# ON UNIPOTENT REPRESENTATIONS OF RAMIFIED $p$-ADIC GROUPS

Maarten Solleveld
IMAPP, Radboud Universiteit
Heyendaalseweg 135, 6525AJ Nijmegen, the Netherlands
email: m.solleveld@science.ru.nl

Abstract. Let $G$ be any connected reductive group over a non-archimedean local field. We analyse the unipotent representations of $G$, in particular in the cases where $G$ is ramified. We establish a local Langlands correspondence for this class of representations, and we show that it satisfies all the desiderata of Borel as well as the conjecture of Hiraga, Ichino and Ikeda about formal degrees.

This generalizes work of Lusztig and of Feng, Opdam and the author, to reductive groups that do not necessarily split over an unramified extension of the ground field.

## Contents



## Introduction

Let $F$ be a non-archimedean local field and let $\mathcal{G}$ be a connected reductive $F$-group. We consider smooth, complex representations of the group $G = \mathcal{G}(F)$. An irreducible smooth $G$-representation $\pi$ is called unipotent if there exists a parahoric subgroup $P_{\mathfrak{f}} \subset G$ and an irreducible $P_{\mathfrak{f}}$-representation $\sigma$, which is inflated from a cuspidal unipotent representation of the finite reductive quotient of $P_{\mathfrak{f}}$, such that $\pi|_{P_{\mathfrak{f}}}$ contains $\sigma$.

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The study of unipotent representations of $p$-adic groups was initiated by Morris [Mor1, Mor2] and Lusztig [Lus2, Lus3]. In a series of papers [FeOp, FOS1, Feng, Sol3, Opd3, FOS2] Yongqi Feng, Eric Opdam and the author investigated various aspects of these representations: Hecke algebras, classification, formal degrees, L-packets. This culminated in a proof of a local Langlands correspondence for this class of representations.

However, all this was worked out under the assumption that $\mathcal{G}$ splits over the maximal unramified extension $F_{\mathrm{nr}}$ of $F$. In that case $\mathcal{G}$ already splits over a finite unramified extension of $F$. In the opposite case, where $\mathcal{G}$ does not split over $F_{\mathrm{nr}}$, we call $\mathcal{G}$ ramified (over $F$).

On the one hand, the assumption that $\mathcal{G}$ is $F_{\mathrm{nr}}$-split is reasonable: unipotent $G$-representations come from unipotent representations over the residue field $k_F$, and extensions of $k_F$ correspond naturally to unramified extensions of $F$. This enables one to regard a cuspidal unipotent representation of $\mathcal{G}(F)$ as a member of a family, indexed by the finite unramified extensions of $F$. Over a finite field an analogue is known from [Lus1].

On the other hand, in examples of ramified simple $F$-groups the unipotent representations look the same as for $F$-groups that are not ramified, see [Mor2]. While general depth zero representations of ramified $F$-groups may very well be more intricate, for unipotent representations it is not easy to spot what difficulties could be created by ramification of the ground field. In any case, many of the nice properties of unipotent representations were already expected to hold for all connected reductive $F$-groups. For instance, an enhanced L-parameter should correspond to a unipotent representation if and only if it is unramified (that is, trivial on the inertia subgroup $\mathbf{I}_F$ of the Weil group $\mathbf{W}_F$).

Summarising, the restriction to $F_{\mathrm{nr}}$-split groups in the study of unipotent representations seems to be made mainly for technical convenience. In the current paper we will prove the main results of [Lus2, Lus3, FOS1, Sol3, FOS2] for ramified simple $p$-adic groups, and then generalize them to arbitrary connected reductive $F$-groups. Before we summarise our main conclusions below, we need to introduce some notations.

We denote the set of irreducible $G$-representations by $\mathrm{Irr}(G)$, and we often add a subscript "unip" for unipotent and subscript "cusp" to indicate cuspidality. Let ${}^L G = G^\vee \rtimes \mathbf{W}_F$ be the dual L-group of $G$. To a Langlands parameter $\phi$ for $G$ we associate a finite group $\mathcal{S}_\phi$ as in [Art2, AMS1]. An enhancement of $\phi$ is an irreducible representation $\rho$ of $\mathcal{S}_\phi$. We denote the collection of $G$-relevant enhanced L-parameters (considered modulo $G^\vee$-conjugation) by $\Phi_e(G)$. Then $\Phi_{\mathrm{nr},e}(G)$ denotes the subset of $\Phi_e(G)$ given by the condition $\phi|_{\mathbf{I}_F} = \mathrm{id}_{\mathbf{I}_F}$.

**Theorem 1.** *Let $\mathcal{G}$ be a connected reductive group over a non-archimedean local field $F$ and write $G = \mathcal{G}(F)$. There exists a bijection*

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{unip}}(G) & \longrightarrow & \Phi_{\mathrm{nr},e}(G) \\ \pi & \mapsto & (\phi_\pi, \rho_\pi) \\ \pi(\phi,\rho) & \leftarrow\!\shortmid & (\phi,\rho) \end{array} .$$

*We can construct such a bijection for every group $G$ of this kind, in a compatible way. The resulting family of bijections has the following properties:*

*(a) Compatibility with direct products of reductive $F$-groups.*

(b) *Equivariance with respect to the canonical actions of the group $X_{\mathrm{wr}}(G)$ of weakly unramified characters of $G$.*
(c) *The central character of $\pi$ equals the character of $Z(G)$ determined by $\phi_\pi$.*
(d) *$\pi$ is tempered if and only if $\phi_\pi$ is bounded.*
(e) *$\pi$ is essentially square-integrable if and only if $\phi_\pi$ is discrete.*
(f) *$\pi$ is supercuspidal if and only if $(\phi_\pi, \rho_\pi)$ is cuspidal.*
(g) *The analogous bijections for the Levi subgroups of $G$ and the cuspidal support maps form a commutative diagram*

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{unip}}(G) & \longrightarrow & \Phi_{\mathrm{nr},e}(G) \\ \downarrow & & \downarrow \\ \bigsqcup_M \mathrm{Irr}_{\mathrm{cusp,unip}}(M)/N_G(M) & \longrightarrow & \bigsqcup_M \Phi_{\mathrm{nr,cusp}}(M)/N_{G^\vee}({}^L M) \end{array}.$$

*Here $M$ runs over a collection of representatives for the conjugacy classes of Levi subgroups of $G$.*
(h) *Suppose that $P = MU$ is a parabolic subgroup of $G$ and that $(\phi^M, \rho^M) \in \Phi_{\mathrm{nr},e}(M)$ is bounded. Let $\phi$ be $\phi^M$ considered as L-parameter for $G$. Then the normalized parabolically induced representation $I_P^G \pi(\phi^M, \rho^M)$ is a direct sum of representations $\pi(\phi, \rho)$, with multiplicities $[\rho^M : \rho]_{\mathcal{S}_\phi^M}$.*
(i) *Compatibility with the Langlands classification for representations of reductive groups and the Langlands classification for enhanced L-parameters* [SiZi].
(j) *Compatibility with restriction of scalars of reductive groups over non-archimedean local fields.*
(k) *Let $\eta : \tilde{\mathcal{G}} \to \mathcal{G}$ be a homomorphism of connected reductive $F$-groups, such that the kernel of $\mathrm{d}\eta : \mathrm{Lie}(\tilde{\mathcal{G}}) \to \mathrm{Lie}(\mathcal{G})$ is central and the cokernel of $\eta$ is a commutative $F$-group. Let ${}^L\eta : {}^L\tilde{G} \to {}^L G$ be a dual homomorphism and let $\phi \in \Phi_{\mathrm{nr}}(G)$.*
*Then the L-packet $\Pi_{{}^L\eta \circ \phi}(\tilde{G})$ consists precisely of the constituents of the completely reducible $\tilde{G}$-representations $\eta^*(\pi)$ with $\pi \in \Pi_\phi(G)$.*
(l) *The HII conjecture* [HII] *holds for tempered unipotent $G$-representations.*

*Moreover the properties (a), (c), (k) and (l) uniquely determine the surjection*

$$\mathrm{Irr}_{\mathrm{unip}}(G) \to \Phi_{\mathrm{nr}}(G) : \pi \mapsto \phi_\pi,$$

*up to twisting by weakly unramified characters of $G$ that are trivial on $Z(G)$.*

We regard Theorem 1 as a local Langlands correspondence (LLC) for unipotent representations. We note that parts (b), (c), (d), (e) and (k) are precisely the desiderata formulated by Borel [Bor, §10]. For the unexplained notions in the other parts we refer to [Sol3].

Let us phrase part (l) about Plancherel densities more precisely. We fix an additive character $\psi : F \to \mathbb{C}^\times$ of level zero (by [HII] that can be done without loss of generality). As in [HII] that gives rise to a Haar measure $\mu_{G,\psi}$ on $G$, which we however normalize as in [FOS1, (A.25)].

Let $\mathcal{P} = \mathcal{M}\mathcal{U}$ be a parabolic $F$-subgroup of $\mathcal{G}$, with Levi factor $\mathcal{M}$ and unipotent radical $\mathcal{U}$. Let $\pi \in \mathrm{Irr}(M)$ be square-integrable modulo centre and let $X_{\mathrm{unr}}(M)$ be the group of unitary unramified characters of $M$. Let $\mathcal{O} = X_{\mathrm{unr}}(M)\pi \subset \mathrm{Irr}(M)$ be the orbit in $\mathrm{Irr}(M)$ of $\pi$, under twists by $X_{\mathrm{unr}}(M)$. We define a Haar measure $\mathrm{d}\mathcal{O}$ on $\mathcal{O}$ as in [Wal, p. 239 and 302]. This also provides a Haar measure on the family of (finite length) $G$-representations $I_P^G(\pi')$ with $\pi' \in \mathcal{O}$.

Let $Z(\mathcal{G})_s$ be the maximal $F$-split central torus of $\mathcal{G}$, with dual group $Z(G^\vee)^{\mathbf{W}_F,\circ}$. We denote the adjoint representation of ${}^L M$ on $\mathrm{Lie}\big(G^\vee\big)/\mathrm{Lie}\big(Z(M^\vee)^{\mathbf{W}_K}\big)$ by $\mathrm{Ad}_{G^\vee,M^\vee}$. We compute $\gamma$-factors with respect to the Haar measure on $F$ that gives the ring of integers $\mathfrak{o}_F$ volume 1.

**Conjecture 2.** [HII, §1.5]
*Suppose that the enhanced L-parameter of $\pi$ is $(\phi_\pi, \rho_\pi) \in \Phi_e(M)$.*
*Then the Plancherel density at $I_P^G(\pi) \in \mathrm{Rep}(G)$ is*

$$c_M \dim(\rho_\pi) \left| \pi_0\big( Z_{(G/Z(G)_s)^\vee}(\phi_\pi) \big) \right|^{-1} |\gamma(0, \mathrm{Ad}_{G^\vee,M^\vee} \circ \phi_\pi, \psi)| \, \mathrm{d}\mathcal{O}(\pi),$$

*for some constant $c_M \in \mathbb{R}_{>0}$ independent of $F$ and $\mathcal{O}$.*
*Moreover, with the above normalizations of Haar measures $c_M$ equals 1.*

It is also interesting to consider Theorem 1 for all inner twists of a given quasi-split group simultaneously. That is done best with the rigid inner twists from [Kal1, Dil]. In that setting we replace $\mathcal{S}_\phi$ by a slightly different component group $\mathcal{S}_\phi^+$ and we write

$$\Phi^+({}^L G) = \big\{ (\phi, \rho^+) : \phi \in \Phi(G), \rho^+ \in \mathrm{Irr}(\mathcal{S}_\phi^+) \big\}.$$

For a rigid inner twist $(\mathcal{G}^z, z)$ of $\mathcal{G}$, we also replace $\Phi_e(G^z)$ by a slightly different set $\Phi^+(G^z, z)$ of relevant enhanced L-parameters. The (disjoint) union of the sets $\Phi^+(G^z, z)$, over all $z$ in a set $H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{G})$ parametrizing the equivalence classes of rigid inner twists of $\mathcal{G}$, is precisely $\Phi^+({}^L G)$.

We check (in Section 7) that the new setup is essentially equivalent to the setup used so far, with the bonus that it is a bit more canonical. It follows that Theorem 1 is also valid in terms of rigid inner twists and the associated enhancements of L-parameters.

**Theorem 3.** (see Theorem 7.4)
*The union of the instances of Theorem 1 for all rigid inner twists of a quasi-split connected reductive $F$-group $\mathcal{G}$ gives a bijection*

$$\Phi_{\mathrm{nr}}^+({}^L G) \longrightarrow \bigsqcup\nolimits_{z \in H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{G})} \mathrm{Irr}_{\mathrm{unip}}(G^z).$$

It is believed (or hoped) that in the local Langlands program enhanced L-parameters are in bijection with the irreducible representations of all rigid inner twists of a given reductive $p$-adic group. Theorem 3 beautifully confirms this for unramified L-parameters and unipotent representations.

Let us explain our strategy to prove Theorem 1. The papers [FeOp, FOS1, Opd3, FOS2] all use reduction to the case of simple (adjoint) $F$-groups, so that is where we start. Like in [Mor1, Mor2, Lus2, Lus3] we want to analyse the parahoric subgroups $P_\mathfrak{f}$ of $G$, their (cuspidal) unipotent representations $\sigma$ and the Hecke algebras determined by a type of the form $(P_\mathfrak{f}, \sigma)$. The main trick stems from a remark of Lusztig [Lus3, §10.13]: for every ramified simple $F$-group $\mathcal{G}$ there exists a $F_{\mathrm{nr}}$-split simple "companion group" $\mathcal{G}'$. which has the same local index and the same relative local Dynkin diagram as $\mathcal{G}$ (up to the direction of some arrows in these diagrams). That determines $\mathcal{G}'$ up to isogeny, and we fix it by requiring that $(Z(G'^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}} \cong (Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}}$. We will construct a LLC for $\mathrm{Irr}(G)_{\mathrm{unip}}$ via $\mathrm{Irr}(G')_{\mathrm{unip}}$ (for which it is known already).

In Section 1 we provide an overview of all possible $\mathcal{G}$ and $\mathcal{G}'$. It turns out that, although $\mathcal{G}'$ is connected when $\mathcal{G}$ is adjoint, sometimes $\mathcal{G}' = \mathcal{G}'^{\circ} \times \{\pm 1\}$.

This setup provides a bijection between the $G$-orbits of facets in the Bruhat–Tits building of $\mathcal{G}(F)$ and the analogous set for $G' = \mathcal{G}'(F)$, say $\mathfrak{f} \mapsto \mathfrak{f}'$. We call a representation of the parahoric subgroup $P_{\mathfrak{f}}$ unipotent (resp. cuspidal) if it arises by inflation from a unipotent (resp. cuspidal) representation of the finite reductive quotient of $P_{\mathfrak{f}}$. We show in Theorem 2.3 that the relation between the ramified simple $F$-group $\mathcal{G}$ and its companion group $\mathcal{G}'$ gives rise to a bijection

$$\mathrm{Irr}(P_{\mathfrak{f}})_{\mathrm{unip}} \longleftrightarrow \mathrm{Irr}(P_{\mathfrak{f}'})_{\mathrm{unip}} : \sigma \mapsto \sigma'. \tag{1}$$

Notice that this actually is a statement about finite reductive groups. Let $\hat{P}_{\mathfrak{f}}$ be the pointwise stabilizer of $\mathfrak{f}$ in $G$. Then (1) can be extended to a bijection

$$\mathrm{Irr}(\hat{P}_{\mathfrak{f}})_{\mathrm{unip}} \longleftrightarrow \mathrm{Irr}(\hat{P}_{\mathfrak{f}'}) : \hat{\sigma} \mapsto \hat{\sigma}'. \tag{2}$$

For cuspidal representations (2) induces to a bijection

$$\mathrm{Irr}(G)_{\mathrm{cusp,unip}} \longleftrightarrow \mathrm{Irr}(G')_{\mathrm{cusp,unip}} \tag{3}$$

which is almost canonical (Corollary 2.5).

In Section 3 we compare the non-cuspidal unipotent representations of $G$ and $G'$. Let $\hat{\sigma} \in \mathrm{Irr}(\hat{P}_{\mathfrak{f}})_{\mathrm{unip,cusp}}$, so that $(\hat{P}_{\mathfrak{f}}, \hat{\sigma})$ is a type for a Bernstein component of unipotent $G$-representations [Mor3]. The Bernstein block $\mathrm{Rep}(G)_{(\hat{P}_{\mathfrak{f}},\hat{\sigma})}$ is equivalent with the module category of the Hecke algebra $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$. We prove in Theorem 3.1 that (2) canonically induces an algebra isomorphism

$$\mathcal{H}(G', \hat{P}_{\mathfrak{f}'}, \hat{\sigma}') \to \mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma}). \tag{4}$$

These Hecke algebras are essential for everything in the non-cuspidal cases. By [Lus2, §1] there are equivalences of categories

$$\mathrm{Rep}(G)_{\mathrm{unip}} = \prod_{\{(\hat{P}_{\mathfrak{f}},\hat{\sigma})\}/G\text{-conjugation}} \mathrm{Rep}(G)_{(\hat{P}_{\mathfrak{f}},\hat{\sigma})} \cong \prod_{\{(\hat{P}_{\mathfrak{f}},\hat{\sigma})\}/G\text{-conjugation}} \mathrm{Mod}\big(\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})\big). \tag{5}$$

Combining that with (4) for all possible $(\hat{P}_{\mathfrak{f}}, \hat{\sigma})$ yields an equivalence of categories

$$\mathrm{Rep}(G)_{\mathrm{unip}} \longrightarrow \mathrm{Rep}(G')_{\mathrm{unip}}. \tag{6}$$

Although (6) is not entirely canonical, we do show that it preserves several properties of representations.

Now that the situation for unipotent representations of simple $F$-groups is under control, we turn to the complex dual groups and L-parameters for $G$ and $G'$. The most important observation (checked case-by-case with the list from Section 1) is Lemma 4.1: there exists a canonical isomorphism $G'^{\vee} \to (G^{\vee})^{\mathbf{I}_F}$. This induces a canonical bijection

$$\Phi_{\mathrm{nr},e}(G') \longrightarrow \Phi_{\mathrm{nr},e}(G), \tag{7}$$

which preserves relevant properties of enhanced L-parameters (Proposition 4.4 and Lemma 4.5). From (6), (7) and Theorem 1 for $G'$ [Sol3, FOS2] we deduce Theorem 1 for ramified simple groups. More precisely, we establish some properties of the bijection

$$\mathrm{Irr}(G)_{\mathrm{unip}} \longrightarrow \Phi_{\mathrm{nr},e}(G), \tag{8}$$

not yet all. In particular equations (4)–(8) mean that the main results of [Lus2, Lus3] are now available for all simple $F$-groups.

With the case of simple $F$-groups settled, we embark on the study of supercuspidal unipotent representations of connected reductive $F$-groups (Section 5). For a $F_{\mathrm{nr}}$-split group $\mathcal{G}$, Theorem 1 was proven for $\mathrm{Irr}(G)_{\mathrm{cusp,unip}}$ in [FOS1] (again with most but not yet all properties). We aim to generalize the arguments from [FOS1] to possibly ramified connected reductive $F$-groups. It is only at this stage that the differences caused by ramification of field extensions force substantial modifications of previous strategies.

Assume for the moment that the centre of $\mathcal{G}$ is $F$-anisotropic. When $\mathcal{G}$ is in addition $F_{\mathrm{nr}}$-split, the derived group $G_{\mathrm{der}}$ has the same supercuspidal unipotent representations as $G$ [FOS1, §15]. That is not true for ramified $F$-groups. Related to that $(G^\vee)^{\mathbf{I}_F}$ need not be connected. Let $\mathsf{q} : \mathcal{G} \to \mathcal{G}_{\mathrm{ad}}$ be the quotient map to the adjoint group and let $\mathsf{q}^\vee : G_{\mathrm{ad}}{}^\vee \to G^\vee$ be the dual homomorphism. The set $\mathsf{q}^\vee(\Phi_{\mathrm{nr}}(G_{\mathrm{ad}}))$ constists precisely of the $\phi \in \Phi_{\mathrm{nr}}(G)$ with $\Phi(\mathrm{Frob}) \in G^{\vee,\mathbf{I}_F,\circ}$.

Similarly, the natural map $X_{\mathrm{wr}}(G_{\mathrm{ad}}) \to X_{\mathrm{wr}}(G)$ need not be surjective for ramified groups. We denote its image by $X_{\mathrm{wr}}(G_{\mathrm{ad}}, G)$. In Lemmas 5.3 and 5.4 we show that there are natural bijections

$$\begin{aligned} &X_{\mathrm{wr}}(G) \underset{X_{\mathrm{wr}}(G_{\mathrm{ad}},G)}{\times} \mathrm{Irr}(G/Z(G))_{\mathrm{unip}} \longrightarrow \mathrm{Irr}(G)_{\mathrm{unip}}, \\ &X_{\mathrm{wr}}(G) \underset{X_{\mathrm{wr}}(G_{\mathrm{ad}},G)}{\times} \mathsf{q}^\vee(\Phi_{\mathrm{nr}}(G_{\mathrm{ad}})) \longrightarrow \Phi_{\mathrm{nr}}(G). \end{aligned} \tag{9}$$

Using (9) and the case of adjoint groups, the proof of Theorem 1 for supercuspidal unipotent representations of $F_{\mathrm{nr}}$-split semisimple $F$-groups in [FOS1] generalizes readily to connected reductive $F$-groups with anisotropic centre. The step from there to arbitrary connected reductive $F$-groups is easy anyway. This completes the proof of Theorem 1 for supercuspidal unipotent representations, except for the properties (d), (e), (g), (h), (i) and (k).

In Section 6 we set out to generalize the local Langlands correspondence for $\mathrm{Irr}(G)_{\mathrm{unip}}$ in [Sol3] from $F_{\mathrm{nr}}$-split to arbitrary connected reductive $F$-groups. With the above results on the adjoint and the cuspidal cases, that is straightforward. The arguments from [Sol3] yield Theorem 1 for $\mathrm{Irr}(G)_{\mathrm{unip}}$, except for the properties (k) and (l).

Property (k), about the behaviour of unipotent representations upon pullback along certain homomorphisms of reductive groups, is an instance of the main results of [Sol2]. We only have to verify that the $F_{\mathrm{nr}}$-split assumption made in [Sol2, §7] can be lifted. That requires a few remarks about the small modifications in the ramified case. We formulate a more precise version of property (k) in Theorem 6.3.

Finally we deal with the essential uniqueness of our LLC and with property (l), the HII conjecture 2. For $F_{\mathrm{nr}}$-split groups the latter is the main result of [FOS2]. We check that the arguments from [FOS2] can be generalized to possibly ramified connected reductive $F$-groups.

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## 1. List of ramified simple groups

Let $F$ be a non-archimedean local field with ring of integers $\mathfrak{o}_F$ and a uniformizer $\varpi_F$. Let $k_F = \mathfrak{o}_F/\varpi_F\mathfrak{o}_F$ be its residue field, of cardinality $q_F$. We fix a separable closure $F_s$ and assume that all separable extensions of $F$ are realized in $F_s$. Let $F_{\mathrm{nr}}$ be the maximal unramified extension of $F$. Let $\mathbf{W}_F \subset \mathrm{Gal}(F_s/F)$ be the Weil group of $F$ and let Frob be a geometric Frobenius element. Let $\mathbf{I}_F = \mathrm{Gal}(F_s/F_{\mathrm{nr}}) \subset \mathbf{W}_F$ be the inertia subgroup, so that $\mathbf{W}_F/\mathbf{I}_F \cong \mathbb{Z}$ is generated by Frob.

Let $\mathcal{G}$ be a connected reductive $F$-group and pick a maximal $F$-split torus $\mathcal{S}$ in $\mathcal{G}$. Let $\mathcal{T}_{\mathrm{nr}}$ be a maximal $F_{\mathrm{nr}}$-split torus in $Z_{\mathcal{G}}(\mathcal{S})$ defined over $F$ – such a torus exists by [Tit2, §1.10]. Then $\mathcal{T} := Z_{\mathcal{G}}(\mathcal{T}_{\mathrm{nr}})$ is a maximal torus of $\mathcal{G}$, defined over $F$ and containing $\mathcal{T}_{\mathrm{nr}}$ and $\mathcal{S}$. Let $\Phi(\mathcal{G},\mathcal{T})$ be the associated root system. We also fix a Borel subgroup $\mathcal{B}$ of $\mathcal{G}$ containing $\mathcal{T}$ and defined over $\mathcal{F}_{\mathrm{nr}}$, which determines bases $\Delta_{\mathcal{T}}$ of $\Phi(\mathcal{G},\mathcal{T})$, $\Delta_{\mathrm{nr}}$ of $\Phi(\mathcal{G},\mathcal{T}_{\mathrm{nr}})$ and $\Delta$ of $\Phi(\mathcal{G},\mathcal{S})$.

We call $G = \mathcal{G}(F)$:

- *unramified* if $\mathcal{G}$ is quasi-split and splits over $F_{\mathrm{nr}}$;
- *ramified* if $\mathcal{G}$ does not split over $F_{\mathrm{nr}}$.

Unfortunately this common terminology does not exhaust the possibilities: some $F_{\mathrm{nr}}$-split groups are neither ramified nor unramified. Earlier work on unipotent representations of reductive $p$-adic groups applied to groups that are not ramified. In this section we present the list of simple ramified $F$-groups of adjoint type, obtained from [Tit1, Tit2]. For each such group we provide some useful data, which we describe next. We follow the conventions and terminology from [Tit2, Sol3].

The Bruhat–Tits building $\mathcal{B}(\mathcal{G},F)$ has an apartment $\mathbb{A}_{\mathcal{S}} = X_*(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ associated to $\mathcal{S}$. The walls of $\mathbb{A}_S$ determine an affine root system $\Sigma$, which naturally projects onto the finite root system $\Phi(\mathcal{G},\mathcal{S})$. Similarly the Bruhat–Tits building $\mathcal{B}(\mathcal{G},F_{\mathrm{nr}})$ has an apartment $\mathbb{A}_{\mathrm{nr}} = X_*(\mathcal{T}_{\mathrm{nr}}) \otimes_{\mathbb{Z}} \mathbb{R}$ associated to $\mathcal{T}_{\mathrm{nr}}$. The walls of $\mathbb{A}_{\mathrm{nr}}$ determine an affine root system $\Sigma_{\mathrm{nr}}$, which naturally projects onto $\Phi(\mathcal{G},\mathcal{T}_{\mathrm{nr}})$. We recall from [Tit2, 2.6.1] that

$$\mathcal{B}(\mathcal{G},F) = \mathcal{B}(\mathcal{G},F_{\mathrm{nr}})^{\mathrm{Gal}(F_{\mathrm{nr}}/F)} = \mathcal{B}(\mathcal{G},F_{\mathrm{nr}})^{\mathrm{Frob}} \quad \text{and} \quad \mathbb{A}_{\mathcal{S}} = \mathbb{A}_{\mathrm{nr}}^{\mathrm{Frob}}. \tag{10}$$

Let $C_{\mathrm{nr}}$ be a Frob-stable chamber in $\mathbb{A}_{\mathrm{nr}}$ whose closure contains 0 and which (as far as possible) lies in the positive Weyl chamber determined by $\mathcal{B}$. The walls of $C_{\mathrm{nr}}$ provide a basis $\Delta_{\mathrm{nr,aff}}$ of $\Sigma_{\mathrm{nr}}$, which naturally surjects to $\Delta_{\mathrm{nr}}$. The group $\mathrm{Gal}(F_s/F_{\mathrm{nr}})$ acts naturally on $C_{\mathrm{nr}}$ and hence on $\Delta_{\mathrm{nr,aff}}$. The Dynkin diagram of $(\Sigma_{\mathrm{nr}}, \Delta_{\mathrm{nr,aff}})$, together with the action of Frob, is called the *local index* of $\mathcal{G}(F)$.

By (10) there exists a unique chamber $C_0$ in $\mathbb{A}_{\mathcal{S}}$ containing $C_{\mathrm{nr}} \cap \mathbb{A}_{\mathcal{S}}$. The walls of $C_0$ yield a basis $\Delta_{\mathrm{aff}}$ of $\Sigma$ which projects onto $\Delta$. By construction $\Delta_{\mathrm{aff}}$ consists of the restrictions of $\Delta_{\mathrm{nr,aff}}$ to $\mathbb{A}_{\mathcal{S}}$. As $\mathcal{G}$ is simple, $|\Delta_{\mathrm{aff}}| = |\Delta| + 1$ and $|\Delta_{\mathrm{nr,aff}}| = |\Delta_{\mathrm{nr}}| + 1$. The relative local Dynkin diagram of $\mathcal{G}(F)$ is defined as the Dynkin diagram of $(\Sigma, \Delta_{\mathrm{aff}})$.

We will also need a group called $\Omega$ or $\Omega_G$, which can be described in several equivalent ways [PaRa, Appendix]:

- $\mathrm{Irr}\big((Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}}\big)$, where $G^\vee$ is the complex dual group of $G$;
- $G$ modulo the kernel of the Kottwitz homomorphism $G \to \mathrm{Irr}\big((Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}}\big)$;
- $G$ modulo the subgroup generated by all parahoric subgroups of $G$;
- the stabilizer of $C_0$ in the group $N_G(S)/(Z_G(S) \cap P_{C_0})$, where $P_{C_0} \subset G$ denotes the Iwahori subgroup associated to $C_0$;

- $\big( (X_*(\mathcal{T})/\mathbb{Z}\Phi^\vee(\mathcal{G},\mathcal{T}))_{\mathbf{I}_F} \big)^{\mathrm{Frob}}$.

The group $\Omega_G$ acts naturally on the relative local Dynkin diagram of $\mathcal{G}(F)$. We say that a character of $G$ is weakly unramified if it is trivial on every parahoric subgroup of $G$. By the above, the group $X_{\mathrm{wr}}(G)$ of all such characters is naturally isomorphic with $\mathrm{Irr}(\Omega_G)$ and with $(Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}}$.

We say that $\mathcal{G}$ is simple if it is simple as $F_s$-group. If it is merely simple as $F$-group, we call it $F$-simple. For every ramified simple $F$-group $\mathcal{G}$ we give a $F_{\mathrm{nr}}$-split "companion" $F$-group $\mathcal{G}'$. It is determined by the following requirements:

- There exists a Frob-equivariant bijection between $\Delta_{\mathrm{nr,aff}}$ for $\mathcal{G}$ and $\mathcal{G}'$, which preserves the number of bonds in the Dynkin diagram(s) of $(\Sigma_{\mathrm{nr}}, \Delta_{\mathrm{nr,aff}})$. Thus the local index of $\mathcal{G}'$ is the same as that of $\mathcal{G}$, except that the directions of some arrows may differ. In particular this gives a bijection from the relative local Dynkin diagram for $\mathcal{G}'(F)$ to that for $\mathcal{G}(F)$.
- There is an isomorphism $\Omega_{G'} \cong \Omega_G$, which renders the bijection between the relative local Dynkin diagrams $\Omega_G$-equivariant.

We specify a bijection between $\Delta_{\mathrm{nr,aff}}$ for $\mathcal{G}$ and $\mathcal{G}'$ by marking one special vertex on both sides. In most cases 0 is a special vertex, then we pick that one. This also determines one marked vertex of $\Delta_{\mathrm{aff}}$ (and one of $\Delta'_{\mathrm{aff}}$). The remainder of the relative local Dynkin diagram $\Delta_{\mathrm{aff}}$ is canonically in bijection with $\Delta$, so the bijection $\Delta_{\mathrm{aff}} \longleftrightarrow \Delta'_{\mathrm{aff}}$ induces a bijection $\Delta \longleftrightarrow \Delta'$.

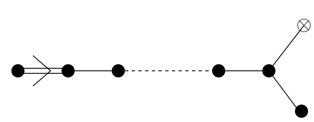
These relations between the groups $\mathcal{G}$ and $\mathcal{G}'$ lead to many similarities. For instance, their parabolic $F$-subgroups can be compared. Namely, it is well-known that the $G$-conjugacy classes of parabolic $F$-subgroups of $\mathcal{G}$ are naturally in bijection with the power set of $\Delta$ [Spr, Theorem 15.4.6]. The same holds for $\mathcal{G}'$. Hence the above bijection $\Delta \longleftrightarrow \Delta'$ induces a bijection from the set of conjugacy classes of parabolic $F$-subgroups of $\mathcal{G}$ to the analogous set for $\mathcal{G}'$. Furthermore every conjugacy class of parabolic subgroups of $\mathcal{G}$ contains a unique standard parabolic subgroup (with respect to $\mathcal{B}$). We will denote the resulting bijection between standard parabolic $F$-subgroups by $\mathcal{P} \mapsto \mathcal{P}'$.

Apart from the above data, the group $\mathcal{G}$ depends on the choice of a suitable field extension of $F$. Below $F^{(2)}$ is the unique unramified quadratic extension of $F$ and $E$ (resp. $E^{(2)}$) denotes a ramified separable quadratic extension of $F$ (resp. of $F^{(2)}$).

We use the names for the local indices from [Tit1, Tit2]. For each name, we start with the adjoint group $\mathcal{G}$ of that type, and its companion group. After that, we list the groups isogenous to $\mathcal{G}$. These have the same local index and relative local Dynkin diagram as $\mathcal{G}$ and their companion groups are isogenous to $\mathcal{G}'$, but they have a smaller group $\Omega_G$.

1.1. **B-C$_{\mathbf{n}}$.**

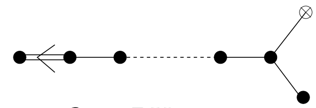
$\mathcal{G} = PU_{2n}$, quasi-split over $F$, split over $E$

Local index and relative local Dynkin diagram: 

Trivial Frob-action

$\Omega_G$ has two elements, it exchanges the two legs on the right hand side

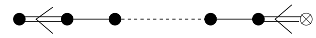
$\mathcal{G}' = SO_{2n+1}$, $F$-split

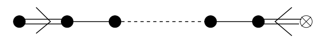
Local index and relative local Dynkin diagram: 

Groups isogenous to $PU_{2n}$ fit in a sequence $SU_{2n} \to \mathcal{G} \to PU_{2n}$.
Such a group is determined by the order of its schematic centre, call that $d$.

$$\mathcal{G}' = \begin{cases} SO_{2n+1} & \text{if } d \text{ is odd} \\ Spin_{2n+1} \times \{\pm 1\} & \text{if } d \text{ is even and } 2n/d \text{ is even} \\ Spin_{2n+1} & \text{if } d \text{ is even and } 2n/d \text{ is odd} \end{cases}$$

In the first two cases $\Omega_G$ has order two and it acts on the diagram as for $PU_{2n}$, in the second case $|\Omega_G| = 2$ and it acts trivially on the diagram, while in the third case $|\Omega_G| = 1$.

1.2. **C-BC$_\mathbf{n}$.**
$\mathcal{G} = PU_{2n+1}$, quasi-split over $F$, split over $E$
Local index and relative local Dynkin diagram: 
Trivial Frob-action
$\Omega_G$ has one element

$\mathcal{G}' = Sp_{2n}$, $F$-split
Local index and relative local Dynkin diagram: 

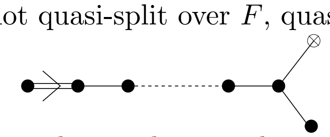
Groups isogenous to $PU_{2n+1}$ fit in a sequence $SU_{2n+1} \to \mathcal{G} \to PU_{2n+1}$.
Then $|\Omega_G| = 1$ and $\mathcal{G}' = Sp_{2n}$.

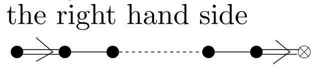
1.3. **C-B$_\mathbf{n}$.**
$\mathcal{G} = PSO^*_{2n+2}$, quasi-split over $F$, split over $E$
Local index and relative local Dynkin diagram: 
Trivial Frob-action
$\Omega_G$ has two elements, it reflects in the middle of the diagram

$\mathcal{G}' = PSp_{2n}$, $F$-split
Local index and relative local Dynkin diagram: 

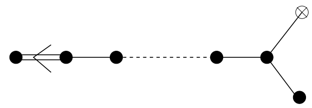
Isogenous group $\mathcal{G} = Spin^*_{2n+2}$: $\Omega_G = 1$ and $\mathcal{G}' = Sp_{2n}$.
The isogenous group $\mathcal{G} = SO^*_{2n+2}$ has $\Omega_G$ of order 2, but acting trivially on the diagram. We take $\mathcal{G}' = Sp_{2n} \times \{\pm 1\}$.

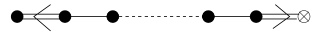
1.4. **$^2$B-C$_\mathbf{n}$.**
$\mathcal{G} = PU_{2n}$, not quasi-split over $F$, quasi-split over $F^{(2)}$, split over $E^{(2)}$

Local index: 

Frob exchanges the two legs on the right hand side
Relative local Dynkin diagram: 
$\Omega_G$ has two elements, it acts trivially on the diagram

$\mathcal{G}' = SO_{2n+1}$, not split over $F$, split over $F^{(2)}$

Local index: 

Relative local Dynkin diagram: 

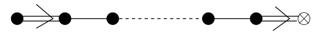
For isogenous groups $\mathcal{G}$, the situation is as for $\mathbf{B}\text{-}\mathbf{C}_\mathbf{n}$,
except that $\mathcal{G}'$ splits over $F^{(2)}$ but not over $F$.

### 1.5. $\mathbf{{}^2C\text{-}B_{2n}}$.

$\mathcal{G} = PSO^*_{4n}$, not quasi-split over $F$, quasi-split over $F^{(2)}$, split over $E^{(2)}$

Local index: 

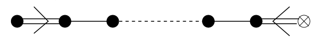
Frob exchanges the upper and the lower row

Relative local Dynkin diagram: 

$\Omega_G$ has two elements, it acts trivially on the diagram

$\mathcal{G}' = PSp_{4n-2}$, not split over $F$, split over $F^{(2)}$

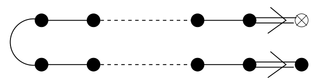
Local index: 

Relative local Dynkin diagram: 

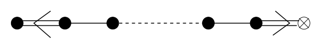
Isogenous group $\mathcal{G} = Spin^*_{4n}$: $|\Omega_G| = 1$ and $\mathcal{G}' = Sp_{4n-2}$.
Isogenous group $\mathcal{G} = SO^*_{4n}$: $|\Omega_G| = 2$ and $\mathcal{G}' = Sp_{4n-2} \times \{\pm 1\}$.

### 1.6. $\mathbf{{}^2C\text{-}B_{2n+1}}$.

$\mathcal{G} = PSO^*_{4n+2}$, not quasi-split over $F$, quasi-split over $F^{(2)}$, split over $E^{(2)}$

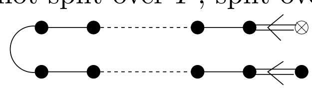
Local index: 

Frob exchanges the upper and the lower row

Relative local Dynkin diagram: 

$\Omega_G$ has two elements, it acts trivially on the diagram

$\mathcal{G}' = PSp_{4n}$, not split over $F$, split over $F^{(2)}$

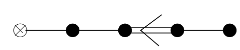
Local index: 

Relative local Dynkin diagram: 

Isogenous group $\mathcal{G} = Spin^*_{4n+2}$: $|\Omega_G| = 1$ and $\mathcal{G}' = Sp_{4n}$.
Isogenous group $\mathcal{G} = SO^*_{4n+2}$: $|\Omega_G| = 2$ and $\mathcal{G}' = Sp_{4n} \times \{\pm 1\}$.

### 1.7. $\mathbf{F_4^I}$.

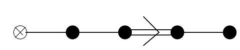
$\mathcal{G} = {}^2E_{6,\mathrm{ad}}$, quasi-split over $F$, split over $E$

Local index and relative local Dynkin diagram: 

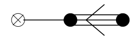
Trivial Frob-action

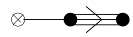
$\Omega_G$ has one element

$\mathcal{G}' = F_4$, $F$-split

Local index and relative local Dynkin diagram: 

Isogenous group $\mathcal{G} = {}^2E_{6,\mathrm{sc}}$: $|\Omega_G| = 1$ and $\mathcal{G}' = F_4$.

1.8. $\mathbf{G_2^I}$.
$\mathcal{G} = {}^rD_{4,\mathrm{ad}}$ with $r = 3$ or $r = 6$, quasi-split over $F$, split over a Galois extension $E'/F$ of degree $r$ such that the unique degree 3 subextension of $F$ is ramified
Local index and relative local Dynkin diagram: 
Trivial Frob-action
$\Omega_G$ has one element

$\mathcal{G}' = G_2$, $F$-split
Local index and relative local Dynkin diagram: 

Isogenous group $\mathcal{G} = {}^rD_{4,\mathrm{sc}}$: $|\Omega_G| = 1$ and $\mathcal{G}' = G_2$.

To fulfill the requirement $\Omega_G \cong \Omega_{G'}$ we sometimes needed a disconnected group $\mathcal{G}' = \mathcal{G}'^{\circ} \times \{\pm 1\}$. All standard operations for connected reductive groups extend naturally such $\mathcal{G}'$. For instance, the Bruhat–Tits building of $(\mathcal{G}', F)$ is that of $(\mathcal{G}'^{\circ}, F)$, with $\{\pm 1\}$ acting trivially. In particular a parahoric subgroup of $\mathcal{G}'(F)$ is a parahoric subgroup of $\mathcal{G}'^{\circ}(F)$. The complex dual group of $\mathcal{G}'^{\circ} \times \{\pm 1\}$ is defined to be $(\mathcal{G}'^{\circ})^{\vee} \times \{\pm 1\}$. In Lemma 4.1 we will see that this fits well, which motivates our choice of $\mathcal{G}'$.

## 2. Matching of unipotent representations

By construction there is a canonical bijection between the set of faces of $C_{\mathrm{nr}}$ and the collection of proper subsets of $\Delta_{\mathrm{nr,aff}}$. Explicitly, it associates to a face $\mathfrak{f}$ the set $J_{\mathfrak{f}}$ of simple affine roots of $\Sigma_{\mathrm{nr}}$ that vanish on $\mathfrak{f}$. With (10) and $\Delta_{\mathrm{aff}} = \Delta_{\mathrm{nr,aff}}/\mathrm{Gal}(F_{\mathrm{nr}}/F)$ this leads to canonical bijections between the following sets:

- proper subsets of $\Delta_{\mathrm{aff}}$;
- Frob-stable proper subsets of $\Delta_{\mathrm{nr,aff}}$;
- Frob-stable faces of $C_{\mathrm{nr}}$;
- faces of $C_0$.

Let $\mathfrak{f}$ be a Frob-stable face of $C_{\mathrm{nr}}$, identified with a face of $C_0$. Bruhat and Tits [BrTi] associated to $\mathfrak{f}$ an $\mathfrak{o}_F$-group $\mathcal{G}_{\mathfrak{f}}$, such that $\mathcal{G}_{\mathfrak{f}}^{\circ}(\mathfrak{o}_F)$ equals the parahoric subgroup $P_{\mathfrak{f}} \subset G$ associated to $\mathfrak{f}$ and $\mathcal{G}_{\mathfrak{f}}(\mathfrak{o}_F) = N_G(P_{\mathfrak{f}})$ equals the $G$-stabilizer of $\mathfrak{f}$.

Let $\overline{\mathcal{G}_{\mathfrak{f}}}$ be the maximal reductive quotient of $\mathcal{G}_{\mathfrak{f}}$ as $k_F$-group. Then $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k_F) = P_{\mathfrak{f}}/P_{\mathfrak{f}}^{+}$, where $P_{\mathfrak{f}}^{+}$ denotes the pro-unipotent radical of $P_{\mathfrak{f}}$.

Let $\Omega_{G,\mathfrak{f}}$ be the stabilizer of $\mathfrak{f}$ in $\Omega_G$ (with respect to its action on $\Delta_{\mathrm{aff}}$). Then

$$\overline{\mathcal{G}_{\mathfrak{f}}}/\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}} \cong N_G(P_{\mathfrak{f}})/P_{\mathfrak{f}} \cong \Omega_{G,\mathfrak{f}}. \tag{11}$$

The algebraic group $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}$ splits over $k_{F_{\mathrm{nr}}}$ (an algebraic closure of $k_F$) and its Dynkin diagram is the subdiagram of $\Delta_{\mathrm{nr,aff}}$ formed by the vertices in $J_{\mathfrak{f}}$. Further, the isogeny class of the $k_F$-group $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}$ is determined by the action of Frob on $J_{\mathfrak{f}}$, so it only depends on $\mathfrak{f}$ and the local index of $\mathcal{G}$.

**Proposition 2.1.** *Let $\mathcal{G}$ be a ramified simple $F$-group and let $\mathcal{G}'$ be its $F_{\mathrm{nr}}$-split companion group, as in Section 1. Let $\mathfrak{f}$ be a* Frob*-stable face of $C_{\mathrm{nr}}$ and let $\mathfrak{f}'$ be the* *face of $C'_{\mathrm{nr}}$ corresponding to it via the bijection between the local indices of $\mathcal{G}$ and $\mathcal{G}'$. Then the $k_F$-groups $\overline{\mathcal{G}_{\mathfrak{f}}}$ and $\overline{\mathcal{G}'_{\mathfrak{f}'}}$ have the following in common:*

- *their Lie type, up to changing the direction of some arrows in the Dynkin diagram;*
- *their dimension;*
- $|\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k_F)| = |\overline{\mathcal{G}'^{\circ}_{\mathfrak{f}'}}(k_F)|$*;*
- $\Omega_{G,\mathfrak{f}} \cong \Omega_{G',\mathfrak{f}'}$.

*Proof.* The setup from Section 1 provides a bijection between $J_{\mathfrak{f}}$ and $J_{\mathfrak{f}'}$, which preserves the number of bonds in the Dynkin diagrams of $\overline{\mathcal{G}_{\mathfrak{f}}}$ and $\overline{\mathcal{G}'_{\mathfrak{f}'}}$. Decomposing into connected components, we get $J_{\mathfrak{f}} = \bigsqcup_i J_{\mathfrak{f}}^i$ and $J_{\mathfrak{f}'} = \bigsqcup_i J_{\mathfrak{f}'}^i$ where the connected Dynkin diagram $J_{\mathfrak{f}}^i$ is isomorphic to $J_{\mathfrak{f}'}^i$ or to its dual.

Write $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}$ as an almost direct product of simple groups $\overline{\mathcal{G}_{\mathfrak{f}}^{i}}$, and similarly for $\overline{\mathcal{G}'^{\circ}_{\mathfrak{f}'}}$. Then $\overline{\mathcal{G}_{\mathfrak{f}}^{i}}$ is isogenous to $\overline{\mathcal{G}'^{i}_{\mathfrak{f}'}}$ or to the dual group of $\overline{\mathcal{G}'^{i}_{\mathfrak{f}'}}$ over $k_F$. Consequently

$$\dim \overline{\mathcal{G}_{\mathfrak{f}}^{\circ}} = \sum\nolimits_i \dim \overline{\mathcal{G}_{\mathfrak{f}}^{i}} = \sum\nolimits_i \dim \overline{\mathcal{G}'^{i}_{\mathfrak{f}'}} = \dim \overline{\mathcal{G}'^{\circ}_{\mathfrak{f}'}}.$$

The number of elements of a connected reductive group over a finite field only depends on the group up to isogeny [GeMa, Proposition 1.4.13.c]. It also does not change if we replace a simple group by its dual group, by Chevalley's counting formulas [GeMa, Theorem 1.6.7]. We deduce

$$\big|\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k_F)\big| = \prod\nolimits_i \big|\overline{\mathcal{G}_{\mathfrak{f}}^{i}}(k_F)\big| = \prod\nolimits_i \big|\overline{\mathcal{G}'^{i}_{\mathfrak{f}'}}(k_F)\big| = \big|\overline{\mathcal{G}'^{\circ}_{\mathfrak{f}'}}(k_F)\big|.$$

The claim about $\Omega_{G,\mathfrak{f}}$ follows from the $\Omega_G$-equivariance of the bijection between $\Delta_{\mathrm{aff}}$ and $\Delta'_{\mathrm{aff}}$. $\square$

It is well-known that the conjugacy classes of parabolic $k_F$-subgroups of $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}$ are naturally in bijection with the subsets of the Dynkin diagram $J_{\mathfrak{f}}$. Every conjugacy class of parabolic $k_F$-subgroups of $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}$ contains a unique parabolic subgroup $\mathcal{P}$ which is standard (with respect to the image of $\mathcal{S}$ and the basis $\Delta$ of $\Phi(\mathcal{G},\mathcal{S})$). We denote the unique standard $k_F$-Levi factor of $\mathcal{P}$ by $\mathcal{M}$ and its unipotent radical by $\mathcal{U}$.

The same holds for $\overline{\mathcal{G}'^{\circ}_{\mathfrak{f}'}}$, and we have a bijection between $J_{\mathfrak{f}}$ and $J_{\mathfrak{f}'}$. Thus $(\mathcal{P},\mathcal{M})$ determines a unique standard parabolic pair $(\mathcal{P}',\mathcal{M}')$ for $\overline{\mathcal{G}'^{\circ}_{\mathfrak{f}'}}$, defined over $k_F$.

Let $\mathrm{Irr}_{\mathrm{unip}}(H)$ denote the collection of irreducible unipotent representations of a group $H$.

**Proposition 2.2.** *Use the notations from Proposition 2.1 and let $\mathcal{P} = \mathcal{M}\mathcal{U}$ be a standard parabolic $k_F$-subgroup $\mathcal{P} = \mathcal{M}\mathcal{U}$ of $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k_F)$.*

($a$) *There exists a natural bijection*

$$\mathrm{Irr}_{\mathrm{unip,cusp}}(\mathcal{M}(k_F)) \longleftrightarrow \mathrm{Irr}_{\mathrm{unip,cusp}}(\mathcal{M}'(k_F)).$$

($b$) *There exists a bijection*

$$\mathrm{Irr}_{\mathrm{unip}}(\mathcal{M}(k_F)) \longleftrightarrow \mathrm{Irr}_{\mathrm{unip}}(\mathcal{M}'(k_F)),$$

*which preserves dimensions.*

($c$) *Extend part (b) to a bijection* $\mathrm{Rep}_{\mathrm{unip}}(\mathcal{M}(k_F)) \longleftrightarrow \mathrm{Rep}_{\mathrm{unip}}(\mathcal{M}'(k_F))$ *by making it additive. The system of such bijections, with $\mathcal{P}$ running over all standard* *parabolic $k_F$-subgroups of $\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)$, is compatible with parabolic induction and Jacquet restriction.*

*Proof.* (a) By [Lus1, Proposition 3.15] $\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F))$ depends only on $\overline{\mathcal{G}_{\mathfrak{f}}^\circ}$ up to isogeny. Using the constructions from the proof of Proposition 2.1, we obtain a natural bijection

$$\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)) \longleftrightarrow \prod\nolimits_i \mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^i}(k_F)), \tag{12}$$

and similarly for $\overline{\mathcal{G}_{\mathfrak{f}'}'^\circ}(k_F)$. The unipotent representations of $\overline{\mathcal{G}_{\mathfrak{f}}^i}(k_F)$ are built from the cuspidal unipotent representations of Levi factors $\mathcal{M}_i(k_F)$ of parabolic subgroups $\mathcal{P}_i(k_F)$ and from Hecke algebras, see [Lus1, Theorem 3.26]. Since $\overline{\mathcal{G}_{\mathfrak{f}}^i}$ and $\overline{\mathcal{G}_{\mathfrak{f}'}'^i}$ have the same Dynkin diagram (up to the direction of arrows), the conjugacy class of $\mathcal{P}_i = \mathcal{M}_i\mathcal{U}_i$ corresponds to a unique conjugacy class of parabolic subgroups $\mathcal{P}_i' = \mathcal{M}_i'\mathcal{U}_i'$ of $\overline{\mathcal{G}_{\mathfrak{f}'}'^i}$. Then $\mathcal{M}_i$ and $\mathcal{M}_i'$ also have the same Dynkin diagram (up to the direction of arrows).

The classification of cuspidal unipotent representations in [Lus1, §3] shows that for each such $\mathcal{P}_i$ there is a canonical bijection

$$\mathrm{Irr}_{\mathrm{unip,cusp}}(\mathcal{M}_i(k_F)) \longleftrightarrow \mathrm{Irr}_{\mathrm{unip,cusp}}(\mathcal{M}_i'(k_F)), \tag{13}$$

which preserves dimensions. Moreover, if $\psi$ is any Frob-equivariant automorphism of the Dynkin diagram of $\mathcal{M}_i$ (and hence also for $\mathcal{M}_i'$) and we lift it to automorphisms of $\mathcal{M}_i(k_F)$ and of $\mathcal{M}_i'(k_F)$, then (13) is $\psi$-equivariant. These claims can be checked case-by-case from [Lus1]. To make that easier, one may note that the list in Section 1 shows that no factors of Lie type $E_n$ are involved. We conclude that the bijection (13) is natural.

Now we combine (13) with (12) for $\mathrm{Irr}_{\mathrm{cusp,unip}}(\mathcal{M}(k_F))$ and for $\mathrm{Irr}_{\mathrm{cusp,unip}}(\mathcal{M}'(k_F))$.
(b) Fix $\mathcal{P}_i = \mathcal{M}_i\mathcal{U}_i$ and $\rho \in \mathrm{Irr}_{\mathrm{unip,cusp}}(\mathcal{M}_i(k_F))$, and let $\rho_i' \in \mathrm{Irr}_{\mathrm{unip,cusp}}(\mathcal{M}_i'(k_F))$ be its image under (13). From [Lus1, Table II] we see that the Hecke algebra $\mathrm{End}_{\overline{\mathcal{G}_{\mathfrak{f}}^i}(k_F)}\big(\mathrm{ind}_{\mathcal{P}_i(k_F)}^{\overline{\mathcal{G}_{\mathfrak{f}}^i}(k_F)}\rho_i\big)$ for $(\overline{\mathcal{G}_{\mathfrak{f}}^i}, \mathcal{P}_i, \rho_i)$ is isomorphic to the Hecke algebra of $(\overline{\mathcal{G}_{\mathfrak{f}'}'^i}, \mathcal{P}_i', \rho_i')$. This works for all $(\mathcal{P}_i, \rho_i)$ and, as described in [Lus1, §3.25], it gives rise to a bijection

$$\mathrm{Irr}_{\mathrm{unip}}\big(\overline{\mathcal{G}_{\mathfrak{f}}^i}(k_F)\big) \longleftrightarrow \mathrm{Irr}_{\mathrm{unip}}\big(\overline{\mathcal{G}_{\mathfrak{f}'}'^i}(k_F)\big). \tag{14}$$

By [Lus1, (3.26.1)] and Proposition 2.1, (14) preserves dimensions. Combine this with (12) to get part (b) for $\overline{\mathcal{G}_{\mathfrak{f}}^i}(k_F)$. For $\mathcal{M}(k_F)$ it can be shown in the same way.
(c) In the constructions for part (b) everything is obtained by parabolic induction from the cuspidal level, followed by selecting suitable subrepresentations by means of Hecke algebras. In view of the transitivity of parabolic induction, this setup entails that the system of bijections $\mathrm{Rep}_{\mathrm{unip}}(\mathcal{M}(k_F)) \longleftrightarrow \mathrm{Rep}_{\mathrm{unip}}(\mathcal{M}'(k_F))$ is compatible with parabolic induction. Since Jacquet restriction is the adjoint functor of parabolic induction, the system of bijections is also compatible with that. $\square$

By (11) the group $\Omega_{G,\mathfrak{f}}$ acts naturally on $\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F))$.

**Theorem 2.3.** (*a*) *The bijection*

$$\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)) \longleftrightarrow \mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}'}'^\circ}(k_F)),$$

*constructed in the proof of Proposition 2.2.b is $\Omega_{G,\mathfrak{f}}$-equivariant.*

(*b*) *It extends to a bijection*

$$\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)) \longleftrightarrow \mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}'_{\mathfrak{f}'}}(k_F)),$$

*which preserves dimensions and cuspidality.*

*Proof.* Recall from Section 1 that $|\Omega_G| \leq 2$. When $|\Omega_{G,\mathfrak{f}}| = 1$, (11) shows that there is nothing to prove. So we may assume that $\Omega_{G,\mathfrak{f}} = \{1, \omega\} \cong \Omega_{G',\mathfrak{f}'}$, where we have picked representatives for $\omega$ in $\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)$ and in $\overline{\mathcal{G}'_{\mathfrak{f}'}}(k_F)$.
(a) The bijections (13) combine to a $\Omega_{G,\mathfrak{f}}$-equivariant bijection between

$$(15) \qquad \left\{ \begin{array}{c} (\mathcal{P} = \mathcal{M}\mathcal{U}, \rho) : \mathcal{P} \text{ parabolic } k_F\text{-subgroup of } \overline{\mathcal{G}_{\mathfrak{f}}^\circ} \\ \text{with Levi factor } \mathcal{M}, \rho \in \mathrm{Irr}_{\mathrm{unip,cusp}}(\mathcal{M}(k_F)) \end{array} \right\} \Big/ \overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)\text{-conjugacy}$$

to its analogue for $\overline{\mathcal{G}'^\circ_{\mathfrak{f}'}}(k_F)$.

When $\omega$ does not stabilize the $\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)$-orbit of $(\mathcal{P}, \rho)$, the subsets of $\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F))$ associated to $(\mathcal{P}, \rho)$ and to $(\omega\mathcal{P}\omega^{-1}, \omega \cdot \rho)$ are disjoint [Lus1, §3.25]. In particular $\omega$ does not stabilize any representation in such a set.

Suppose now that $(\omega\mathcal{P}\omega^{-1}, \omega \cdot \rho)$ is $\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)$-conjugate to $(\mathcal{P}, \rho)$. Choosing another representative for $\omega$ in $\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)$, we may assume that $(\omega\mathcal{P}\omega^{-1}, \omega \cdot \rho) = (\mathcal{P}, \rho)$. Since the parabolic subgroup $\mathcal{P}$ is its own normalizer in $\overline{\mathcal{G}_{\mathfrak{f}}}$, this choice of $\omega$ is unique up to inner automorphisms of $\mathcal{P}(k_F)$. Now $\rho$ extends to a representation $\tilde{\rho}$ of

$$\langle \mathcal{P}(k_F), \omega \rangle = \mathcal{P}(k_F) \cup \omega\mathcal{P}(k_F).$$

In fact there are two choices for $\tilde{\rho}$, differing by a quadratic character, but which one does not matter because we only need conjugation by $\tilde{\rho}(\omega)$.

Then $\Pi := \mathrm{ind}_{\mathcal{P}(k_F)}^{\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)} \rho$ extends to the $\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)$-representation $\tilde{\Pi} := \mathrm{ind}_{\langle \mathcal{P}(k_F), \omega \rangle}^{\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)} \tilde{\rho}$. Conjugation by $\tilde{\Pi}(\omega)$ provides an automorphism $\psi_\omega$ of the Hecke algebra

$$\mathcal{H} := \mathrm{End}_{\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)}(\Pi) = \mathrm{End}_{\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)}(\tilde{\Pi}).$$

A $\pi \in \mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F))$ associated to $(\mathcal{P}, \rho)$ corresponds to the irreducible $\mathcal{H}$-module $\mathrm{Hom}_{\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)}(\Pi, \pi)$. Conversely, any $\pi_{\mathcal{H}} \in \mathrm{Irr}(\mathcal{H})$ gives rise to $E \otimes_{\mathcal{H}} \pi_{\mathcal{H}} \in \mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F))$. Under this correspondence the action of $\omega$ on $\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F))$ translates to the action of $\psi_\omega$ on $\mathrm{Irr}(\mathcal{H})$.

Given that $\omega$ stabilizes $(\mathcal{P}, \omega)$, the entire setup is canonical up to inner automorphisms. The $\Omega_{G,\mathfrak{f}}$-action can be described entirely with data coming from the cuspidal level. Of course the same applies to $\overline{\mathcal{G}'^\circ_{\mathfrak{f}'}}(k_F)$. Together with the $\Omega_{G,\mathfrak{f}}$-equivariance of the bijection involving (15), we deduce $\Omega_{G,\mathfrak{f}}$-equivariance in the desired generality.
(b) To extend the bijection from Proposition 2.2.b to $\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)$ and $\overline{\mathcal{G}'_{\mathfrak{f}'}}(k_F)$, we need Clifford theory with respect to the action of $\Omega_{G,\mathfrak{f}} \cong \Omega_{G',\mathfrak{f}'}$ on $\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F))$ and $\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}'^\circ_{\mathfrak{f}'}}(k_F))$.

When $\omega$ does not stabilize $\pi \in \mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F))$, the $\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)$-representation $\pi \oplus \omega \cdot \pi$ extends to an irreducible $\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)$-representation $\tilde{\pi}$. In this way the pair $\{\pi, \omega \cdot \pi\}$ accounts for one element of $\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}}(k_F))$.

When $\omega$ stabilizes $\pi \in \mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F))$, $\pi$ extends in precisely two ways to an irreducible representation of $\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)$. The two extensions $\pi_+, \pi_-$ are related by

$\pi_-(\omega) = -\pi_+(\omega)$. Thus $\pi$ gives rise to a pair $\{\pi_+, \pi_-\}$ in $\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}}(k_F))$. Clifford theory tells us that every element of $\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}}(k_F))$ arises in a unique way from one of these two constructions.

In view of part (a), Clifford theory works in the exactly same way for $\overline{\mathcal{G}'_{\mathfrak{f}'}}(k_F)$. Denoting the bijection from Proposition 2.2.b by $\pi \mapsto \pi'$, we can extend it to

$$\mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)) \longleftrightarrow \mathrm{Irr}_{\mathrm{unip}}(\overline{\mathcal{G}'_{\mathfrak{f}'}}(k_F))$$

by sending either sending $\tilde{\pi}$ to $\tilde{\pi}'$ or sending $\pi_+$ to $\pi'_+$ and $\pi_-$ to $\pi'_-$. (Notice that this is not canonical, for we could just as well exchange $\pi'_+$ and $\pi'_-$.) As dimensions and cuspidality are preserved in Proposition 2.2, they are preserved here as well. $\square$

It will be handy to know how the bijections in Theorem 2.3 behave with respect to outer automorphisms of $\mathcal{G}$ and $\mathcal{G}'$. From the list in Section 1 we see that $\mathcal{G}'$ has Lie type $B_n, C_n, F_4$ or $G_2$, so all its automorphisms are inner. On the other hand, the group $\mathcal{G}$ does admit outer automorphisms. Requiring that they fix a pinning, outer automorphisms can be classified in terms of $\mathbf{W}_F$-equivariant automorphisms of the (absolute) Dynkin diagram of $(\mathcal{G}, \mathcal{T})$, see [Sol2, Corollary 3.3]. Then we call them *diagram automorphisms* of $\mathcal{G}(F)$.

In paragraphs 1.1–1.7 the absolute root datum of $\mathcal{G}$ admits exactly one nontrivial automorphism $\tau$. In paragraph 1.8 there is no such automorphism for ${}^6D_4$, and there are two for ${}^3D_4$, say $\tau$ and $\tau^2$. In all these cases $\tau$ lifts to an automorphism of $\mathcal{G}(F)$ because $\mathcal{G}$ is either quasi-split or the unique inner twist of its quasi-split form.

Recall that the $F$-points of $\mathcal{G}$ can be obtained from its $F_s$-points by taking the invariants with respect to the $\mathbf{W}_F$-action that defines the structure as $F$-group. This $\mathbf{W}_F$-action is a combination of the natural Galois action on matrix coefficients and algebraic group automorphisms. In the cases under consideration, one element of $\mathbf{W}_F$ acts as $g \mapsto \overline{\tau(g)}$, where the overline indicates a field automorphism. It follows that on $\mathcal{G}(F)$, $\tau$ works out as

- the nontrivial field automorphism of $E/F$, applied to matrix coefficients, for $\mathbf{B\text{-}C_n}, \mathbf{C\text{-}BC_n}, \mathbf{C\text{-}B_n}, \mathbf{F_4^I}$;
- the nontrivial field automorphism of $E^{(2)}/F^{(2)}$, applied to matrix coefficients, for $\mathbf{{}^2B\text{-}C_n}, \mathbf{{}^2C\text{-}B_{2n}}, \mathbf{{}^2C\text{-}B_{2n+1}}$;
- one of the two nontrivial field automorphisms of $E'/F$, applied to matrix coefficients, for $\mathbf{G_2^I}$ with $\mathcal{G}$ of type ${}^3D_4$.

**Lemma 2.4.** *The diagram automorphism $\tau$ of $\mathcal{G}(F)$ stabilizes the groups $P_{\mathfrak{f}}, N_G(P_{\mathfrak{f}})$, $\overline{\mathcal{G}^\circ_{\mathfrak{f}}}(k_F)$ and $\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)$, as well as all their unipotent representations.*

*Proof.* The local index of $\mathcal{G}$ is obtained from the Dynkin diagram of $(\mathcal{G}, \mathcal{T})$ by dividing out the $\mathbf{I}_F$-action. Here $\mathbf{I}_F$ acts via powers of $\tau$, so $\tau \in \mathrm{Aut}(\mathcal{G}(F))$ acts trivially on the local index of $\mathcal{G}$. It follows that $\tau$ stabilizes every face of $C_{\mathrm{nr}}$, and hence acts on the four indicated groups.

The absolute Dynkin diagram of $\overline{\mathcal{G}^\circ_{\mathfrak{f}}}$ is a subdiagram of $\Delta_{\mathrm{nr,aff}}$, and $\tau$ fixes that pointwise. Consequently $\tau$ acts on $\overline{\mathcal{G}^\circ_{\mathfrak{f}}}$ by an inner $k_F$-automorphism, that is, as conjugation by an element of the adjoint group $(\overline{\mathcal{G}^\circ_{\mathfrak{f}}})_{\mathrm{ad}}(k_F)$. It is known from [Lus1, Proposition 3.15] that every unipotent representation $\pi$ of $\overline{\mathcal{G}^\circ_{\mathfrak{f}}}(k_F)$ extends to a representation of $(\overline{\mathcal{G}^\circ_{\mathfrak{f}}})_{\mathrm{ad}}(k_F)$. This shows that $\tau$ stabilizes all unipotent representations of $\overline{\mathcal{G}^\circ_{\mathfrak{f}}}(k_F)$ and of $P_{\mathfrak{f}}$.

In the proof of Theorem 2.3.b we saw how Clifford theory produces irreducible unipotent representations of $\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)$ from those of $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k_F)$. The constructions over there work just as well when we consider $\pi$ as $(\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}})_{\mathrm{ad}}(k_F)$-representation. The extension $\overline{\mathcal{G}_{\mathfrak{f}}}$ of $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}$ by $\Omega_{G,\mathfrak{f}}$ naturally induces an extension $(\overline{\mathcal{G}_{\mathfrak{f}}})_{\mathrm{ad}}$ of $(\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}})_{\mathrm{ad}}$ by $\Omega_{G,\mathfrak{f}}$. It follows that $\tilde{\pi}$, $\pi_+$ and $\pi_-$ are also representations of $(\overline{\mathcal{G}_{\mathfrak{f}}})_{\mathrm{ad}}(k_F)$. In particular $\tau$ acts on them via an element of $(\overline{\mathcal{G}_{\mathfrak{f}}})_{\mathrm{ad}}(k_F)$, so these representations are stabilized by $\tau$. Clifford theory tells us that these account for all irreducible unipotent representations of $\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)$ and of $N_G(P_{\mathfrak{f}})$. $\square$

Let $P_{\mathfrak{f}}$ be a maximal parahoric subgroup of $G$ and let $\sigma \in \mathrm{Irr}(P_{\mathfrak{f}})$ be inflated from a cuspidal unipotent representation of $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k_F) = P_{\mathfrak{f}}/P_{\mathfrak{f}}^+$. As noted for instance in [Lus2, Mor1, Mor2, MoPr], $\mathrm{ind}_{P_{\mathfrak{f}}}^G \sigma$ is a direct sum of finitely many supercuspidal $G$-representations.

For a more precise description we choose an extension $\sigma^N$ of $\sigma$ to $N_G(P_{\mathfrak{f}})$. That is always possible [Mor2, Proposition 4.6], and any two such extensions differ by a character of $N_G(P_{\mathfrak{f}})/P_{\mathfrak{f}} \cong \Omega_{G,\mathfrak{f}}$:

$$\mathrm{ind}_{P_{\mathfrak{f}}}^{N_G(P_{\mathfrak{f}})}(\sigma) = \bigoplus\nolimits_{\chi \in \mathrm{Irr}(\Omega_{G,\mathfrak{f}})} \sigma^N \otimes \chi. \tag{16}$$

Every supercuspidal unipotent $G$-representation is of the form

$$\mathrm{ind}_{N_G(P_{\mathfrak{f}})}^G (\sigma^N). \tag{17}$$

Given a supercuspidal unipotent $G$-representation, the pair $(N_G(P_{\mathfrak{f}}), \sigma^N)$ is unique up to conjugation.

Let $G^\vee$ be the complex dual group of $\mathcal{G}$, endowed with an action of $\mathrm{Gal}(F_s/F)$ (by pinned automorphisms) coming from the $F$-structure of $\mathcal{G}$. Then $\mathfrak{g}^\vee = \mathrm{Lie}(G^\vee)$ is a representation of $\mathrm{Gal}(F_s/F)$ and of $\mathbf{W}_F$. We denote its Artin conductor by $\mathbf{a}(\mathfrak{g}^\vee)$. We note that by [GrRe, (18) and §3.4] this equals the Artin conductor of the motive of $\mathcal{G}$. For $F_{\mathrm{nr}}$-split groups $\mathbf{a}(\mathfrak{g}^\vee) = 0$, while for ramified groups $\mathbf{a}(\mathfrak{g}^\vee) \in \mathbb{Z}_{>0}$.

Let $|\omega_G|$ be the canonical Haar measure on $G$ from [GaGr, §5]. Let $\psi : F \to \mathbb{C}^\times$ be an additive character. Recall that the order of $\psi$ is the largest $n \in \mathbb{Z}$ such that $\psi(f) = 1$ for all $f \in F$ of valuation $\geq -n$. Following [FOS1, (A.25)] we normalize the Haar measure on $G$ as

$$\mu_{G,\psi} = q_F^{-(\mathbf{a}(\mathfrak{g}^\vee) + \mathrm{ord}(\psi)\dim(\mathcal{G}/Z(\mathcal{G})_s))/2} |\omega_G|. \tag{18}$$

Actually this is a correction on [FOS1, (A.25)], because there it said $\dim(\mathcal{G})$ instead of $\dim(\mathcal{G}/Z(\mathcal{G})_s)$. In [FOS1] the authors took the reduction from $G$ to $G/Z(G)_s$ (a standard way to handle formal degrees when the centre of $G$ is not compact) for granted, so that [FOS1, (A.25)] should only be used when $Z(G)$ is compact. For [FOS1, Lemmas A.5 and A.6] this does not matter, they work equally well with (18) and with [FOS1, (A.25)]. Apart from that, [FOS1] only uses these normalized Haar measures in Proposition A.7. The proof of that, especially the part about reduction to the case where $\psi$ has order 0, is actually based on the normalization (18).

Unless explicitly mentioned otherwise, we assume that $\psi$ has order 0. For $F_{\mathrm{nr}}$-split groups (18) agrees with the normalizations in [Gro, GaGr, HII], while for ramified groups the correction term $q_F^{-\mathbf{a}(\mathfrak{g}^\vee)/2}$ is needed to relate formal degrees to adjoint $\gamma$-factors as in [HII].

The computation of the volume of the Iwahori subgroup of $G$ in [Gro, (4.11)] gives:

$$\mathrm{vol}(P_{\mathfrak{f}}) = \big|\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k_F)\big|\, q_F^{-\big(\mathbf{a}(\mathfrak{g}^\vee)+\dim \overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}+\dim (G^\vee)^{\mathbf{I}_F}\big)/2}. \tag{19}$$

By [DeRe, §5.1] this formula actually holds for every facet $\mathfrak{f}$ and every connected reductive $F$-group.

For a ramified simple group, we will see in Lemma 4.1 that

$$\dim (G^\vee)^{\mathbf{I}_F} = \dim (G'^\vee)^{\mathbf{I}_F} = \dim G'^\vee = \dim \mathcal{G}'. \tag{20}$$

With (19), (20) and Proposition 2.1 we can compare the Haar measures on $G$ and $G'$:

$$\mathrm{vol}(P_{\mathfrak{f}'}) = \big|\overline{\mathcal{G}'^{\circ}_{\mathfrak{f}'}}(k_F)\big|\, q_F^{-\big(\dim \overline{\mathcal{G}'^{\circ}_{\mathfrak{f}'}}+\dim (G'^\vee)^{\mathbf{I}_F}\big)/2} = q_F^{\mathbf{a}(\mathfrak{g}^\vee)/2}\mathrm{vol}(P_{\mathfrak{f}}). \tag{21}$$

By (11) the formal degree of (17) is

$$\mathrm{fdeg}\big(\mathrm{ind}_{N_G(P_{\mathfrak{f}})}^G \sigma^N\big) = \frac{\dim(\sigma^N)}{\mathrm{vol}(N_G(P_{\mathfrak{f}}))} = \frac{\dim(\sigma) q_F^{\big(\mathbf{a}(\mathfrak{g}^\vee)+\dim \overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}+\dim (G^\vee)^{\mathbf{I}_F}\big)/2}}{|\Omega_{G,\mathfrak{f}}|\,|\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k_F)|}. \tag{22}$$

**Corollary 2.5.** *Every diagram automorphism of $\mathcal{G}(F)$ or $\mathcal{G}'(F)$ stabilizes every irreducible supercuspidal unipotent representation of that group.*

*The bijection from Theorem 2.3.b induces a bijection*

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{unip,cusp}}(G) & \longleftrightarrow & \mathrm{Irr}_{\mathrm{unip,cusp}}(G') \\ \pi & \leftrightarrow & \pi' \end{array}$$

*which relates formal degrees as*

$$\mathrm{fdeg}(\pi') = q_F^{-\mathbf{a}(\mathfrak{g}^\vee)/2}\mathrm{fdeg}(\pi)$$

*This bijection is canonical up to choosing extensions of cuspidal unipotent representations of $P_{\mathfrak{f}}$ to $N_G(P_{\mathfrak{f}})$ (or equivalently: from $\overline{\mathcal{G}_{\mathfrak{f}}^{\circ}}(k_F)$ to $\overline{\mathcal{G}_{\mathfrak{f}}}(k_F)$).*

*Proof.* The first claim follows from Lemma 2.4 and the discussion preceding it. The bijectivity is a consequence of Theorem 2.3 and the bijection between $\Delta_{\mathrm{nr,aff}}$ and $\Delta'_{\mathrm{nr,aff}}$. The indicated canonicity comes from Proposition 2.2. The relation between the formal degrees follows (22), (21) and the dimension preservation in Theorem 2.3.b. □

## 3. Matching of Hecke algebras

To analyse the non-supercuspidal unipotent $G$-representations, we need types and Hecke algebras, following [BuKu]. This was worked out for general depth zero representations in [Mor1, Mor3], and for representations of $F_{\mathrm{nr}}$-split simple groups of adjoint type in [Lus2, §1]. Fortunately the arguments from [Lus2, §1] also apply to ramified simple groups, see [Sol3, §3]. We recall the main points, in the notation from [Sol3, §3].

Let $\hat{P}_{\mathfrak{f}}$ be the pointwise stabilizer of $\mathfrak{f}$ in $G$, so $P_{\mathfrak{f}}^+ \subset P_{\mathfrak{f}} \subset \hat{P}_{\mathfrak{f}} \subset N_G(P_{\mathfrak{f}})$. Then

$$\hat{P}_{\mathfrak{f}}/P_{\mathfrak{f}} \cong \Omega_{G,\mathfrak{f},\mathrm{tor}},$$

where the right hand side denotes the pointwise stabilizer of $\mathfrak{f}$ in $\Omega_G$ (or equivalently the pointwise stabilizer in $\Omega_G$ of all vertices of $\mathfrak{f}$). As $\ker(\Omega_G \to \Omega_{G_{\mathrm{ad}}})$ acts trivially on the relative local Dynkin diagram of $\mathcal{G}$, it is contained in $\Omega_{G,\mathfrak{f},\mathrm{tor}}$.

Let $\hat\sigma$ be an extension of a cuspidal unipotent representation $\sigma$ of $P_{\mathfrak f}$ to $\hat P_{\mathfrak f}$. Then $(\hat P_{\mathfrak f},\hat\sigma)$ is a type for a single Bernstein block of $G$, say $\mathrm{Rep}(G)_{(\hat P_{\mathfrak f},\hat\sigma)}$. We denote the associated Hecke algebra by

$$\mathcal{H}(G,\hat P_{\mathfrak f},\hat\sigma)=\mathrm{End}_G\big(\mathrm{ind}_{\hat P_{\mathfrak f}}^G\hat\sigma\big). \tag{23}$$

There is an equivalence of categories

$$\begin{array}{ccc} \mathrm{Rep}(G)_{(\hat P_{\mathfrak f},\hat\sigma)} & \longrightarrow & \mathrm{Mod}(\mathcal{H}(G,\hat P_{\mathfrak f},\hat\sigma)) \\ \pi & \mapsto & \mathrm{Hom}_{\hat P_{\mathfrak f}}(\hat\sigma,\pi) \end{array}. \tag{24}$$

Let $J_{\mathfrak f}\subset\Delta_{\mathrm{aff}}$ be the set of simple affine roots that vanish on $\mathfrak f$. If $|J_{\mathfrak f}|=|\Delta_{\mathrm{aff}}|-1$, then $\mathrm{ind}_{\hat P_{\mathfrak f}}^G(\hat\sigma)$ is irreducible, supercuspidal and $\mathcal{H}(G,\hat P_{\mathfrak f},\hat\sigma)\cong\mathbb{C}$.

Henceforth we assume that $|J_{\mathfrak f}|<|\Delta_{\mathrm{aff}}|-1$, so that $\mathfrak f$ is not a vertex of $\mathcal{B}(\mathcal{G},F)$ and $P_{\mathfrak f}$ is not a maximal parahoric subgroup of $G$. The set $\Delta_{\mathfrak f,\mathrm{aff}}:=\Delta_{\mathrm{aff}}\setminus J_{\mathfrak f}$ indexes a set of generators $S_{\mathfrak f,\mathrm{aff}}$ for an affine Weyl group $W_{\mathrm{aff}}(J_{\mathfrak f},\sigma)$ contained in $N_G(S)/(N_G(S)\cap P_{C_0})$. Let $\ell$ be the length function of the Coxeter system $(W_{\mathrm{aff}}(J_{\mathfrak f},\sigma),S_{\mathfrak f,\mathrm{aff}})$. Together with a parameter function $q^{\mathcal N}:\Delta_{\mathfrak f,\mathrm{aff}}\to\mathbb{R}_{>0}$ this gives rise to an Iwahori–Hecke algebra $\mathcal{H}(W_{\mathrm{aff}}(J_{\mathfrak f},\sigma),q^{\mathcal N})$. As a $\mathbb{C}$-vector space it has a basis $\{N_w : w\in W_{\mathrm{aff}}(J_{\mathfrak f},\sigma)\}$, every generator $N_s$ (with $s\in S_{\mathfrak f,\mathrm{aff}}$) satisfies a quadratic relation

$$\big(N_s-q^{\mathcal N(s)/2}\big)\big(N_s+q^{\mathcal N(s)/2}\big)=0 \tag{25}$$

and there are braid relations

$$N_wN_v=N_{wv}\quad\text{whenever}\quad \ell(w)+\ell(v)=\ell(wv). \tag{26}$$

Moreover the relations (25) and (26) provide a presentation of $\mathcal{H}(W_{\mathrm{aff}}(J_{\mathfrak f},\sigma),q^{\mathcal N})$.

The group $\Omega_{G,\mathfrak f}/\Omega_{G,\mathfrak f,\mathrm{tor}}$ (which in our setting has order one or two) acts naturally on $(W_{\mathrm{aff}}(J_{\mathfrak f},\sigma),S_{\mathfrak f,\mathrm{aff}})$ and on $\mathcal{H}(W_{\mathrm{aff}}(J_{\mathfrak f},\sigma),q^{\mathcal N})$. With these notations there is an algebra isomorphism

$$\mathcal{H}(G,\hat P_{\mathfrak f},\hat\sigma)\cong\mathcal{H}(W_{\mathrm{aff}}(J_{\mathfrak f},\sigma),q^{\mathcal N})\rtimes\Omega_{G,\mathfrak f}/\Omega_{G,\mathfrak f,\mathrm{tor}}. \tag{27}$$

When $\Omega_{G,\mathfrak f}/\Omega_{G,\mathfrak f,\mathrm{tor}}$ is represented by $\{1,\omega\}\subset N_G(P_{\mathfrak f})$, the basis element $N_\omega$ of (27) acts on $\mathrm{ind}_{\hat P_{\mathfrak f}}^G\hat\sigma$ by

$$(N_\omega f)(g)=\sigma^N(\omega)f(g\omega)\qquad f\in\mathrm{ind}_{\hat P_{\mathfrak f}}^G\hat\sigma, \tag{28}$$

where $\sigma^N\in\mathrm{Irr}(N_G(P_{\mathfrak f}))$ is an extension of $\hat\sigma$. Hence (27) is canonical up choosing such an extension, or equivalently up to a character of $N_G(P_{\mathfrak f})/\hat P_{\mathfrak f}$.

As Lusztig noted in [Lus3, §10.13], all these constructions depend only on the local index of $G$ and on the action of $\Omega_G$ on the relative local Dynkin diagram.

Let $\mathcal{G}'$ be the $F_{\mathrm{nr}}$-split companion group of $\mathcal{G}$, as in Section 1. Applying the proof of Theorem 2.3.b to Theorem 2.3.a with $\Omega_{G,\mathfrak f,\mathrm{tor}}$ instead of $\Omega_{G,\mathfrak f}$, we obtain a bijection

$$\mathrm{Irr}_{\mathrm{unip}}(\hat P_{\mathfrak f})\longleftrightarrow\mathrm{Irr}_{\mathrm{unip}}(\hat P_{\mathfrak f'}), \tag{29}$$

which preserves dimensions and cuspidality. In fact, as $|\Omega_{G,\mathfrak f,\mathrm{tor}}|\le 2$, we can take for (29) just an instance of Theorem 2.3.a if $\Omega_{G,\mathfrak f,\mathrm{tor}}=1$ and an instance of Theorem 2.3.b otherwise.

**Theorem 3.1.** *Let $\hat\sigma \in \mathrm{Irr}_{\mathrm{unip,cusp}}(\hat P_{\mathfrak f})$ and let $\hat\sigma' \in \mathrm{Irr}_{\mathrm{unip,cusp}}(\hat P_{\mathfrak f'})$ be its image under* (29)*.*

(*a*) *The bijection $\Delta'_{\mathrm{nr,aff}} \longleftrightarrow \Delta_{\mathrm{nr,aff}}$ induces an isomorphism of Coxeter systems*

$$\begin{array}{ccc} (W_{\mathrm{aff}}(J_{\mathfrak f'},\sigma'), S_{\mathfrak f',\mathrm{aff}}) & \longrightarrow & (W_{\mathrm{aff}}(J_{\mathfrak f},\sigma), S_{\mathfrak f,\mathrm{aff}}) \\ (w',s') & \mapsto & (w,s) \end{array}.$$

(*b*) *The linear map*

$$\begin{array}{ccc} \mathcal H(W_{\mathrm{aff}}(J_{\mathfrak f'},\sigma'), q^{\mathcal N'}) & \longrightarrow & \mathcal H(W_{\mathrm{aff}}(J_{\mathfrak f},\sigma), q^{\mathcal N}) \\ N_{w'} & \mapsto & N_w \end{array}$$

*is an algebra isomorphism.*

(*c*) *Part (b) and Theorem 2.3.b induce an algebra isomorphism*

$$\mathcal H(G', \hat P_{\mathfrak f'}, \hat\sigma') \longrightarrow \mathcal H(G, \hat P_{\mathfrak f}, \hat\sigma).$$

*Proof.* (a) Recall from Section 1 that the local indices of $G$ and $G'$ are isomorphic up to changing some arrows. From [Lus2, §1.15 and §2.28–2.30] we see that $(W_{\mathrm{aff}}(J_{\mathfrak f},\sigma), S_{\mathfrak f,\mathrm{aff}})$ depends only on $\mathfrak f$ and on the local index of $G$, and that it does not change if we reverse some arrows in $\Delta_{\mathrm{nr,aff}}$. This gives the isomorphism of Coxeter systems.
(b) Similarly, [Lus2, §1.18] and [Lus1, Table II] show that $q^{\mathcal N} : S_{\mathfrak f,\mathrm{aff}} \to \mathbb R_{>0}$ depends on $\mathfrak f$ and the local index of $G$ (modulo changing the direction of arrows). From the relations (25) and (26) we see that part (a) extends linearly to an isomorphism of Iwahori–Hecke algebras.
(c) Choose an extension $\sigma^N$ of $\hat\sigma$ to $N_G(P_{\mathfrak f})$ and use it to get (27). Analogously, we use the image of $\sigma^N$ under Theorem 2.3.b to construct (27) for $G'$. Then the group isomorphism

$$\Omega_{G,\mathfrak f}/\Omega_{G,\mathfrak f,\mathrm{tor}} \cong \Omega_{G',\mathfrak f'}/\Omega_{G',\mathfrak f',\mathrm{tor}} \tag{30}$$

extends the isomorphism from part (b) to the indicated affine Hecke algebras.

We still need to check that this isomorphism does not depend on the choice of $\sigma^N$. The only other possible extension of $\hat\sigma$ is $\sigma^N \otimes \chi_-$, where $\chi_-$ denotes the unique nontrivial character of $N_G(P_{\mathfrak f})/\hat P_{\mathfrak f}$. Notice that the latter group is naturally isomorphic with (30) and with $N_{G'}(P_{\mathfrak f'})/\hat P_{\mathfrak f'}$. Then the image in $\mathrm{Irr}(N_{G'}(P_{\mathfrak f'}))$ is also adjusted by tensoring with $\chi_-$, and (28) shows that $\sigma^N \otimes \chi_-$ leads to the same isomorphism of affine Hecke algebras as $\sigma^N$. □

**Corollary 3.2.** *There are equivalences of categories*

$$\begin{array}{ccccccc} \mathrm{Rep}(G)_{(\hat P_{\mathfrak f},\hat\sigma)} & \to & \mathrm{Mod}(\mathcal H(G,\hat P_{\mathfrak f},\hat\sigma)) & \to & \mathrm{Mod}(\mathcal H(G',\hat P_{\mathfrak f'},\hat\sigma')) & \leftarrow & \mathrm{Rep}(G')_{(\hat P_{\mathfrak f'},\hat\sigma')} \\ \pi & \mapsto & \mathrm{Hom}_{\hat P_{\mathfrak f}}(\hat\sigma,\pi) & \mapsto & \mathrm{Hom}_{\hat P_{\mathfrak f'}}(\hat\sigma',\pi') & \leftarrow\!\shortmid & \pi' \end{array}.$$

*With* (5) *these combine to an equivalence between the categories* $\mathrm{Rep}(G)_{\mathrm{unip}}$ *and* $\mathrm{Rep}(G')_{\mathrm{unip}}$.

*Proof.* The equivalences with the Bernstein block $\mathrm{Rep}(G)_{(\hat P_{\mathfrak f},\hat\sigma)}$ are a consequence of Theorem 3.1 and (24). By Theorem 2.3 and the bijection $\Delta_{\mathrm{nr,aff}} \leftrightarrow \Delta'_{\mathrm{nr,aff}}$ the indexing set in (5) is in bijection with $\{(\hat P_{\mathfrak f'},\hat\sigma')\}/G'$-conjugation. Hence the above equivalence of categories for one Bernstein block combine, in the same way for $G$ and $G'$, to all unipotent Bernstein blocks. □

We aim to show that Corollary 3.2 preserves many relevant properties. Let $\mathcal{L}_\mathfrak{f}$ be the standard $F$-Levi subgroup of $\mathcal{G}$ such that $\Phi(\mathcal{L}_\mathfrak{f}, \mathcal{S})$ consists precisely of the roots in $\Phi(\mathcal{G}, \mathcal{S})$ that are constant on $\mathfrak{f}$. Then the cuspidal supports of elements of $\mathrm{Irr}(G)_{(\hat{P}_\mathfrak{f}, \hat{\sigma})}$ are contained in a Bernstein component of $\mathrm{Rep}(L_\mathfrak{f})$. We define $\mathcal{L}'_{\mathfrak{f}'} \subset \mathcal{G}'$ in the same way.

By definition, a character of $G$ is weakly unramified if it is trivial on every parahoric subgroup of $G$. Via the Kottwitz map $G \to \Omega_G$, these characters can be identified with the characters of $\Omega_G$. In particular Section 1 provides a canonical bijection between the weakly unramified characters of $G$ and of $G'$.

**Lemma 3.3.** *The equivalence between the categories* $\mathrm{Rep}(G)_{\mathrm{unip}}$ *and* $\mathrm{Rep}(G')_{\mathrm{unip}}$ *from Corollary 3.2 is compatible with twisting by weakly unramified characters.*

*Proof.* When $\Omega_G = 1$, also $\Omega_{G'} = 1$, all weakly unramified characters are trivial and there is nothing to prove. Otherwise $|\Omega_G| = |\Omega_{G'}| = 2$. Then we identify the nontrivial weakly unramified character of $G$ with that of $G'$ and we call it $\chi$. There are three cases to consider:

- When $|\Omega_{G,\mathfrak{f}} / \Omega_{G,\mathfrak{f},\mathrm{tor}}| = 2$, tensoring by $\chi$ stabilizes the four categories in the first part of Corollary 3.2. It is clear from (28) that its effect is compatible with the equivalences between these four categories.
- When $|\Omega_{G,\mathfrak{f}}| = |\Omega_{G,\mathfrak{f},\mathrm{tor}}| = 2$, tensoring by $\chi$ identifies $\mathcal{H}(G, \hat{P}_\mathfrak{f}, \hat{\sigma})$ with $\mathcal{H}(G, \hat{P}_\mathfrak{f}, \chi \otimes \hat{\sigma})$ and $\mathcal{H}(G', \hat{P}_{\mathfrak{f}'}, \hat{\sigma}')$ with $\mathcal{H}(G', \hat{P}_{\mathfrak{f}'}, \chi \otimes \hat{\sigma}')$. If $\pi$ is mapped to $\pi'$, then by the complete analogy on both sides ($G$ and $G'$), $\chi \otimes \pi$ is mapped to $\chi \otimes \pi'$.
- When $|\Omega_{G,\mathfrak{f}}| = |\Omega_{G,\mathfrak{f},\mathrm{tor}}| = 1$, the representations $\mathrm{ind}_{\hat{P}_\mathfrak{f}}^G (\hat{\sigma})$ and $\mathrm{ind}_{\hat{P}_{\mathfrak{f}'}}^{G'} (\hat{\sigma}')$ are unaffected by tensoring with $\chi$. These are progenerators of the categories $\mathrm{Rep}(G)_{(\hat{P}_\mathfrak{f}, \hat{\sigma})}$ and $\mathrm{Rep}(G')_{(\hat{P}_{\mathfrak{f}'}, \hat{\sigma}')}$, so all elements of those categories are stable under tensoring by $\chi$. $\Box$

Recall that any $\pi \in \mathrm{Rep}(G)$ is called essentially square-integrable if its restriction to the derived group $G_{\mathrm{der}}$ is square-integrable. In particular this forces $\pi$ to be admissible. For the definitions of various kinds of representations of affine Hecke algebras we refer to [Sol1].

**Lemma 3.4.** ($a$) *The equivalences of categories in Corollary 3.2 preserve temperedness of representations.*
($b$) *The equivalence between the categories* $\mathrm{Rep}(G)_{(\hat{P}_\mathfrak{f}, \hat{\sigma})}$ *and* $\mathrm{Rep}(G')_{(\hat{P}_{\mathfrak{f}'}, \hat{\sigma}')}$ *preserves essential square-integrability.*

*Proof.* (a) The isomorphism from Theorem 3.1.c comes from isomorphisms between all the data used to construct these affine Hecke algebras, so it extends to an isomorphism between their respective Schwartz completions. By definition [Sol1, §1], this means that the middle map in Corollary 3.2 preserves temperedness.

For the two outer maps in Corollary 3.2 the statement is a consequence of [Sol1, Theorem 3.12 and Corollary 4.4].
(b) Suppose that $\mathrm{Rep}(G)_{(\hat{P}_\mathfrak{f}, \hat{\sigma})}$ contains an essentially square-integrable representation $\pi$, necessarily of finite length. Then [Sol1, Proposition 3.10.a and Corollary 4.4] tell us that the root systems for $(G, L_\mathfrak{f})$ and for $\mathcal{H}(G, \hat{P}_\mathfrak{f}, \hat{\sigma})$ have the same rank. By isomorphism, the root system underlying $\mathcal{H}(G', \hat{P}_{\mathfrak{f}'}, \hat{\sigma}')$ also has that rank. The rank

of the root system of $(G, L_\mathfrak{f})$ is simply $|\Delta_{\mathrm{aff}} \setminus J_\mathfrak{f}| - 1 = |\Delta'_{\mathrm{aff}} \setminus J_{\mathfrak{f}'}| - 1$, so equal to the rank of the root system of $(G', L'_{\mathfrak{f}'})$.

This argument works just as well from the other side: if $\mathrm{Rep}(G')_{(\hat{P}_{\mathfrak{f}'}, \hat{\sigma}')}$ contains an essentially square-integrable representation, then the root systems underlying the four terms in Corollary 3.2 all have the same rank. Knowing that, [Sol1, Theorem 3.9 and Corollary 4.4] prove the statement. □

Let us investigate the effect of Corollary 3.2 on formal degrees of square-integrable unipotent $G$-representations. Recall that we normalized the Haar measure on $G$ in (19). We endow the affine Hecke algebra $\mathcal{H}(G, \hat{P}_\mathfrak{f}, \hat{\sigma})$ with the unique trace such that

$$\mathrm{tr}(N_w) = \left\{ \begin{array}{ll} N_e(1) = \dim(\hat{\sigma}) \mathrm{vol}(\hat{P}_\mathfrak{f})^{-1} & w = e \\ 0 & w \neq e \end{array} \right. .$$

The Plancherel decomposition of this trace [Opd1] determines a density on the set of irreducible tempered $\mathcal{H}(G, \hat{P}_\mathfrak{f}, \hat{\sigma})$-representations, and in particular provides a normalization of formal degrees. Similarly we normalize the trace on $\mathcal{H}(G', \hat{P}_{\mathfrak{f}'}, \hat{\sigma}')$ by $\mathrm{tr}(N_e) = \dim(\hat{\sigma}') \mathrm{vol}(\hat{P}_{\mathfrak{f}'})^{-1}$, and we use the Plancherel density derived from that.

**Lemma 3.5.** *The equivalences of categories in Corollary 3.2 relate formal degrees of square-integrable representations as*

$$\mathrm{fdeg}(\pi') = q_F^{-\mathbf{a}(\mathfrak{g}^\vee)/2} \mathrm{fdeg}(\pi).$$

*It multiplies Plancherel densities of irreducible tempered representations by the same factor* $q_F^{-\mathbf{a}(\mathfrak{g}^\vee)/2}$.

*Proof.* By [BHK] the two outer maps in Corollary 3.2, with the indicated normalizations, preserve formal degrees. By Theorem 2.3.b and (21) the Hecke algebra isomorphism from Theorem 3.1.c multiplies the traces by a factor $q_F^{-\mathbf{a}(\mathfrak{g}^\vee)/2}$. Hence it adjusts formal degrees by the same factor.

The same argument applies to Plancherel densities. □

Finally we consider the diagram automorphism $\tau$ of $\mathcal{G}(F)$ from page 15. By Lemma 2.4 it stabilizes $\hat{P}_\mathfrak{f}$ and $\hat{\sigma}$, so it acts canonically on $\mathcal{H}(G, \hat{P}_\mathfrak{f}, \hat{\sigma})$ by an algebra automorphism.

**Lemma 3.6.** *For every type $(\hat{P}_\mathfrak{f}, \hat{\sigma})$ as above, the action of $\tau$ on $\mathcal{H}(G, \hat{P}_\mathfrak{f}, \hat{\sigma})$ is the identity. Hence $\tau$ stabilizes all unipotent representations of $G$.*

*Proof.* By Lemma 2.4 $\tau$ fixes the identity element $N_e$ of $\mathcal{H}(G, \hat{P}_\mathfrak{f}, \hat{\sigma})$, and it fixes $N_G(P_\mathfrak{f})/P_\mathfrak{f} \cong \Omega_{G,\mathfrak{f}}$ pointwise. Further, we observed in the proof of Lemma 2.4 that $\tau$ acts trivially on the local index of $G$. Together with $\mathfrak{f}$, these objects determine $W_{\mathrm{aff}}(J_\mathfrak{f}, \sigma) \rtimes \Omega_{G,\mathfrak{f}}$. Hence $\tau$ stabilizes the double coset $\hat{P}_\mathfrak{f} w \hat{P}_\mathfrak{f}$, for every $w \in W_{\mathrm{aff}}(J_\mathfrak{f}, \sigma) \rtimes \Omega_{G,\mathfrak{f}} / \Omega_{G,\mathfrak{f},\mathrm{tor}}$.

In particular $\tau(N_s) \in \mathbb{C} N_s$ for every $s \in S_{\mathfrak{f},\mathrm{aff}}$. The quadratic relation (25) and $\tau(N_e) = N_e$ force $\tau(N_s) = N_s$. Hence $\tau$ is the identity on $\mathcal{H}(W_{\mathrm{aff}}(J_\mathfrak{f}, \sigma), q^\mathcal{N})$.

Suppose that $N_G(P_\mathfrak{f})/P_\mathfrak{f} = \{1, \omega\}$ and $\sigma \in \mathrm{Irr}_{\mathrm{unip}}(P_\mathfrak{f})$. Then $\sigma$ can be extended in two ways to $N_G(P_\mathfrak{f})$, differing by a character of $N_G(P_\mathfrak{f})/P_\mathfrak{f}$. In the proof of Lemma 2.4 we saw that $\tau$ stabilizes the two extensions $\sigma_+, \sigma_-$. On the other hand, if $\tau(N_\omega) = -N_\omega$, then $\tau$ would exchange $\sigma_+$ and $\sigma_-$. As $N_\omega^2 = N_e$, we conclude that $\tau(N_\omega) = N_\omega$. Now we see from (27) that $\tau$ fixes $\mathcal{H}(G, \hat{P}_\mathfrak{f}, \hat{\sigma})$ entirely.

Then (24) implies that $\tau$ stabilizes all elements of $\mathrm{Rep}(G)_{(\hat P_{\mathfrak f},\hat\sigma)}$. This holds for all types $(\hat P_{\mathfrak f},\hat\sigma)$, so by (5) for the whole of $\mathrm{Rep}(G)_{\mathrm{unip}}$. $\square$

Let $\mathcal P$ be a standard parabolic $F$-subgroup of $\mathcal G$ and let $\mathcal P'$ be the associated standard parabolic $F$-subgroup of $\mathcal G'$, as explained in Section 1. Let $\mathcal M$ and $\mathcal M'$ be their respective standard Levi factors. Then $\mathcal M$ and $\mathcal M'$ stand in the same relation to each other as $\mathcal G$ and $\mathcal G'$, except that they need not be simple or adjoint.

When $\mathcal M$ contains $\mathcal L_{\mathfrak f}$, [Mor3, Theorem 2.1] says that

$$(\hat P_{\mathfrak f}\cap L_{\mathfrak f})/(P_{\mathfrak f}\cap L_{\mathfrak f})\cong(\hat P_{\mathfrak f}\cap M)/(P_{\mathfrak f}\cap M)\cong\hat P_{\mathfrak f}/P_{\mathfrak f}.$$

Then $\hat\sigma$ can also be considered as a representation of $\hat P_{\mathfrak f}\cap L_{\mathfrak f}$ or of $\hat P_{M,\mathfrak f}:=\hat P_{\mathfrak f}\cap M$. Thus Theorem 2.3 induces a bijection

$$\mathrm{Irr}(\hat P_{M,\mathfrak f})_{\mathrm{cusp,unip}}\to\mathrm{Irr}(\hat P_{M',\mathfrak f'})_{\mathrm{cusp,unip}}:\hat\sigma\mapsto\hat\sigma'. \tag{31}$$

We construct an equivalence of categories

$$\mathrm{Rep}(M)_{(\hat P_{M,\mathfrak f},\hat\sigma)}\longleftrightarrow\mathrm{Rep}(M')_{(\hat P_{M',\mathfrak f'},\hat\sigma')}$$

as in Corollary 3.2. Let

$$\mathrm{pr}_{(\hat P_{M,\mathfrak f},\hat\sigma)}:\mathrm{Rep}(M)\to\mathrm{Rep}(M)_{(\hat P_{M,\mathfrak f},\hat\sigma)}$$

be the natural projection coming from the Bernstein decomposition. We denote the normalized parabolic induction functor and the normalized Jacquet restriction functor associated to $P$ by

$$I_P^G:\mathrm{Rep}(M)\to\mathrm{Rep}(G)\quad\text{and}\quad J_P^G:\mathrm{Rep}(G)\to\mathrm{Rep}(M).$$

Let $\overline{\mathcal P}$ be the parabolic $F$-subgroup of $\mathcal G$ which is opposite to $\mathcal P$ with respect to $\mathcal M$.

**Lemma 3.7.** *The following diagrams commute:*

$$\begin{array}{ccc} \mathrm{Rep}(G)_{(\hat P_{\mathfrak f},\hat\sigma)} & \longleftrightarrow & \mathrm{Rep}(G')_{(\hat P_{\mathfrak f'},\hat\sigma')} \\ \uparrow I_P^G & & \uparrow I_{P'}^{G'} \\ \mathrm{Rep}(M)_{(\hat P_{M,\mathfrak f},\hat\sigma)} & \longleftrightarrow & \mathrm{Rep}(M')_{(\hat P_{M',\mathfrak f'},\hat\sigma')} \end{array}
\qquad
\begin{array}{ccc} \mathrm{Rep}(G)_{(\hat P_{\mathfrak f},\hat\sigma)} & \longleftrightarrow & \mathrm{Rep}(G')_{(\hat P_{\mathfrak f'},\hat\sigma')} \\ \downarrow \mathrm{pr}_{(\hat P_{M,\mathfrak f},\hat\sigma)}\circ J_{\overline P}^G & & \downarrow \mathrm{pr}_{(\hat P_{M',\mathfrak f'},\hat\sigma')}\circ J_{\overline{P'}}^{G'} \\ \mathrm{Rep}(M)_{(\hat P_{M,\mathfrak f},\hat\sigma)} & \longleftrightarrow & \mathrm{Rep}(M')_{(\hat P_{M',\mathfrak f'},\hat\sigma')} \end{array}$$

*Proof.* By [Mor3, Corollary 3.10] and [BuKu, Proposition 8.5] the type $(\hat P_{\mathfrak f},\hat\sigma)$ is a cover of $(\hat P_{M,\mathfrak f},\hat\sigma)$. In this setting [BuKu, Corollary 7.12] gives a canonical algebra monomorphism

$$t_{\overline P}:\mathcal H(M,\hat P_{M,\mathfrak f},\hat\sigma)\to\mathcal H(G,\hat P_{\mathfrak f},\hat\sigma),$$

which implements unnormalized Jacquet restriction with respect to $\overline P$ [BuKu, Corollary 8.4]. We adjust it by the square root of a modular character as in the proof of [Sol1, Lemma 4.1], and call the result $\lambda_{MG}$. In terms of the presentation from (25) and (26), this works out as

$$\lambda_{MG}(N_w)=N_w\qquad\text{for all } w\in W_{\mathrm{aff}}(J_{\mathfrak f},\sigma)\rtimes\Omega_{G,\mathfrak f}/\Omega_{G,\mathfrak f,\mathrm{tor}}\text{ with }\hat P_{\mathfrak f}w\hat P_{\mathfrak f}\cap M\neq\emptyset.$$

Via $\lambda_{MG}$ we regard $\mathcal H(M,\hat P_{M,\mathfrak f},\hat\sigma)$ as a subalgebra of $\mathcal H(G,\hat P_{\mathfrak f},\hat\sigma)$. Then [Sol1, Condition 3.1 and Lemma 4.1] say that restriction of representations from $\mathcal H(G,\hat P_{\mathfrak f},\hat\sigma)$

to $\mathcal{H}(M, \hat{P}_{M,\mathfrak{f}}, \hat\sigma)$ fits in a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Rep}(G)_{(\hat{P}_{\mathfrak{f}}, \hat\sigma)} & \longleftrightarrow & \mathrm{Mod}(\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat\sigma)) \\
\downarrow \mathrm{pr}_{(\hat{P}_{M,\mathfrak{f}}, \hat\sigma)} \circ J_P^G & & \downarrow \mathrm{Res} \\
\mathrm{Rep}(M)_{(\hat{P}_{M,\mathfrak{f}}, \hat\sigma)} & \longleftrightarrow & \mathrm{Mod}(\mathcal{H}(M, \hat{P}_{M,\mathfrak{f}}, \hat\sigma))
\end{array} . \tag{32}
$$

The same holds if the vertical arrows are replaced by $I_P^G$ and by induction from $\mathcal{H}(M, \hat{P}_{M,\mathfrak{f}}, \hat\sigma)$ to $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat\sigma)$.

Of course that applies equally well to $G'$ and $M'$. Clearly the Hecke algebra isomorphism from Theorem 3.1.c transfers $\lambda_{MG}$ to $\lambda_{M'G'}$. Hence the diagram

$$
\begin{array}{ccc}
\mathrm{Mod}(\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat\sigma)) & \longleftrightarrow & \mathrm{Mod}(\mathcal{H}(G', \hat{P}_{\mathfrak{f}'}, \hat\sigma')) \\
\downarrow \mathrm{Res} & & \downarrow \mathrm{Res} \\
\mathrm{Mod}(\mathcal{H}(M, \hat{P}_{M,\mathfrak{f}}, \hat\sigma)) & \longleftrightarrow & \mathrm{Mod}(\mathcal{H}(M', \hat{P}_{M',\mathfrak{f}'}, \hat\sigma'))
\end{array}
$$

commutes, and similarly with the vertical arrows replaced by induction functors. $\square$

In particular Lemma 3.7 shows that Corollary 3.2 respects supercuspidality – which we knew already from Corollary 2.5.

## 4. Comparison of Langlands parameters

For the moment, $\mathcal{G}$ denotes any connected reductive $F$-group, and $G = \mathcal{G}(F)$. Let $G^\vee$ be the complex dual group of $\mathcal{G}$ and let ${}^L G = G^\vee \rtimes \mathbf{W}_F$ be a Langlands dual group. Recall [Bor] that a Langlands parameter for $G$ is a homomorphism $\phi : \mathbf{W}_F \times SL_2(\mathbb{C}) \to {}^L G$ satisfying certain conditions. We denote the set of $G^\vee$-equivalence classes of Langlands parameters for $G$ by $\Phi(G)$. We call $\phi$:

- bounded if $\phi(\mathrm{Frob}) = (s, \mathrm{Frob})$ with $s$ in a compact subgroup of $G^\vee$;
- discrete if the image of $\phi$ is not contained in ${}^L M$ for any proper $F$-Levi subgroup $\mathcal{M}$ of $\mathcal{G}$;
- unramified if $\phi(i) = (1, i)$ for all $i \in \mathbf{I}_F$.

We denote the corresponding subsets of $\Phi(G)$ by, respectively, $\Phi_{\mathrm{bdd}}(G), \Phi^2(G)$ and $\Phi_{\mathrm{nr}}(G)$. We note that an unramified L-parameter is determined up to $G^\vee$-conjugacy by the semisimple element $\phi(\mathrm{Frob}) \in {}^L G$ and the unipotent element

$$u_\phi := \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big) \in G^\vee .$$

Let ${G^\vee}_{\mathrm{sc}}$ be the simply connected cover of the derived group ${G^\vee}_{\mathrm{der}}$ of $G^\vee$. The image of $Z_{G^\vee}(\phi)$ in ${G^\vee}_{\mathrm{der}}$ is $Z_{G^\vee}(\phi) Z(G^\vee)/Z(G^\vee)$. Let $Z^1_{{G^\vee}_{\mathrm{sc}}}(\phi)$ be the preimage of that in ${G^\vee}_{\mathrm{sc}}$ and define

$$\mathcal{S}_\phi := \pi_0\big(Z^1_{{G^\vee}_{\mathrm{sc}}}(\phi)\big).$$

This is the component group of $\phi$ used in [Art2, AMS1]. An enhancement of $\phi$ is an irreducible representation $\rho$ of $\mathcal{S}_\phi$. The group $G^\vee$ acts naturally on the set of enhanced L-parameters by

$$g \cdot (\phi, \rho) = (g \phi g^{-1}, g \cdot \rho) \qquad (g \cdot \rho)(h) = \rho(g^{-1} h g).$$

Via the canonical map $Z({G^\vee}_{\mathrm{sc}}) \to Z(\mathcal{S}_\phi)$, every enhancement $\rho$ determines a character $\chi_\rho$ of $Z({G^\vee}_{\mathrm{sc}})$. On the other hand, $\mathcal{G}$ is an inner twist of a unique quasi-split $F$-group $\mathcal{G}^*$. The parametrization of equivalence classes of inner twists of $\mathcal{G}^*$ by

$$H^1(\mathbf{W}_F, \mathcal{G}^*_{\mathrm{ad}}) \cong \mathrm{Irr}\big(Z({G^\vee}_{\mathrm{sc}})^{\mathbf{W}_F}\big)$$

provides a character $\chi_{\mathcal{G}}$ of $Z(G^\vee{}_{\mathrm{sc}})^{\mathbf{W}_F}$. We choose an extension to a character $\chi^e_{\mathcal{G}}$ of $Z(G^\vee{}_{\mathrm{sc}})$. (Such an extension is related to an explicit construction of $\mathcal{G}$ as inner twist of $\mathcal{G}^*$, compare with Section 7.) Then we say that $\rho$ or $(\phi, \rho)$ is $G$-relevant if $\chi_\rho = \chi^e_{\mathcal{G}}$. We denote the collection of $G^\vee$-orbits of $G$-relevant enhanced L-parameters by $\Phi_e(G)$.

When $\mathcal{G}$ is ramified $F$-simple and $\mathcal{G}' = \mathcal{G}'^\circ \times \{\pm 1\}$, we have $Z(G'^{\vee,\circ}) = 1$ and $Z(G'^\vee) = \{\pm 1\}$. Then we define $\chi_{\mathcal{G}'} \in \mathrm{Irr}(Z(G'^\vee))$ to be trivial if $\mathcal{G}$ is quasi-split over $F$ and nontrivial otherwise.

**Lemma 4.1.** *Let $\mathcal{G}$ be a ramified simple $F$-group and $\mathcal{G}'$ be its companion group from Section 1. There exists a $\mathbf{W}_F$-equivariant isomorphism*

$$\lambda^\vee_{G'G} : G'^\vee \to (G^\vee)^{\mathbf{I}_F},$$

*which is unique up to inner automorphisms.*

*Proof.* This boils down to one quick check for every entry in the list in Section 1. In all cases $\mathbf{W}_F$ acts trivially on $(G^\vee)^{\mathbf{I}_F}$ (and on $G'^\vee$) because $\mathcal{G}$ is an inner twist of the quasi-split $F$-group given by $\mathcal{G}(F_{\mathrm{nr}})$.

- $\mathbf{B\text{-}C_n}$. Let $A^{-T}$ be the inverse transpose of an invertible matrix and let $J$ be an anti-diagonal square matrix whose nonzero entries are alternatingly 1 and -1. Then $\mathbf{I}_F$ acts on $G^\vee = SL_{2n}(\mathbb{C})$ via $A \mapsto JA^{-T}J^{-1}$ and

$$(G^\vee)^{\mathbf{I}_F} = SL_{2n}(\mathbb{C})^{\mathbf{I}_F} = Sp_{2n}(\mathbb{C}) = G'^\vee.$$

  The same $\mathbf{I}_F$-action is well-defined on any group $G^\vee$ isogenous to $SL_{2n}(\mathbb{C})$. Such a group is determined by the order $d^\vee$ of $Z(G^\vee)$. Then $2n/d^\vee$ is the order of the schematic centre of $\mathcal{G}$. We find

$$(G^\vee)^{\mathbf{I}_F} = G'^\vee = \begin{cases} Sp_{2n}(\mathbb{C}) & \text{if } 2n/d^\vee \text{ is odd} \\ PSp_{2n}(\mathbb{C}) \times \{\pm 1\} & \text{if } 2n/d^\vee \text{ is even and } d^\vee \text{ is even} \\ PSp_{2n}(\mathbb{C}) & \text{if } 2n/d^\vee \text{ is even and } d^\vee \text{ is odd} \end{cases}$$

- $\mathbf{C\text{-}BC_n}$. Similarly to the previous case, for every group $G^\vee$ isogenous to $SL_{2n+1}(\mathbb{C})$:

$$(G^\vee)^{\mathbf{I}_F} = SO_{2n+1}(\mathbb{C}) = G'^\vee.$$

- $\mathbf{C\text{-}B_n}$. We endow $\mathbb{C}^{2n+2}$ with the symmetric bilinear form given by $\langle e_i, e_j \rangle = \delta_{j,2n+3-i}$. We let $\mathbf{I}_F$ act on $G^\vee = Spin_{2n+2}(\mathbb{C})$ via conjugation by $I_n \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus I_n \in O_{2n+2}(\mathbb{C})$. There are three cases:

$$\begin{array}{rcccccl} (PSO^{*,\vee}_{2n+2})^{\mathbf{I}_F} & = & Spin_{2n+2}(\mathbb{C})^{\mathbf{I}_F} & = & Spin_{2n+1}(\mathbb{C}) & = & G'^\vee, \\ (SO^{*,\vee}_{2n+2})^{\mathbf{I}_F} & = & SO_{2n+2}(\mathbb{C})^{\mathbf{I}_F} & = & O_{2n+1}(\mathbb{C}) & = & G'^\vee, \\ (Spin^{*,\vee}_{2n+2})^{\mathbf{I}_F} & = & PSO_{2n+2}(\mathbb{C})^{\mathbf{I}_F} & = & SO_{2n+1}(\mathbb{C}) & = & G'^\vee. \end{array}$$

- $\mathbf{^2B\text{-}C_n}$. These cases are the same as for $\mathbf{B\text{-}C_n}$.
- $\mathbf{^2C\text{-}B_{2n}}$. Here $\mathbf{I}_F$ acts on $G^\vee$ as in $\mathbf{C\text{-}B_n}$. There are three cases:

$$\begin{array}{rcccccl} (PSO^{*,\vee}_{4n})^{\mathbf{I}_F} & = & Spin_{4n}(\mathbb{C})^{\mathbf{I}_F} & = & Spin_{4n-1}(\mathbb{C}) & = & G'^\vee, \\ (SO^{*,\vee}_{4n})^{\mathbf{I}_F} & = & SO_{4n}(\mathbb{C})^{\mathbf{I}_F} & = & O_{4n-1}(\mathbb{C}) & = & G'^\vee, \\ (Spin^{*,\vee}_{4n})^{\mathbf{I}_F} & = & PSO_{4n}(\mathbb{C})^{\mathbf{I}_F} & = & SO_{4n-1}(\mathbb{C}) & = & G'^\vee. \end{array}$$

- $\mathbf{{}^2C\text{-}B_{2n+1}}$. Again $\mathbf{I}_F$ acts on $G^\vee$ as in $\mathbf{C\text{-}B_n}$, and
$$\begin{array}{ccccccc} (PSO^{*,\vee}_{4n+2})^{\mathbf{I}_F} & = & Spin_{4n+2}(\mathbb{C})^{\mathbf{I}_F} & = & Spin_{4n+1}(\mathbb{C}) & = & G'^\vee, \\ (SO^{*,\vee}_{4n+2})^{\mathbf{I}_F} & = & SO_{4n+2}(\mathbb{C})^{\mathbf{I}_F} & = & O_{4n+1}(\mathbb{C}) & = & G'^\vee, \\ (Spin^{*,\vee}_{4n+2})^{\mathbf{I}_F} & = & PSO_{4n+2}(\mathbb{C})^{\mathbf{I}_F} & = & SO_{4n+1}(\mathbb{C}) & = & G'^\vee. \end{array}$$
- $\mathbf{F_4^I}$. The group $\mathbf{I}_F$ acts on $G^\vee = E_{6,\mathrm{sc}}(\mathbb{C})$ via an outer automorphism which stabilizes a pinning, and
$$(G^\vee)^{\mathbf{I}_F} = E_{6,\mathrm{sc}}(\mathbb{C})^{\mathbf{I}_F} = F_4(\mathbb{C}) = G'^\vee.$$
The same holds with $\mathcal{G} = E_{6,\mathrm{sc}}$ and $G^\vee = E_{6,\mathrm{ad}}(\mathbb{C})$.
- $\mathbf{G_2^I}$. In this case some elements of $\mathbf{I}_F$ act on $G^\vee = Spin_8(\mathbb{C})$ via an automorphism $\tau$ of order three which stabilizes a pinning, and maybe some other elements of $\mathbf{I}_F$ act via an outer automorphism which stabilizes the same pinning. The $\mathbf{I}_F$-invariants are already determined by $\tau$:
$$(G^\vee)^{\mathbf{I}_F} = Spin_8(\mathbb{C})^\tau = G_2(\mathbb{C}) = G'^\vee.$$
The same holds with $\mathcal{G} = {}^r D_{4,\mathrm{sc}}$ and $G^\vee(\mathbb{C}) = PSO_8(\mathbb{C})$.

Any two isomorphisms $G'^\vee \to (G^\vee)^{\mathbf{I}_F}$ differ by an automorphism of $G'^\vee$. As $G'^\vee$ has type $B_n, C_n, F_4$ or $G_2$, all its automorphisms are inner. □

Let us compare the unramified L-parameters for $G$ and $G'$.

**Lemma 4.2.** *Lemma 4.1 induces a canonical bijection $\lambda^\Phi_{G'G} : \Phi_{\mathrm{nr}}(G') \to \Phi_{\mathrm{nr}}(G)$.*

*Proof.* For $G'$ the group $\mathbf{I}_F$ acts trivially on $G'^\vee$. Hence the data for an unramified L-parameter are simple: a group homomorphism
$$\phi' : \mathbf{W}_F/\mathbf{I}_F \times SL_2(\mathbb{C}) \to G'^\vee \rtimes \mathbf{W}_F/\mathbf{I}_F$$
which is algebraic on $SL_2(\mathbb{C})$ and with $\phi'(\mathrm{Frob}) \in G'^\vee\mathrm{Frob}$ semisimple. To get $\Phi_{\mathrm{nr}}(G')$, we consider such $\phi'$ up to conjugation by $G'^\vee$.

In an unramified L-parameter $\phi$ for $G$, $\phi(i) = (1, i)$ for all $i \in \mathbf{I}_F$. The semisimple element $\phi(\mathrm{Frob}) = (s, \mathrm{Frob}) \in {}^L G$ must satisfy
$$\begin{aligned} (1, \mathrm{Frob}\, i\, \mathrm{Frob}^{-1}) &= \phi(\mathrm{Frob}\, i\, \mathrm{Frob}^{-1}) \\ &= \phi(\mathrm{Frob})\phi(i)\phi(\mathrm{Frob})^{-1} = (s, \mathrm{Frob})(1, i)(s, \mathrm{Frob})^{-1} \\ &= (s, \mathrm{Frob}\, i\, \mathrm{Frob}^{-1})(s^{-1}, 1) = (s(\mathrm{Frob}\, i\, \mathrm{Frob}^{-1})(s^{-1}), \mathrm{Frob}\, i\, \mathrm{Frob}^{-1}). \end{aligned}$$
Hence $(\mathrm{Frob}\, i\, \mathrm{Frob}^{-1})(s)$ must equal $s$ for $i \in \mathbf{I}_F$, which says that $s \in (G^\vee)^{\mathbf{I}_F}$. The remaining content of $\phi$ is an algebraic group homomorphism
$$SL_2(\mathbb{C}) \to Z_{(G^\vee)^{\mathbf{I}_F}}(\phi(\mathrm{Frob})).$$
In principle unramified L-parameters for $G$ are considered up to conjugation by elements of $G^\vee$. But if they must stay unramified, we may only conjugate by elements that centralize $(1, i)$ for all $i \in \mathbf{I}_F$, that is, by elements of $(G^\vee)^{\mathbf{I}_F}$.

In view of the above, Lemma 4.1 provides a bijection
$$\Phi_{\mathrm{nr}}(G') \to \Phi_{\mathrm{nr}}(G) : \phi' \mapsto \lambda^\vee_{G'G} \circ \phi', \tag{33}$$
where $\lambda^\vee_{G'G}$ is extended to $G'^\vee \rtimes \mathbf{W}_F \to (G^\vee)^{\mathbf{I}_F} \rtimes \mathbf{W}_F$ by making it the identity on $\mathbf{W}_F$. By definition inner automorphisms have no effect on $\Phi(G')$, they are already divided out. Hence the unicity property in Lemma 4.1 entails that the bijection (33) is canonical. □

Enhancements of unramified L-parameters for $G$ and $G'$ can be compared in the same way.

**Lemma 4.3.** *Let $\phi' \in \Phi_{\rm nr}(G')$ and let $\phi \in \Phi_{\rm nr}(G)$ be its image under Lemma 4.2.*

(*a*) *Lemma 4.1 induces an isomorphism $Z_{G'^\vee}(\phi') \to Z_{G^\vee}(\phi)$, which sends $Z(G'^\vee)^{\mathbf{W}_F}$ to $Z(G^\vee)^{\mathbf{W}_F}$.*

(*b*) *Part (a) gives a canonical bijection from the set of $G'$-relevant enhancements of $\phi'$ to the set of $G$-relevant enhancements of $\phi$.*

*Proof.* (a) As $Z_{G^\vee}(\phi) \subset (G^\vee)^{\mathbf{I}_F}$, this is a direct consequence of Lemmas 4.1 and 4.2.
(b) We find it easiest to proceed by classification. For $\mathbf{B}\text{-}\mathbf{C_n}, \mathbf{C}\text{-}\mathbf{BC_n}, \mathbf{C}\text{-}\mathbf{B_n}, \mathbf{F_4^I}$ and $\mathbf{G_2^I}$ the $F$-groups $\mathcal{G}$ and $\mathcal{G}'$ are quasi-split, so $\chi^e_{\mathcal{G}} = \mathrm{triv} = \chi^e_{\mathcal{G}'}$. Then the relevant enhancements of $\phi$ are $\mathrm{Irr}(\mathcal{S}_\phi / Z(G^\vee{}_{\rm sc})) = \mathrm{Irr}\big(Z_{G^\vee}(\phi)/Z(G^\vee)^{\mathbf{W}_F}\big)$, and similarly for $\phi'$. Clearly part (a) induces a group isomorphism $Z_{G'^\vee}(\phi)/Z(G'^\vee)^{\mathbf{W}_F} \to Z_{G^\vee}(\phi)/Z(G^\vee)^{\mathbf{W}_F}$, which settles these cases.

For ${}^\mathbf{2}\mathbf{B}\text{-}\mathbf{C_n}, {}^\mathbf{2}\mathbf{C}\text{-}\mathbf{B_{2n}}$ and ${}^\mathbf{2}\mathbf{C}\text{-}\mathbf{B_{2n+1}}$, the $F$-groups $\mathcal{G}$ and $\mathcal{G}'$ are the unique non-quasi-split inner twists of a quasi-split group, so $\chi_{\mathcal{G}}$ and $\chi_{\mathcal{G}'}$ both equal the unique nontrivial character of

$$Z(G'^\vee{}_{\rm sc})^{\mathbf{W}_F} \cong Z(G^\vee{}_{\rm sc})^{\mathbf{W}_F} \cong \mathbb{Z}/2\mathbb{Z}.$$

In these cases, by inspection, $(\mathcal{G}_{\rm ad})'$ equals $\mathcal{G}'_{\rm ad}$. From that and Lemma 4.1 for $\mathcal{G}_{\rm ad}$ we deduce

$$G'^\vee{}_{\rm sc} = G_{\rm ad}{}'^\vee \cong (G_{\rm ad}{}^\vee)^{\mathbf{I}_F} = (G^\vee{}_{\rm sc})^{\mathbf{I}_F}. \tag{34}$$

One checks directly that $\mathbf{W}_F$ acts trivially on $Z(G'^\vee{}_{\rm sc}))$, so that $\mathcal{S}_{\phi'} = \pi_0(Z_{G'^\vee{}_{\rm sc}}(\phi'))$. The $G'$-relevant enhancements of $\phi'$ are the irreducible representations of $\mathcal{S}_\phi$ with nontrivial $Z(G'^\vee{}_{\rm sc}$-character. By (34) and part (a) for $G_{\rm ad}$, these are matched bijectively with the irreducible representations of $\pi_0(Z_{G^\vee{}_{\rm sc}}(\phi))$ whose $Z(G^\vee{}_{\rm sc})^{\mathbf{W}_F}$-character is nontrivial. By Lemma 7.2.c (the part that we need is elementary, it relies only on (**??**)) that set of representations is naturally in bijection with the set of irreducible representations of $\mathcal{S}_\phi$ whose $Z(G^\vee{}_{\rm sc})$-character equals $\chi^e_{\mathcal{G}}$. $\square$

Recall that the group of weakly unramified characters of $X_{\rm wr}(G)$ is naturally isomorphic with $(Z(G^\vee)^{\mathbf{I}_F})_{\rm Frob}$. The latter group acts on $\Phi_e(G)$ by

$$z \cdot (\phi, \rho) = (z\phi, \rho), \qquad (z\phi)(\mathrm{Frob}) = z(\phi(\mathrm{Frob})). \tag{35}$$

where $z\phi = \phi$ on $\mathbf{I}_F \times SL_2(\mathbb{C})$. The constructions in Section 1 entail that

$$(Z(G^\vee)^{\mathbf{I}_F})_{\rm Frob} \cong \mathrm{Irr}(\Omega_G) \cong \mathrm{Irr}(\Omega_{G'}) \cong (Z(G'^\vee)^{\mathbf{I}_F})_{\rm Frob}. \tag{36}$$

We call this twisting a Langlands parameter by a weakly unramified character. Recall the definition of cuspidality for enhanced L-parameters from [AMS1, §6].

**Proposition 4.4.** *Lemma 4.1 induces a canonical bijection*

$$\lambda^{\Phi_e}_{G'G} : \Phi_{{\rm nr},e}(G') \to \Phi_{{\rm nr},e}(G).$$

*This map and its inverse preserve boundedness, discreteness, cuspidality and twists by $(Z(G'^\vee)^{\mathbf{I}_F})_{\rm Frob} \cong (Z(G^\vee)^{\mathbf{I}_F})_{\rm Frob}$ as in* (35).

*Proof.* The map $\lambda^{\Phi_e}_{G'G}$, its bijectivity and its canonicity come from Lemmas 4.1, 4.2 and 4.3. It is clear from the construction that this bijection preserves weakly unramified twists by the group (36).

We consider an arbitrary $(\phi', \rho') \in \Phi_{\mathrm{nr},e}(G')$ and we write $(\phi, \rho) = \lambda^{\Phi_e}_{G'G}(\phi', \rho')$. Boundedness of $\phi$ depends only on $s = \phi(\mathrm{Frob})\mathrm{Frob}^{-1} \in G^\vee$. We saw in the proof of Lemma 4.2 that $s \in (G^\vee)^{\mathbf{I}_F}$, while by construction

$$\lambda^{\vee\,-1}_{G'G}(s) = \phi'(\mathrm{Frob})\mathrm{Frob}^{-1} \in G'^\vee.$$

Hence $\phi$ is bounded if and only if $\phi'$ is bounded.

As observed in [GrRe, §3.2], $\phi$ is discrete if and only if $Z_{G^\vee}(\phi)/Z(G^\vee)^{\mathbf{W}_F}$ is finite. By Lemma 4.3.a this is equivalent to finiteness of $Z_{G'^\vee}(\phi)/Z(G'^\vee)^{\mathbf{W}_F}$.

The construction in Lemma 4.2 entails that

$$Z_{G^\vee}(\phi(\mathbf{W}_F)) = \lambda^\vee_{G'G}\big(Z_{G'^\vee}(\phi'(\mathbf{W}_F))\big) \quad \text{and} \quad \phi|_{SL_2(\mathbb{C})} = \lambda^\vee_{G'G} \circ \phi'|_{SL_2(\mathbb{C})}.$$

Similarly, in Lemma 4.3.b we defined $\rho = \rho' \circ \lambda^{\vee\,-1}_{G'G}$. Hence cuspidality of $(\phi, \rho)$ depends only on the pair $(u_\phi, \rho)$ for the group $Z_{G^\vee}(\phi(\mathbf{W}_F))$. The situation for $(\phi', \rho')$ is entirely analogous, with objects isomorphic to those for $(\phi, \rho)$. Hence one of these enhanced L-parameters if cuspidal if and only if the other is so. $\square$

Let $\mathcal{P} = \mathcal{M}\mathcal{U}$ be a standard parabolic $F$-subgroup of $\mathcal{G}$ and let $\mathcal{P}' = \mathcal{M}'\mathcal{U}'$ be the associated standard parabolic $F$-subgroup of $\mathcal{G}'$. Then $\lambda^\vee_{G'G}$ restricts to an isomorphism $M'\vee \to (M^\vee)^{\mathbf{I}_F}$. (When $\mathcal{G}' = \mathcal{G}'^\circ \times \{\pm 1\}$, we take $\mathcal{M}' = \mathcal{M}'^\circ \times \{\pm 1\}$ and $M'^\vee = M'^{\vee,\circ} \times \{\pm 1\}$.) Then $\mathbf{W}_F$ acts trivially on $M'^\vee$ and the character $\chi_{\mathcal{M}'}$ of $Z(M'^\vee)$ can be deduced from $\chi_{\mathcal{G}'}$ via [AMS1, Lemma 6.6]. As in Lemmas 4.1–4.3 and Proposition 4.4 we obtain a canonical bijection

$$\lambda^{\Phi_e}_{M'M} : \Phi_{\mathrm{nr},e}(M') \to \Phi_{\mathrm{nr},e}(M). \tag{37}$$

The cuspidal support of an element of $\Phi_{\mathrm{nr},e}(G)$ [AMS1, §7] can be realized as an element of $\Phi_{\mathrm{nr,cusp}}(M)$ for a standard $F$-Levi subgroup $\mathcal{M}$ of $\mathcal{G}$.

**Lemma 4.5.** *The system of bijections $\lambda^{\Phi_e}_{M'M}$ (running over all standard $F$-Levi subgroups) commutes with the cuspidal support map for enhanced L-parameters.*

*Proof.* Recall from [AMS1, Proposition 7.3 and Definition 7.7] that the cuspidal support $\mathbf{Sc}(\phi, \rho)$ of $(\phi, \rho) \in \Phi_{\mathrm{nr},e}(G)$

- has the same $\phi|_{\mathbf{I}_F}$,
- is a cuspidal L-parameter for a (standard) $F$-Levi subgroup $M$ of $G$,
- is determined entirely by a construction in the complex reductive group $Z^1_{G_{\mathrm{ad}}{}^\vee}(\phi|_{\mathbf{I}_F}) = G_{\mathrm{ad}}{}^{\vee,\mathbf{I}_F}$, with $\phi(\mathrm{Frob}), u_\phi$ and $\rho$ as input.

Via Lemmas 4.1–4.3 and Proposition 4.4 all this is canonically transferred to analogous objects with primes. It follows that

$$\mathbf{Sc}\big(\lambda^{\Phi_e}_{G'G}(\phi', \rho')\big) = \lambda^{\Phi_e}_{M'M}(\mathbf{Sc}(\phi', \rho')). \tag{38}$$

$\square$

We denote the adjoint action of ${}^L G$ on $\mathrm{Lie}(G^\vee)/\mathrm{Lie}(Z(G^\vee)^{\mathbf{W}_F})$ by $\mathrm{Ad}_{G^\vee}$. The adjoint $\gamma$-factor of $\phi \in \Phi(G)$ is related to $\epsilon$-factors and L-functions as

$$\gamma(s, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) = \epsilon(s, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) L(1 - s, \mathrm{Ad}_{G^\vee} \circ \phi) L(s, \mathrm{Ad}_{G^\vee} \circ \phi)^{-1}. \tag{39}$$

Here $s \in \mathbb{C}$ and $\psi : F \to \mathbb{C}^\times$ is an additive character, which by our conventions from (18) must have order 0. For the definitions of the local factors in (39) we refer to [Tate, GrRe]. Let $\mathrm{Ad}_{(G^\vee)^{\mathbf{I}_F}}$ denote the adjoint action of $(G^\vee)^{\mathbf{I}_F} \rtimes \mathbf{W}_F$ on $\mathrm{Lie}((G^\vee)^{\mathbf{I}_F})/\mathrm{Lie}(Z(G^\vee)^{\mathbf{W}_F})$. For an unramified L-parameter $\phi$, $\mathrm{Ad}_{G^\vee} \circ \phi$ can be restricted to $\mathrm{Ad}_{G^{\vee,\mathbf{I}_F}} \circ \phi$.

**Lemma 4.6.** *Let $\mathcal{G}$ be any connected reductive $F$-group and let $\phi \in \Phi_{\mathrm{nr}}(G)$. There exists $\varepsilon \in \{\pm 1, \pm\sqrt{-1}\}$, with $\varepsilon^2$ depending only on* $\mathrm{Lie}(G^\vee)$*, such that*

$$\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) = \varepsilon q_F^{\mathbf{a}(\mathrm{Lie}(G^\vee))/2} \gamma(0, \mathrm{Ad}_{(G^\vee)^{\mathbf{I}_F}} \circ \phi, \psi).$$

*Proof.* Let $\mathrm{Lie}(G^\vee)^{ram}$ be the "ramified part" of $\mathrm{Lie}(G^\vee)$, that is, the sum of the nontrivial irreducible $\mathbf{I}_F$-subrepresentations. Since $\mathbf{I}_F$ is normal in $\mathbf{W}_F$, this gives a decomposition of $\mathbf{W}_F$-representations

$$\mathrm{Lie}(G^\vee) = \mathrm{Lie}(G^\vee)^{\mathbf{I}_F} \oplus \mathrm{Lie}(G^\vee)^{ram} \tag{40}$$

As unramified L-parameters act via $\mathbf{W}_F$ and $(G^\vee)^{\mathbf{I}_F}$, (40) can also be considered as a decomposition of $\mathbf{W}_F \times SL_2(\mathbb{C})$-representations, which we write as

$$\mathrm{Ad}_{G^\vee} \circ \phi = \mathrm{Ad}_{\mathrm{Lie}(G^\vee)^{\mathbf{I}_F}} \circ \phi \oplus \mathrm{Ad}_{\mathrm{Lie}(G^\vee)^{ram}} \circ \phi.$$

By the additivity of $\gamma$-factors

$$\gamma(s, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) = \gamma(s, \mathrm{Ad}_{\mathrm{Lie}(G^\vee)^{\mathbf{I}_F}} \circ \phi, \psi) \gamma(s, \mathrm{Ad}_{\mathrm{Lie}(G^\vee)^{ram}} \circ \phi, \psi) \quad \forall s \in \mathbb{C}. \tag{41}$$

Further $(\mathrm{Lie}(G^\vee)^{ram})^{\mathbf{I}_F} = 0$, so

$$L(s, \mathrm{Lie}(G^\vee)^{ram}) = 1 \quad \text{and} \quad \epsilon(s, \mathrm{Ad}_{\mathrm{Lie}(G^\vee)^{ram}} \circ \phi, \psi) = \epsilon(s, \mathrm{Ad}_{\mathrm{Lie}(G^\vee)^{ram}} \circ \phi|_{\mathbf{W}_F}, \psi).$$

With that (41) becomes

$$\gamma(s, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) = \gamma(s, \mathrm{Ad}_{\mathrm{Lie}(G^\vee)^{\mathbf{I}_F}} \circ \phi, \psi) \epsilon(s, \mathrm{Ad}_{\mathrm{Lie}(G^\vee)^{ram}} \circ \phi|_{\mathbf{W}_F}, \psi) \quad \forall s \in \mathbb{C}. \tag{42}$$

It was observed in [GrRe, §3.2] that $\mathrm{Ad}_{G^\vee} \circ \phi$ and $\mathrm{Ad}_{G^\vee}|_{\mathfrak{g}^{\vee,ram}} \circ \phi$ are self-dual with respect to the Killing form. By [GrRe, (15)] this implies

$$\epsilon(1/2, \mathrm{Ad}_{G^\vee}|_{\mathfrak{g}^{\vee,ram}} \circ \phi|_{\mathbf{W}_F}, \psi) = \varepsilon$$

for some $\varepsilon \in \{\pm 1, \pm\sqrt{-1}\}$ with $\varepsilon^2$ depending only on $\mathrm{Lie}(G^\vee)$. Then [Tate, (3.4.5)] says

$$\epsilon(0, \mathrm{Ad}_{\mathrm{Lie}(G^\vee)^{ram}} \circ \phi|_{\mathbf{W}_F}, \psi) = \varepsilon q_F^{\mathbf{a}(\mathrm{Lie}(G^\vee)^{ram} \circ \phi|_{\mathbf{W}_F})/2}. \tag{43}$$

By definition the Artin conductor of a $\mathbf{W}_F$-representation $V$ depends only on the restriction to $\mathbf{I}_F$, and $\mathbf{a}(V) = 0$ if $V^{\mathbf{I}_F} = V$. As $\phi$ is unramified and by (40)

$$\mathbf{a}(\mathrm{Lie}(G^\vee)^{ram} \circ \phi|_{\mathbf{W}_F}) = \mathbf{a}(\mathrm{Lie}(G^\vee)^{ram}) = \mathbf{a}(\mathrm{Lie}(G^\vee)), \tag{44}$$

where $\mathrm{Lie}(G^\vee)$ and $\mathrm{Lie}(G^\vee)^{ram}$ are endowed with the $\mathbf{W}_F$-action derived from conjugation inside ${}^L G$. Now combine (42), (43) and (44). $\square$

For discrete L-parameters, the conjectures from [HII] assert that $|\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi, \psi)|$ is related to the formal degree of any member of the L-packet $\Pi_\phi(G)$.

We return to ramified simple groups. We know from Lemma 3.5 how formal degrees behave under the transfer from $\mathrm{Rep}(G)_{\mathrm{unip}}$ to $\mathrm{Rep}(G')_{\mathrm{unip}}$. It turns out that adjoint $\gamma$-factors behave in the same way. Let $\phi' \in \Phi_{\mathrm{nr}}(G')$ and write $\phi = \lambda^\Phi_{G'G}(\phi)$. By Lemmas 4.1 and 4.2

$$\gamma(s, \mathrm{Ad}_{\mathrm{Lie}(G^\vee)^{\mathbf{I}_F}} \circ \phi, \psi) = \gamma(s, \mathrm{Ad}_{G'^\vee} \circ \phi', \psi) \qquad \forall s \in \mathbb{C}.$$

Then Lemma 4.6 says

$$\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi, \psi) = \varepsilon q_F^{\mathbf{a}(\mathrm{Lie}(G^\vee))/2} \gamma(0, \mathrm{Ad}_{G'^\vee} \circ \phi', \psi). \tag{45}$$

With the material from Sections 3 and 4 we obtain a good candidate for a local Langlands correspondence for unipotent representations of ramified simple groups. Recall that a LLC

$$\mathrm{Irr}(G')_{\mathrm{unip}} \to \Phi_{\mathrm{nr},e}(G') : \pi' \mapsto (\phi_{\pi'}, \rho_{\pi'}) \tag{46}$$

with many nice properties was constructed in [Sol3] and [FOS2, Theorem 2.1]. For supercuspidal representations of adjoint groups this agrees with [Lus2, Mor2], for other unipotent representations it differs from the earlier constructions of Lusztig. For supercuspidal representations of ramified simple groups there are some arbitrary choices in [Sol3], which stem from [FOS1]. From [Opd3, §4.5.1] we get some additional requirements related to suitable spectral transfer morphisms. As in [FOS2], we use those requirements to fix some of the choices in [FOS1].

Consider the composition of Corollary 3.2, (46) and Proposition 4.4:

$$\begin{array}{ccccccc} \mathrm{Rep}(G)_{\mathrm{unip}} & \to & \mathrm{Rep}(G')_{\mathrm{unip}} & \to & \Phi_{\mathrm{nr},e}(G') & \to & \Phi_{\mathrm{nr},e}(G) \\ \pi & \mapsto & \pi' & \mapsto & (\phi_{\pi'}, \rho_{\pi'}) & \mapsto & \lambda^{\Phi_e}_{G'G}(\phi_{\pi'}, \rho_{\pi'}). \end{array} \tag{47}$$

All involved maps are bijective, so we obtain a bijection

$$\begin{array}{ccc} \mathrm{Rep}(G)_{\mathrm{unip}} & \longrightarrow & \Phi_{\mathrm{nr},e}(G) \\ \pi & \mapsto & (\phi_\pi, \rho_\pi) \end{array}. \tag{48}$$

**Theorem 4.7.** *Let $\mathcal{G}$ be a ramified simple group. The bijection* (48) *satisfies:*

(*a*) *$\pi$ is tempered if and only if $\phi_\pi$ is bounded.*
(*b*) *$\pi$ is essentially square-integrable if and only if $\phi_\pi$ is discrete.*
(*c*) *$\pi$ is supercuspidal if and only if $(\phi_\pi, \rho_\pi)$ is cuspidal.*
(*d*) *Let $\chi \in X_{\mathrm{wr}}(G)$ correspond to $\hat{\chi} \in (Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}}$. Then $(\phi_{\chi\otimes\pi}, \rho_{\chi\otimes\pi}) = (\hat{\chi}\phi_\pi, \rho_\pi)$.*
(*e*) *The HII conjectures hold for tempered $\pi \in \mathrm{Irr}(G)_{\mathrm{unip}}$.*
(*f*) *Equivariance with respect to $\mathbf{W}_F$-automorphisms of the Dynkin diagram of $\mathcal{G}$.*

*Proof.* (a) and (b) These follow from Lemma 3.4, Proposition 4.4 and [Sol3, Lemmas 5.6 and 5.7].
(c) Follows from Corollary 2.5, [Sol3, Lemma 5.4] and Lemma 4.5.
(d) This is a consequence of Lemma 3.3, [Sol3, Lemma 5.3] and Proposition 4.4.
(e) For $G'$ this was shown in [FOS2, Theorem 5.3]. When we pass from $G$ to $G'$, Lemma 3.5 says that the formal degree a square-integrable modulo centre representation picks up a factor $q_F^{\mathbf{a}(\mathfrak{g}^\vee)/2}$. By (45) the absolute value of the involved adjoint $\gamma$-factor picks up the same factor. The other ingredients of the HHI-formula for formal degrees (the size of a component group and the dimension of an irreducible representation thereof) do not change when passing from $G$ to $G'$, by Lemma 4.3.
(f) From page 15 we know that it suffices to consider the single diagram automorphism $\tau$. In Lemma 3.6 we showed that it fixes every $\pi \in \mathrm{Irr}_{\mathrm{unip}}(G)$. It remains to show that $\tau$ acts trivially on $\Phi_{\mathrm{nr},e}(G)$. From the classification on page 15 we also see that, on $G^\vee$, $\tau$ coincides with an element of $\mathbf{I}_F$. By Lemma 4.2 $\mathrm{im}(\phi_\pi) \subset (G^\vee)^{\mathbf{I}_F} \rtimes \mathbf{W}_F$ and by Lemma 4.3 $Z_{G^\vee}(\phi_\pi) \subset (G^\vee)^{\mathbf{I}_F}$. Hence $\tau$ indeed fixes $(\phi_\pi, \rho_\pi)$. $\square$

## 5. Supercuspidal unipotent representations

In [FOS1] it was assumed that all reductive $F$-groups under consideration split over an unramified extension of $F$. In this section we will lift that condition, and we generalize all the results from that paper to arbitrary connected reductive $F$-groups. Let us formulate the generalization of the main results of [FOS1] that we are after.

**Theorem 5.1.** *Let $\mathcal{G}$ be a connected reductive $F$-group. There exists a bijection*

$$\begin{array}{ccc} \mathrm{Irr}(G)_{\mathrm{cusp,unip}} & \longrightarrow & \Phi_{\mathrm{nr}}(G)_{\mathrm{cusp}} \\ \pi & \mapsto & (\phi_\pi, \rho_\pi) \end{array}$$

*with the following properties:*

(*a*) *Equivariance with respect to twisting by weakly unramified characters.*
(*b*) *Equivariance with respect to $\mathbf{W}_F$-automorphisms of the root datum of $\mathcal{G}$.*
(*c*) *Compatibility with almost direct products of reductive groups.*
(*d*) *Suppose that $\pi \in \mathrm{Irr}(G)_{\mathrm{cusp,unip}}$ is a constituent of the pullback of $\pi_{\mathrm{ad}} \in \mathrm{Irr}(G_{\mathrm{ad}})_{\mathrm{cusp,unip}}$ to $G$. Then the canonical map*

$$G_{\mathrm{ad}}{}^\vee \rtimes \mathbf{W}_F \to G^\vee \rtimes \mathbf{W}_F$$

*sends $\phi_{\pi_{\mathrm{ad}}}$ to $\phi_\pi$.*
(*e*) *Let $Z(\mathcal{G})_s$ be the maximal $F$-split central torus of $\mathcal{G}$. When $\pi$ is unitary:*

$$\mathrm{fdeg}(\pi, d\mu_{G,\psi}) = \dim(\rho_\pi) \big| \pi_0\big( Z_{(\mathcal{G}/Z(\mathcal{G})_s)^\vee}(\phi) \big) \big|^{-1} |\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\pi, \psi)|.$$

*For a given $\pi$ the properties (a), (c), (d) and (e) determine $\phi_\pi$ uniquely, up to twists by weakly unramified characters of $(\mathcal{G}/Z(\mathcal{G})_s)(F)$.*

Most of the time we will assume that the centre of $\mathcal{G}$ is $F$-anisotropic. For such groups we recall the definitions of a few relevant numbers from [FOS1, §2]. Let $\phi \in \Phi_{\mathrm{nr}}(G)$ and $\sigma \in \mathrm{Irr}_{\mathrm{cusp,unip}}(P_{\mathfrak{f}})$.

- $\mathsf{a}$ is the number of $\tilde{\phi} \in \Phi^2_{\mathrm{nr}}(G)$ which admit a $G$-relevant cuspidal enhancement and for each $F$-simple factor $\mathcal{G}_i$ of $\mathcal{G}$ satisfy
$$\gamma(0, \mathrm{Ad}_{G_i^\vee} \circ \tilde{\phi}, \psi) = c_i \gamma(0, \mathrm{Ad}_{G_i^\vee} \circ \phi, \psi)$$
for some $c_i \in \mathbb{Q}^\times$ (as rational functions of $q_F$).
- $\mathsf{b}$ is the number of $G$-relevant cuspidal enhancements of $\lambda$.
- $\mathsf{a}'$ is defined as $|\Omega_{G,\mathfrak{f}}|$ times the number of $G$-conjugacy classes of maximal parahoric subgroups $P_{\tilde{\mathfrak{f}}} \subset G$ for which there exists a $\tilde{\sigma} \in \mathrm{Irr}_{\mathrm{cusp,unip}}(P_{\tilde{\mathfrak{f}}})$ such that the components $\sigma_i, \tilde{\sigma}_i$ corresponding to any $F$-simple factor $\mathcal{G}_i$ of $\mathcal{G}$ satisfy
$$\mathrm{fdeg}(\mathrm{ind}_{P_{\mathfrak{f}',i}}^{G}(\tilde{\sigma}_i) = \tilde{c}_i \mathrm{fdeg}(\mathrm{ind}_{P_{\mathfrak{f},i}}^{G}(\sigma_i)$$
for some $\tilde{c}_i \in \mathbb{Q}^\times$ (as rational functions of $q_F$).
- $\mathsf{b}'$ is the number of $\tilde{\sigma} \in \mathrm{Irr}_{\mathrm{cusp,unip}}(P_{\mathfrak{f}})$ with $\dim(\tilde{\sigma}) = \dim(\sigma)$.

Suppose now that $\mathcal{G}$ is semisimple. From [FOS2, Theorem 2], Theorem 4.7 and compatibility with direct products of simple groups we get a map

$$\mathrm{Irr}(G)_{\mathrm{cusp,unip}} \to \Phi^2_{\mathrm{nr}}(G)/X_{\mathrm{wr}}(G), \tag{49}$$

where $X_{\mathrm{wr}}(G)$ is identified with $(Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}}$. Fix an irreducible direct summand $\pi$ of $\mathrm{ind}_{P_{\mathfrak{f}}}^{G}(\sigma)$ and let $X_{\mathrm{wr}}(G)\phi_\pi$ be its image under (49).

**Lemma 5.2.** *Let $\mathcal{G}$ be a ramified simple $F$-group and let $\mathcal{G}'$ be its $F_{\mathrm{nr}}$-split companion group. Let $\mathfrak{f}, \sigma, \pi, \phi_\pi$ be as above and let $\mathfrak{f}', \sigma', \pi', \phi_{\pi'}$ be their images under the maps from Proposition 2.1, Theorem 2.3, Corollary 2.5 and Lemma 4.2. Then the numbers* $\mathsf{a}, \mathsf{b}, \mathsf{a}', \mathsf{b}'$ *for $G, \mathfrak{f}, \sigma, \lambda_\pi$ are the same as their counterparts for $G', \mathfrak{f}', \sigma', \phi_{\pi'}$.*

*Proof.* For $\mathsf{a}$ and $\mathsf{b}$ this follows from Proposition 4.4 and Lemma 4.6. For $\mathsf{a}'$ and $\mathsf{b}'$ it is a consequence of Proposition 2.1, Theorem 2.3 and Corollary 2.5. □

*Proof of Theorem 5.1 for adjoint groups*
For simple adjoint groups Theorem 5.1 is established case-by-case, as explained in [FOS1, §12]. The ramified simple adjoint groups are not considered in [FOS1], for those we use Theorem 4.7 to associate enhanced L-parameters to $\mathrm{Irr}(G)_{\mathrm{cusp,unip}}$. By Corollary 2.5, Lemma 4.6 and [FOS1, Theorem 1], these are essentially (in a sense specified in that paper) the only L-parameters that make the HII conjectures true for $\mathrm{Irr}(G)_{\mathrm{cusp,unip}}$.

With Sections 2 and 4 and Lemma 5.2, we transfer all further issues in the proof of [FOS1, Proposition 12.1] to the group $G'$, which is treated in [FOS1]. The generalization from simple adjoint groups to all adjoint groups in [FOS1, Proposition 12.2 and Lemma 16.1] works equally well for ramified groups. We note that restriction of scalars is dealt with in [FOS1, Appendix], which is already written in the generality of reductive groups. □

For non-adjoint reductive $F$-groups we have to be more careful. It appears that for semisimple $F$-groups the proof of Theorem 5.1 in [FOS1, §13–14] can be modified without too much trouble. However, the arguments for reductive $F$-groups with anisotropic centre in [FOS1, §15] do not easily carry over to ramified groups. The main difference is that in the $F_{\mathrm{nr}}$-split case the inclusion $G_{\mathrm{der}} \to G$ induces a bijection $\mathrm{Irr}(G)_{\mathrm{cusp,unip}} \to \mathrm{Irr}(G_{\mathrm{der}})_{\mathrm{cusp,unip}}$ [FOS1, Lemma 15.3]. For ramified groups this is just false, firstly because ramified anisotropic tori can admit nontrivial weakly unramified characters, secondly because the pullback map $\mathrm{Irr}(G)_{\mathrm{cusp,unip}} \to \mathrm{Rep}(G_{\mathrm{der}})$ need not preserve irreducibility, and thirdly because not all elements of $\mathrm{Irr}(G_{\mathrm{der}})_{\mathrm{cusp,unip}}$ are contained in a representation pulled back from $G$. In view of this, we rather aim to extend the arguments from [FOS1, §13–14] to all (possibly ramified) reductive $F$-groups $\mathcal{G}$ with anisotropic centre.

Thus, we want to reduce Theorem 5.1 for $\mathcal{G}$ to Theorem 5.1 for its adjoint group $\mathcal{G}_{\mathrm{ad}}$. One problem is that, in contrast with the $F_{\mathrm{nr}}$-split case, the natural map $\Omega_G \to \Omega_{G_{\mathrm{ad}}}$ need not be injective. Equivalently, the natural map

$$X_{\mathrm{wr}}(G_{\mathrm{ad}}) \cong (Z({G^\vee}_{\mathrm{sc}})^{\mathbf{I}_F})_{\mathrm{Frob}} \to (Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}} \cong X_{\mathrm{wr}}(G) \tag{50}$$

need not be surjective. Clearly, the image of (50) is contained in the kernel of the natural map to

$$X_{\mathrm{wr}}(Z(\mathcal{G})^\circ(F)) \cong \big(Z(G^\vee/{G^\vee}_{\mathrm{der}})^{\mathbf{I}_F}\big)_{\mathrm{Frob}}.$$

But even for ramified simple groups, (50) fails to be surjective in two cases:

- $\mathcal{G} = SU_{2n}/\mu_{2n/d}$, where $\mu_k$ denotes the group scheme of $k$-th roots of unity, $d \in 2\mathbb{N}$ and $2n/d \in 2\mathbb{N}$. Then $G^\vee = SL_{2n}(\mathbb{C})/\mu_d(\mathbb{C})$, ${G^\vee}_{\mathrm{sc}} = SL_{2n}(\mathbb{C})$, $(Z({G^\vee}_{\mathrm{sc}})^{\mathbf{I}_F})_{\mathrm{Frob}} = \{1,-1\}$ and $(Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}} = \{1, \exp(\pi i/d)\}$.
- $\mathcal{G} = SO^*_{2n}$. Then $G^\vee = SO_{2n}(\mathbb{C})$, ${G^\vee}_{\mathrm{sc}} = Spin_{2n}(\mathbb{C})$, $(Z({G^\vee}_{\mathrm{sc}})^{\mathbf{I}_F})_{\mathrm{Frob}} = \ker(Spin_{2n}(\mathbb{C}) \to SO_{2n}(\mathbb{C}))$ and $(Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}} = \{1,-1\}$.

We note that in the first case

$$(G^\vee)^{\mathbf{I}_F} = PSp_{2n}(\mathbb{C}) \times \langle \exp(\pi i/d)\rangle / \langle \exp(2\pi i/d)\rangle$$

and in the second case

$$(G^\vee)^{\mathbf{I}_F} = O_{2n-1}(\mathbb{C}) = SO_{2n-1}(\mathbb{C}) \times \{1,-1\}.$$

In both cases there are natural isomorphisms

$$(Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}} = Z(G^\vee)^{\mathbf{W}_F} = Z(G^{\vee,\mathbf{W}_F}) \cong \pi_0(G^{\vee,\mathbf{I}_F}) = \pi_0(G^{\vee,\mathbf{I}_F})_{\mathrm{Frob}}. \tag{51}$$

For all other simple groups $\mathcal{G}$, $G^{\vee,\mathbf{I}_F}$ is connected, see the proof of Lemma 4.1. With the list in Section 1, one checks that for any simple $F$-group $\mathcal{G}$, $\ker(\Omega_G \to \Omega_{G_{\mathrm{ad}}})$ is naturally isomorphic to $\pi_0(\mathcal{G}')$ (and it is trivial when $\mathcal{G}$ is $F_{\mathrm{nr}}$-split).

For any homomorphism of connected reductive $F$-groups $\mathcal{G} \to \mathcal{H}$, we define

$$\Omega_{G,H} := \mathrm{im}(\Omega_G \to \Omega_H) \quad \text{and} \quad X_{\mathrm{wr}}(H,G) := \mathrm{im}(X_{\mathrm{wr}}(H) \to X_{\mathrm{wr}}(G)).$$

For $\mathcal{H} = \mathcal{G}_{\mathrm{ad}}$ we obtain short exact sequences (dual to each other):

$$\begin{array}{ccccccccc} 1 & \to & \ker(\Omega_G \to \Omega_{G_{\mathrm{ad}}}) & \to & \Omega_G & \to & \Omega_{G,G_{\mathrm{ad}}} & \to & 1, \\ 1 & \leftarrow & \mathrm{Irr}(\ker(\Omega_G \to \Omega_{G_{\mathrm{ad}}})) & \leftarrow & X_{\mathrm{wr}}(G) & \leftarrow & X_{\mathrm{wr}}(G_{\mathrm{ad}},G) & \leftarrow & 1. \end{array} \tag{52}$$

We note also that the image of $\ker(G^\vee{}_{\mathrm{sc}} \to G^\vee)$ in $X_{\mathrm{wr}}(G_{\mathrm{ad}}) \cong (Z(G^\vee{}_{\mathrm{sc}})^{\mathbf{I}_F})_{\mathrm{Frob}}$ is

$$\mathrm{Irr}(\Omega_{G_{\mathrm{ad}}} / \Omega_{G,G_{\mathrm{ad}}}) = \ker(X_{\mathrm{wr}}(G_{\mathrm{ad}}) \to X_{\mathrm{wr}}(G)). \tag{53}$$

For semisimple groups, the method from (51) yields a group $\pi_0(G^{\vee,\mathbf{I}_F})_{\mathrm{Frob}}$ isomorphic to $X_{\mathrm{wr}}(G)/X_{\mathrm{wr}}(G_{\mathrm{ad}},G)$. It is naturally represented in $X_{\mathrm{wr}}(G) \cong (Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}}$, and forms a complement to $X_{\mathrm{wr}}(G_{\mathrm{ad}},G)$. Thus both sequences (52) split for semisimple groups (but not necessarily for reductive groups).

For $\sigma_{\mathrm{ad}} \in \mathrm{Irr}_{\mathrm{cusp,unip}}(P_{\mathfrak{f},\mathrm{ad}})$, let $\sigma \in \mathrm{Irr}_{\mathrm{cusp,unip}}(P_{\mathfrak{f}})$ be its pullback. Morris showed in [Mor2, Proposition 4.6] that $\sigma_{\mathrm{ad}}$ can be extended to a representation $\sigma_{\mathrm{ad}}^N$ of $N_{G_{\mathrm{ad}}}(P_{\mathfrak{f},\mathrm{ad}})$ (on the same vector space). Let $\sigma^N \in \mathrm{Irr}(N_G(P_{\mathfrak{f}}))$ be the pullback of $\sigma_{\mathrm{ad}}^N$ along $G \to G_{\mathrm{ad}}$. This construction shows that

$$\sigma \text{ can be extended to } N_G(P_{\mathfrak{f}}), \text{ via } N_{G_{\mathrm{ad}}}(P_{\mathfrak{f},\mathrm{ad}}). \tag{54}$$

Another issue with Theorem 5.1 concerns the pullback of supercuspidal unipotent representations along the canonical map $\mathsf{q} : G \to G_{\mathrm{ad}}$. Such a pullback is trivial on $Z(G)$ and it can never involve elements of $X_{\mathrm{wr}}(G)$ outside $X_{\mathrm{wr}}(G,G_{\mathrm{ad}})$. Notice that in general $G/Z(G)$ is a proper subgroup of $G_{\mathrm{ad}} = \mathcal{G}_{\mathrm{ad}}(F)$. We consider $\mathrm{Irr}(G/Z(G))$ as a subset of $\mathrm{Irr}(G)$, endowed all the relevant notions from $\mathrm{Rep}(G)$.

We note also that a $G_{\mathrm{ad}}$-orbit of facets of $\mathcal{B}(\mathcal{G},F)$ can decompose into several $G$-orbits. These are parametrized by $G_{\mathrm{ad}}/N_{G_{\mathrm{ad}}}(P_{\mathfrak{f},\mathrm{ad}})G$ (in such a quotient $G$ is a shorthand for its image in $G_{\mathrm{ad}}$). This can cause the pullback of an irreducible $G_{\mathrm{ad}}$-representation to become reducible (as $G$-representation).

**Lemma 5.3.** (*a*) *The $G$-constituents of* $\mathsf{q}^*(\mathrm{ind}_{P_{\mathfrak{f},\mathrm{ad}}}^{G_{\mathrm{ad}}} \sigma_{\mathrm{ad}})$ *are the representations*

$$\mathrm{Ad}(g)^* \mathrm{ind}_{N_G(P_{\mathfrak{f}})}^G (\sigma^N) = \mathrm{ind}_{N_G(P_{g\mathfrak{f}})}^G (g \cdot \sigma^N),$$

*where $g \in G_{\mathrm{ad}}$ and $\sigma^N$ is an extension of $\sigma$ to $N_G(P_{\mathfrak{f}})$ such that $Z(G) \subset \ker(\sigma^N)$.*

(*b*) *Tensoring with weakly unramified characters provides a bijection*

$$X_{\mathrm{wr}}(G) \times_{X_{\mathrm{wr}}(G_{\mathrm{ad}},G)} \mathrm{Irr}(G/Z(G))_{(P_{\mathfrak{f}},\sigma)} \to \mathrm{Irr}(G)_{(P_{\mathfrak{f}},\sigma)}.$$

*In particular* $\mathrm{Irr}(\ker(\Omega_G \to \Omega_{G_{\mathrm{ad}}}))$ *acts freely on* $\mathrm{Irr}(G)_{(P_{\mathfrak{f}},\sigma)}/X_{\mathrm{wr}}(G_{\mathrm{ad}},G)$.

*Proof.* (a) This follows from (16) and (17).
(b) By part (a) tensoring with $\chi \in X_{\mathrm{wr}}(G)$ maps any element of $\mathrm{Irr}(G/Z(G))_{(P_{\mathfrak{f}},\sigma)}$ to $\mathrm{Irr}(G/Z(G))_{(P_{\mathfrak{f}},\sigma)}$ if and only if $\chi \in X_{\mathrm{wr}}(G_{\mathrm{ad}},G)$, and then $\chi$ stabilizes that set entirely. Combine that with (16) and (52). $\square$

On the Galois side something similar happens. Not every unramified $\phi \in \Phi(G)$ can be lifted along $\mathsf{q}^\vee : G_{\mathrm{ad}}{}^\vee \to G^\vee$ to an element of $\Phi(G_{\mathrm{ad}})$.

**Lemma 5.4.** *(a) $\phi \in \Phi_{\mathrm{nr}}(G)$ can be lifted to an element of $\Phi_{\mathrm{nr}}(G_{\mathrm{ad}})$ if and only if $\phi(\mathrm{Frob})\mathrm{Frob}^{-1}$ lies in $({G^\vee}_{\mathrm{der}})^{\mathbf{I}_F,\circ}(1-\mathrm{Frob})(G^{\vee,\mathbf{I}_F})$.*
*(b) The action of any $\chi \in X_{\mathrm{wr}}(G)$ maps any element of $\mathsf{q}^\vee(\Phi_{\mathrm{nr}}(G_{\mathrm{ad}}))$ to $\mathsf{q}^\vee(\Phi_{\mathrm{nr}}(G_{\mathrm{ad}}))$ if and only if $\chi \in X_{\mathrm{wr}}(G_{\mathrm{ad}}, G)$. This provides a bijection*

$$X_{\mathrm{wr}}(G) \times_{X_{\mathrm{wr}}(G_{\mathrm{ad}},G)} \mathsf{q}^\vee(\Phi_{\mathrm{nr}}(G_{\mathrm{ad}})) \to \Phi_{\mathrm{nr}}(G).$$

*Proof.* (a) By reduction to the absolutely simple case and classification one sees that $G_{\mathrm{ad}}{}^{\vee,\mathbf{I}_F} = ({G^\vee}_{\mathrm{sc}})^{\mathbf{I}_F}$ is always connected. Hence its image in $G^\vee$ is precisely $({G^\vee}_{\mathrm{der}})^{\mathbf{I}_F,\circ}$, and $\mathsf{q}^\vee(\phi_{\mathrm{ad}})(\mathrm{Frob})\mathrm{Frob}^{-1}$ always lies in $({G^\vee}_{\mathrm{der}})^{\mathbf{I}_F,\circ}$. The equivalence relation in $\Phi_{\mathrm{nr}}(G)$ still allows for conjugation by elements of $G^{\vee,\mathbf{I}_F}$. That can change $\mathsf{q}^\vee(\phi_{\mathrm{ad}})(\mathrm{Frob})\mathrm{Frob}^{-1}$ by elements of $(1-\mathrm{Frob})(G^{\vee,\mathbf{I}_F})$.
(b) Let $\phi \in \mathsf{q}^\vee(\Phi_{\mathrm{nr}}(G_{\mathrm{ad}}))$ and let $\chi \in X_{\mathrm{wr}}(G)$ such that $\chi\phi \in \mathsf{q}^\vee(\Phi_{\mathrm{nr}}(G_{\mathrm{ad}}))$. From the proof of part (a) we see that $\chi$ can be represented by an element $z \in G_{\mathrm{der}}{}^{\vee,\mathbf{I}_F,\circ} \cap Z(G^\vee)^{\mathbf{I}_F}$. Then $z$ can be lifted to an element of $Z({G^\vee}_{\mathrm{sc}})^{\mathbf{I}_F}$, so $\chi$ lies in the image of $X_{\mathrm{wr}}(G_{\mathrm{ad}}) \to X_{\mathrm{wr}}(G)$.

From (51) we see that $Z({G^\vee}_{\mathrm{der}})^{\mathbf{I}_F} \to \pi_0\big(({G^\vee}_{\mathrm{der}})^{\mathbf{I}_F}\big)$ is surjective. Hence $Z(G^\vee)^{\mathbf{I}_F} \to G^{\vee,\mathbf{I}_F}/({G^\vee}_{\mathrm{der}})^{\mathbf{I}_F,\circ}$ is surjective as well. It follows that every $\phi \in \Phi_{\mathrm{nr}}(G)$ can be written as an element of $X_{\mathrm{wr}}(G)$ times an element of $\mathsf{q}^\vee(\Phi_{\mathrm{nr}}(G_{\mathrm{ad}}))$. Combine that with part (a). $\square$

*Proof of Theorem 5.1 for reductive $F$-groups with anisotropic centre*
We analyse [FOS1, §13] in detail. Let $\pi_{\mathrm{ad}}$ be an irreducible constituent of $\mathrm{ind}_{P_{\mathfrak{f},\mathrm{ad}}}^{G_{\mathrm{ad}}}(\sigma_{\mathrm{ad}})$. Let $(\phi_{\mathrm{ad}}, \rho_{\mathrm{ad}})$ be the enhanced L-parameter of $\pi_{\mathrm{ad}}$, via Theorem 5.1 for $G_{\mathrm{ad}}$. Let $\pi$ be an irreducible constituent of $\mathsf{q}^*(\pi_{\mathrm{ad}})$ and put $\phi = \mathsf{q}^\vee(\phi_{\mathrm{ad}})$.

Write $g' = [\Omega_{G_{\mathrm{ad}}}/\Omega_{G_{\mathrm{ad}},\mathfrak{f}} : \Omega_{G,G_{\mathrm{ad}}}/\Omega_{G,G_{\mathrm{ad}},\mathfrak{f}}]$. It is checked in [FOS1, p. 29] that $\mathsf{a}'_{\mathrm{ad}} = |\Omega_{G_{\mathrm{ad}},\mathfrak{f}}|$, $\mathsf{b}' = \mathsf{b}'_{\mathrm{ad}}$ and

$$\mathsf{a}' = |\Omega_{G,\mathfrak{f}}| g' = |\ker(\Omega_G \to \Omega_{G_{\mathrm{ad}}})|\, |\Omega_{G_{\mathrm{ad}},\mathfrak{f}} : \Omega_{G,G_{\mathrm{ad}},\mathfrak{f}}]^{-1} \mathsf{a}'_{\mathrm{ad}}.$$

As in [FOS1, (13.3)], let $N_{\phi_{\mathrm{ad}}} \subset \Omega_{G_{\mathrm{ad}},\mathfrak{f}}$ be such that

$$X_{\mathrm{wr}}(G_{\mathrm{ad}})_{\phi_{\mathrm{ad}}} = \mathrm{Irr}(\Omega_{G_{\mathrm{ad}}})_{\phi_{\mathrm{ad}}} = \mathrm{Irr}(\Omega_{G_{\mathrm{ad}}}/N_{\phi_{\mathrm{ad}}}).$$

Then $\mathsf{a}_{\mathrm{ad}} = |N_{\phi_{\mathrm{ad}}}|$. Taking the above into account, [FOS1, Lemma 13.1] generalizes with almost the same proof. It says:

**Lemma 5.5.** *Suppose that $\phi_{\mathrm{ad}} \in \Phi^2_{\mathrm{nr}}(G_{\mathrm{ad}})$.*
*(a) $X_{\mathrm{wr}}(G) = \mathrm{Irr}(\Omega_G)$ acts transitively on the collection of elements $\phi' \in \Phi^2_{\mathrm{nr}}(G)$ which, for every $F$-simple factor $\mathcal{G}_i$ of $\mathcal{G}$, have the same $\gamma$-factor (at $s = 0$) $\gamma(0, \mathrm{Ad}_{G_i^\vee)} \circ \phi', \psi)$ as $\phi = \mathsf{q}^\vee(\phi_{\mathrm{ad}})$.*
*(b) The stabilizer of $\phi \in \Phi^2_{\mathrm{nr}}(G)$ in $X_{\mathrm{wr}}(G)$ equals*

$$\mathrm{Irr}\big(\Omega_{G,G_{\mathrm{ad}}}/(\Omega_{G,G_{\mathrm{ad}}} \cap N_{\phi_{\mathrm{ad}}})\big),$$

*and it contains $\mathrm{Irr}(\Omega_{G,G_{\mathrm{ad}}}/\Omega_{G,G_{\mathrm{ad}},\mathfrak{f}})$.*
*(c) $\mathsf{a} = |N_{\phi_{\mathrm{ad}}} \cap \Omega_{G,G_{\mathrm{ad}}}|\, |\ker(\Omega_G \to \Omega_{G_{\mathrm{ad}}}|$.*

The other arguments from [FOS1, §13] also generalize, the main difference is that we often have to replace $\Omega_G$ by $\Omega_{G,G_{\mathrm{ad}}}$. In particular [FOS1, Lemma 13.2] becomes

$$\mathcal{A}_\phi/\mathcal{A}_{\phi_{\mathrm{ad}}} \cong \mathrm{Irr}(\Omega_{G_{\mathrm{ad}}}/\Omega_{G,G_{\mathrm{ad}}} N_{\phi_{\mathrm{ad}}})$$

and [FOS1, Lemma 13.4] becomes

$$\mathsf{b} = g'[\Omega_{G_{\mathrm{ad}},\mathfrak{f}} : \Omega_{G,G_{\mathrm{ad}}} N_{\phi_{\mathrm{ad}}}]^{-1} \mathsf{b}_{\mathrm{ad}}.$$

We turn to [FOS1, §14]. With the above modifications to [FOS1, §13], the proof in [FOS1, §14] extends directly to possibly ramified reductive $F$-groups with anisotropic centre. The enhanced L-parameters are constructed first for $G$-representations contained in $\mathsf{q}^*(\pi_{\mathrm{ad}})$ for some $\pi_{\mathrm{ad}} \in \mathrm{Irr}_{\mathrm{cusp,unip}}(G_{\mathrm{ad}})$, and then extended $X_{\mathrm{wr}}(G)$-equivariantly to the whole of $\mathrm{Irr}_{\mathrm{cusp,unip}}(G)$ by means of Lemmas 5.3.b and 5.4.b. This establishes Theorem 5.1 for $\mathcal{G}$, except part (e) and (when $\mathcal{G}$ is not semisimple) parts (b) and (c).

We note that for an anisotropic $F$-torus $\mathcal{T}$ the above parametrization agrees with the natural isomorphism

$$\mathrm{Irr}_{\mathrm{unip}}(T) = X_{\mathrm{wr}}(T) \cong (T^{\mathbf{I}_F})_{\mathrm{Frob}} \cong \Phi_{\mathrm{nr}}(T), \tag{55}$$

which is a special case of the LLC for tori. Since (55) is natural, we obtain property (b) for all reductive $F$-groups with anisotropic centre. For property (c), the compatibility with almost direct products, we refer to the proof of [FOS1, Proposition 15.6] in combination with (55).

Next we consider the proof of the HII conjecture for $\mathrm{Irr}_{\mathrm{cusp,unip}}(G)$ in [FOS1, Lemmas 16.2 and 16.3]. This also goes by reduction to adjoint groups. Both formal degrees of $G$-representations and adjoint $\gamma$-factors of L-parameters for $G$ are invariant under the action of $X_{\mathrm{wr}}(G)$. In view of Lemmas 5.3.b and 5.4.b, this means that it suffices to check the HII conjectures for $\mathrm{Irr}(G/Z(G))_{\mathrm{cusp,unip}}$ and $\mathsf{q}^\vee(\Phi_{\mathrm{nr}}(G_{\mathrm{ad}}))$.

Let $Z(G)_1^\circ$ be the unique parahoric subgroup of $Z(\mathcal{G})^\circ(F)$ and let $\overline{Z(\mathcal{G})^\circ}(k_F)$ be its finite reductive quotient. We note that $\overline{\mathcal{G}_{\mathfrak{f}}^\circ}$ is isogenous to $\overline{\mathcal{G}_{\mathrm{ad},\mathfrak{f}}^\circ} \times \overline{Z(\mathcal{G})^\circ}$. By [GeMa, Proposition 1.4.12.c] these two groups have the same number of $k_F$-points. That and (21) lead to

$$\begin{aligned} \text{(56)}\quad \mathrm{vol}(P_{\mathfrak{f}}) = |\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)| q_F^{-(\mathbf{a}(\mathfrak{g}^\vee) + \dim \overline{\mathcal{G}_{\mathfrak{f}}^\circ} + \dim G^{\vee,\mathbf{I}_F})/2} = \\ |\overline{\mathcal{G}_{\mathrm{ad},\mathfrak{f}}^\circ}(k_F)| q_F^{-(\mathbf{a}(\mathfrak{g}_{\mathrm{ad}}^\vee) + \dim \overline{\mathcal{G}_{\mathrm{ad},\mathfrak{f}}^\circ} + \dim G_{\mathrm{ad}}^{\vee,\mathbf{I}_F})/2} |\overline{Z(\mathcal{G})^\circ}(k_F)| q_F^{-(\mathbf{a}(Z(\mathfrak{g}^\vee)) + \dim \overline{Z(\mathcal{G})^\circ} + \dim Z(\mathcal{G})^\circ)/2} \\ = \mathrm{vol}(P_{\mathfrak{f},\mathrm{ad}}) \mathrm{vol}(Z(G)_1^\circ). \end{aligned}$$

Write $\pi_{\mathrm{ad}} = \mathrm{ind}_{N_{G_{\mathrm{ad}}}(P_{\mathfrak{f},\mathrm{ad}})}^{G_{\mathrm{ad}}} (\sigma_{\mathrm{ad}}^N)$ and let $\pi \in \mathrm{Irr}(G)_{\mathrm{cusp,unip}}$ be a direct summand of $\mathsf{q}^*(\pi_{\mathrm{ad}})$. Then (22) and (56) yield

$$\begin{aligned} \text{(57)}\quad \frac{\mathrm{fdeg}(\pi, \mu_{G,\psi})}{\mathrm{fdeg}(\pi_{\mathrm{ad}}, \mu_{G_{\mathrm{ad}},\psi})} &= \frac{\dim(\sigma) |\overline{\mathcal{G}_{\mathrm{ad},\mathfrak{f}}^\circ}(k_F)| q_F^{-(\mathbf{a}(\mathfrak{g}_{\mathrm{ad}}^\vee) + \dim \overline{\mathcal{G}_{\mathrm{ad},\mathfrak{f}}^\circ} + \dim G_{\mathrm{ad}}^{\vee,\mathbf{I}_F})/2} |\Omega_{G_{\mathrm{ad}},\mathfrak{f}}|}{\dim(\sigma_{\mathrm{ad}}) |\overline{\mathcal{G}_{\mathfrak{f}}^\circ}(k_F)| q_F^{-(\mathbf{a}(\mathfrak{g}^\vee) + \dim \overline{\mathcal{G}_{\mathfrak{f}}^\circ} + \dim G^{\vee,\mathbf{I}_F})/2} |\Omega_{G,\mathfrak{f}}|} \\ &= \frac{q_F^{(\mathbf{a}(Z(\mathfrak{g}^\vee)) + \dim \overline{Z(\mathcal{G})^\circ} + \dim Z(\mathcal{G})^\circ)/2} |\Omega_{G_{\mathrm{ad}},\mathfrak{f}}|}{|\Omega_{G,\mathfrak{f}}| \, |\overline{Z(\mathcal{G})^\circ}(k_F)|} = \frac{|\Omega_{G_{\mathrm{ad}},\mathfrak{f}}|}{|\Omega_{G,\mathfrak{f}}| \mathrm{vol}(Z(G)_1^\circ)}. \end{aligned}$$

By construction $\mathsf{q}^\vee(\phi_{\pi_{\mathrm{ad}}}) = \phi_\pi$ and $\mathcal{A}_{\phi_{\pi_{\mathrm{ad}}}} \subset \mathcal{A}_{\phi_\pi}$. Equations [FOS1, (16.7) and (16.8)] must be modified to

$$\frac{\dim(\rho_\pi)}{\dim(\rho_{\pi_{\mathrm{ad}}})} = \frac{|\Omega_{G_{\mathrm{ad}},f}| \, |\Omega_{G,\mathfrak{f}}^\circ \cap N_{\phi_{\pi_{\mathrm{ad}}}}|}{|\Omega_{G,\mathfrak{f}}^\circ| \, |N_{\phi_{\pi_{\mathrm{ad}}}}|} \tag{58}$$

while [FOS1, (16.9) and (16.10)] become

$$\frac{|S_{\phi_\pi}|}{|S^\sharp_{\phi_\pi}|} = \frac{|Z({G^\vee}_{\mathrm{sc}})^{\mathbf{W}_F}|}{|Z(G^\vee)^{\mathbf{W}_F}|} = \frac{|\Omega_{G_{\mathrm{ad}}}|}{|\Omega_G|}. \tag{59}$$

Following (59), one obtains

$$\frac{|S^\sharp_{\phi_\pi}|}{|S^\sharp_{\rho_{\pi_{\mathrm{ad}}}}|} = \frac{|\Omega_G|}{|\Omega_{G_{\mathrm{ad}}}|}[\Omega_{G_{\mathrm{ad}}} : \Omega_G^\circ N_{\phi_{\pi_{\mathrm{ad}}}}] = |\ker(\Omega_G \to \Omega_{G_{\mathrm{ad}}})| \frac{|\Omega_G^\circ \cap N_{\phi_{\pi_{\mathrm{ad}}}}|}{|N_{\phi_{\pi_{\mathrm{ad}}}}|}. \tag{60}$$

By [FOS1, (16.14)]

$$\begin{aligned} \gamma(s, \mathrm{Ad}_{G^\vee} \circ \phi_\pi, \psi) &= \gamma(s, \mathrm{Ad}_{G^\vee_{\mathrm{der}}} \circ \phi_\pi, \psi)\gamma(s, \mathrm{Ad}_{Z(G^\vee)^\circ} \circ \phi_\pi, \psi) \\ &= \gamma(s, \mathrm{Ad}_{G_{\mathrm{ad}}^\vee} \circ \phi_{\pi_{\mathrm{ad}}}, \psi)\gamma(s, \mathrm{Ad}_{G^\vee/{G^\vee}_{\mathrm{der}}} \circ \mathrm{id}_{\mathbf{W}_F}, \psi). \end{aligned} \tag{61}$$

We note that the formal degree of a unitary character of $Z(\mathcal{G})^\circ(F)$ is

$$\mathrm{vol}(Z(\mathcal{G})^\circ(F))^{-1} = |\Omega_{Z(\mathcal{G})^\circ(F)}|^{-1}\mathrm{vol}(Z(G)_1^\circ)^{-1}.$$

It was shown in [HII, Lemma 3.5 and Correction] that

$$|\gamma(0, \mathrm{Ad}_{G^\vee/{G^\vee}_{\mathrm{der}}} \circ \mathrm{id}_{\mathbf{W}_F}, \psi)| = |\Omega_{Z(\mathcal{G})^\circ(F)}| \,\mathrm{fdeg}(\mathrm{triv}_{Z(\mathcal{G})^\circ(F)}) = \mathrm{vol}(Z(G)_1^\circ)^{-1}. \tag{62}$$

From (58)–(62) we deduce

$$\begin{aligned} &\frac{\dim(\rho_\pi)|S^\sharp_{\phi_{\pi_{\mathrm{ad}}}}|\,|\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\pi, \psi)|}{\dim(\rho_{\pi_{\mathrm{ad}}})|S^\sharp_{\phi_\pi}|\,|\gamma(0, \mathrm{Ad}_{G_{\mathrm{ad}}^\vee} \circ \phi_{\pi_{\mathrm{ad}}}, \psi)|} = \\ &\qquad \frac{|\Omega_{G_{\mathrm{ad}},\mathfrak{f}}|\,|\Omega_{G,G_{\mathrm{ad}},\mathfrak{f}} \cap N_{\phi_{\pi_{\mathrm{ad}}}}|}{|\Omega_{G,G_{\mathrm{ad}},\mathfrak{f}}|\,|\Omega_{G,G_{\mathrm{ad}}} \cap N_{\phi_{\pi_{\mathrm{ad}}}}|\,|\ker(\Omega_G \to \Omega_{G_{\mathrm{ad}}})|\mathrm{vol}(Z(G)_1^\circ)}. \end{aligned} \tag{63}$$

As $N_{\phi_{\pi_{\mathrm{ad}}}} \subset \Omega_{G_{\mathrm{ad}},\mathfrak{f}}$, this simplifies to

$$\frac{|\Omega_{G_{\mathrm{ad}},\mathfrak{f}}|}{|\Omega_{G,G_{\mathrm{ad}},\mathfrak{f}}|\,|\ker(\Omega_G \to \Omega_{G_{\mathrm{ad}}})|\mathrm{vol}(Z(G)_1^\circ)} = \frac{|\Omega_{G_{\mathrm{ad}},\mathfrak{f}}|}{|\Omega_{G,\mathfrak{f}}|\mathrm{vol}(Z(G)_1^\circ)}. \tag{64}$$

By (57), the expressions (64) and (63) also equal $\mathrm{fdeg}(\pi, \mu_{G,\psi})/\mathrm{fdeg}(\pi_{\mathrm{ad}}, \mu_{G_{\mathrm{ad}},\psi})$. From the already established HII conjectures for $\mathrm{Irr}(G_{\mathrm{ad}})_{\mathrm{unip,cusp}}$ we know that

$$\mathrm{fdeg}(\pi_{\mathrm{ad}}, \mu_{G_{\mathrm{ad}},\psi}) = \dim(\rho_{\pi_{\mathrm{ad}}})|S^\sharp_{\phi_{\pi_{\mathrm{ad}}}}|^{-1}|\gamma(0, \mathrm{Ad}_{G_{\mathrm{ad}}^\vee} \circ \phi_{\pi_{\mathrm{ad}}}, \psi)|.$$

With (57), (63) and (64) we conclude the analogous equality for $\mathrm{Irr}(G)_{\mathrm{cusp,unip}}$ holds. $\square$

In fact the above shows more, namely that [FOS1, Theorem 2.2] holds for all reductive $F$-groups with anisotropic centre. This concerns precise statements about the numbers $\mathsf{a}, \mathsf{b}, \mathsf{a}', \mathsf{b}'$, in terms of subquotients of $\Omega_G$. In contrast with [FOS1, §13–14], the formulation of [FOS1, Theorem 2.2] does not have to be adjusted to accomodate for ramified groups, it generalizes exactly as written.

*Proof of Theorem 5.1 for reductive groups*

This can be derived from the case of reductive $F$-groups with anisotropic centre, see [FOS1, p. 38–41 and p. 44]. For these arguments it does not matter whether $\mathcal{G}$ is ramified or not. The only small difference is that in one of the steps on [FOS1, p. 41] we should not restrict from $(\mathcal{G}_{\mathrm{der}} Z(\mathcal{G})_a)(F)$ to $G_{\mathrm{der}}$, with our proof for reductive $F$-groups with anisotropic centre that step already works with $(\mathcal{G}_{\mathrm{der}} Z(\mathcal{G})_a)(F)$.

For later use we recall the main idea of the proof. Let $Z(\mathcal{G})_s$ be the maximal $F$-split torus in $Z(\mathcal{G})$. Then $\mathcal{G}/Z(\mathcal{G})_s$ has $F$-anisotropic centre and $(\mathcal{G}/Z(\mathcal{G})_s)(F) = G/Z(G)_s$ [FOS1, (15.6)]. Tensoring with weakly unramified characters yields a natural bijection [FOS1, (15.8)]

$$X_{\mathrm{wr}}(G) \underset{X_{\mathrm{wr}}(G/Z(G)_s)}{\times} \mathrm{Irr}(G/Z(G)_s)_{\mathrm{cusp,unip}} \to \mathrm{Irr}(G)_{\mathrm{cusp,unip}}. \tag{65}$$

Similarly twisting by $X_{\mathrm{wr}}(G) \cong (Z(G^\vee)^{\mathbf{I}_F})_{\mathrm{Frob}}$ provides a natural bijection [FOS1, (15.12)]

$$X_{\mathrm{wr}}(G) \underset{X_{\mathrm{wr}}(G/Z(G)_s)}{\times} \Phi_{\mathrm{nr},e}(G/Z(G)_s) \to \Phi_{\mathrm{nr},e}(G). \tag{66}$$

Combining (65) and (66) with Theorem 5.1 for $G/Z(G)_s$, one obtains the desired $X_{\mathrm{wr}}(G)$-equivariant bijection $\mathrm{Irr}(G)_{\mathrm{cusp,unip}} \to \Phi_{\mathrm{nr},e}(G)$. $\square$

## 6. A local Langlands correspondence

We want to generalize the results leading to a local Langlands correspondence for unipotent representations in [Sol3] from $F_{\mathrm{nr}}$-split to arbitrary connected reductive $F$-groups $\mathcal{G}$. In the non-supercuspidal case these results rely mainly on [AMS1, AMS2, AMS3], in which no restriction on $\mathcal{G}$ is placed. Sections 1, 2 and 3 of [Sol3] were also written in that generality.

In [Sol3, §4] it is assumed that the reductive groups split over $F_{\mathrm{nr}}$, but that is only to apply the main result of [FOS1]. If we replace the input for [Sol3, Theorem 4.1 and Proposition 4.2] by Theorem 5.1, they apply to ramified connected reductive $F$-groups as well.

In [Sol3, Lemma 4.4] a Hecke algebra $\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma})$ as in (23) is compared with Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ constructed from enhanced L-parameters for an absolutely simple adjoint group $G$. When $\mathcal{G}$ is moreover ramified and $\mathcal{G}'$ is its $F_{\mathrm{nr}}$-split companion group, we showed in Theorem 3.1 that Theorem 2.3.b induces an algebra isomorphism

$$\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma}) \cong \mathcal{H}(G', \hat{P}_{\mathfrak{f}'}, \hat{\sigma}').$$

The algebra $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ (see [AMS3, §3.3] and [Sol3, §2]) is constructed from the group $Z^1_{G^\vee{}_{\mathrm{sc}}}(\phi(\mathbf{I}_F))$ with the data $\phi(\mathrm{Frob}), u_\phi$ and $\rho$. Here $(\phi, \rho)$ comes from the Bernstein component in $\Phi_e(G)$ associated to $\mathrm{Irr}(G)_{(\hat{P}_{\mathfrak{f}}, \hat{\sigma})}$ by [Sol3, Proposition 4.2], so $\phi$ is unramified. As $\mathcal{G}$ is adjoint, $Z^1_{G^\vee{}_{\mathrm{sc}}}(\phi(\mathbf{I}_F)) = (G^\vee)^{\mathbf{I}_F}$. In view of the comparison results Lemma 4.1 and Proposition 4.4, the data underlying $\mathcal{H}(\mathfrak{s}'^\vee, \vec{z})$ are canonically isomorphic to those for $\mathcal{H}(\mathfrak{s}^\vee, \vec{z})$. Hence $\mathcal{H}(\mathfrak{s}^\vee, \vec{v})$ is canonically isomorphic to the Hecke algebra $\mathcal{H}(\mathfrak{s}'^\vee, \vec{v})$ constructed in the same way for $G'$.

Recall from (47) that the transfer between enhanced unramified L-parameters for $G$ and $G'$ reflects the transfer between cuspidal unipotent representations in Theorem 2.3. The group $G'$ was already treated in [Sol3, Lemma 4.5] and [Lus2, Lus3]. In this way we obtain algebra isomorphisms

$$\mathcal{H}(G, \hat{P}_{\mathfrak{f}}, \hat{\sigma}) \cong \mathcal{H}(G', \hat{P}_{\mathfrak{f}'}, \hat{\sigma}') \cong \mathcal{H}(\mathfrak{s}'^\vee, \vec{v}) \cong \mathcal{H}(\mathfrak{s}^\vee, \vec{v}). \tag{67}$$

This means that [Sol3, Lemma 4.5] holds for ramified $F$-groups. With that the entire Section 4 of [Sol3] works for arbitrary connected reductive $F$-groups. Now [Sol3, Theorem 5.1] gives:

**Theorem 6.1.** *There exists a bijective local Langlands correspondence*

$$\begin{array}{ccc} \mathrm{Irr}(G)_{\mathrm{unip}} & \longleftrightarrow & \Phi_{\mathrm{nr},e}(G) \\ \pi & \mapsto & (\phi_\pi, \rho_\pi) \\ \pi(\phi,\rho) & \leftarrow\!\shortmid & (\phi,\rho) \end{array} \tag{68}$$

In [Sol3, §5] several properties of Theorem 6.1 were checked. These arguments generalize readily to possibly ramified connected reductive $F$-groups, if we take the following into account for the cases with $\mathcal{G}$ simple:

- For the $X_{\mathrm{wr}}(G)$-equivariance from [Sol3, Lemma 5.3] we use Lemma 3.3 and Proposition 4.4.
- For the cuspidality and the compatibility with cuspidal supports from [Sol3, Lemmas 5.4 and 5.5] we use Lemmas 3.7 and 4.5.
- For temperedness and boundedness in [Sol3, Lemma 5.6] we use Lemma 3.4.a and Proposition 4.4.
- For square-integrability and discreteness in [Sol3, Lemma 5.7] we use Lemma 3.4.b and Proposition 4.4.
- For the considerations with parabolic induction in [Sol3, Lemmas 5.9 and 5.10] we use Lemma 3.7 and (32).

The central characters associated to both sides of Theorem 6.1, as discussed in [Sol3, Lemma 5.8], need more attention. Recall from [Lan, p. 20–23] and [Bor, §10.1] that every $\phi \in \Phi(G)$ determines a character $\chi_\phi$ of $Z(G)$. For the construction, one first embeds $\mathcal{G}$ in a connected reductive $F$-group $\overline{\mathcal{G}}$ with $\mathcal{G}_{\mathrm{der}} = \overline{\mathcal{G}}_{\mathrm{der}}$, such that $Z(\overline{\mathcal{G}})$ is connected. Then one lifts $\phi$ to a L-parameter $\overline{\phi}$ for $\overline{G} = \overline{\mathcal{G}}(F)$. The natural projection ${}^L\overline{G} \to {}^L Z(\overline{G})$ produces an L-parameter $\overline{\phi}_z$ for $Z(\overline{G}) = Z(\overline{\mathcal{G}})(F)$, and via the local Langlands correspondence for tori $\overline{\phi}_z$ determines a character $\chi_{\overline{\phi}}$ of $Z(\overline{G})$. Then $\chi_\phi$ is given by restricting $\chi_{\overline{\phi}}$ to $Z(G)$. Langlands [Lan, p. 23] checked that $\chi_\phi$ does not depend on the choices made above.

**Lemma 6.2.** *In Theorem 6.1 the central character of $\pi$ equals $\chi_{\phi_\pi}$.*

*Proof.* By construction $G^\vee$ is the quotient of $\overline{G}^\vee$ by a central subgroup. Then $\overline{G}^\vee{}_{\mathrm{der}}$ projects onto $G^\vee{}_{\mathrm{der}}$.

In the cuspidal support $(M, \phi_M, \rho_M) := \mathbf{Sc}(\phi_\pi, \phi_\pi)$, the difference between $\phi_\pi$ and $\phi_M$ lies entirely in $G^\vee{}_{\mathrm{der}}$. Hence $\phi_\pi$ and $\phi_M$ give the same map $\mathbf{W}_F \to (G^\vee / G^\vee{}_{\mathrm{der}}) \rtimes \mathbf{W}_F$. Then their lifts $\overline{\phi_\pi}$ and $\overline{\phi_c}$ project to the same map

$$\mathbf{W}_F \to (\overline{G}^\vee / \overline{G}^\vee{}_{\mathrm{der}}) \rtimes \mathbf{W}_F = {}^L Z(\overline{G}).$$

Consequently $\phi_\pi$ and $\phi_c$ determine the same character of $Z(\overline{G})$, and $\chi_{\phi_\pi} = \chi_{\phi_c}|_{Z(G)}$.

Similarly, the central character of $\pi$ equals that of its supercuspidal support (restricted to $Z(G)$). Together with [Sol3, Lemma 5.5] (generalized above to possibly ramified $F$-groups), this means that it suffices to consider the case where $\pi$ is supercuspidal and $(\phi_\pi, \rho_\pi)$ is cuspidal.

We specialize further to the case where $Z(\mathcal{G})^\circ$ is $F$-anisotropic. When $\pi$ is contained in $\mathsf{q}^*(\pi_{\mathrm{ad}})$ for some $\pi_{\mathrm{ad}} \in \mathrm{Irr}(G_{\mathrm{ad}})_{\mathrm{cusp,unip}}$, its central character is obviously trivial. By the construction in Section 5, $\phi_\pi = \mathsf{q}^\vee(\pi_{\mathrm{ad}})$. Hence $\phi_\pi(\mathbf{W}_F) \subset G^\vee{}_{\mathrm{der}} \rtimes \mathbf{W}_F$ and $\overline{\phi_\pi}_z$ is the trivial parameter $\mathrm{id}_{\mathbf{W}_F}$ for $Z(\overline{G})$. Then $\chi_{\phi_\pi} = \mathrm{triv}_{Z(G)}$, as required.

Other $\tilde\pi \in \mathrm{Irr}(G)_{\mathrm{cusp,unip}}$ are obtained from such a $\pi$ by tensoring with a suitable $\chi \in X_{\mathrm{wr}}(G)$, see Lemma 5.3. This is mimicked in Lemma 5.4, and $\phi_{\tilde\pi} = \phi_{\chi\pi} = \chi\phi_\pi$. Then the central character of $\tilde\pi$ is $\chi$ and $\overline{\phi_{\tilde\pi}}_z = \overline{\chi}_z \overline{\phi_\pi}_z = \overline{\chi}_z$, so $\chi_{\phi_{\tilde\pi}}$ is also $\chi$.

Finally we consider the case where $\pi$ is supercuspidal and $Z(\mathcal{G})^\circ$ is $F$-isotropic. Since $\mathcal{G}/Z(\mathcal{G})_s$ has $F$-anisotropic centre, we already know the claim for $G/Z(G)_s$. But with (65) and (66) the LLC for $\mathrm{Irr}(G)_{\mathrm{cusp,unip}}$ is deduced from its analogue for $G/Z(G)_s$ by twisting with $X_{\mathrm{wr}}(G)$ on both sides of the correspondence. Explicitly, every $\pi \in \mathrm{Irr}(G)_{\mathrm{cusp,unip}}$ can be written as $\chi \otimes \tilde\pi$ with $\tilde\pi \in \mathrm{Irr}(G/Z(G)_s)_{\mathrm{cusp,unip}}$, and then $\phi_\pi = \chi\phi_{\tilde\pi}$. By the lemma for $G/Z(G)_s$, the central character of $\pi$ equals $\chi \otimes \chi_{\phi_{\tilde\pi}}$. On the other hand

$$\overline{\phi_\pi}_z = \overline{\chi\phi_{\tilde\pi}}_z = \overline{\chi}_z \overline{\phi_{\tilde\pi}}_z,$$

so $\chi_{\phi_\pi} = \chi \otimes \chi_{\phi_{\tilde\pi}}$ as well. $\square$

Summarising: we generalized the entire paper [Sol3] from $F_{\mathrm{nr}}$-split to arbitrary connected reductive $F$-groups. In particular we may now use its main result [Sol3, Theorem 1] in that generality.

Next we investigate the functoriality of Theorem 6.1, as in [Sol2]. The larger part of that paper (namely Sections 1–5) is written in complete generality, for all connected reductive groups. Only [Sol2, §7] deals exclusively with unipotent representations. There it is assumed that the groups are $F_{\mathrm{nr}}$-split, following [FOS1, Sol3].

Fortunately all the arguments from [Sol2, §7] are also valid for ramified groups. There are only two small points to note:

- In the proof of Lemma [Sol2, Lemma 7.1] for possibly ramified connected reductive $F$-groups, we must omit the reduction step from $G$ (with $F$-anisotropic centre) to $G_{\mathrm{der}}$. With our proof of Theorem 5.1 for reductive groups with anisotropic centre, the arguments for [Sol2, Lemma 7.1] apply directly.
- In [Sol2, (7.21)] it is claimed that

  $$\mathsf{q} : \hat{P}_{\mathfrak{f}} / P_{\mathfrak{f}} \to \hat{P}_{\mathfrak{f},\mathrm{ad}} / P_{\mathfrak{f},\mathrm{ad}}$$

  is injective, which need not be true when $\mathcal{G}$ is ramified. To overcome that, we can take $\hat\sigma \in \mathrm{Irr}(\hat{P}_{\mathfrak{f}})_{\mathrm{cusp}}$ of the form

  $$\chi \otimes \mathsf{q}^*(\hat\sigma_{\mathrm{ad}}) \text{ with } \hat\sigma_{\mathrm{ad}} \in \mathrm{Irr}(P_{\mathfrak{f},\mathrm{ad}})_{\mathrm{cusp}} \text{ and } \chi \in \mathrm{Irr}(\Omega_{G,\mathfrak{f}}) \cong \mathrm{Irr}(\hat{P}_{\mathfrak{f}}/P_{\mathfrak{f}}),$$

  as in Lemma 5.3. For $g \in \hat{P}_{\mathfrak{f},\mathrm{ad}}$ and $p \in \hat{P}_{\mathfrak{f}}$ we have $\mathrm{Ad}(g)^*\chi = \chi$ because $\Omega_{G,\mathfrak{f}}$ and $\Omega_{G_{\mathrm{ad}},\mathfrak{f}}$ are abelian. The equation following [Sol2, (7.21)] becomes

  $$\begin{aligned}&\mathrm{Ad}(g)^*(\hat\sigma)(p) = \chi(p)\big(\mathrm{Ad}(g)^*\mathsf{q}^*(\sigma_{\mathrm{ad}})\big)(p) = \\ &\qquad \chi(p)\hat{\sigma_{\mathrm{ad}}}(g)\hat{\sigma_{\mathrm{ad}}}(\mathsf{q}(p))\hat{\sigma_{\mathrm{ad}}}(g^{-1}) = \hat{\sigma_{\mathrm{ad}}}(g)\hat\sigma(p)\hat{\sigma_{\mathrm{ad}}}(g^{-1}) \qquad p \in \hat{P}_{\mathfrak{f}}.\end{aligned}$$

  With that, the proof of [Sol2, Lemma 7.5.b] works fine.

This means that the results of [Sol2] hold for unipotent representations of any connected reductive $F$-group. To formulate this precisely, let $\eta : \tilde{\mathcal{G}} \to \mathcal{G}$ be a homomorphism of connected reductive $F$-groups such that

- the kernel of $\mathrm{d}\eta : \mathrm{Lie}(\tilde{\mathcal{G}}) \to \mathrm{Lie}(\mathcal{G})$ is central,
- the cokernel of $\eta$ is a commutative $F$-group.

Let ${}^L = \eta^\vee \rtimes \mathrm{id} : G^\vee \rtimes \mathbf{W}_F \to \tilde{G}^\vee \rtimes \mathbf{W}_F$ be a L-homomorphism dual to $\eta$. For $\eta \in \Phi(G)$ we get ${}^L\eta \circ \phi \in \Phi(\tilde{G})$. Then $\eta$ gives rise to an injective algebra homomorphism

$$ {}^S\eta : \mathbb{C}[\mathcal{S}_\phi] \to \mathbb{C}[\mathcal{S}_{{}^L\eta\circ\phi}], \tag{69}$$

which under mild assumptions is canonical. It is a twist of the injection ${}^L\eta : \mathcal{S}_\phi \to \mathcal{S}_{{}^L\eta\circ\phi}$ by a character of $\mathcal{S}_\phi$, see [Sol2, Proposition 5.4].

Decomposing $\eta$ as in [Sol2, (5.2)], we see that the pullback $\eta^*$ sends unipotent $G$-representations to unipotent $\tilde{G}$-representations and that ${}^L\eta$ maps $\Phi_{\mathrm{nr}}(G)$ to $\Phi_{\mathrm{nr}}(\tilde{G})$. Then [Sol2, Conjecture 2 and Theorem 3], applied with the LLC from Theorem 6.1 say:

**Theorem 6.3.** *For any $(\phi, \rho) \in \Phi_{\mathrm{nr},e}(G)$:*

$$\eta^*(\pi(\phi,\rho)) = \bigoplus\nolimits_{\tilde{\rho}\in \mathrm{Irr}(\mathcal{S}_{{}^L\eta\circ\phi})} \mathrm{Hom}_{\mathcal{S}_\phi}\big(\rho, {}^S\eta^*(\tilde{\rho})\big) \otimes \pi({}^L\eta\circ\phi, \tilde{\rho}).$$

Finally we come to Conjecture 2 by Hiraga, Ichino and Ikeda [HII]. To prove it we will generalize the arguments from [FOS2], which was designed for $F_{\mathrm{nr}}$-split groups.

**Proposition 6.4.** (*a*) *The LLC from Theorem 6.1 satisfies the HII conjecture 2, up to some rational constants that depend only on an orbit $\mathcal{O}$.*

(*b*) *Part (a), Lemma 6.2, Theorem 6.3 and compatibility with direct products of reductive groups determine this LLC uniquely, up to twists by $X_{\mathrm{wr}}(G_{\mathrm{ad}}, G)$.*

*Proof.* Part (a) is shown in [Opd3, Theorem 4.5.1], for the "Langlands parametrization" from that paper. We proved it for ramified simple groups in Theorem 4.7.e, which in combination with [Opd3, §4.5] gives part (a) for all adjoint $F$-groups. The proof in the case of $F_{\mathrm{nr}}$-split groups with anisotropic centre in [Opd3] proceeds via reduction to adjoint groups. It relies on spectral transfer morphisms for affine Hecke algebras [Opd2]. We showed in Theorem 3.1 that the Hecke algebras for ramified adjoint $F$-groups have exactly the same shape and the same parameters as those for suitable $F_{\mathrm{nr}}$-split adjoint $F$-groups, so that Opdam's arguments with spectral transfer morphisms remain valid. This means that the Langlands parametrization from [Opd3] can be constructed for all connected reductive $F$-groups, and that it satisfies Conjecture 2 up to constants.

Our LLC from Theorem 6.1 extends [Sol3] to possibly ramified groups. In [FOS2, Theorem 2.1] it is checked that the LLC from [Sol3] agrees with the Langlands parametrization from [Opd3] (in the sense that the latter can be obtained from the former by forgetting the enhancements of L-parameters). We need to extend this compatibility to Theorem 6.1 and the above generalization of Opdam's Langlands parametrization.

By Lemmas 5.3 and 5.4, it suffices to do so for L-parameters $\phi \in \mathsf{q}^\vee(\Phi_{\mathrm{nr}}(G_{\mathrm{ad}}))$ and for unipotent $G$-representations with trivial central character. For those we saw in the proof of Lemma 5.4 that $\phi(\mathrm{Frob}) \in G^{\vee,\mathbf{I}_F,\circ}$. By [Bor, Lemma 6.4] that element corresponds to a unique $W(G^{\vee,\mathbf{I}_F,\circ}, T^{\vee,\mathbf{I}_F,\circ})^{\mathrm{Frob}}$-orbit in $(T^{\vee,\mathbf{I}_F,\circ})_{\mathrm{Frob}}$. With this modification in mind, the proof of [FOS2, Theorem 2.1] works for such L-parameters and $G$-representations. The first part of that proof establishes part (a) of the current proposition, while the last part deals with the essential uniqueness asserted in part (b). $\square$

With Proposition 6.4, everything in [FOS2, Sections 1–4] works equally well for ramified $F$-groups. Recall that we proved the HII conjecture for square-integrable representations of ramified simple $F$-groups in Theorem 4.7.e. Together with [FOS2, §5.1] that establishes Conjecture 2 for square-integrable representations of adjoint $F$-groups.

**Lemma 6.5.** *Let $\mathcal{G}$ be semisimple and let $\delta \in \mathrm{Irr}_{\mathrm{unip}}(G)$ be square-integrable. Then there exists a $\chi \in X_{\mathrm{wr}}(G)$ and a square-integrable $\delta_{\mathrm{ad}} \in \mathrm{Irr}_{\mathrm{unip}}(G_{\mathrm{ad}})$ such that $\chi \otimes \delta$ is a constituent of the pullback $\eta^*(\delta_{\mathrm{ad}})$.*

*Proof.* Let $\mathcal{M}$ be a $F$-Levi subgroup of $\mathcal{G}$ and let $\mathcal{M}_{AD} = \mathcal{M}/Z(\mathcal{G})$ be the image of $\mathcal{M}$ in $\mathcal{G}_{\mathrm{ad}}$. In Lemma 5.3 we showed that for every $\pi_M \in \mathrm{Irr}_{\mathrm{cusp,unip}}(M)$ there exist $\chi_M \in X_{\mathrm{wr}}(M)$ and $\pi_{M_{AD}} \in \mathrm{Irr}_{\mathrm{cusp,unip}}(M_{AD})$ such that $\chi_M \otimes \pi_M$ is a constituent of the pullback $\eta_M^*(\pi_{M_{AD}})$. We recall from [FOS2, (37)] that parabolic induction is compatible with pullback from $G_{\mathrm{ad}}$ (resp. $M_{AD}$). This implies that every $\pi \in \mathrm{Irr}_{\mathrm{unip}}(G)$ is, up to twisting by a $\chi \in X_{\mathrm{wr}}(G)$, contained in the pullback of a $\pi_{\mathrm{ad}} \in \mathrm{Irr}(G_{\mathrm{ad}})_{\mathrm{unip}}$.

Since $G$ is semisimple, $\chi$ is automatically unitary and tensoring by $\chi$ preserves square-integrability. We can regard $\pi \otimes \chi$ as a representation of the cocompact subgroup $G/Z(G)$ of $G_{\mathrm{ad}}$. Then a small variation on [Tad, Proposition 2.7] says that $\pi \otimes \chi$ is square-integrable if and only if $\pi_{\mathrm{ad}}$ is square-integrable. □

We note that the tensoring with the unitary character $\chi$ in Lemma 6.5 does not change the Plancherel densities. Therefore $\chi$ may be ignored in the subsequent computations of formal degrees. With Lemma 6.5 at hand, the proofs in [FOS2, §5.2] apply to all connected reductive $F$-groups with anisotropic centre, if we make the following modifications:

- By [FOS2, (34)] we must multiply the right hand side of [FOS2, (47)] with $\mathrm{vol}(Z(G)_1^\circ)^{-1}$. (This factor is invisible in the semisimple setting of [FOS2, §5.2].) For consistency, we must multiply [FOS2, (48)] and the right hand sides of [FOS2, Theorem 5.4.a and (51)] with the same factor.
- Using (62) we replace the equality
$$\gamma(s, \mathrm{Ad}_{G^\vee} \circ \phi_\delta, \psi) = \gamma(s, \mathrm{Ad}_{G^\vee{}_{\mathrm{sc}}} \circ \phi_{\delta_{\mathrm{ad}}}, \psi)$$
in the proof of [FOS2, Theorem 5.4.b] by
$$\left| \frac{\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\delta, \psi)}{\gamma(0, \mathrm{Ad}_{G^\vee{}_{\mathrm{sc}}} \circ \phi_{\delta_{\mathrm{ad}}}, \psi)} \right| = |\gamma(0, \mathrm{Ad}_{Z(G^\vee)^\circ} \circ \mathrm{id}_{\mathbf{W}_F}, \psi)| = \mathrm{vol}(Z(G)_1^\circ)^{-1}.$$
Then the next line of that proof becomes
$$\frac{\mathrm{fdeg}(\delta)}{\mathrm{fdeg}(\delta_{\mathrm{ad}})} = \frac{\dim(\rho_\delta) \left| S^\sharp_{\phi_{\delta_{\mathrm{ad}}}} \right| |\gamma(0, \mathrm{Ad}_{G^\vee} \circ \phi_\delta, \psi)|}{\dim(\rho_{\delta_{\mathrm{ad}}}) \left| S^\sharp_{\phi_\delta} \right| |\gamma(0, \mathrm{Ad}_{G^\vee{}_{\mathrm{sc}}} \circ \phi_{\delta_{\mathrm{ad}}}, \psi)|}$$
Combining that with the adjoint case yields the HII conjecture for formal degrees of square-integrable representations of connected reductive $F$-groups with anisotropic centre.

That renders most of [FOS2, §5.3] superfluous, except for the last part of the proof of [FOS2, Theorem 5.6]. That achieves the generalization (from $F$-anisotropic centre) to square-integrable modulo centre representations of arbitrary connected reductive $F$-groups.

We move on to Plancherel densities for tempered unipotent representation of possibly ramified $F$-groups. Some statements in [FOS2, §6.2] need to be modified:

- The adjoint $\gamma$-factors no longer need to be real-valued, as in [FOS2, Lemma A.5], because of $\epsilon$-factors of ramified $\mathbf{W}_F$-subrepresentations of $\mathrm{Lie}(G^\vee)$. To compensate for that, one can include fourth roots of unity as in Lemma 4.6, or one can replace $\pm\gamma(0, \mathrm{Ad} \circ \phi, \psi)$ everywhere by $|\gamma(0, \mathrm{Ad} \circ \phi, \psi)|$.
- In view of (19), [FOS2, (67)] becomes

$$\frac{\tau(N_e)}{\tau_{\mathcal{H}^M}(N_e)} = \frac{\mathrm{vol}(\hat{P}_{\mathfrak{f},M})}{\mathrm{vol}(\hat{P}_{\mathfrak{f}})} = \frac{|\overline{\mathcal{M}}_{\mathfrak{f}}(k_F)|\, q_F^{(\dim \overline{\mathcal{G}}_{\mathfrak{f}}^\circ + \dim \mathcal{G} + \mathbf{a}(\mathrm{Lie}\, G^\vee))/2}}{|\overline{\mathcal{G}}_{\mathfrak{f}}(k_F)|\, q_F^{(\dim \overline{\mathcal{M}}_{\mathfrak{f}}^\circ + \dim \mathcal{M} + \mathbf{a}(\mathrm{Lie}\, M^\vee))/2}}.$$

  This entails that in [FOS2, Lemma 6.4] one also gets an extra factor $q_F^{(\mathbf{a}(\mathrm{Lie}\, G^\vee) - \mathbf{a}(\mathrm{Lie}\, M^\vee))/2}$.
- The computation of adjoint $\gamma$-factors in [FOS2, Appendix A.2] applies only to the $\mathbf{I}_F$-fixed points in the involved complex Lie algebras. With Lemma 4.3 we can obtain similar formulas based on $\mathrm{Lie}(G^\vee)$ and $\mathrm{Lie}(M^\vee)$. It follows that [FOS2, (69)] must be replaced by

$$\gamma(0, \mathrm{Ad}_{G^\vee, M^\vee} \circ t\phi_M, \psi) = \epsilon \gamma(0, \mathrm{Ad}_{M^\vee} \circ t\phi_M, \psi) m^{M^*}(tr_M) \frac{q_F^{(\dim \mathcal{G} + \mathbf{a}(\mathrm{Lie}\, G^\vee))/2}}{q_F^{(\dim \mathcal{M} + \mathbf{a}(\mathrm{Lie}\, M^\vee))/2}},$$

  where $\epsilon^2 \in \{\pm 1\}$ depends only on the $\mathbf{W}_F$-representation $\mathrm{Lie}(G^\vee)/\mathrm{Lie}(M^\vee)$.

With these adjustments [FOS2, §6] becomes valid for all connected reductive $F$-groups. In particular [FOS2, Theorem 6.5] then establishes the HII conjecture for all tempered irreducible unipotent $G$-representations.

## 7. Rigid inner twists

So far we adhered to the conventions of Arthur [Art2, ABPS] for the setup with inner forms, components groups of L-parameters and relevance of enhancements. In this paragraph we take a different point of view, that of rigid inner twists. This notion was developed for reductive groups over local fields of characteristic zero by Kaletha [Kal1]. Recently Dillery extended it to reductive groups over local function fields [Dil].

The main point is to replace $H^1(\mathbf{W}_F, \mathcal{G}_{\mathrm{ad}})$, which parametrizes inner twists of $\mathcal{G}$, by a new cohomology set $H^1(\mathcal{E}, \mathcal{Z} \to \mathcal{G})$. Here $\mathcal{Z}$ is a fixed finite central $F$-subgroup of $\mathcal{G}$. When $\mathrm{char}(F) = 0$, this is based on a canonical extension of topological groups

$$1 \to u \to W \to \mathrm{Gal}(F_s/F) \to 1.$$

Let $Z^1(\mathcal{E}, \mathcal{Z} \to \mathcal{G})$ be the set of those continuous cocycles of $W$ in $\mathcal{G}(F_s)$, whose restriction to $u$ is a homomorphism $u \to \mathcal{Z}(F_s)$. In [Kal1, §3.2], $H^1(\mathcal{E}, \mathcal{Z} \to \mathcal{G})$ is defined as a subset of $H^1(W, \mathcal{G})$, namely the image of $Z^1(\mathcal{E}, \mathcal{Z} \to \mathcal{G})$. The construction of $H^1(\mathcal{E}, \mathcal{Z} \to \mathcal{G})$ is similar, but considerably more involved, when $\mathrm{char}(F) > 0$ [Dil, §3.2].

For our purposes it is best to take $\mathcal{Z} = Z(\mathcal{G}_{\mathrm{der}})$, as suggested in [Kal1]. We write

$$\overline{\mathcal{G}} = \mathcal{G}/Z(\mathcal{G}_{\mathrm{der}}) = \mathcal{G}_{\mathrm{der}}/Z(\mathcal{G}_{\mathrm{der}}) \times Z(\mathcal{G})/Z(\mathcal{G}_{\mathrm{der}}) = \mathcal{G}_{\mathrm{ad}} \times Z(\overline{\mathcal{G}}).$$

The canonical homomorphisms $\mathcal{G} \to \overline{\mathcal{G}} \to \mathcal{G}_{\mathrm{ad}}$ induce natural maps

$$H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{G}) \to H^1(\mathbf{W}_F, \overline{\mathcal{G}}) \to H^1(\mathbf{W}_F, \mathcal{G}_{\mathrm{ad}}) \tag{70}$$

which are surjective [Dil, Proposition 5.12].

Recall that an inner twist $(\mathcal{G}', \psi)$ of $\mathcal{G}$ is determined by an isomorphism of $F_s$-groups $\psi : \mathcal{G} \to \mathcal{G}'$, such that for every $\gamma \in \mathrm{Gal}(F_s/F)$ the $F$-automorphism $\psi^{-1} \circ \gamma \circ \psi \circ \gamma^{-1}$ of $\mathcal{G}$ is inner. Via (70) every $z \in H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{G})$ determines an element of $H^1(\mathbf{W}_F, \mathcal{G}_{\mathrm{ad}})$ and hence a unique equivalence class of inner twists $\mathcal{G}$. By the surjectivity of (70) every inner twist of $\mathcal{G}$ (up to equivalence) can be obtained in this way.

By definition [Kal1, §5.1] a rigid inner twist of $\mathcal{G}$ is a triple $(\mathcal{G}', \psi, z)$ where $(\mathcal{G}', \psi)$ is an inner twist of $\mathcal{G}$ and $z \in Z^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{G})$ such that

$$\psi^{-1} \circ \gamma \circ \psi \circ \gamma^{-1} = \mathrm{Ad}(z(\gamma)) \text{ for all } \gamma \in \mathrm{Gal}(F_s/F).$$

This applies to local fields of characteristic zero, and it probably it works well for most connected reductive groups over local functions field also. Nevertheless, the actual definition of rigid inner twistsis more complicated when $F$ has positive characteristic [Dil, §7.1]. In any case, $H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{G})$ can be regarded as the set of equivalence classes of rigid inner twists of $\mathcal{G}$. The big advantage of this setup is that it allows canonical transfer factors [Dil, §6].

Let $\overline{\mathcal{G}}^\vee = \mathcal{G}^\vee{}_{\mathrm{sc}} \times Z(\mathcal{G}/Z(\mathcal{G}_{\mathrm{der}}))^\vee$ be the dual group of $\overline{\mathcal{G}}$ and write

$$\overline{G}^\vee = \overline{\mathcal{G}}^\vee(\mathbb{C}) = G^\vee{}_{\mathrm{sc}} \times Z(\overline{G}^\vee)^\circ, \tag{71}$$

a cover of $G^\vee$. The $\mathbf{W}_F$-stable pinning of $G^\vee$ can be lifted to a pinning of $\overline{G}^\vee$ and we use the latter to define an action of $\mathbf{W}_F$ on $\overline{G}^\vee$. That furnishes a surjection ${}^L\overline{G} \to {}^L G$. Let $Z(\overline{G}^\vee)^+$ be the preimage of $Z(G^\vee)^{\mathbf{W}_F}$ in $Z(\overline{G}^\vee)$. In [Dil, Corollary 7.11] a natural isomorphism

$$H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{G}) \cong \mathrm{Irr}\big(\pi_0\big(Z(\overline{G}^\vee)^+\big)\big) \tag{72}$$

was established. Via (70) it extends the Kottwitz isomorphism

$$H^1(\mathbf{W}_F, \mathcal{G}_{\mathrm{ad}}) \cong \mathrm{Irr}\big(Z(G^\vee{}_{\mathrm{sc}})^{\mathbf{W}_F}\big) \tag{73}$$

from [Tha]. This shows that the fibers of (70) carry simply transitive actions of

$$\mathrm{Irr}\big(\pi_0\big(Z(\overline{G}^\vee)^+\big)/Z(G^\vee{}_{\mathrm{sc}})^{\mathbf{W}_F}\big). \tag{74}$$

In particular the number of rigid inner twists lying over a given inner twist (both considered up to equivalence) is finite and equals $\big[\pi_0\big(Z(\overline{G}^\vee)^+\big) : Z(G^\vee{}_{\mathrm{sc}})^{\mathbf{W}_F}\big]$. From (71) we see that the group (74) is trivial when $\mathcal{G}$ is an inner form of a split group. However, when $\mathcal{G}$ is an outer form of a split group (e.g. a non-split torus), (74) can very well be nontrivial. In such cases the quasi-split rigid inner twist of a reductive group is not unique anymore – apparently a price one has to pay for canonical transfer factors.

For issues involving parabolic induction we need to understand how (72) is related to its versions for Levi subgroups. To this end we assume that $G$ is quasi-split. Let $(\mathcal{G}^z, \psi, z)$ be a rigid inner twist of $\mathcal{G}$ and let $\mathcal{L}^z$ be a Levi $F$-subgroup of $\mathcal{G}^z$. Any 1-cocycle used to construct $\mathcal{L}^z$ as inner twist of a Levi $F$-subgroup $\mathcal{L}$ of $\mathcal{G}$ can also be used for $\mathcal{G}^z$. Therefore we may assume (upon replacing $z$ by an equivalent element) that $z \in Z^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{L})$ and that $\psi$ restricts to an $F_s$-isomorphism $\mathcal{L} \to \mathcal{L}^z$.

Consider the Levi subgroup $\tilde{\mathcal{L}} = \mathcal{L}/Z(\mathcal{G}_{\mathrm{der}})$ of $\overline{\mathcal{G}}$ and the inclusion

$$\mathcal{L}/Z(\mathcal{G}_{\mathrm{der}}) = \tilde{\mathcal{L}} \to \overline{\mathcal{G}}. \tag{75}$$

We note that in general there no canonical homomorphism between $\tilde{\mathcal{L}}$ and $\overline{\mathcal{L}}$.

**Lemma 7.1.** *(a) The map (75) induces a surjection* $\pi_0\big(Z(\overline{G}^\vee)^+\big) \to \pi_0\big(Z(\tilde{L}^\vee)^+\big)$.

*(b) Let* $\chi_z \in \mathrm{Irr}(Z(\overline{G}^\vee)^+)$ *be associated to* $z \in H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{G})$ *via (72) and let* $\chi_{z,\mathcal{L}}$ *be the character of* $Z(\tilde{L}^\vee)^+$ *associated to* $z \in H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{L})$ *via (72). Then* $\chi_z$ *is the pullback of* $\chi_{z,\mathcal{L}}$ *along the map from part (a).*

*Proof.* (a) Since $\tilde{\mathcal{L}}$ is a Levi subgroup of $\overline{\mathcal{G}} = \mathcal{G}_{\mathrm{ad}} \times Z(\overline{\mathcal{G}})$, we can express it as $\mathcal{L}_{AD} \times Z(\overline{\mathcal{G}})$, where $\mathcal{L}_{AD}$ is a Levi subgroup of $\mathcal{G}_{\mathrm{ad}}$. We see that $\tilde{L}^\vee = L_c^\vee \times Z(\overline{G}^\vee)^\circ$, where $L_c^\vee$ is the preimage of $L^\vee$ in $G^\vee{}_{\mathrm{sc}}$. By [Art1, Lemma 1.1]

$$Z(G^\vee)^{\mathbf{W}_F} Z(L^\vee)^{\mathbf{W}_F,\circ} = Z(L^\vee)^{\mathbf{W}_F}.$$

Taking preimages in $\overline{G}^\vee$, we find

$$Z(\overline{G}^\vee)^+ Z(\tilde{L}^\vee)^{\mathbf{W}_F,\circ} = Z(\tilde{L}^\vee)^+.$$

Now it is clear that $\pi_0\big(Z(\overline{G}^\vee)^+\big)$ surjects onto $\pi_0\big(Z(\tilde{L}^\vee)^+\big)$.

(b) The map (75) and the naturality of (72) give rise to a commutative diagram

$$\begin{array}{ccc} H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{G}) & \leftarrow & H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{L}) \\ \downarrow & & \downarrow \\ \mathrm{Irr}(Z(\overline{G}^\vee)^+) & \leftarrow & \mathrm{Irr}(Z(\tilde{L}^\vee)^+) \end{array} \quad . \tag{76}$$

Use that the same data $(\psi, z)$ realizes $\mathcal{G}^z$ as inner twist of $\mathcal{G}$ and $\mathcal{L}^z$ as an inner twist of $\mathcal{L}$. □

Consider a Langlands parameter $\phi : \mathbf{W}_F \times SL_2(\mathbb{C}) \to {}^L G$. Let $Z_{\overline{G}^\vee}(\phi)$ be the preimage of $Z_{G^\vee}(\phi)$ under ${}^L \overline{G} \to {}^L G$. Notice that

$$Z_{\overline{G}^\vee}(\phi) \cap Z(\overline{G}^\vee) = Z(\overline{G}^\vee)^+.$$

In this context the appropriate component group of $\phi$ is

$$\mathcal{S}_\phi^+ := \pi_0(Z_{\overline{G}^\vee}(\phi)).$$

Now an enhancement of $\phi$ is defined to be an irreducible representation $\rho^+$ of $\mathcal{S}_\phi^+$. The groups $\overline{G}^\vee$ and $G^\vee$ act naturally on the set of such enhanced L-parameters $(\phi, \rho^+)$, with the same orbits. The set of $G^\vee$-association classes of such enhanced L-parameters depends only on ${}^L G$, so we denote it by $\Phi^+({}^L G)$.

Recall from [Bor, §3] and [ABPS, Definition 1.3] that $G$-relevance of a Langlands parameter $\phi$ can be formulated in terms of parabolic subgroups of ${}^L G$ that contain the image of $\phi$. A nice aspect of rigid inner twists is that they enable a canonical definition of $G$-relevance of $\phi$ in terms of enhancements.

In view of the surjectivity of (70), we may assume without loss of generality that $G$ is quasi-split. We abbreviate a rigid inner twist $(\mathcal{G}^z, \psi, z)$ of $\mathcal{G}$ to $(\mathcal{G}^z, z)$. By the compatibility of (72) and (73), $\chi_z\big|_{Z(G^\vee{}_{\mathrm{sc}})^{\mathbf{W}_F}} = \chi_{\mathcal{G}^z}$. We say that $(\phi, \rho^+) \in \Phi^+({}^L G)$, or just the enhancement $\rho^+$, is relevant for $(\mathcal{G}^z, z)$ if the character of $Z(\overline{G}^\vee)^+$ determined by $\rho^+$ via the natural map $Z(\overline{G}^\vee)^+ \to \mathcal{S}_\phi^+$ equals $\chi_z$.

We denote the subset of $\Phi^+({}^L G)$ that is relevant for $(\mathcal{G}^z, z)$ by $\Phi^+(G^z, z)$. By design every enhancement $\rho^+$ is relevant for exactly one rigid inner twist of $\mathcal{G}$ (up to equivalence). That yields a natural decomposition

$$\Phi^+({}^L G) = \bigsqcup\nolimits_{z \in H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{G})} \Phi^+(G^z, z). \tag{77}$$

For comparison, in Section 4 we fixed an inner twist $\mathcal{G}$ of a quasi-split group and we let $\chi_{\mathcal{G}}$ be the associated character of $Z(G^\vee{}_{\mathrm{sc}})^{\mathbf{W}_F}$. To define relevance of enhancements there, we picked some extension $\chi_{\mathcal{G}}^e$ of $\chi_{\mathcal{G}}$ to $Z(G^\vee{}_{\mathrm{sc}})$ and used that to pin down central characters of $\mathcal{S}_\phi$-representations. That works fine, but the freedom in the choice of $\chi_{\mathcal{G}}^e$ means that it is not entirely canonical when $Z(G^\vee{}_{\mathrm{sc}})^{\mathbf{W}_F} \neq Z(G^\vee{}_{\mathrm{sc}})$. In that case, if one picks a set of representatives for the inner forms of $\mathcal{G}$, not all characters of $Z(G^\vee{}_{\mathrm{sc}})$ will occur in the union of the $\Phi_e(G')$ over the selected $G'$. That prevents a decomposition like (77) with $\mathcal{S}_\phi$ instead of $\mathcal{S}_\phi^+$, in the most general case at least.

Let us compare the two kinds of relevant enhancements of $\phi$. For a group $S$ with a central subgroup $Z$ and $\chi \in \mathrm{Irr}(Z)$ we write

$$\mathrm{Irr}(S,\chi) = \{(\pi, V) \in \mathrm{Irr}(S) : \pi(z) = \chi(z)\,\mathrm{id}_V \ \forall z \in Z\}.$$

**Lemma 7.2.** [Kal2, (4.6) and (4.7)]
*Let $(\mathcal{G}^z, z)$ be a rigid inner twist of a quasi-split connected reductive $F$-group $\mathcal{G}$. Choose an extension $\chi_{\mathcal{G}^z}^e \in \mathrm{Irr}(Z(G^\vee{}_{\mathrm{sc}}))$ of $\chi_{\mathcal{G}^z} \in \mathrm{Irr}\big(Z(G^\vee{}_{\mathrm{sc}})^{\mathbf{W}_F}\big)$ which agrees with $\chi_z$ on $Z(G^\vee{}_{\mathrm{sc}})^+$.*

($a$) *There is a canonical group isomorphism*

$$\mathcal{S}_\phi \to \mathcal{S}_\phi^+ \underset{Z(G^\vee{}_{\mathrm{sc}})^+}{\times} Z(G^\vee{}_{\mathrm{sc}})$$

($b$) *When $\phi$ is $G^z$-relevant, part (a) induces a bijection*

$$\mathrm{Irr}\big(\mathcal{S}_\phi, \chi_{\mathcal{G}^z}^e\big) \longleftrightarrow \mathrm{Irr}\big(\mathcal{S}_\phi^+, \chi_z\big).$$

*Here $\chi_{\mathcal{G}^z}^e$ is considered as a character on the image of $Z(G^\vee{}_{\mathrm{sc}})$ in $\mathcal{S}_\phi$, and similarly for $\chi_z$. When $\phi$ is not $G^z$-relevant, these two sets are empty.*

As a consequence of Lemma 7.2.b, there is a natural bijection

$$\Phi^+(G^z, z) \to \Phi_e(G^z), \tag{78}$$

whenever $\mathcal{G}^z, z$ and $\chi_{\mathcal{G}^z}^e$ are as in Lemma 7.2. By Lemma 7.1 the relevance condition is compatible with passage to Levi subgroups. Thus all aspects of the local Langlands correspondence discussed in this paper can be viewed equally well in terms of rigid inner twists $(\mathcal{G}^z, z)$ and enhanced L-parameters $\Phi^+(G^z, z)$. The only topic that needs some further clarification is the cuspidal support map for enhanced L-parameters figuring in Theorem 1.g.

**Lemma 7.3.** *The cuspidal support map for $\Phi_e(G^z)$ can also be defined for $\Phi^+(G^z, z)$, retaining all its properties.*

*Proof.* The construction of this map in [AMS1, §7] and Lemma 7.2 show us how to proceed:

- translate $(\phi, \rho^+) \in \Phi^+(G^z, z)$ to $(\phi, \rho) \in \Phi_e(G_z)$ with Lemma 7.2,
- compute its cuspidal support $(L, \phi_v, q\epsilon)$ as in [AMS1, Definition 7.7],
- translate $q\epsilon$ to $q\epsilon^+ \in \mathrm{Irr}(\mathcal{S}_{\phi_v}^+)$, using Lemma 7.2 and $\chi_{\mathcal{L},z}$,
- define the cuspidal support of $(\phi, \rho^+)$ to be $(L, \phi_v, q\epsilon^+)$.

The composition of these steps preserves the $Z(\overline{G}^\vee)^+$-character $\chi_z$. Together with Lemma 7.1 that means that $(\phi_v, q\epsilon^+)$ is relevant for the Levi subgroup $L$ of $G^z$. $\square$

We write $\Phi_{\mathrm{nr}}^+(G^z, z) = \{(\phi, \rho^+) \in \Phi^+(G^z, z) : \phi \in \Phi_{\mathrm{nr}}(G^z)\}$ and define $\Phi_{\mathrm{nr}}^+({}^LG) \subset \Phi^+({}^LG)$ similarly.

**Theorem 7.4.** *Let $\mathcal{G}$ be a quasi-split connected reductive $F$-group and let $(\mathcal{G}^z, z)$ be a rigid inner twist of $\mathcal{G}$. There exists a bijection*

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{unip}}(G^z) & \longleftrightarrow & \Phi^+_{\mathrm{nr}}(G^z, z) \\ \pi & \mapsto & (\phi_\pi, \rho^+_\pi) \end{array}$$

*with all the properties from Theorem 1.*

*When we let $(\mathcal{G}^z, z)$ run over all rigid inner twists of $\mathcal{G}$ up to equivalence, we obtain a bijection*

$$\bigsqcup\nolimits_{z \in H^1(\mathcal{E}, Z(\mathcal{G}_{\mathrm{der}}) \to \mathcal{G})} \mathrm{Irr}_{\mathrm{unip}}(G^z) \longleftrightarrow \Phi^+_{\mathrm{nr}}({}^L G).$$

*Proof.* The first bijection is the composition of Theorem 1 and (78). By Lemmas 7.1, 7.2 and 7.3 it enjoys the same properties as in Theorem 1.

Then the second bijection follows from (77). $\square$

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