau\otimes\nu)$.

The Langlands classification for (enhanced) L-parameters [SiZi] was already discussed before Proposition 2.3 – we use the notations from over there.

Lemma 5.10. *Let* $(\phi,\rho) \in \Phi_{\mathrm{nr},e}(G)$ *and let* (QU_Q,ϕ_b,z) *be the triple associated to* ϕ *by* [SiZi, Theorem 4.6]. *Recall from* (20) *that* ρ *can also be considered as enhancement of* ϕ *or* ϕ_b *as L-parameters for Q.*

(a) $\pi(\phi,\rho)$ *is the unique irreducible quotient of* $I_{QU_Q}^G \pi^Q(\phi,\rho)$.

(b) $\pi^Q(\phi,\rho) = z \otimes \pi^Q(\phi_b,\rho)$ *with* $\pi^Q(\phi_b,\rho) \in \mathrm{Irr}_{\mathrm{unip}}(Q)$ *tempered and* $z \in X_{\mathrm{nr}}(Q)$ *strictly positive with respect to* QU_Q. *The data for* $\pi(\phi,\rho)$ *in the Langlands classification for Rep(G) are* $(QU_Q,\pi^Q(\phi_b,\rho),z)$, *up to G-conjugacy.*

Proof. (a) By Lemma 4.5 (and the proof of Theorem 4.4) $\vec{v} \in \mathbb{R}_{>1}^d$. Thus we may apply Proposition 2.3, which says that the analogous statement for $\mathcal{H}(\mathfrak{s}^\vee,\vec{v})$-modules holds. With Theorem 4.4, Theorem 3.2.b and (56) we transfer that to $\mathrm{Rep}(G)_{\mathfrak{s}_\psi}$.
(b) The first part follows from Lemmas 5.3 and 5.6. The second part is a consequence of the uniqueness (up to conjugacy) of the Langlands data of $\pi(\phi,\rho)$. □

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