RESOLUTIONS FOR REPRESENTATIONS OF REDUCTIVE p-ADIC GROUPS VIA THEIR BUILDINGS

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Abstract. Schneider–Stuhler and Vignéras have used cosheaves on the affine Bruhat–Tits building to construct natural projective resolutions of finite type for admissible representations of reductive p-adic groups in characteristic not equal to p. We use a system of idempotent endomorphisms of a representation with certain properties to construct a cosheaf and a sheaf on the building and to establish that these are acyclic and compute homology and cohomology with these coefficients. This implies Bernstein's result that certain subcategories of the category of representations are Serre subcategories. Furthermore, we also get results for convex subcomplexes of the building. Following work of Korman, this leads to trace formulas for admissible representations.

Contents

1. Introduction

Let $\mathcal{G}_{\mathbb{K}}$ be a reductive p-adic group, that is, the group of $\mathbb{K}$-rational points of a reductive linear algebraic group over a non-Archimedean local field $\mathbb{K}$. Let $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ be the affine Bruhat–Tits building of $\mathcal{G}_{\mathbb{K}}$. Work related to the Baum–Connes conjecture has shown that $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ knows a lot about topological properties of the category of smooth representations of $\mathcal{G}_{\mathbb{K}}$ (see [2, 14]). This article follows earlier work by Peter Schneider and Ulrich Stuhler [13] who use the building to construct natural projective resolutions of finite type for admissible representations of $\mathcal{G}_{\mathbb{K}}$.

Supported by the German Research Foundation (Deutsche Forschungsgemeinschaft (DFG)) through the Institutional Strategy of the University of Göttingen.

This was extended by Marie-France Vignéras [17] to representations on vector spaces over fields of characteristic not equal to p.

These resolutions may be used to study Euler–Poincaré functions of representations, to compute formal dimensions of discrete series representations, and to compute the inverse of the Baum–Connes assembly map on the K-theory classes of discrete series representations (see [12, 13]).

The input data for the resolutions of Schneider–Stuhler, besides a representation $\pi\colon \mathcal{G}_{\mathbb{K}} \to \mathrm{Aut}(V)$, is a carefully chosen system of compact open subgroups K_σ for all polysimplices σ in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$, depending on a parameter $e \in \mathbb{N}$. Let V_σ be the subspace of K_σ-fixed points in V. Then $(V_\sigma)_{\sigma\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})}$ is a cosheaf on $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$, which gives rise to a cellular chain complex $C_*\bigl(\mathcal{BT}(\mathcal{G}_{\mathbb{K}}), (V_\sigma)\bigr)$. This is shown to be a projective resolution of V if the subspaces V_x for vertices x in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ span V. The proof is indirect and depends on Joseph Bernstein's deep theorem that the category of representations V that are generated by the subspaces V_x is a Serre subcategory in the category of smooth representations of $\mathcal{G}_{\mathbb{K}}$ (see [3]).

One goal of this article is to obtain a Lefschetz fixed point formula for the character of an admissible representation of $\mathcal{G}_{\mathbb{K}}$. This issue was studied by Jonathan Korman in [10]. He could not get results in the higher rank case because this would require more information about the resolutions of Schneider and Stuhler. In order to compute the value of the character on a compact regular element g of $\mathcal{G}_{\mathbb{K}}$, we need the cellular chain complex $C_*\bigl(\Sigma, (V_\sigma)\bigr)$ to remain acyclic if Σ is a finite convex subcomplex of $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$. We may choose Σ invariant under g, and then the trace of g on $C_*\bigl(\Sigma, (V_\sigma)\bigr)$ agrees with the trace on V for sufficiently large Σ.

We are going to prove directly that $C_*\bigl(\Sigma, (V_\sigma)\bigr)$ is a resolution of $\sum_{x\in\Sigma^\circ} V_x$ for convex subcomplexes $\Sigma \subseteq \mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ and certain cosheaves (V_σ); here Σ° denotes the set of vertices of Σ. This implies immediately that the category of representations with $V = \sum_{x\in\Sigma^\circ} V_x$ is a Serre subcategory of the category of all smooth representations. Moreover, we can complete Korman's program and formulate a Lefschetz fixed point formula for character values of admissible representations. We do not yet spend much time to discuss this formula because we hope to establish a more powerful trace formula in a forthcoming article.

The main innovation in this article is the axiomatic formulation of the properties of the cosheaf (V_σ) that are needed for the homology computation. Our starting point is a system of idempotent endomorphisms $e_x\colon V \to V$ for vertices x in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ with the following three properties:

- e_x and e_y commute if x and y are adjacent vertices in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$;
- $e_x e_z e_y = e_x e_y$ if $z \in \mathcal{H}(x, y)$, and the vertices x and z are adjacent; here x, y and z are vertices in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ and $\mathcal{H}(x, y)$ denotes the intersection of all apartments containing x and y;
- $e_{gx} = \pi_g e_x \pi_g^{-1}$ for all $g \in \mathcal{G}_{\mathbb{K}}$ and all vertices x in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$.

Given such a system of idempotents, we let e_σ for a polysimplex σ in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ be the product of the commuting idempotents e_x for the vertices x of σ, and we let $V_\sigma := e_\sigma(V)$. This defines a cosheaf on $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$, and we show that $C_*\bigl(\Sigma, (V_\sigma)\bigr)$ for a convex subcomplex Σ of $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ is a resolution of $\sum_{x\in\Sigma^\circ} e_x(V)$, where Σ° denotes the set of vertices of Σ.

The system (e_x) provides a sheaf with the same spaces V_σ, using the projections $e_\sigma\colon V \to V_\sigma$. We show that the cochain complex $C^*\bigl(\Sigma, (V_\sigma)\bigr)$ for this sheaf is a resolution of $V \,\big/\, \bigcap_{x\in\Sigma^\circ} \ker e_x$. Furthermore, if Σ is finite then

$$V \cong \sum_{x\in\Sigma^\circ} e_x(V) \oplus \bigcap_{x\in\Sigma^\circ} \ker e_x.$$

The idempotent endomorphism u_Σ of V that effects this decomposition is given by the remarkably simple formula

$$u_\Sigma := \sum_{\sigma\in\Sigma} (-1)^{\dim\sigma} e_\sigma.$$

This fact plays an important role in our proof.

In characteristic 0, the cellular chain complex $C_*\big(\mathcal{BT}(\mathcal{G}_\mathbb{K}), (V_\sigma)\big)$ consists of finitely generated projective modules if V is admissible, so that we get a projective resolution of finite type of $\sum e_x(V)$. Since $V \mapsto C_*\big(\mathcal{BT}(\mathcal{G}_\mathbb{K}), (V_\sigma)\big)$ is an exact functor, the class of representations for which it provides a resolution of V is a Serre subcategory. Thus the class of smooth representations with $\sum e_x(V) = V$ is a Serre subcategory in the category of all smooth representations of $\mathcal{G}_\mathbb{K}$. A corresponding statement holds in the cohomological case, provided we use rough representations instead of smooth ones. By definition, a representation is smooth if it is the *inductive* limit of the subspaces of K_n-invariants, where (K_n) is a decreasing sequence of compact open subgroups with $\bigcap_{n\in\mathbb{N}} K_n = \{1\}$; it is rough if it is the *projective* limit of the same subspaces of K_n-invariants, where we map K_{n+1}-invariants to K_n-invariants by averaging.

Let V be an admissible $\mathbb{F}$-linear representation for a field $\mathbb{F}$ whose characteristic is not p. Assume $V = \sum V_x$, and let $f\colon \mathcal{G}_\mathbb{K} \to \mathbb{F}$ be a locally constant function supported in a compact subgroup $K \subseteq \mathcal{G}_\mathbb{K}$. Then $f(V)$ is finite-dimensional and hence contained in $V|_\Sigma := \sum_{x\in\Sigma^\circ} V_x$ for some finite convex subcomplex Σ in $\mathcal{BT}(\mathcal{G}_\mathbb{K})$, which we may take K-invariant. Then $C_*\big(\Sigma, (V_\sigma)\big)$ is a resolution of $V|_\Sigma$ by finite-dimensional representations of K. Hence the character of $V|_\Sigma$, restricted to K, is equal to the sum

$$\chi_\Sigma(g) = \sum_{\substack{\sigma\in\Sigma\\ g\sigma=\sigma}} (-1)^{\deg\sigma} \chi_{V_\sigma}(g),$$

where χ_{V_σ} denotes the trace of the g-action on V_σ, with a sign if g reverses the orientation of σ. For the chosen function $f \in \mathcal{H}(\mathcal{G}_\mathbb{K})$, the trace of f on V agrees with the trace on $V|_\Sigma$ because $f(V) \subseteq V|_\Sigma$. For arbitrary f, the trace on V will be a limit of such traces on $V|_\Sigma$.

The above recipe provides a formula for the values of the character on regular elements. For regular elliptic elements, this is already contained in [13], and for $\mathcal{G}_\mathbb{K}$ of rank 1 such character formulas are established in [10].

1.1. **Notation and basic setup.** The following notation will be used throughout this article.

Let $\mathbb{K}$ be a non-Archimedean local field, that is, a finite extension of $\mathbb{Q}_p$ for some prime p or the field of Laurent series $\mathbb{F}_q[\![t, t^{-1}]$ over the finite field $\mathbb{F}_q$ with q elements for a prime power q. Let p be the characteristic of the residue field of $\mathbb{K}$. Let $\mathcal{O}$ be the maximal compact subring of $\mathbb{K}$ and let $\mathcal{P}$ be the maximal ideal in $\mathcal{O}$. Let q be the cardinality of the residue field $\mathcal{O}/\mathcal{P}$.

Let $\mathcal{G}$ be a reductive linear algebraic group defined over $\mathbb{K}$. We write $\mathcal{G}_\mathbb{K}$ for its set of $\mathbb{K}$-rational points and briefly call $\mathcal{G}_\mathbb{K}$ a *reductive p-adic group.*

Recall that $\mathcal{G}_{\mathbb{K}}$ is a second countable, totally disconnected, locally compact group. That is, its topology may be defined by a decreasing sequence of compact open subgroups $(K_n)_{n\in\mathbb{N}}$.

1.1.1. *Representations as modules over a Hecke algebra.* Smooth representations of $\mathcal{G}_{\mathbb{K}}$ on $\mathbb{Q}$-vector spaces are equivalent to non-degenerate modules over the Hecke algebra $\mathcal{H}(\mathcal{G}_{\mathbb{K}},\mathbb{Q})$ of locally constant, compactly supported $\mathbb{Q}$-valued functions of $\mathcal{G}_{\mathbb{K}}$. Following Vignéras [16, Section I.3] we replace $\mathcal{H}(\mathcal{G}_{\mathbb{K}},\mathbb{Q})$ by a Hecke algebra with $\mathbb{Z}[1/p]$-coefficients. This allows us to extend the correspondence between representations of $\mathcal{G}_{\mathbb{K}}$ and $\mathcal{H}(\mathcal{G}_{\mathbb{K}})$-modules to representations on $\mathbb{Z}[1/p]$-modules, thus covering vector spaces over fields of characteristic different from p. Besides the non-degenerate $\mathcal{H}$-modules, which we call smooth here, we also need a dual class of rough $\mathcal{H}$-modules, which we introduce here (see also [11]).

Lemma 1.1. *There is a compact open subgroup $K \subseteq \mathcal{G}_{\mathbb{K}}$ that is a pro-p-group, that is, the index $[K : K']$ is a power of p for all compact open subgroups K' of K.*

Proof. Closed subgroups of pro-p-groups are again pro-p-groups. Since any linear algebraic group is contained in Gl_d by definition, it suffices to prove the assertion for $\mathrm{Gl}_d(\mathbb{K})$. The subgroups $K_n := 1 + \mathbb{M}_d(\mathcal{P}^n)$ for $n \geq 1$ form a decreasing sequence of compact open subgroups and a neighbourhood basis of 1. Since $[K_n : K_{n+1}] = q^{d^2}$ is a power of p for each n and any open subgroup of K_1 contains K_n for some $n \in \mathbb{N}$, K_1 is a pro-p-group. □

The lemma allows us to choose a Haar measure μ on $\mathcal{G}_{\mathbb{K}}$ with $\mu(K) \in \mathbb{Z}[1/p]$ for all compact open subgroups K. Let $\mathcal{H}$ or, more precisely, $\mathcal{H}\big(\mathcal{G}_{\mathbb{K}}, \mathbb{Z}[1/p]\big)$ be the $\mathbb{Z}[1/p]$-module of all locally constant, compactly supported functions $\mathcal{G}_{\mathbb{K}} \to \mathbb{Z}[1/p]$. Define the convolution of $f_1, f_2 \in \mathcal{H}$ by

$$f_1 * f_2(g) := \int_{\mathcal{G}_{\mathbb{K}}} f_1(h) f_2(h^{-1}g)\,\mathrm{d}\mu(h) \qquad \text{for } g \in \mathcal{G}_{\mathbb{K}}.$$

We claim that this belongs to $\mathcal{H}$ again. To see this, choose a compact open subgroup K that is so small that f_2 is left K-invariant and f_1 is right K-invariant; then f_1 is a $\mathbb{Z}[1/p]$-linear combinations of characteristic functions of cosets gK for $g \in \mathcal{G}_{\mathbb{K}}$, and $\chi_{gK} * f_2 = \mu(K) \cdot \lambda_g f_2$, where $\lambda_g f_2(x) = f_2(g^{-1}x)$ as usual.

Thus $\mathcal{H}$ is a ring over $\mathbb{Z}[1/p]$. It is a subring of the $\mathbb{Q}$-valued Hecke algebra $\mathcal{H}(\mathcal{G}_{\mathbb{K}},\mathbb{Q}) := \mathcal{H} \otimes_{\mathbb{Z}} \mathbb{Q}$. The group $\mathcal{G}_{\mathbb{K}}$ is embedded in the multiplier algebra of $\mathcal{H}$, that is, products of the form gf or fg with $g \in \mathcal{G}_{\mathbb{K}}$ and $f \in \mathcal{H}$ are well-defined and satisfy the expected properties.

For a compact open pro-p-subgroup $K \subseteq \mathcal{G}_{\mathbb{K}}$, let

$$\langle K\rangle := \mu(K)^{-1}\chi_K.$$

This is an idempotent element in the ring $\mathcal{H}$. Let $(K_n)_{n\in\mathbb{N}}$ be a decreasing sequence of compact open pro-p-subgroups of $\mathcal{G}_{\mathbb{K}}$ with $\bigcap K_n = \{1\}$. Then $\big(\langle K_n\rangle\big)_{n\in\mathbb{N}}$ is an increasing approximate unit of projections in $\mathcal{G}_{\mathbb{K}}$.

Definition 1.2. An $\mathcal{H}$-module V is called *smooth* if $V \cong \varinjlim \langle K_n\rangle V$, that is, for each $v \in V$ there is $n \in \mathbb{N}$ with $\langle K_n\rangle \cdot v = v$.

It is called *rough* if $V \cong \varprojlim \langle K_n\rangle V$, that is, for any $(v_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} V$ with $\langle K_n\rangle v_{n+1} = v_n$ for all $n \in \mathbb{N}$, there is a unique $v \in V$ with $v_n = \langle K_n\rangle \cdot v$ for all $n \in \mathbb{N}$.

We define the *smoothening* $\mathrm{S}(V)$ and the *roughening* $\mathrm{R}(V)$ of an $\mathcal{H}$-module V by

$$\mathrm{S}(V) := \varinjlim \langle K_n\rangle V, \qquad \mathrm{R}(V) := \varprojlim \langle K_n\rangle V,$$

using the embedding $\langle K_n\rangle V \to \langle K_{n+1}\rangle V$ and the projection $\langle K_{n+1}\rangle V \to \langle K_n\rangle V$ induced by $\langle K_n\rangle$ as structure maps.

Since $\langle K_n\rangle V$ is a unital $\langle K_n\rangle\mathcal{H}\langle K_n\rangle$-module and $\mathcal{H} = \varinjlim \langle K_n\rangle\mathcal{H}\langle K_n\rangle$, both $\mathrm{S}(V)$ and $\mathrm{R}(V)$ are modules over $\mathcal{H}$. Even more, the multiplier algebra $\mathcal{M}(\mathcal{H})$ of $\mathcal{H}$ acts on $\mathrm{S}(V)$ and $\mathrm{R}(V)$ because for any multiplier μ of $\mathcal{H}$, both $\mu\langle K_n\rangle$ and $\langle K_n\rangle\mu$ belong to $\langle K_m\rangle\mathcal{H}\langle K_m\rangle$ for some $m \in \mathbb{N}$. This allows us to well-define $\mu\langle K_n\rangle v \in \langle K_m\rangle V$ for $v \in V$ and $\langle K_m\rangle\mu v \in \langle K_m\rangle V$ for $v \in \mathrm{R}(V)$. The canonical maps $\mathrm{S}(V) \to V \to \mathrm{R}(V)$ are $\mathcal{M}(\mathcal{H})$-module homomorphisms. In particular, since $\mathcal{G}_{\mathbb{K}} \subseteq \mathcal{M}(\mathcal{H})$, smooth and rough $\mathcal{H}$-modules both carry natural representations of $\mathcal{G}_{\mathbb{K}}$.

We may alternatively define smoothenings and roughenings as $\mathrm{S}(V) \cong \mathcal{H} \otimes_{\mathcal{H}} V$ and $\mathrm{R}(V) \cong \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}, V)$. These definitions are used in [11] and can be extended to all locally compact groups.

Proposition 1.3. *The category of smooth $\mathcal{H}$-modules is equivalent to the category of smooth representations of $\mathcal{G}_{\mathbb{K}}$ on $\mathbb{Z}[1/p]$-modules.*

Let V and W be two $\mathcal{H}$-modules. If V is smooth, then the map $\mathrm{S}(W) \to W$ induces an isomorphism $\mathrm{Hom}_{\mathcal{H}}\big(V, \mathrm{S}(W)\big) \cong \mathrm{Hom}_{\mathcal{H}}(V, W)$. If W is rough, then the map $V \to \mathrm{R}(V)$ induces an isomorphism $\mathrm{Hom}_{\mathcal{H}}(\mathrm{R}(V), W) \cong \mathrm{Hom}_{\mathcal{H}}(V, W)$.

Let V be an $\mathcal{H}$-module. The natural maps $\mathrm{S}(V) \to V \to \mathrm{R}(V)$ induce natural isomorphisms

$$\mathrm{S}\big(\mathrm{S}(V)\big) \cong \mathrm{S}(V) \cong \mathrm{S}\big(\mathrm{R}(V)\big), \qquad \mathrm{R}\big(\mathrm{S}(V)\big) \cong \mathrm{R}(V) \cong \mathrm{R}\big(\mathrm{R}(V)\big).$$

The smoothening and roughening functors restrict to equivalences of categories between the subcategories of rough and smooth $\mathcal{H}$-modules, respectively.

Proof. The first statement is well-known for $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, \mathbb{Q})$, and the proof carries over literally to the $\mathbb{Z}[1/p]$-linear case.

Any $\mathcal{H}$-module homomorphism $f\colon V \to W$ maps $\langle K_n\rangle V$ to $\langle K_n\rangle W$. If V is smooth, then $V = \varinjlim \langle K_n\rangle V$, so that f factors through $\varinjlim \langle K_n\rangle W = \mathrm{S}(W)$. Thus $\mathrm{Hom}_{\mathcal{H}}\big(V, \mathrm{S}(W)\big) \cong \mathrm{Hom}_{\mathcal{H}}(V, W)$. We also get induced maps

$$\mathrm{R}(V) = \varprojlim_n \langle K_n\rangle V \to \langle K_n\rangle V \to \langle K_n\rangle W$$

for all n. These piece together to a map $\mathrm{R}(V) \to \mathrm{R}(W)$. If W is rough, this shows that f extends uniquely to a map $\mathrm{R}(V) \to W$, so that $\mathrm{Hom}_{\mathcal{H}}(\mathrm{R}(V), W) \cong \mathrm{Hom}_{\mathcal{H}}(V, W)$. The assertions in the third paragraph follow because $\langle K_n\rangle\,\mathrm{S}(V) = \langle K_n\rangle V = \langle K_n\rangle\,\mathrm{R}(V)$ for all $n \in \mathbb{N}$. They show that S and R are inverse to each other as functors between the subcategories of rough and smooth representations, respectively, whence the equivalence of categories. □

Recall that both smooth and rough $\mathcal{H}(\mathcal{G}_{\mathbb{K}})$-modules carry an induced group representation of $\mathcal{G}_{\mathbb{K}}$. Conversely, this representation of $\mathcal{G}_{\mathbb{K}}$ determines the module structure, by integration. Thus we may also speak of smooth and rough group representations of $\mathcal{G}_{\mathbb{K}}$. A representation is rough if and only if it is the projective limit of the subspaces of K_n-invariants with respect to the averaging maps.

1.1.2. *Cellular chain complexes of equivariant cosheaves.* For any reductive p-adic group, Bruhat and Tits [4, 5, 15] constructed an affine building. More precisely, they constructed two buildings, one for $\mathcal{G}_{\mathbb{K}}$ and one for its maximal semisimple quotient $\mathcal{G}_{\mathbb{K}}^{\mathrm{ss}}$. We shall use the building for $\mathcal{G}_{\mathbb{K}}^{\mathrm{ss}}$, which we call the *Bruhat–Tits building* of $\mathcal{G}_{\mathbb{K}}$ and denote by $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$.

Recall that $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ is a locally finite polysimplicial complex of dimension equal to the rank of $\mathcal{G}^{\mathrm{ss}}$. It carries a canonical metric, for which it becomes a CAT(0)-space. The group $\mathcal{G}_{\mathbb{K}}^{\mathrm{ss}}$ acts on $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$, properly, cocompactly and isometrically. Being a CAT(0)-space, it follows that $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ is K-equivariantly contractible for any compact subgroup K of $\mathcal{G}_{\mathbb{K}}^{\mathrm{ss}}$, so that $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ is a classifying space for proper actions of $\mathcal{G}_{\mathbb{K}}^{\mathrm{ss}}$ (see [1]). The action of $\mathcal{G}_{\mathbb{K}}^{\mathrm{ss}}$ induces one of $\mathcal{G}_{\mathbb{K}}$ because of the quotient map $\mathcal{G} \twoheadrightarrow \mathcal{G}^{\mathrm{ss}}$.

We mostly treat polysimplicial complexes such as $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ as purely combinatorial objects and view $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ as the set of polysimplices, partially ordered by $\tau \prec \sigma$ if τ is a face of σ. (Hence it would make no big difference if we used the building for $\mathcal{G}_{\mathbb{K}}$ instead of the building for $\mathcal{G}_{\mathbb{K}}^{\mathrm{ss}}$.) A polysimplex of dimension 0 is called a *vertex*, and a polysimplex of maximal dimension is called a *chamber*. For a polysimplicial complex Σ, we let Σ° be its set of vertices. Two vertices or polysimplices x and y are called *adjacent* if there is a polysimplex σ with $x, y \prec \sigma$; adjacent vertices need not be connected by an edge unless Σ is a simplicial complex. The *star* of a polysimplex is the set of all polysimplices adjacent to it. If σ and τ are adjacent, then we let $[\sigma, \tau]$ be the smallest polysimplex containing $\sigma \cup \tau$.

The action of $\mathcal{G}_{\mathbb{K}}$ on $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ preserves the polysimplicial structure, so that we get an induced action on the set of polysimplices.

Now we recall how to construct chain and cochain complexes of representations using (simplicial) cosheaves and sheaves on $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ (see also [13, Section II.1]). Cosheaves are also called coefficient systems.

Let Σ be a polysimplicial complex. A *sheaf* on Σ is a system of Abelian groups $(V_\sigma)_{\sigma \in \Sigma}$ with maps $\varphi_\sigma^\tau \colon V_\sigma \to V_\tau$ for $\tau \succ \sigma$ that satisfy $\varphi_\sigma^\sigma = \mathrm{Id}_{V_\sigma}$ and $\varphi_\tau^\omega \circ \varphi_\sigma^\tau = \varphi_\sigma^\omega$ for $\omega \succ \tau \succ \sigma$. In other words, a sheaf is a functor on the category associated to the partially ordered set $(\Sigma, \prec)$. Dually, a *cosheaf* on Σ is contravariant functor on this category, that is, a system of Abelian groups $(V_\sigma)_{\sigma \in \Sigma}$ with maps $\varphi_\sigma^\tau \colon V_\sigma \to V_\tau$ for $\tau \prec \sigma$ that satisfy $\varphi_\sigma^\sigma = \mathrm{Id}_{V_\sigma}$ and $\varphi_\tau^\omega \circ \varphi_\sigma^\tau = \varphi_\sigma^\omega$ for $\omega \prec \tau \prec \sigma$.

To form cellular chain complexes, we equip each simplex with an orientation. This induces orientations on its boundary faces. We define

$$\varepsilon_{\tau\sigma} := \begin{cases} 1 & \text{if } \tau \prec \sigma \text{ with compatible orientations,} \\ -1 & \text{if } \tau \prec \sigma \text{ with opposite orientations,} \\ 0 & \text{if } \tau \text{ is not a face of } \sigma. \end{cases}$$

Let $\Gamma = (V_\sigma, \varphi_\sigma^\tau)$ be a cosheaf on a polysimplicial complex Σ. The *cellular chain complex* $C_*\bigl(\Sigma, \Gamma\bigr)$ of Σ with coefficients Γ is the $\mathbb{N}$-graded chain complex $\bigoplus_{\sigma \in \Sigma} V_\sigma$ with V_σ in degree $\deg(\sigma)$ and with the boundary map

$$\partial\bigl((v_\sigma)_{\sigma \in \Sigma}\bigr)_\tau := \sum_{\sigma \in \Sigma} \varepsilon_{\tau\sigma} \varphi_\sigma^\tau(v_\sigma).$$

The homology of $C_*(\Sigma, \Gamma)$ is denoted by $\mathrm{H}_*(\Sigma, \Gamma)$ and called the *homology of* Σ *with coefficients* Γ.

Dually, let $\Gamma = (V_\sigma, \varphi_\sigma^\tau)$ be a sheaf on Σ and assume that Σ is locally finite – this holds for subcomplexes of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$. The *cellular cochain complex* $C^*\big(\Sigma, \Gamma\big)$ of Σ with coefficients Γ is the $\mathbb{N}$-graded cochain complex $\prod_{\sigma\in\Sigma} V_\sigma$ with V_σ in degree $\deg(\sigma)$ and with the boundary map

$$\partial\big((v_\sigma)_{\sigma\in\Sigma}\big)_\tau := \sum_{\sigma\in\Sigma} \varepsilon_{\sigma\tau}\varphi_\sigma^\tau(v_\sigma),$$

which is well-defined because Σ is locally finite. The cohomology of $C^*(\Sigma,\Gamma)$ is denoted by $\mathrm{H}^*(\Sigma,\Gamma)$ and called the *cohomology of* Σ *with coefficients* Γ.

A $\mathcal{G}_\mathbb{K}$*-equivariant cosheaf or sheaf* on $\mathcal{BT}(\mathcal{G}_\mathbb{K})$ is a cosheaf or sheaf Γ on $\mathcal{BT}(\mathcal{G}_\mathbb{K})$ with isomorphisms $\alpha_g \colon V_\sigma \xrightarrow{\cong} V_{g\cdot\sigma}$ for all $g \in \mathcal{G}_\mathbb{K}$, $\sigma \in \mathcal{BT}(\mathcal{G}_\mathbb{K})$ compatible with the maps φ_σ^τ, such that $\alpha_1 = \mathrm{Id}$, $\alpha_g \circ \alpha_h = \alpha_{gh}$. The cellular (co)chain complex of a $\mathcal{G}_\mathbb{K}$-equivariant (co)sheaf inherits a representation of $\mathcal{G}_\mathbb{K}$ by

$$\alpha_g\big((v_\sigma)_{\sigma\in\Sigma}\big)_\tau := \sum_{\sigma\in\Sigma} g_{\tau\sigma}\alpha_g(v_\sigma),$$

where

$$g_{\tau\sigma} = \begin{cases} 1 & \text{if } g(\sigma) = \tau \text{ and } g|_\sigma \colon \sigma \to \tau \text{ preserves orientations,} \\ -1 & \text{if } g(\sigma) = \tau \text{ and } g|_\sigma \colon \sigma \to \tau \text{ reverses orientations,} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Each V_σ inherits a representation of the stabiliser

$$P_\sigma^\dagger := \{g \in \mathcal{G}_\mathbb{K} \mid g\sigma = \sigma\}.$$

Notice that this group may be strictly larger than the pointwise stabiliser

$$P_\sigma := \{g \in \mathcal{G}_\mathbb{K} \mid gx = x \text{ for each vertex } x \text{ of } \sigma\}.$$

Lemma 1.4. *The representation of $\mathcal{G}_\mathbb{K}$ on $C_*(\mathcal{BT}(\mathcal{G}_\mathbb{K}),\Gamma)$ is smooth if and only if P_σ acts smoothly on V_σ for each polysimplex σ in $\mathcal{BT}(\mathcal{G}_\mathbb{K})$.*

The representation of $\mathcal{G}_\mathbb{K}$ on $C^*(\mathcal{BT}(\mathcal{G}_\mathbb{K}),\hat{\Gamma})$ is rough if and only if P_σ acts roughly on V_σ for each polysimplex σ in $\mathcal{BT}(\mathcal{G}_\mathbb{K})$.

Proof. We may replace P_σ by $P_\sigma^\dagger$ in both statements because the former is an open subgroup of $P_\sigma^\dagger$.

Let S be a set of representatives for the orbits of $\mathcal{G}_\mathbb{K}$ on $\mathcal{BT}(\mathcal{G}_\mathbb{K})$. This set is finite because $\mathcal{G}_\mathbb{K}$ acts transitively on the set of chambers. As a representation of $\mathcal{G}_\mathbb{K}$

$$C_*(\mathcal{BT}(\mathcal{G}_\mathbb{K}),\Gamma) = \bigoplus_{\sigma\in S} \mathrm{cInd}_{P_\sigma^\dagger}^{\mathcal{G}_\mathbb{K}} V_\sigma,$$

where we equip V_σ with the induced representation of $P_\sigma^\dagger$, twisted by the orientation character in (1), and where $\mathrm{cInd}_{P_\sigma^\dagger}^{\mathcal{G}_\mathbb{K}} V_\sigma$ is the space of all functions $f \colon \mathcal{G}_\mathbb{K} \to V_\sigma$ with $gf(xg) = f(x)$ for $g \in P_\sigma^\dagger$ and $f(x) = 0$ for x outside a compact subset of $\mathcal{G}_\mathbb{K}/P_\sigma^\dagger$. The group $\mathcal{G}_\mathbb{K}$ acts on this by left translation. It is easy to see that this representation is smooth if $P_\sigma^\dagger$ acts smoothly on V_σ. Similarly,

$$C^*(\mathcal{BT}(\mathcal{G}_\mathbb{K}),\Gamma) = \prod_{\sigma\in S} \mathrm{Ind}_{P_\sigma^\dagger}^{\mathcal{G}_\mathbb{K}} V_\sigma,$$

where V_σ carries the same representation as above and $\mathrm{Ind}_{P_\sigma^\dagger}^{\mathcal{G}_\mathbb{K}} V_\sigma$ is defined like $\mathrm{cInd}_{P_\sigma^\dagger}^{\mathcal{G}_\mathbb{K}} V_\sigma$ but without the support restriction. Such a representation of $\mathcal{G}_\mathbb{K}$ is usually not smooth, even if $P_\sigma^\dagger$ acts smoothly on V_σ, because there is no uniformity in the smoothness of functions in $\operatorname{Ind}_{P_\sigma^\dagger}^{\mathcal{G}_\mathbb{K}} V_\sigma$.

Let (X_n) be an increasing sequence of $P_\sigma^\dagger$-biinvariant subsets of $\mathcal{G}_\mathbb{K}$ with $\mathcal{G}_\mathbb{K} = \bigcup X_n$. By definition, $\operatorname{Ind}_{P_\sigma^\dagger}^{\mathcal{G}_\mathbb{K}} V_\sigma$ is the *projective* limit of the spaces of functions in $\operatorname{Ind}_{P_\sigma^\dagger}^{\mathcal{G}_\mathbb{K}} V_\sigma$ that are supported in X_n. The group $P_\sigma^\dagger$ acts smoothly or roughly on this subspace if and only if it acts smoothly or roughly on V_σ. The induced representation of $P_\sigma^\dagger$ on the projective limit of these rough representations remains rough. This is equivalent to roughness as a representation of $\mathcal{G}_\mathbb{K}$ because $P_\sigma^\dagger$ is open in $\mathcal{G}_\mathbb{K}$. □

Definition 1.5. We define the hull $\mathcal{H}(\sigma,\tau)$ of two polysimplices σ and τ in a building as the intersection of all apartments containing $\sigma \cup \tau$ (see Figure 1 for some examples).

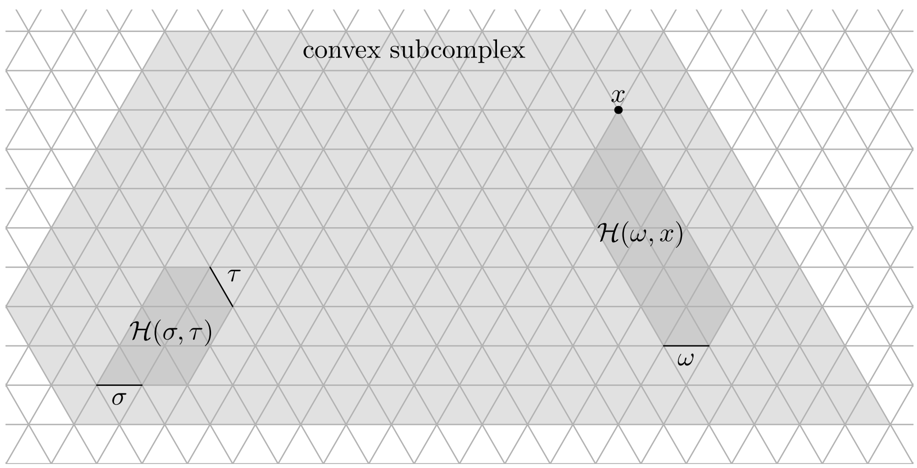

FIGURE 1. Hulls of facets and a convex subcomplex in an $\tilde{A}_2$-apartment

This notion generalizes the hull of two chambers, as defined in [9, §16.2].

Definition 1.6. A *subcomplex* of a polysimplicial complex Σ is a subset Σ' of Σ with $\tau \in \Sigma'$ if $\tau \prec \sigma$ and $\sigma \in \Sigma'$.

Definition 1.7. A subcomplex Σ of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$ is called *convex* if any polysimplex contained in $\mathcal{H}(\sigma,\tau)$ for $\sigma, \tau \in \Sigma$, is contained in Σ.

By definition, $\mathcal{H}(\sigma,\tau)$ is the smallest convex subcomplex containing $\sigma \cup \tau$. A subcomplex Σ of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$ is convex in this combinatorial sense if and only if its geometric realisation $|\Sigma|$ is convex in $|\mathcal{BT}(\mathcal{G}_\mathbb{K})|$ in the geometric sense: $x \in |\Sigma|$ if x lies on the geodesic segment between two points of $|\Sigma|$.

Example 1.8. The star of a polysimplex in $\mathcal{BT}(\mathcal{G}_\mathbb{K})$ is a convex subcomplex.

Let $K \subseteq \mathcal{G}_\mathbb{K}$ be a compact subgroup. Define $\mathcal{BT}(\mathcal{G}_\mathbb{K})^K \subseteq \mathcal{BT}(\mathcal{G}_\mathbb{K})$ by $\sigma \in \mathcal{BT}(\mathcal{G}_\mathbb{K})^K$ if and only if all vertices of σ are fixed by K. This is a non-empty convex subcomplex of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$. It is finite if the subgroup K is compact and open.

2. Natural resolutions of representations

Peter Schneider and Ulrich Stuhler [13] associated a certain cosheaf to an admissible $\mathbb{Q}$-linear representation V of a reductive p-adic group $\mathcal{G}_{\mathbb{K}}$ and showed that the cellular chain complex with coefficients in this cosheaf is a resolution of V. Their proof was indirect and based on a deep result of Joseph Bernstein about Serre subcategories of the category of smooth representations ([3, Corollaire 3.9]). Marie-France Vignéras [17] extended the constructions in [13] to representations over fields of characteristic different from p, based on the results of [16].

We are going to prove directly that cellular (co)chain complexes with certain (co)sheaves as coefficients are acyclic and compute their (co)homology in degree 0. It is important for the proof and for some applications, such as the character computations below, to allow finite convex subcomplexes of the building.

The cosheaves considered in [13,17] are of the form $V_\sigma := V^{K_\sigma}$ for certain compact open subgroups $K_\sigma \subseteq \mathcal{G}_{\mathbb{K}}$; here $V^{K_\sigma} \subseteq V$ denotes the subspace of K_σ-invariants. Since the subgroup K_σ can be computed from the groups K_x for the vertices of σ, it suffices to describe the subgroups K_x for vertices $x \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ$.

The subgroups used in [13] are small enough to be pro-p-groups (see Lemma 1.1) and hence give rise to idempotents $\langle K_\sigma\rangle \in \mathcal{H} = \mathcal{H}\big(\mathcal{G}_{\mathbb{K}}, \mathbb{Z}[1/p]\big)$. The subspace V^{K_σ} is the range $\langle K_\sigma\rangle V$ of this idempotent on V. These idempotents are more relevant than the subgroups for our proofs, which therefore break down in characteristic p. In the next section, we formalise the required properties of the idempotents $\langle K_x\rangle$.

2.1. Consistent systems of idempotents.

Definition 2.1. A system $(e_x)_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ}$ of idempotent endomorphisms $e_x\colon V \to V$ is called *consistent* if it has the following properties:

(a) e_x and e_y commute if x and y are adjacent;

(b) $e_x e_z e_y = e_x e_y$ for $x, y, z \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ$ with $z \in \mathcal{H}(x, y)$ and z is adjacent to x.

If $\pi\colon \mathcal{G}_{\mathbb{K}} \to \mathrm{Aut}(V)$ is a group representation, then a system of idempotents is called *equivariant* if

(c) $e_{gx} = \pi_g e_x \pi_g^{-1}$ for all $g \in \mathcal{G}_{\mathbb{K}}$, $x \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ$.

The idempotents e_x for vertices x yield idempotents e_σ for polysimplices σ, which inherit analogues of the consistency properties:

Proposition 2.2. *Let $(e_x)_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ}$ be a consistent system of idempotents. For a polysimplex $\sigma \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ$,*

$$e_\sigma := \prod_{\substack{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ \\ x\prec\sigma}} e_x$$

is a well-defined idempotent endomorphism of V.

(d) *$e_\sigma e_\tau = e_{[\sigma,\tau]}$ if the polysimplices τ and σ are adjacent; here $[\sigma, \tau]$ denotes the smallest polysimplex containing σ and τ;*

(e) *$e_\sigma e_\omega e_\tau = e_\sigma e_\tau$ if σ, τ, and ω are polysimplices in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ with $\omega \in \mathcal{H}(\sigma, \tau)$;*

if $(e_x)_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ}$ is equivariant, then

(f) *$\pi_g e_\sigma \pi_g^{-1} = e_{g\cdot\sigma}$ for all $g \in \mathcal{G}_{\mathbb{K}}$, $\sigma \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})$.*

Proposition 2.2 is interesting from an axiomatic point of view although our (co)homology computations only require a small part of it and checking (d)–(f) is not much harder in examples than checking (a)–(c). Since the proof of Proposition 2.2 is rather complicated, we postpone it to Section 2.4.

Given a consistent system of idempotents, we define

$$V_\sigma := e_\sigma(V) \subseteq V$$

and let $\varphi_\sigma^\tau\colon V_\sigma \to V_\tau$ for $\tau \prec \sigma$ be the inclusion map (here we use (d)). This defines a cosheaf on $\mathcal{BT}(\mathcal{G}_\mathbb{K})$, which we denote by Γ. If the system (e_x) is equivariant, Γ is a $\mathcal{G}_\mathbb{K}$-equivariant cosheaf by (f).

The cellular chain complex $C_*(\mathcal{BT}(\mathcal{G}_\mathbb{K}),\Gamma)$ is augmented by the map

$$\alpha\colon C_0(\mathcal{BT}(\mathcal{G}_\mathbb{K}),\Gamma) = \bigoplus_{x\in\mathcal{BT}(\mathcal{G}_\mathbb{K})^\circ} V_x \to V,$$

taking the embedding $V_x \rightarrowtail V$ on the summand V_x. Clearly, α is $\mathcal{G}_\mathbb{K}$-equivariant and satisfies $\alpha\circ\partial = 0$.

We also let $\hat\varphi_\tau^\sigma\colon V_\tau \to V_\sigma$ for $\tau \prec \sigma$ be the projection e_σ; this is a split surjection by (d). This defines a sheaf $\hat\Gamma = (V_\sigma, \hat\varphi_\tau^\sigma)$ on $\mathcal{BT}(\mathcal{G}_\mathbb{K})$. It is $\mathcal{G}_\mathbb{K}$-equivariant if (e_x) is equivariant. Its cellular chain complex is augmented by the equivariant chain map

$$\alpha\colon V \to C^0(\mathcal{BT}(\mathcal{G}_\mathbb{K}),\hat\Gamma) = \prod_{x\in\mathcal{BT}(\mathcal{G}_\mathbb{K})^\circ} V_x, \qquad v \mapsto \bigl(e_x(v)\bigr)_{x\in\mathcal{BT}(\mathcal{G}_\mathbb{K})^\circ}.$$

Condition (b) is not necessary for Γ and $\hat\Gamma$ to be equivariant simplicial (co)sheaves, but to prove acyclicity of $C_*(\mathcal{BT}(\mathcal{G}_\mathbb{K}),\Gamma)$ and $C^*(\mathcal{BT}(\mathcal{G}_\mathbb{K}),\hat\Gamma)$.

Although the sheaf and cosheaf Γ and $\hat\Gamma$ seem unrelated at first sight, these two constructions become equivalent when we allow V to be an object of a general Abelian category $\mathcal{C}$.

In this setting, the collection of endomorphisms of V is still a ring, so that idempotents in $\operatorname{End}(V)$ make sense. A consistent system of idempotents in $\operatorname{End}(V)$ for an object V of an Abelian category $\mathcal{C}$ yields a cosheaf Γ and a sheaf $\hat\Gamma$ with values in $\mathcal{C}$ exactly as above. We may form the cellular chain and cochain complexes $C_*(\Sigma,\Gamma)$ and $C^*(\Sigma,\hat\Gamma)$ provided $\mathcal{C}$ has countable coproducts and products. For (co)homology computations, we require these coproducts and products to be exact.

Lemma 2.3. *The passage from $\mathcal{C}$ to its opposite category $\mathcal{C}^{\mathrm{op}}$ exchanges the roles of Γ and $\hat\Gamma$ and hence of $C_*(\Sigma,\Gamma)$ and $C^*(\Sigma,\hat\Gamma)$.*

This is why it is useful to allow general categories in the following, although we are mainly interested in representations on $\mathbb{Z}[1/p]$-modules or on vector spaces over some field.

Proof. Since $\operatorname{End}_{\mathcal{C}^{\mathrm{op}}}(V)$ is the opposite ring of $\operatorname{End}_{\mathcal{C}}(V)$, both rings $\operatorname{End}_{\mathcal{C}^{\mathrm{op}}}(V)$ and $\operatorname{End}_{\mathcal{C}}(V)$ have the same idempotents. Thus the constructions in $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ use the same data. Conditions (d)–(f) in Proposition 2.2 are manifestly invariant under passage to the opposite ring, so that we get the same consistent or equivariant systems of idempotents in $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$. Now consider an idempotent endomorphism p of V as an endomorphism in $\mathcal{C}^{\mathrm{op}}$. Its range remains $p(V)$, and the embedding $p(V) \to V$ becomes the quotient map $V \to p(V)$ induced by p. As a consequence, the construction of Γ in $\mathcal{C}^{\mathrm{op}}$ yields precisely $\hat\Gamma$. Furthermore, the passage to opposite category exchanges products and coproducts, so that $C_*(\Sigma,\Gamma)$ becomes $C^*(\Sigma,\hat\Gamma)$ in the opposite category, for any subcomplex Σ of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$. $\square$

Theorem 2.4. *Let $\mathcal{C}$ be an Abelian category with exact countable products and coproducts. Let V be an object of $\mathcal{C}$ and let $(e_x)_{x\in\mathcal{BT}(\mathcal{G}_\mathbb{K})^\circ}$ be a consistent system of idempotents in its endomorphism ring $\operatorname{End}(V)$. Let $\mathcal{G}_\mathbb{K}$ be a reductive p-adic group and let Σ be a convex subcomplex of its affine Bruhat–Tits building $\mathcal{BT}(\mathcal{G}_\mathbb{K})$. Let I denote the directed set of finite convex subcomplexes of Σ.*

- *The cellular chain complex $C_*(\Sigma,\Gamma)$ is exact except in degree 0, where the augmentation map induces an isomorphism*
$$\mathrm{H}_0(\Sigma,\Gamma) \cong \varinjlim_{\Sigma_\mathrm{f}\in I} \sum_{x\in\Sigma_\mathrm{f}^\circ} e_x(V).$$
- *The cellular cochain complex $C^*(\Sigma,\hat\Gamma)$ is exact except in degree 0, where the augmentation map induces an isomorphism*
$$\varprojlim_{\Sigma_\mathrm{f}\in I} \left(V \Big/ \bigcap_{x\in\Sigma_\mathrm{f}^\circ} \ker e_x \right) \cong \mathrm{H}^0(\Sigma,\hat\Gamma).$$
- *If Σ is itself finite, then the composite map*
$$\sum_{x\in\Sigma^\circ} e_x(V) \cong \mathrm{H}_0(\Sigma,\Gamma) \to V \to \mathrm{H}^0(\Sigma,\hat\Gamma) \cong V \Big/ \bigcap_{x\in\Sigma^\circ} \ker e_x$$
is an isomorphism, that is,
$$V \cong \sum_{x\in\Sigma^\circ} e_x(V) \oplus \bigcap_{x\in\Sigma^\circ} \ker e_x.$$

Here we define $\sum_{x\in\Sigma^\circ} e_x(V)$ as the image of the map $\bigoplus_{x\in\Sigma^\circ} e_x(V) \to V$ and $\bigcap_{x\in\Sigma^\circ} \ker(e_x)$ as the infimum of $\ker(e_x)$ for $x\in\Sigma^\circ$, which is the kernel of the map $V \to \prod_{x\in\Sigma^\circ} e_x(V)$.

Remark 2.5. Although $\bigcap_{x\in\Sigma^\circ} \ker(e_x) \cong \varprojlim_{\Sigma_\mathrm{f}\in I} \bigcap_{x\in\Sigma_\mathrm{f}^\circ} \ker(e_x)$, we usually have
$$\varprojlim_{\Sigma_\mathrm{f}\in I} \left(V \Big/ \bigcap_{x\in\Sigma_\mathrm{f}^\circ} \ker e_x \right) \ncong V \Big/ \bigcap_{x\in\Sigma^\circ} \ker(e_x),$$
already for irreducible smooth representations on $\mathbb{Q}$-vector spaces. The right hand side is a smooth representation in this case, while the cohomology of $C^*(\Sigma,\hat\Gamma)$ is a rough representation of $\mathcal{G}_\mathbb{K}$ by Lemma 1.4. But infinite-dimensional irreducible smooth representations are not rough (see also Proposition 3.6).

Theorem 2.4 is the main result of this article. Its proof fills Section 2.5. The first assertion is the most important one and generalises results in [13, 17]. The assertions about sheaf cohomology and its comparison with cosheaf homology appear to be new. In our categorical formulation, they are equivalent to the corresponding statements about cosheaf homology.

2.2. **Some examples of consistent systems of idempotents.**

Now we consider some special cases of Theorem 2.4. In these applications, $\mathcal{C}$ is a category of modules or vector spaces. First we consider the case where $e_x = \langle K_x\rangle$ for compact open subgroups $K_x \subseteq \mathcal{G}_\mathbb{K}$.

Lemma 2.6. *Let $(K_x)_{x\in\mathcal{BT}(\mathcal{G}_\mathbb{K})^\circ}$ be a system of compact open pro-p-subgroups of $\mathcal{G}_\mathbb{K}$ and let V be $\mathbb{Z}[1/p]$-linear, so that the representation of $\mathcal{G}_\mathbb{K}$ on V integrates to a representation $\mathcal{H} \to \operatorname{End}(V)$. Assume*

(g) $K_x \cdot K_y = K_y \cdot K_x$ *if* x *and* y *are adjacent;*
(h) $K_z \subseteq K_x \cdot K_y$ *if* $x, y, z \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ$, $z \in \mathcal{H}(x, y)$, *and* z *is adjacent to* x*;*
(i) $gK_xg^{-1} = K_{gx}$ *for all* $g \in \mathcal{G}_{\mathbb{K}}$, $x \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})$.

Then the system of idempotents $e_x := \langle K_x \rangle$ *is consistent and equivariant, and*

$$K_\sigma = \prod_{\substack{x \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ \\ x \prec \sigma}} K_x$$

for a polysimplex σ *in* $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ *is a compact open subgroup of* $\mathcal{G}_{\mathbb{K}}$ *with* $e_\sigma = \langle K_\sigma \rangle$.

Conversely, (g)–(i) *are necessary for* (e_x) *to be consistent as left multiplication operators on* $\mathcal{H}$.

Here we use the naive product of subsets

$$A \cdot B := \{a \cdot b \mid a \in A, b \in B\} \qquad \text{for } A, B \subseteq \mathcal{G}_{\mathbb{K}}.$$

Proof. Since V is a module over $\mathcal{H}$ and the latter acts faithfully on itself, it suffices to show that the idempotents e_x satisfy Conditions (a)–(c) if and only if the subgroups K_x satisfy (g)–(i).

Since e_xe_y and e_ye_x are supported on K_xK_y and K_yK_x, respectively, (g) is necessary for Condition (a) in Definition 2.1. Conversely, if $K_xK_y = K_yK_x$, then this is a compact open subgroup and $e_xe_y = \langle K_xK_y \rangle$. The same argument yields the description of e_σ for a polysimplex σ. The equivalence between (c) and (i) is manifest. Condition (h) implies (b) because $\langle K_x \rangle \cdot gh \cdot \langle K_y \rangle = \langle K_x \rangle \langle K_y \rangle$ if $g \in K_x$, $h \in K_y$. Conversely, the convolutions $\langle K_x \rangle \langle K_y \rangle$ and $\langle K_x \rangle \langle K_z \rangle \langle K_y \rangle$ are supported in $K_x \cdot K_y$ and $K_x \cdot K_z \cdot K_y$, respectively. Hence $e_xe_ze_y = e_xe_y$ implies $K_x \cdot K_y = K_x \cdot K_z \cdot K_y \supseteq K_z$. □

Conditions (g) and (i) are enough to get a cosheaf on the building. We need (h) to compute the homology of $C_*(\mathcal{BT}(\mathcal{G}_{\mathbb{K}}), \Gamma)$.

It is rather easy to find systems of subgroups satisfying only (g) and (i). Let $P_A \subseteq \mathcal{G}_{\mathbb{K}}$ for a subcomplex A of $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ denote the subgroup of all $g \in \mathcal{G}_{\mathbb{K}}$ that fix all simplices in A. For each orbit in $\mathcal{G}_{\mathbb{K}} \backslash \mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ$, pick a representative $x \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ$ and a subgroup K_x of $P_{\mathrm{Star}\,x}$ that is normal in P_x (recall that the star of x consists of all polysimplices in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ that are adjacent to x); extend this to all of $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ$ by $K_{gx} = gK_xg^{-1}$ for $g \in \mathcal{G}_{\mathbb{K}}$. This makes sense because K_x is normal in P_x, and satisfies (i) by construction. If x and y are adjacent, then $K_y \subseteq P_{\mathrm{Star}\,y} \subseteq P_x$ normalises K_x, so that $gK_x = K_xg$ for all $g \in K_y$. This yields (g).

The subgroups considered by Schneider and Stuhler satisfy (h); [17, Lemma 1.28] checks this only for points on the straight line between x and y, but the same argument works if we merely assume $z \in \mathcal{H}(x, y)$.

Now let $\mathcal{G}_{\mathbb{K}}$ be the general linear group $\mathrm{Gl}_d(\mathbb{K})$ for some $d \in \mathbb{N}$. We denote its affine Bruhat–Tits building by $\mathcal{BT}$. A special feature of this group is that it acts transitively on the vertices of $\mathcal{BT}$. Hence an equivariant system of idempotents $(e_x)_{x \in \mathcal{BT}^\circ}$ is already specified by a single idempotent.

First we recall the structure of $\mathcal{BT}$. Let $\mathcal{O}$ be the maximal compact subring of $\mathbb{K}$ and let $\mathcal{P}$ be the maximal ideal in $\mathcal{O}$. Let q be the cardinality of the residue field $\mathcal{O}/\mathcal{P}$. Let $\varpi \in \mathcal{P}$ be a uniformiser, that is, $\mathcal{P} = \varpi \cdot \mathcal{O}$. We write $\mathcal{P}^n := \varpi^n \cdot \mathcal{O}$ for $n \in \mathbb{Z}$.

A *lattice* in $\mathbb{K}^d$ is an $\mathcal{O}$-submodule of $\mathbb{K}^d$ isomorphic to the *standard lattice* $\mathcal{O}^d$. Two lattices Λ and Λ' are *equivalent*, $\Lambda \simeq \Lambda'$, if there is $x \in \mathbb{K}^\times$ with $x \cdot \Lambda = \Lambda'$;

they are *adjacent* if there is $x \in \mathbb{K}^\times$ with $\varpi x \cdot \Lambda \subseteq \Lambda' \subseteq x \cdot \Lambda$. The group $\mathrm{Gl}_d(\mathbb{K})$ acts on the set of lattices in an obvious way, preserving the relations of equivalence and adjacency.

Vertices of $\mathcal{BT}$ are equivalence classes $[\Lambda]$ of such lattices. The vertices $[\Lambda_1], \dots, [\Lambda_k]$ form a (non-degenerate) simplex in $\mathcal{BT}$ if and only if Λ_i is adjacent but not equivalent to Λ_j for all $i, j = 1, \dots, k$ with $i \neq j$. A k-simplex adjacent to $[\Lambda]$ is equivalent to a flag $V_0 \subsetneq V_2 \subsetneq \dots \subsetneq V_k$ in the $\mathcal{O}/\mathcal{P}$-vector space $\mathcal{O}/\mathcal{P} \cdot \Lambda$.

Since any lattice is of the form $g \cdot \mathcal{O}^d$ for some $g \in \mathrm{Gl}_d(\mathbb{K})$, the group $\mathrm{Gl}_d(\mathbb{K})$ acts transitively on the set of vertices. The stabiliser of $[\mathcal{O}^d]$ is $Z \cdot \mathrm{Gl}_d(\mathcal{O})$, where

$$Z := \{x \cdot 1_d \mid x \in \mathbb{K}^\times\} \cong \mathrm{Gl}_1(\mathbb{K})$$

denotes the centre of $\mathrm{Gl}_d(\mathbb{K})$. An equivariant system of idempotents $(e_{[\Lambda]})_\Lambda$ is already specified by the single idempotent $e := e_{[\mathcal{O}^d]}$. This projection must commute with $\mathrm{Gl}_d(\mathcal{O})$, and any $\mathrm{Gl}_d(\mathcal{O})$-equivariant idempotent endomorphism e of V generates an equivariant system of idempotent endomorphisms by $e_{[g\mathcal{O}^n]} := geg^{-1}$. It remains to analyse when this equivariant system of idempotent endomorphisms is consistent. In all applications we care about, e comes from an idempotent in $\mathcal{H}\big(\mathrm{Gl}_d(\mathcal{O})\big)$ and the consistency conditions already hold in the ring $\mathcal{H}\big(\mathrm{Gl}_d(\mathbb{K})\big)$. We assume this from now on.

Apartments in the building $\mathcal{BT}$ correspond to unordered bases $\{b_1, \dots, b_d\}$ in $\mathbb{K}^d$; the vertices of the apartment for the basis $\{b_1, \dots, b_d\}$ are the lattices of the form $\sum_{j=1}^d \mathcal{P}^{n_j} b_j$ for $n_1, \dots, n_d \in \mathbb{Z}$. The inequalities $n_1 \geq n_2 \geq \dots \geq n_d$ define a positive chamber in this apartment. Let D^+ be the set of all diagonal matrices in Gl_d in the chosen basis with entries $(\varpi^{n_1}, \dots, \varpi^{n_d})$ with $n_1 \geq n_2 \geq \dots \geq n_d$. Then the vertices of this positive chamber are $[g\mathcal{O}^n]$ with $g \in D^+$.

Since the consistency conditions are compatible with the group actions, it suffices to check Conditions (a) and (b) in the special case $x = [\mathcal{O}^n]$. We may also assume that y belongs to the positive chamber in the apartment associated to the standard basis of $\mathbb{K}^d$, that is, $y = \big[\sum_{j=1}^d \mathcal{P}^{n_j} b_j\big]$ with $n_1, \dots, n_d \in \mathbb{Z}$ and $n_1 \geq n_2 \geq \dots \geq n_d$, because the $\mathrm{Gl}_d(\mathcal{O})$-orbit of y contains a lattice of this form. The vertex $y = \big[\sum_{j=1}^d \mathcal{P}^{n_j} b_j\big]$ is adjacent to $[\mathcal{O}^n]$ if and only if $n_1 - n_d = 1$ or, equivalently, we are dealing with the lattice $[\Omega_l \mathcal{O}^d]$, where Ω_l is the diagonal matrix with l entries ϖ and $d - l$ entries 1 for some l. Thus (a) amounts to

$$e\Omega_l e\Omega_l^{-1} = \Omega_l e\Omega_l^{-1} e \qquad \text{for } l = 1, \dots, d-1. \tag{2}$$

Actually, it suffices to establish this for $l \leq d/2$ because if $l > d/2$ there is $g \in \mathrm{Gl}_d$ with $g\mathcal{O}^n = \Omega_{d-l}\mathcal{O}^n$ and $g\Omega^l\mathcal{O}^n = \varpi \cdot \mathcal{O}^n$.

Condition (2) holds if e is supported in the normal subgroup $1 + \mathbb{M}_d(\mathcal{P})$ of $\mathrm{Gl}_d(\mathcal{O})$. Then $\Omega_l e \Omega_l^{-1}$ is supported in $\mathrm{Gl}_d(\mathcal{O})$ for all l. Thus $\Omega_l e \Omega_l^{-1}$ commutes with e because the latter is assumed central in $\mathrm{Gl}_d(\mathcal{O})$. It is unclear whether there is an idempotent e not supported in $1 + \mathbb{M}_d(\mathcal{P})$ that satisfies (2).

If z is a vertex in $\mathcal{H}(x, y)$, then z belongs to the same positive chamber in the same apartment, that is, $z = \big[\sum_{j=1}^d \mathcal{P}^{m_j} b_j\big]$ with $m_1, \dots, m_d \in \mathbb{Z}$ and $m_1 \geq m_2 \geq \dots \geq m_d$. It belongs to $\mathcal{H}(x, y)$ if $m_i - m_j \leq n_i - n_j$ for $1 \leq i \leq j \leq d$. Equivalently, there are $g, h \in D^+$ with $z = h[\mathcal{O}^n]$ and $y = gz$. Therefore, the condition $e_x e_z e_y = e_x e_y$ for all $x, y, z \in \mathcal{BT}^\circ$ with $z \in \mathcal{H}(x, y)$ is equivalent to

$$ege \cdot ehe = eghe \qquad \text{for all } g, h \in D^+, \tag{3}$$

that is, the map $D^+ \to \mathrm{End}(V)$, $g \mapsto ege$ is multiplicative. We may restrict here to z adjacent to x, that is, $h = \Omega_l$ for some l because these elements generate the monoid D^+.

Equation (3) for special projections is frequently used in representation theory; for instance, see [7, Lemma 4.1.5]. In particular, it is well-known and easy to check that the idempotent $\langle U^{(r)}\rangle$ associated to the compact open subgroup

$$U^{(r)} := 1 + \mathbb{M}_d(\mathcal{P}^{r+1}) \qquad \text{for } r \in \mathbb{Z}_{\geq 0}$$

satisfies these conditions. These subgroups are pro-p-groups, so that $\langle U^{(r)}\rangle$ belongs to $\mathcal{H}\big(\mathrm{Gl}_d, \mathbb{Z}[1/p]\big)$, and they are normal in $\mathrm{Gl}_d(\mathcal{O})$ and contained in $1 + \mathbb{M}_d(\mathcal{P}^{r+1})$, the stabiliser of the star of $[\mathcal{O}^d]$. We have already remarked above that this is enough to get a system of open subgroups $(e_x)_{x\in\mathcal{BT}^\circ}$ satisfying (i) and (g). It is known also that $U^{(r)}gU^{(r)}hU^{(r)} = U^{(r)}ghU^{(r)}$ for $g, h \in D^+$. This implies (3) and hence (h), that is, we have a consistent equivariant system of subgroups $(U_x^{(r)})_{x\in\mathcal{BT}^\circ}$.

More explicitly, these subgroups for vertices are

$$U_{[\Lambda]}^{(r)} := \{g \in \mathrm{Gl}_d(\mathbb{K}) \mid (g-1)(\Lambda) \subseteq \mathcal{P}^{r+1} \cdot \Lambda\}$$

because this system of subgroups is equivariant and $U_{[\mathcal{O}^d]}^{(r)} = U^{(r)}$. Let σ be an l-simplex in $\mathcal{BT}(\mathcal{G}_\mathbb{K})$, let $\Lambda_0 \supsetneq \Lambda_1 \supsetneq \cdots \supsetneq \Lambda_{l-1} \supseteq \Lambda_l = \mathcal{P}\cdot\Lambda_0$ be lattices in $\mathbb{K}^d$ that represent its vertices. Then

$$U_\sigma^{(r)} = \{g \in \mathrm{Gl}_d(\mathbb{K}) \mid (g-1)(\Lambda_{j-1}) \subseteq \mathcal{P}^r \cdot \Lambda_j \text{ for } j = 1, \dots, l\}.$$

To check this, notice first that this system of subgroups is $\mathrm{Gl}_d(\mathbb{K})$-equivariant, so that it suffices to treat one representative in each orbit. We pick the representatives $\Lambda_j = \Omega_{k_j} \cdot \mathcal{O}^d$ with $0 = k_0 < k_1 < \cdots < k_l = d$. In this case, (g_{ij}) belongs to $\prod_{j=0}^{l-1} \Omega_l U^{(r)} \Omega_l^{-1}$ if and only if $g_{ij} - \delta_{ij} \in \mathcal{P}^r$ for $i \leq k_n < j$ for some n and $g_{ij} - \delta_{ij} \in \mathcal{P}^{r+1}$ otherwise. This yields exactly $U_\sigma^{(r)}$.

Finally, we let $\mathcal{G}_\mathbb{K}$ be a semi-simple p-adic group (the generalisation to reductive groups is easy but complicates notation). The following situation is considered in the theory of types (see [6]).

Let $x \in \mathcal{BT}(\mathcal{G}_\mathbb{K})^\circ$ be a vertex. Let $P_x \subseteq \mathcal{G}_\mathbb{K}$ be its stabiliser; this is a compact open subgroup of $\mathcal{G}_\mathbb{K}$ because $\mathcal{G}$ is semi-simple. Let ρ be an irreducible representation of P_x; assume that the central projection e_x in $\mathcal{H}(P_x, \mathbb{Q})$ associated to ρ acts on V (this is the case if V is a $\mathbb{Q}$-vector space or if $e_x \in \mathcal{H}\big(P_x, \mathbb{Z}[1/p]\big)$ and V is a $\mathbb{Z}[1/p]$-module).

We view $\mathcal{H}(P_x)$ as a subalgebra of $\mathcal{H}$. Assume that $e_x g e_x = 0$ if $g \in \mathcal{G}_\mathbb{K}$ and $gx \neq x$. This ensures that $e_{gx} := g e_x g^{-1}$ for $g \in \mathcal{G}_\mathbb{K}$ and $e_y = 0$ for other vertices defines an equivariant consistent system of idempotents with $e_\sigma = 0$ for all polysimplices of dimension at least 1. The conditions of Proposition 2.2 are obvious here because $e_\sigma e_\tau = 0$ unless $\sigma = \tau$ and $\dim\sigma = 0$.

The cellular chain complex $C_*(\Sigma, \Gamma)$ is concentrated in dimension 0 and has

$$\mathrm{H}_0(\Sigma, \Gamma) = C_0(\Sigma, \Gamma) \cong \mathrm{cInd}_{P_x}^{\mathcal{G}_\mathbb{K}} e_x(V),$$

where cInd denotes the compactly supported induction functor:

$$\mathrm{cInd}_{P_x}^{\mathcal{G}_\mathbb{K}} e_x(V) = \{f\colon \mathcal{G}_\mathbb{K} \to e_x(V) \mid \operatorname{supp} f \text{ is compact}\}^{P_x},$$

where P_x acts by $(g\cdot f)(x) := \pi_g f(xg)$ for $x \in \mathcal{G}_\mathbb{K}$, $g \in P_x$, and $\mathcal{G}_\mathbb{K}$ acts by $(g\cdot f)(x) := f(g^{-1}x)$. Thus the first half of Theorem 2.4 asserts that $\mathrm{cInd}_{P_x}^{\mathcal{G}_\mathbb{K}} e_x(V)$

is isomorphic to the subrepresentation of V generated by $e_x(V)$; for the sheaf cohomology, we get

$$\mathrm{H}^0(\Sigma, \hat{\Gamma}) = C^0(\Sigma, \hat{\Gamma}) \cong \mathrm{Ind}_{P_x}^{\mathcal{G}_{\mathbb{K}}} e_x(V),$$

where Ind denotes the induction functor without support restrictions (and without smoothening). Notice that $\mathcal{G}_{\mathbb{K}}$ acts roughly and not smoothly on $\mathrm{H}^0(\Sigma, \hat{\Gamma})$.

2.3. **Support projections.** This section prepares the proof of Theorem 2.4 by computing support projections for certain finite subcomplexes of the building. These projections are interesting in their own right and will be used in Section 3. We fix V and a consistent system of idempotents (e_x) in $\mathrm{End}(V)$.

Definition 2.7. Let Σ be a subcomplex of the building. A *support projection* for Σ is an idempotent element $u_\Sigma \in \mathrm{End}(V)$ with

$$\mathrm{im}(u_\Sigma) = \sum_{x \in \Sigma^\circ} \mathrm{im}(e_x), \qquad \ker(u_\Sigma) = \bigcap_{x \in \Sigma^\circ} \ker(e_x).$$

Since $\mathrm{im}(p) \oplus \ker(p) = V$ for any idempotent endomorphism p of V, a support projection exists if and only if

$$V = \sum_{x \in \Sigma^\circ} \mathrm{im}(e_x) \oplus \bigcap_{x \in \Sigma^\circ} \ker(e_x),$$

and it is unique if it exists. It is clear that $u_\Sigma \le u_{\Sigma'}$ for $\Sigma \subseteq \Sigma'$.

It is not clear whether a support projection exists for general Σ. If, say, p and q are two rank 1 idempotent 2×2-matrices with $\ker p = \ker q$ but $\mathrm{im}\, p \neq \mathrm{im}\, q$, then there is no idempotent 2×2-matrix with kernel $\ker p \bigcap \ker q$ and image $\mathrm{im}\, p + \mathrm{im}\, q$. For a set $\{p_i\}_{i \in I}$ of self-adjoint projections on Hilbert space, there is a unique self-adjoint projection with kernel $\bigcap \ker(p_i)$; but its image is the *closure* of $\sum \mathrm{im}(p_i)$, and there is no simple formula that expresses it using the given projections p_i.

We are going to show that the support projection of a finite convex subcomplex exists and is given by a straightforward formula. Our method applies to more general subcomplexes of the building. To understand the necessary and sufficient condition for this, we first need two geometric lemmas about hulls.

Lemma 2.8. *Let σ and x be a polysimplex and a vertex in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$. Then there is a unique minimal face τ of σ with $\sigma \in \mathcal{H}(x, \tau)$. That is, a face ω of σ satisfies $\sigma \in \mathcal{H}(x, \omega)$ if and only if $\omega \succ \tau$.*

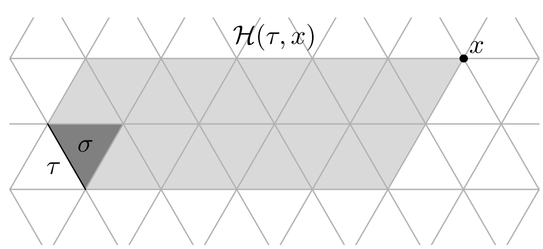

FIGURE 2. Illustration of Lemma 2.8 and the first half of Lemma 2.9

Proof. Fix an apartment A containing σ and x. If the underlying affine root system is reducible, then we split $A = \prod_{i=1}^n A_i$ and $\sigma = \prod_{i=1}^n \sigma_i$. If $\tau_i \subseteq \sigma_i$ solves the problem in A_i, then the polysimplex $\tau := \prod \tau_i$ solves it in A. Hence we may assume that A is irreducible.

Let Δ be a chamber containing σ and let $a_0, \dots, a_d$ be the corresponding simple affine roots with $\Delta = \bigcap_{j=0}^d a_j^{\geq 0}$. If there is j with $a_j|_\sigma = 0$ and $a_j(x) < 0$, then we reflect Δ at the corresponding wall. The new chamber has fewer j with $a_j|_\sigma = 0$ and $a_j(x) > 0$. After finitely many steps, we achieve that $a_j(x) \geq 0$ for all j with $a_j|_\sigma = 0$. Faces of Δ correspond to subsets I of $\{0, \dots, d\}$ via $I \mapsto \Delta \cap \bigcap_{j \in I} \ker(a_j)$. This yields a face of σ if $j \in I$ for all j with $a_j|_\sigma = 0$. Let I be the subset of all j with $a_j(x) > 0$ or $a_j|_\sigma = 0$. We claim that the corresponding face τ of σ satisfies $\sigma \in \mathcal{H}(\tau, x)$ and is minimal with this property.

Let $\omega \prec \sigma$ satisfy $a_j|_\omega = 0$ for some $j \notin I$. Then $a_j(x) \leq 0$ and hence $a_j|_{\mathcal{H}(x,\omega)} \leq 0$, so that $\sigma \notin \mathcal{H}(x, \omega)$. Therefore, if $\sigma \in \mathcal{H}(x, \omega)$ then $\tau \prec \omega$. Conversely, we claim that $\sigma \in \mathcal{H}(x, \tau)$. Let $\tilde{a}$ be any affine root with $\tilde{a}(x) \geq 0$ and $\tilde{a}|_\tau \geq 0$. We must show $\tilde{a}|_\sigma \geq 0$. This is clear if $\tilde{a}|_\tau > 0$ or $\tilde{a}|_\sigma = 0$, so that we may assume that $\tilde{a}$ vanishes on τ but not on σ. We have $\tilde{a} = \sum_{j=0}^d \lambda_j a_j$ with coefficients λ_j of the same sign. Since $\tilde{a}|_\tau = 0$, we have $\lambda_j = 0$ for $j \notin I$. Since $\tilde{a}|_\sigma \neq 0$, some λ_j with $a_j|_\sigma \neq 0$ is non-zero. Since $a_j(x) > 0$ for this j and $a_k(x) \geq 0$ all $k \in I$, we have $\lambda_j \geq 0$ for $j \in I$. Hence $\tilde{a}|_\sigma \geq 0$. □

Lemma 2.9. *Let τ and x be a polysimplex and a vertex in $\mathcal{BT}(\mathcal{G}_\mathbb{K})$. Then there is a unique maximal polysimplex $\sigma \in \mathcal{H}(x, \tau)$ with $\tau \prec \sigma$ (see also Figure 2). That is, a polysimplex ω satisfies $\omega \in \mathcal{H}(x, \tau)$ and $\tau \prec \omega$ if and only if $\tau \prec \omega \prec \sigma$.*

Moreover, if $\sigma = \tau$ then there is a proper face ω of τ with $\tau \in \mathcal{H}(x, \omega)$.

Proof. Let ω_1 and ω_2 be two polysimplices contained in $\mathcal{H}(x, \tau)$ and containing τ. We must show that they are adjacent, that is, they are both faces of a polysimplex ω. If not, then they are separated by an affine root a, say, $a|_{\omega_1} > 0$ and $a|_{\omega_2} < 0$. This implies $a|_\tau = 0$. If $a(x) \geq 0$, then a separates x and τ from ω_2, contradicting $\omega_2 \in \mathcal{H}(x, \tau)$. If $a(x) < 0$, then a separates x and τ from ω_1, contradicting $\omega_1 \in \mathcal{H}(x, \tau)$. Hence ω_1 and ω_2 are adjacent. Of course, $[\omega_1, \omega_2]$ is still contained in $\mathcal{H}(x, \tau)$.

Now assume $\sigma = \tau$. Let $\varphi\colon [0, 1] \to \Sigma$ be the geodesic between x and an interior point of τ. The points $\varphi(t)$ for $t \approx 1$ belong to a polysimplex σ' with $\sigma' \in \mathcal{H}(x, \tau)$ and $\tau \prec \sigma'$. Since $\sigma = \tau$, we have $\varphi(t) \in |\tau|$ for $t \approx 1$. Then we may prolong φ to a geodesic beyond $\varphi(1)$ until it hits a proper face ω of τ. Since $\mathcal{H}(\omega, x)$ contains $\varphi(1)$, an interior point of τ, we get $\tau \in \mathcal{H}(\omega, x)$ for some $\omega \prec \tau$ with $\omega \neq \tau$. □

Our criterion for support projections requires an analogue of Lemma 2.9 for the subcomplex Σ instead of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$:

Definition 2.10. A subcomplex $\Sigma \subseteq \mathcal{BT}(\mathcal{G}_\mathbb{K})$ is called *admissible* if it has the following two properties:

- For any polysimplex $\tau \in \mathcal{BT}(\mathcal{G}_\mathbb{K})$, $\Sigma \cap \tau$ is again a polysimplex or empty.
- Let $x \in \Sigma^\circ$ and $\tau \in \Sigma$. If $\tau \neq x$ and τ has no proper face ω with $\tau \in \mathcal{H}(x, \omega)$, then τ is a proper face of a polysimplex in $\Sigma \cap \mathcal{H}(x, \tau)$.

The first condition is equivalent to the following requirement: if $x_1, \dots, x_n$ are adjacent vertices in Σ, then Σ contains the polysimplex $[x_1, \dots, x_n]$ that they span. Thus admissible subcomplexes are determined by the vertices they contain.

Figures 3 and 4 illustrate admissible subcomplexes. The first figure shows an example of an admissible subcomplex and two examples of non-admissible subcomplexes that violate the first and second condition in Definition 2.10, respectively. Figure 4 illustrates the second condition. If an admissible subcomplex of an $\tilde{A}_2$-apartment contains a but not b and c, then it may not contain any points in the first forbidden region. If it contains x and y but not z, then it may not contain any points in the second forbidden region.

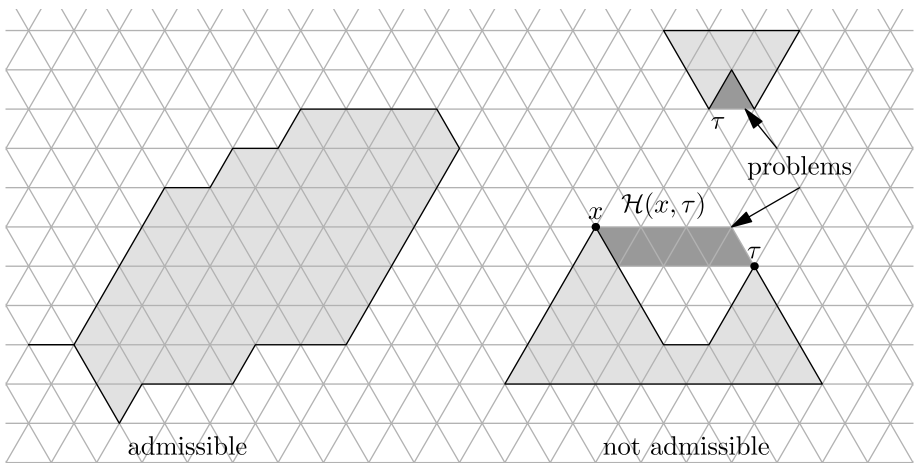

Figure 3. Admissible and non-admissible subcomplexes of an $\tilde{A}_2$-apartment

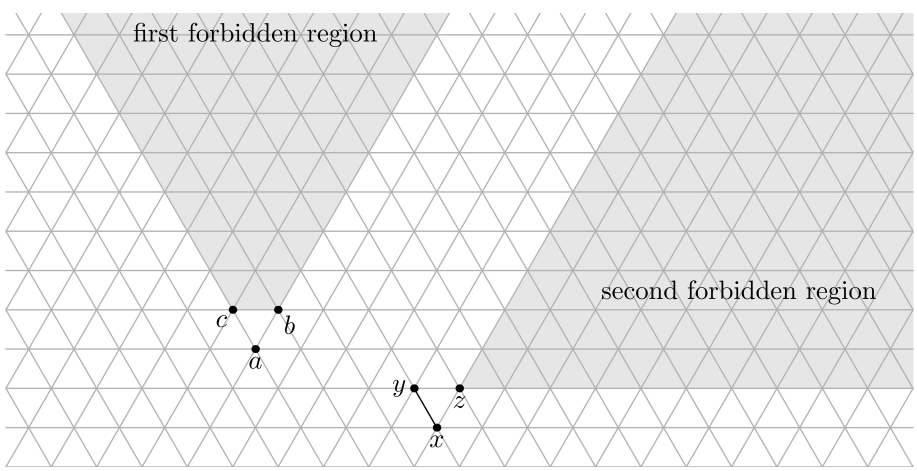

Figure 4. Forbidden regions for admissible subcomplexes of an apartment

Lemma 2.11. *Convex subcomplexes are admissible. If Σ_1 is convex and Σ_2 is admissible, then $\Sigma_1 \cap \Sigma_2$ is admissible.*

Proof. Since an intersection of two polysimplices is again a polysimplex or empty, the first property is hereditary for intersections. Moreover, it is trivial for convex subcomplexes. The second property is inherited by intersections with convex subcomplexes because all $\omega \in \Sigma_2 \cap \mathcal{H}(x,\tau)$ belong to Σ_1 if $x \in \Sigma_1^\circ$ and $\tau \in \Sigma_1$.

We check that a convex subcomplex Σ satisfies the second condition for admissible subcomplexes. Let $x \in \Sigma^\circ$ and $\tau \in \Sigma$. Let $\varphi\colon [0,1] \to \Sigma$ be the geodesic between x and an interior point of τ. If $\varphi(1-\varepsilon) \in \tau$ for sufficiently small $\varepsilon > 0$, we may prolong φ to a geodesic beyond $\varphi(1)$ until it hits a face ω of τ. Then $\mathcal{H}(\omega, x)$ contains an interior point of τ, so that $\tau \in \mathcal{H}(\omega, x)$ for some $\omega \prec \tau$ with $\omega \neq \tau$. If $\varphi(1-\varepsilon) \notin \tau$ for all $\varepsilon > 0$, let ω be the minimal polysimplex containing $\varphi(1-\varepsilon)$ for all sufficiently small $\varepsilon > 0$. Then τ is a proper face of ω, and $\omega \in \mathcal{H}(\tau, x)$ because an interior point of ω lies on a geodesic between an interior point of τ and x. □

Theorem 2.12. *Let (e_x) be a consistent system of idempotents in* $\operatorname{End}(V)$ *and let Σ be a finite convex subcomplex of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$ or, more generally, a finite admissible subcomplex of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$. Then*

$$u_\Sigma := \sum_{\sigma\in\Sigma} (-1)^{\dim(\sigma)} e_\sigma$$

is the support projection for Σ.

It is remarkable that this simple formula for u_Σ works although the idempotents e_σ do not commute.

Proof. Define u_Σ by the above formula. Since $e_\sigma \geq e_\tau$ for $\sigma \prec \tau$, we clearly have

$$\operatorname{im}(u_\Sigma) \subseteq \sum_{\sigma\in\Sigma} \operatorname{im}(e_\sigma) = \sum_{x\in\Sigma^\circ} \operatorname{im}(e_x), \qquad \ker(u_\Sigma) \supseteq \bigcap_{\sigma\in\Sigma} \ker(e_\sigma) = \bigcap_{x\in\Sigma^\circ} \ker(e_x).$$

We will prove $e_x u_\Sigma = e_x = u_\Sigma e_x$ for all $x \in \Sigma^\circ$. This implies $e_\sigma u_\Sigma = e_\sigma = u_\Sigma e_\sigma$ for all $\sigma \in \Sigma$ using the definition of e_σ in Proposition 2.2, and then

$$u_\Sigma \cdot u_\Sigma = \sum_{\sigma\in\Sigma} (-1)^{|\sigma|} e_\sigma u_\Sigma = \sum_{\sigma\in\Sigma} (-1)^{|\sigma|} e_\sigma = u_\Sigma.$$

Furthermore, it follows that

$$\operatorname{im}(u_\Sigma) \supseteq \sum_{x\in\Sigma^\circ} \operatorname{im}(e_x), \qquad \ker(u_\Sigma) \subseteq \bigcap_{x\in\Sigma^\circ} \ker(e_x),$$

so that u_Σ is the support projection of Σ. Thus it remains to establish $e_x u_\Sigma = e_x = u_\Sigma e_x$ for all $x \in \Sigma^\circ$. We only write down the proof of $e_x u_\Sigma = e_x$; the other equation is obtained by working in the opposite category.

Let $m(\sigma)$ for a polysimplex σ be the minimal face τ of σ with $\sigma \in \mathcal{H}(x,\tau)$. This map is idempotent, that is, $m(\sigma)$ has the property that $m(\sigma) \notin \mathcal{H}(x,\tau)$ for any proper face τ of $m(\sigma)$ because otherwise $\sigma \in \mathcal{H}\big(x, m(\sigma)\big) = \mathcal{H}(x,\tau)$. Let $M \subseteq \Sigma$ be the set of all polysimplices of the form $m(\sigma)$. The consistency conditions in Proposition 2.2 imply $e_x e_\sigma = e_x e_\sigma e_{m(\sigma)} = e_x e_{m(\sigma)}$. Hence we may rewrite

$$e_x u_\Sigma = \sum_{\tau\in M} \sum_{\substack{\sigma\in\Sigma \\ m(\sigma)=\tau}} (-1)^{\dim(\sigma)} e_x e_\tau. \tag{4}$$

For each $\tau \in M$, Lemma 2.9 and the first admissibility assumption on Σ yield $\omega \in \Sigma$ such that the set of $\sigma \in \Sigma$ with $m(\sigma) = \tau$ is exactly the set of all $\sigma \in \mathcal{BT}(\mathcal{G}_\mathbb{K})$ with $\tau \prec \sigma \prec \omega$: first construct such a maximal ω in $\mathcal{BT}(\mathcal{G}_\mathbb{K})$, then its intersection with Σ works. The second admissibility assumption about Σ yields $\omega \neq \tau$ or $\tau = x$ because $\tau \in M$. The alternating sum of the dimensions of all polysimplices σ with

$\tau \prec \sigma \prec \omega$ vanishes for $\tau \neq \omega$ and is 1 if $\tau = \omega$. For simplicial complexes, this is because such intermediate faces correspond bijectively to subsets of $\omega^\circ \setminus \tau^\circ$. For polysimplicial complexes, we use the product decomposition to reduce the assertion to the simplicial case. Hence the summand for $\tau \in M$ vanishes unless $\tau = \omega$, that is, $\tau = x$. Thus $e_x u_\Sigma = e_x$. □

This Theorem implies several properties of support projections and hence of the subspaces $\sum_{x\in\Sigma^\circ} \operatorname{im}(e_x)$ and $\bigcap_{x\in\Sigma^\circ} \ker(e_x)$.

Corollary 2.13. *Let Σ_+ and Σ_- be two finite subcomplexes and let $\Sigma_0 := \Sigma_+ \cap \Sigma_-$ and $\Sigma = \Sigma_+ \cup \Sigma_-$. Assume that all four subcomplexes Σ_+, Σ_-, Σ_0 and Σ are admissible. Then $u_\Sigma = u_{\Sigma_+} + u_{\Sigma_-} - u_{\Sigma_0}$, $u_{\Sigma_+} u_{\Sigma_-} = u_{\Sigma_-} u_{\Sigma_+} = u_{\Sigma_0}$, and*

$$\sum_{x\in\Sigma_+^\circ} \operatorname{im}(e_x) \cap \sum_{x\in\Sigma_-^\circ} \operatorname{im}(e_x) = \sum_{x\in\Sigma_0^\circ} \operatorname{im}(e_x),$$

$$\bigcap_{x\in\Sigma_+^\circ} \ker(e_x) + \bigcap_{x\in\Sigma_-^\circ} \ker(e_x) = \bigcap_{x\in\Sigma_0^\circ} \ker(e_x).$$

Proof. The formula for u_Σ follows immediately from Theorem 2.12. Since u_Σ, u_{Σ_+}, and $u_{\Sigma_-} - u_{\Sigma_0}$ are idempotent, it follows that u_{Σ_+} and $u_{\Sigma_-} - u_{\Sigma_0}$ are orthogonal idempotents, so that $u_{\Sigma_+} u_{\Sigma_-} = u_{\Sigma_-} u_{\Sigma_+} = u_{\Sigma_0}$. The assertions about subspaces are special cases of assertions about commuting idempotent operators. □

Let Σ_+, Σ_0 and Σ_- be finite admissible subcomplexes of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$. We say that Σ_0 *separates* Σ_+ and Σ_- if there are finite admissible subcomplexes Σ'_+ and Σ'_- with $\Sigma_\pm \subseteq \Sigma'_\pm$, $\Sigma_0 = \Sigma'_+ \cap \Sigma'_-$, and $\Sigma'_+ \cup \Sigma'_-$ admissible.

Corollary 2.14. *If Σ_0 separates Σ_+ and Σ_-, then $u_{\Sigma_+} u_{\Sigma_-} = u_{\Sigma_+} u_{\Sigma_0} u_{\Sigma_-}$.*

Proof. We have $u_{\Sigma_\pm} \le u_{\Sigma'_\pm}$. By the previous corollary, $u_{\Sigma'_+} u_{\Sigma'_-} = u_{\Sigma_0}$. Hence $u_{\Sigma_+} u_{\Sigma_-} = u_{\Sigma_+} u_{\Sigma'_+} u_{\Sigma'_-} u_{\Sigma_-} = u_{\Sigma_+} u_{\Sigma_0} u_{\Sigma_-}$. □

In particular, this applies to $e_x = u_x$ for a vertex x of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$, and we get $e_x u_\Sigma e_y = e_x e_y$ if Σ separates x and y.

2.4. **Proof of Proposition 2.2.** This section establishes that consistency conditions for the idempotents $(e_x)_{x\in\mathcal{BT}(\mathcal{G}_\mathbb{K})^\circ}$ for vertices imply consistency conditions for the idempotents $(e_\sigma)_{\sigma\in\mathcal{BT}(\mathcal{G}_\mathbb{K})}$ for all polysimplices. Most of the argument deals with the geometry of the building: we need chains of adjacent vertices or polysimplices in hulls of polysimplices.

Condition (a) in Definition 2.1 implies that the order in the product

$$e_\sigma := \prod_{\substack{x\in\mathcal{BT}(\mathcal{G}_\mathbb{K})^\circ \\ x\prec\sigma}} e_x$$

defining e_σ does not matter. Hence e_σ is a well-defined idempotent endomorphism of V. The same argument yields $e_\sigma e_\tau = e_{[\sigma,\tau]}$ for adjacent polysimplices τ and σ. Condition (f) follows immediately from (c). We will spend the remainder of this section to check that (a) and (b) imply (e). We begin with two geometric lemmas.

Lemma 2.15. *Let τ, σ, ω be polysimplices in the building with $\omega \in \mathcal{H}(\sigma, \tau)$. There is a finite sequence of polysimplices $\tau_0 = \tau$, τ_1, ..., τ_{m-1}, $\tau_m = \omega$ such that $\tau_i \in \mathcal{H}(\omega, \tau_{i-1})$, $\omega \in \mathcal{H}(\sigma, \tau_i)$, and either $\tau_{i-1} \prec \tau_i$ or $\tau_{i-1} \succ \tau_i$ for $i = 1, \dots, m$ (see Figure 5).*

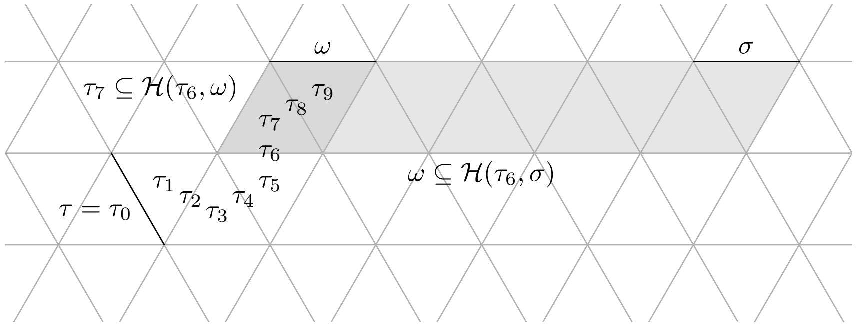

FIGURE 5. Illustration of Lemma 2.15

Proof. Let $\varphi\colon [0,1] \to \mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ be a geodesic between interior points of τ and ω. Each $\varphi(t)$ is an interior point of some polysimplex $\tau(t)$. The function $t \mapsto \tau(t)$ is piecewise constant. Let $0 = t_0 < t_2 < t_4 < \cdots < t_{2n-2} < t_{2n} = 1$ be the points where $\tau(t)$ jumps and choose $t_1, \dots, t_{2n-1}$ with $t_0 < t_1 < t_2 < \cdots < t_{2n-1} < t_{2n}$. Let $\tau_i = \tau(t_i)$, so that $\tau_0 = \tau$ and $\tau_{2n} = \omega$. Then τ_{2j} and τ_{2j+2} must be faces of τ_{2j+1} for $j = 0, \dots, n-1$, so that either $\tau_i \prec \tau_{i+1}$ or $\tau_i \succ \tau_{i+1}$ for $i = 0, \dots, 2n$. Since some interior point of τ_i lies on a geodesic between interior points of τ_{i-1} and ω, we have $\tau_i \in \mathcal{H}(\omega, \tau_{i-1})$.

It remains to check $\omega \in \mathcal{H}(\sigma, \tau_i)$. Let A be an apartment containing τ and σ. Then A also contains ω because $\omega \in \mathcal{H}(\sigma, \tau)$. Hence A contains all polysimplices τ_i. If not $\omega \in \mathcal{H}(\sigma, \tau_i)$, then there is an affine root a on A with $a|_{\tau_i} \geq 0$ and $a|_\sigma \geq 0$, but $a|_\omega < 0$. Since $a \circ \varphi(t) = \lambda t + \mu$ for some $\lambda, \mu \in \mathbb{R}$ and $a \circ \varphi$ changes sign between t_i and 1, it cannot change sign between 0 and t_i, so that $a \circ \varphi(0) \geq 0$ as well, that is, $a|_\tau \geq 0$. But then a separates ω from $\tau \cup \sigma$, contradicting $\omega \in \mathcal{H}(\sigma, \tau)$. Hence $\omega \in \mathcal{H}(\sigma, \tau_i)$. □

Lemma 2.16. *Let σ and τ be polysimplices in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ and let y be a vertex adjacent to σ with $y \in \mathcal{H}(\sigma, \tau)$. Then there is a finite sequence of vertices $z_0, \dots, z_m$ with $z_m = y$ and $z_0 \prec \tau$ such that z_i is adjacent to z_{i-1}, $z_i \in \mathcal{H}(y, z_{i-1})$ and $y \in \mathcal{H}(\sigma, z_i)$ for $i = 1, \dots, m$ (see Figure 6). In particular, there is a vertex z of τ with $y \in \mathcal{H}(\sigma, z)$.*

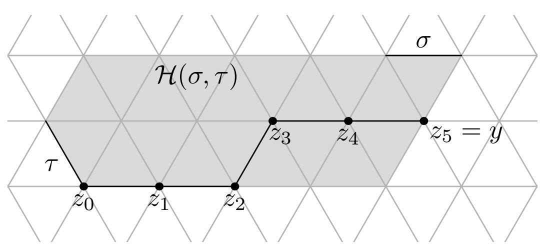

FIGURE 6. Illustration of Lemma 2.16

Proof. Let A be an apartment containing σ and τ. Since $y \in \mathcal{H}(\sigma, \tau)$ implies $y \in A$, we may restrict our attention to A. If the affine root system underlying A is decomposable, then $A = \prod_{i=1}^n A_i$ with apartments of indecomposable affine root systems A_i. Each affine root factors through the projection to A_i for some i. Hence vertices in A are nothing but families of vertices in A_i for all i, and two vertices are adjacent if and only if their A_i-components are adjacent for all i; polysimplices in A are the same as products of simplices in A_i. Moreover, $\mathcal{H}(\sigma,\tau) = \prod_{i=1}^n \mathcal{H}(\sigma_i,\tau_i)$ if $\sigma = \prod_{i=1}^n \sigma_i$ and $\tau = \prod_{i=1}^n \tau_i$. Therefore, if we can solve the problem for σ_i, τ_i and y_i in A_i for each i, we can solve it for σ, τ and y in A. We may assume without loss of generality that the affine root system of A is indecomposable. Then A is a simplicial complex.

An affine root a of the apartment A defines a closed half space

$$a^{\leq} := \{v \in A \mid a(v) \leq 0\}$$

We define $a^{\geq}$ and $a^{>}$ by the same recipe. The hull of σ and τ is the intersection of all $a^{\leq}$ with $\sigma \cup \tau \subseteq a^{\leq}$. Let z be a vertex. If $y \notin \mathcal{H}(\sigma, z)$, then there is an affine root a with $\sigma, z \subseteq a^{\leq}$ and $y \in a^{>}$. Since y is adjacent to σ, $y \in a^{>}$ implies $\sigma \subseteq a^{\geq}$, so that $a|_\sigma = 0$. Hence a vertex z satisfies $y \in \mathcal{H}(\sigma, z)$ if and only if $a(z) > 0$ for all affine roots a with $a|_\sigma = 0$ and $a(y) > 0$. Since $y \in \mathcal{H}(\sigma,\tau)$, the same reasoning shows that for each affine root a with $a|_\sigma = 0$ and $a(y) > 0$, there is a vertex z_a of τ with $a(z_a) > 0$. Our first task is to find $z_0 \prec \tau$ with $y \in \mathcal{H}(\sigma, z_0)$. This is trivial if $y \prec \sigma$, so that we may assume that y does not belong to σ.

Let $d = \dim A$. Since A is simplicial, any chamber is bounded by exactly $d+1$ walls. Let $a_0, \dots, a_d$ be affine roots such that $\gamma := \bigcap_{j=0}^d a_j^{\geq}$ is a chamber that contains $[\sigma, y]$. Order them so that $a_0(y) > 0$ and $a_j(y) = 0$ for $j \neq 0$, and $a_j|_\sigma = 0$ if and only if $j \leq k$, where k is the codimension of σ. We now modify this chamber until $a_j|_\tau \geq 0$ for $j = 1, \dots, k$. If there are j in this range and $z \in \tau$ with $a_j(z) < 0$, then we replace γ by $s_{a_j}(\gamma)$, where s_{a_j} denotes the reflection at the wall $\ker a_j$. This yields another chamber containing $[\sigma, y]$ because a_j vanishes on $[\sigma, y]$. Each reflection reduces the number of j between 1 and k with $a_j|_\tau \ngeq 0$ at least by 1. Finitely many such steps achieve $a_j|_\tau \geq 0$ for $j = 1, \dots, k$. Since $a_0|_\sigma = 0$, $a_0(y) > 0$, and $y \in \mathcal{H}(\sigma,\tau)$, there is a vertex $z_0 \prec \tau$ with $a_0(z_0) > 0$. We claim that $y \in \mathcal{H}(\sigma, z_0)$.

Since the roots $a_0, \dots, a_d$ bound a chamber, any affine root $\tilde{a}$ is of the form $\tilde{a} = \sum_{j=0}^d \lambda_j a_j$ with either $\lambda_j \geq 0$ for all j or $\lambda_j \leq 0$ for all j. Since $a_j|_\sigma \geq 0$ for all j, we have $\tilde{a}|_\sigma = 0$ if and only if $\lambda_j = 0$ for $j > k$. Furthermore, $\tilde{a}(y) > 0$ if and only if $\lambda_0 > 0$, forcing $\lambda_j \geq 0$ for all j. Since $a_j|_\tau \geq 0$ for $1 \leq j \leq k$ by construction, we get $\tilde{a}(z) \geq \lambda_0 a_0(z) > 0$. Therefore, $y \in \mathcal{H}(\sigma, z_0)$.

Furthermore, if $z \in \mathcal{H}(y, z_0)$, then $a_0(z) > 0$ because $a_0(y) > 0$ and $a_0(z_0) > 0$, and $a_j(z) \geq 0$ for $j = 1, \dots, k$ because $a_j(y) \geq 0$ and $a_j(z_0) \geq 0$. Since $z_i \in \mathcal{H}(y, z_{i-1})$ implies $z_i \in \mathcal{H}(y, z_0)$, we conclude that the property $y \in \mathcal{H}(\sigma, z_i)$ follows from the others. This remains so in the case $y \prec \sigma$ excluded above.

Thus it remains to find a finite sequence of vertices $z_1, \dots, z_m = y$ such that z_{i-1} and z_i are adjacent and $z_i \in \mathcal{H}(y, z_{i-1})$ for $i = 1, \dots, m$. To construct z_i given $z_{i-1} \neq y$, we consider the geodesic φ between z_{i-1} and y. Let ω be the simplex such that $\varphi(t)$ is an interior point of ω for $t \in (0, \varepsilon)$ for some $\varepsilon > 0$. Then $\omega \in \mathcal{H}(y, z_{i-1})$, so that we may let z_i be another vertex of ω. Since the passage from z_{i-1} to z_i decreases the (finite) number of walls that separate z_i from y, this construction will lead to $z_m = y$ after finitely many steps. □

Remark 2.17. The adjacency assumption in Lemma 2.16 is necessary. In buildings, say, of type $\tilde{A}_3$, it can happen that there is no vertex $z_0 \prec \tau$ with $y \in \mathcal{H}(\sigma, z_0)$

although $y \in \mathcal{H}(\sigma, \tau)$. This is why we only get a sequence of polysimplices in Lemma 2.15. This phenomenon cannot occur in 2-dimensional buildings.

The counterexample involves simplices in a single apartment of type $\tilde{A}_3$. Let $V := \mathbb{R}^4/\mathbb{R} \cdot (1,1,1,1)$. The roots on this apartment are the affine maps

$$a_{ijk}(x_1, x_2, x_3, x_4) := x_i - x_j - k \qquad \text{for } 1 \le i \ne j \le 4,\ k \in \mathbb{Z}$$

on V. Let

$$x := (0,0,0,0), \qquad y := (4,2,2,0), \qquad z_1 := (4,2,1,0), \qquad z_2 := (4,3,2,0).$$

The points z_1 and z_2 are adjacent, that is, there is no affine root a_{ijk} with $a_{ijk}(z_1) < 0 < a_{ijk}(z_2)$. The point y belongs to the hull $\mathcal{H}(x, [z_1, z_2])$, but neither to $\mathcal{H}(x, z_1)$ nor to $\mathcal{H}(x, z_2)$. To check this, we compute the hulls. Let

$$V_+ := \{(x_1, x_2, x_3, x_4) \in V \mid x_0 \ge x_1 \ge x_2 \ge x_3\}.$$

Since $x, y, z_1, z_2 \in V_+$, all hulls are contained in V_+.

$$\begin{aligned}
\mathcal{H}(x, z_1) &= \{(x_1, x_2, x_3, x_4) \in V_+ \mid x_0 \le x_1 + 2 \le x_2 + 3 \le x_3 + 4\},\\
\mathcal{H}(x, z_2) &= \{(x_1, x_2, x_3, x_4) \in V_+ \mid x_0 \le x_1 + 1 \le x_2 + 2 \le x_3 + 4\},\\
\mathcal{H}(x, [z_1, z_2]) &= \{(x_1, x_2, x_3, x_4) \in V_+ \mid x_0 \le x_1 + 2 \le x_2 + 3 \le x_3 + 5\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and } x_0 \le x_3 + 4\}.
\end{aligned}$$

After these geometric preparations, we can now reduce (e) to (b) and (d) in four steps. First, the second statement in Lemma 2.16 implies $e_x e_y e_\tau = e_x e_\tau$ if x and y are adjacent vertices with $y \in \mathcal{H}(x, \tau)$: let z be a vertex of τ with $y \in \mathcal{H}(x, z)$; then (d) yields $e_\tau = e_z e_\tau$ and (b) yields

$$e_x e_\tau = e_x e_z e_\tau = e_x e_y e_z e_\tau = e_x e_y e_\tau.$$

Secondly, we claim that $e_\tau e_\sigma = e_\tau e_y e_\sigma$ if $y \in \mathcal{H}(\sigma, \tau)$ and y is adjacent to σ. Here we use the sequence of adjacent points (z_i) from Lemma 2.16. The first step yields $e_{z_{i-1}} e_{z_i} e_\sigma = e_{z_{i-1}} e_\sigma$ and $e_{z_{i-1}} e_{z_i} e_y = e_{z_{i-1}} e_y$ because z_i and z_{i-1} are adjacent vertices with $z_i \in \mathcal{H}(y, z_{i-1})$ and $z_i \in \mathcal{H}(\sigma, z_{i-1})$; here we use that $\mathcal{H}(\sigma, z_{i-1})$ contains $\mathcal{H}(y, z_{i-1})$ because $y \in \mathcal{H}(\sigma, z_{i-1})$. Hence the first step yields

$$\begin{aligned}
e_\tau e_\sigma = e_\tau e_{z_0} e_\sigma = e_\tau e_{z_0} e_{z_1} e_\sigma = \dots = e_\tau e_{z_0} e_{z_1} \cdots e_{z_{m-1}} e_y e_\sigma\\
= e_\tau e_{z_0} e_{z_1} \cdots e_{z_{m-2}} e_y e_\sigma = \dots = e_\tau e_y e_\sigma.
\end{aligned}$$

Thirdly, we claim that $e_\sigma e_\tau = e_\sigma e_\omega e_\tau$ if $\omega \in \mathcal{H}(\sigma, \tau)$ and ω is adjacent to τ. Each vertex y of ω is adjacent to τ, so that e_y commutes with e_τ by (d). Hence the second step yields

$$e_\sigma e_\tau = e_\sigma e_y e_\tau = e_\sigma e_{[y,\tau]} = e_\sigma e_\tau e_y$$

for each vertex y of ω. Repeating this argument, we get

$$e_\sigma e_\tau = e_\sigma e_\tau \cdot \prod_{y \prec \omega} e_y = e_\sigma e_\tau e_\omega = e_\sigma e_{[\omega,\tau]} = e_\sigma e_\omega e_\tau.$$

Finally, we use Lemma 2.15 to reduce the general case of (e) to the third step. Let $\omega \in \mathcal{H}(\tau, \sigma)$ be arbitrary and choose a sequence of polysimplices $\tau_0, \dots, \tau_m$ as in Lemma 2.15. Then the third step yields

$$e_\sigma e_\tau = e_\sigma e_{\tau_1} e_\tau = \dots = e_\sigma e_\omega e_{\tau_{m-1}} \cdots e_{\tau_1} e_\tau = e_\sigma e_\omega e_{\tau_{m-2}} \cdots e_{\tau_1} e_\tau = \dots = e_\sigma e_\omega e_\tau.$$

This finishes the proof of Proposition 2.2.

2.5. **Proof of exactness.** In this section, we prove Theorem 2.4. In fact, the theorem remains valid for the admissible subcomplexes introduced in Definition 2.10. We will prove it in that generality.

We first assume that Σ is finite. Later, we will reduce infinite Σ to this special case. Theorem 2.12 yields the last assertion,

$$V \cong \sum_{x\in\Sigma^\circ} e_x(V) \oplus \bigcap_{x\in\Sigma^\circ} \ker e_x.$$

We still have to prove

$$\mathrm{H}_n(\Sigma,\Gamma) \cong \begin{cases} \sum_{x\in\Sigma^\circ} V_x & \text{for } n=0,\\ 0 & \text{for } n>0. \end{cases} \tag{5}$$

The remaining assertion about cohomology follows by the same argument applied to the opposite category, see Lemma 2.3.

We prove (5) for all admissible finite subcomplexes by a divide and conquer method.

Lemma 2.18. *Let Σ be a finite admissible subcomplex and assume that it can be decomposed as $\Sigma = \Sigma_+ \cup \Sigma_-$ with admissible $\Sigma_\pm$ and $\Sigma_0 = \Sigma_+ \cap \Sigma_-$. If (5) holds for Σ_+, Σ_-, and Σ_0, then it holds for Σ as well.*

Proof. The cellular chain complexes for these subcomplexes form an exact sequence

$$C_*(\Sigma_0) \rightarrowtail C_*(\Sigma_+) \oplus C_*(\Sigma_-) \twoheadrightarrow C_*(\Sigma),$$

which generates a Mayer–Vietoris long exact sequence for their homology groups. This long exact sequence combined with (5) for Σ_0, Σ_+ and Σ_- yields $\mathrm{H}_n(\Sigma) = 0$ for $n \geq 2$ and the injectivity of the map $\mathrm{H}_0(\Sigma_0) \to \mathrm{H}_0(\Sigma_+)$, so that $\mathrm{H}_1(\Sigma) = 0$ as well. Furthermore, we have a short exact sequence

$$\sum_{x\in\Sigma_0^\circ} V_x \rightarrowtail \sum_{x\in\Sigma_+^\circ} V_x \oplus \sum_{x\in\Sigma_-^\circ} V_x \twoheadrightarrow \mathrm{H}_0(\Sigma).$$

Now Corollary 2.13 yields

$$\sum_{x\in\Sigma_+^\circ} V_x \cap \sum_{x\in\Sigma_-^\circ} V_x = \sum_{x\in\Sigma_0^\circ} V_x, \qquad \sum_{x\in\Sigma_+^\circ} V_x + \sum_{x\in\Sigma_-^\circ} V_x = \sum_{x\in\Sigma^\circ} V_x.$$

Hence $\mathrm{H}_0(\Sigma) \cong \sum_{x\in\Sigma^\circ} V_x$, so that Σ verifies (5). □

Next we consider the special case where Σ is a single polysimplex, so that the idempotents e_σ for $\sigma \in \Sigma$ all commute. For each subset $I \subseteq \Sigma^\circ$, let e_I^0 be the product of e_x for $x \in I$ and $1 - e_x$ for $x \notin I$. Since the idempotents e_x commute, this is again an idempotent endomorphism of V, and its action on $C_*(\Sigma)$ commutes with the boundary map. Since $V \cong \bigoplus_{I\subseteq\Sigma^\circ} e_I^0(V)$, the chain complex $C_*(\Sigma)$ is a resolution of $\sum_{x\in\Sigma} V_x$ if and only if $e_I^0 C_*(\Sigma)$ is a resolution of

$$e_I^0\left(\sum_{x\in\Sigma} V_x\right) = \begin{cases} e_I^0(V) & \text{if } I \text{ is non-empty},\\ 0 & \text{if } I \text{ is empty} \end{cases}$$

for each subset $I \subseteq \Sigma^\circ$. This is clear for empty I, so that we may assume $I \neq \emptyset$.

The chain complex $e_I^0 C_*(\Sigma)$ has a very simple structure: the contribution from a polysimplex σ is $e_I^0(V)$ if all vertices of σ belong to I, and 0 otherwise. Hence $e_I^0 C_*(\Sigma)$ computes the homology of the subcomplex Σ_I of Σ spanned by I with *constant* coefficients in $e_I^0(V)$. This homology agrees with $e_I^0(V)$ if Σ_I is contractible. But what if Σ_I is not contractible? Here our consistency conditions enter: we claim that Σ_I is a face of Σ or $e_I^0 = 0$, so that $e_I^0(V) = 0$ and $e_I^0 C_*(\Sigma) = 0$. If $x, y \in I$ and $z \in \mathcal{H}(x, y)$ then $e_x e_z e_y = e_x e_y$ by Condition (e). Since the idempotents involved commute, this means that $e_z \geq e_x e_y$, that is, $1 - e_z$ vanishes on the range of $e_x e_y$. Hence $e_I^0 = 0$ if $x, y \in I$ and $z \notin I$. Thus $e_I^0 \neq 0$ forces I to be convex, that is, a single face of Σ. Thus (5) holds if Σ is a single polysimplex.

If (5) failed for some admissible finite subcomplex Σ, then there would be a minimal such Σ, which we pick. The previous argument shows that Σ cannot be a single polysimplex. Lemma 2.18 shows that we cannot cut Σ into smaller admissible subcomplexes. We are going to show that any finite admissible subcomplex that is not a single polysimplex may be cut as in Lemma 2.18. This will show that no counterexample to (5) can exist.

Since Σ is not a single polysimplex, there exists a chamber Δ in an apartment A, and an affine root a corresponding to a wall of Δ, such that Σ contains both a point $x_+ \in \Delta$ with $a(x_+) > 0$ and an $x \in A$ with $a(x) < 0$. Let $\varrho\colon \mathcal{BT}(\mathcal{G}_\mathbb{K}) \to A$ be the retraction centered at Δ. We claim that

$$\begin{aligned}
\mathcal{BT}(\mathcal{G}_\mathbb{K})_+ &:= \{\sigma \in \mathcal{BT}(\mathcal{G}_\mathbb{K}) \mid a|_{\varrho\sigma} \geq 0\},\\
\mathcal{BT}(\mathcal{G}_\mathbb{K})_- &:= \{\sigma \in \mathcal{BT}(\mathcal{G}_\mathbb{K}) \mid a|_{\varrho\sigma} \leq 0\},\\
\mathcal{BT}(\mathcal{G}_\mathbb{K})_0 &:= \{\sigma \in \mathcal{BT}(\mathcal{G}_\mathbb{K}) \mid a|_{\varrho\sigma} = 0\}
\end{aligned}$$

are convex subcomplexes of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$. Indeed, suppose that Δ_1 and Δ_2 are chambers in $\mathcal{BT}(\mathcal{G}_\mathbb{K})_-$, and consider some gallery between them. If it contains a chamber in $\mathcal{BT}(\mathcal{G}_\mathbb{K})_+$, then it must cross the wall corresponding to a twice, and hence the gallery is not minimal. The geodesic between two points $x_1 \in \Delta_1$ and $x_2 \in \Delta_2$ lies inside the union of all such minimal galleries, and therefore entirely in $\mathcal{BT}(\mathcal{G}_\mathbb{K})_-$. The same reasoning shows that $\mathcal{BT}(\mathcal{G}_\mathbb{K})_+$ is convex, and $\mathcal{BT}(\mathcal{G}_\mathbb{K})_0 = \mathcal{BT}(\mathcal{G}_\mathbb{K})_+ \cap \mathcal{BT}(\mathcal{G}_\mathbb{K})_-$.

Lemma 2.11 yields that $\Sigma_? := \mathcal{BT}(\mathcal{G}_\mathbb{K})_? \cap \Sigma$ for $? \in \{+, 0, -\}$ are admissible subcomplexes of Σ. Hence Lemma 2.18 applies and leads to a contradiction. This finishes the proof of Theorem 2.4 for admissible finite subcomplexes Σ.

It remains to reduce the assertions in Theorem 2.4 for infinite Σ to the finite case. This requires an increasing filtration of $\mathcal{BT}(\mathcal{G}_\mathbb{K})$ by finite convex subcomplexes B_n with $\bigcup B_n = \mathcal{BT}(\mathcal{G}_\mathbb{K})$. For instance, we may let B_n be the fixed point subcomplex of K_n for a decreasing sequence of compact open subgroups K_n in $\mathcal{G}_\mathbb{K}$ with $\bigcap K_n = \{1\}$ (Example 1.8). Then $\Sigma_n := \Sigma \cap B_n$ for $n \in \mathbb{N}$ is an increasing sequence of finite admissible subcomplexes of Σ with $\bigcup \Sigma_n = \Sigma$, and

$$C_*(\Sigma) \cong \varinjlim C_*(\Sigma_n).$$

If we work with modules, then we can now use the exactness of inductive limits to finish the proof in the homological case very quickly. The cohomological case requires more work and is understood best in the setting of general Abelian categories, where the arguments in the homological and cohomological case are equivalent by Lemma 2.3.

The maps $C_*(\Sigma_n) \to C_*(\Sigma_{n+1})$ are split monomorphisms by definition. Hence $C_*(\Sigma)$ is not just a colimit but also a homotopy colimit of the sequence of chain complexes $C_*(\Sigma_n)$. This means that there is an exact sequence of chain complexes

$$0 \to \bigoplus_{n\in\mathbb{N}} C_*(\Sigma_n) \xrightarrow{\mathrm{Id}-S} \bigoplus_{n\in\mathbb{N}} C_*(\Sigma_n) \to C_*(\Sigma) \to 0;$$

here S is the shift that embeds the summand $C_*(\Sigma_n)$ into $C_*(\Sigma_{n+1})$. This exact sequence of chain complexes induces a long exact homology sequence, which we may rewrite as a short exact sequence

$$0 \to \varinjlim \mathrm{H}_*(\Sigma_n) \to \mathrm{H}_*(\Sigma) \to \varinjlim{}^1 \mathrm{H}_{*-1}(\Sigma_n) \to 0.$$

Equation (5) implies that the induced maps $\mathrm{H}_0(\Sigma_n) \to \mathrm{H}_0(\Sigma_{n+1})$ are split monomorphisms for all $n \in \mathbb{N}$, and we have already seen that the homology vanishes in other degrees. Finally, we use that the derived inductive limit functor vanishes for inductive systems of *split* monomorphisms $\alpha_n \colon X_n \to X_{n+1}$ because such an inductive limit is equivalent to the coproduct of X_{n+1}/X_n and coproducts in $\mathcal{C}$ are assumed to be exact. Hence our exact sequence shows that the map $\varinjlim \mathrm{H}_*(\Sigma_n) \to \mathrm{H}_*(\Sigma)$ is an isomorphism. The arguments in the cohomological case are dual.

3. Serre subcategories of smooth representations

Let R be a ring with $1/p \in R$. For instance, R may be $\mathbb{Z}[1/p]$ or a field of characteristic not equal to p. We define a Hecke algebra $\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$ with coefficients in R as in Section 1.1.1. Let $(e_x)_{x\in\mathcal{BT}(\mathcal{G}_\mathbb{K})^\circ}$ be an equivariant and consistent system of idempotents in $\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$, that is, the conditions in Definition 2.1 hold in $\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$.

Let $\mathcal{C}$ be an R-linear category with *exact* countable inductive limits. The main example is the category of R-modules. The opposite category of R-modules does not work because its inductive limits correspond to projective limits of modules, which are not exact.

An $\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$*-module in* $\mathcal{C}$ is an object of $\mathcal{C}$ equipped with a ring homomorphism $\mathcal{H}(\mathcal{G}_\mathbb{K}, R) \to \mathrm{End}(V)$. We let $\mathfrak{Rep}$ be the category of $\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$-modules in $\mathcal{C}$. We define *smooth* $\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$-modules in $\mathcal{C}$ exactly as in Definition 1.2.

If V is an $\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$-module in $\mathcal{C}$, then the idempotents e_x in $\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$ are represented by an equivariant consistent system of idempotents in $\mathrm{End}(V)$, which we still denote by (e_x). This construction is natural in the formal sense, so that the resulting cosheaf $\Gamma(V)$ and its cellular chain complex depend functorially on V.

The exactness of inductive limits in $\mathcal{C}$ means that inductive limits of monomorphisms in $\mathcal{C}$ are again monomorphisms. In particular, since the natural maps $\sum_{x\in\Sigma_\mathrm{f}^\circ} e_x(V) \to V$ for finite convex subcomplexes $\Sigma_\mathrm{f} \subseteq \Sigma$ are monomorphisms by definition, so is the induced map $\varinjlim_{\Sigma_\mathrm{f}} \sum_{x\in\Sigma_\mathrm{f}^\circ} e_x(V) \to V$. Its image is

$$\sum_{x\in\Sigma^\circ} e_x(V) := \mathrm{im}\biggl(\bigoplus_{x\in\Sigma^\circ} e_x(V) \to V\biggr).$$

This is the supremum of $\{e_x(V) \mid x \in \Sigma^\circ\}$ in the directed set of subobjects of V. Hence Theorem 2.4 yields

$$\mathrm{H}_0\bigl(\Sigma, \Gamma(V)\bigr) \cong \sum_{x\in\Sigma^\circ} e_x(V) \subseteq V.$$

Theorem 3.1. *Let R be a ring with $1/p \in R$ and let $(e_x)_{x\in\mathcal{BT}(\mathcal{G}_\mathbb{K})^\circ}$ be an equivariant consistent system of idempotents in $\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$. Let $\mathcal{C}$ be an R-linear category with exact countable inductive limits.*

The class $\mathfrak{Rep}(e_x)$ of all $\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)$-modules V in $\mathcal{C}$ with

$$V = \sum_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V)$$

is a Serre subcategory, that is, it is hereditary for extensions, quotients and subobjects (and closed under isomorphism, anyway). Furthermore, this class is closed under coproducts and hence under arbitrary colimits, and all $V \in \mathfrak{Rep}(e_x)$ are smooth.

Proof. We abbreviate $\mathcal{S} := \mathfrak{Rep}(e_x)$. We have $V \in \mathcal{S}$ if and only if the augmentation map $\alpha_V \colon \bigoplus_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V) \to V$ is an epimorphism. If $V_1 \twoheadrightarrow V_2$ is an epimorphism, then so is the induced map $\bigoplus_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V_1) \twoheadrightarrow \bigoplus_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V_2)$. Hence α_{V_2} is an epimorphism if α_{V_1} is one. Thus quotients of objects in $\mathcal{S}$ remain in $\mathcal{S}$. Similarly, coproducts of objects in $\mathcal{S}$ remain in $\mathcal{S}$. Since colimits are quotients of coproducts, this implies that $\mathcal{S}$ is closed under arbitrary colimits.

Let $V_1 \rightarrowtail V_2 \twoheadrightarrow V_3$ be an extension of $\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)$-modules in $\mathcal{C}$. Then we get an extension

$$\bigoplus_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V_1) \rightarrowtail \bigoplus_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V_2) \twoheadrightarrow \bigoplus_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V_3)$$

as well. The Snake Lemma shows that α_{V_2} is an epimorphism if α_{V_1} and α_{V_3} are. Thus $\mathcal{S}$ is closed under extensions.

For any $x \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ$, there is a compact open subgroup K_x such that e_x is K_x-biinvariant. Given a finite subcomplex Σ, we let $K_\Sigma := \bigcap_{x\in\Sigma^\circ} K_x$. Then K_Σ acts trivially on $\sum_{x\in\Sigma^\circ} e_x(V)$. Since $\sum_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V)$ is the inductive limit of such subspaces, any $\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)$-module in $\mathcal{S}$ is smooth.

Finally, it remains to show that subobjects of objects in $\mathcal{S}$ are again in $\mathcal{S}$. Let $V_1 \rightarrowtail V_2 \twoheadrightarrow V_3$ be an extension in $\mathcal{S}$. The augmented cellular chain complexes

$$C_j := \bigl(C_*\bigl(\mathcal{BT}(\mathcal{G}_{\mathbb{K}}), \Gamma(V_j)\bigr) \to V_j\bigr)$$

for $j = 1,2,3$ form an extension of chain complexes $C_1 \rightarrowtail C_2 \twoheadrightarrow C_3$ as well because taking the range of an idempotent in $\mathcal{H}$ is an exact functor on $\mathfrak{Rep}$. Theorem 2.4 yields that $V_j \in \mathcal{S}$ if and only if C_j is exact. Now the long exact homology sequence shows that all three of C_1, C_2 and C_3 are exact once two of them are. If $V_2 \in \mathcal{S}$, then $V_3 \in \mathcal{S}$ because $\mathcal{S}$ is hereditary for quotients; the two-out-of-three property yields $V_1 \in \mathcal{S}$ as well, that is, $\mathcal{S}$ is closed under subobjects. □

Let V be an $\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)$-module. Then $\sum_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V) \subseteq V$ is an $\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)$-module as well because it is the image of a morphism between $\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)$-modules. Thus we may define a functor

$$\Phi\colon \mathfrak{Rep} \to \mathfrak{Rep}, \qquad V \mapsto \sum_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V),$$

which comes with a natural transformation $\Phi(V) \to V$.

Proposition 3.2. *The functor Φ is a retraction from $\mathfrak{Rep}$ onto the full subcategory $\mathfrak{Rep}(e_x)$, that is, $\Phi(V) \in \mathfrak{Rep}(e_x)$ for all V and the natural map $\Phi(V) \to V$ is an isomorphism for $V \in \mathfrak{Rep}(e_x)$. The functor Φ is right adjoint to the embedding functor $\mathfrak{Rep}(e_x) \to \mathfrak{Rep}$, that is, the natural map $\Phi(W) \to W$ induces an isomorphism* $\operatorname{Hom}(V,W) \cong \operatorname{Hom}\bigl(V,\Phi(W)\bigr)$ *for all $V \in \mathfrak{Rep}(e_x)$, $W \in \mathfrak{Rep}$.*

Proof. We have $e_x\big(\Phi(V)\big) = e_x(V)$ because $\Phi(V) \subseteq V$ and $e_x(V) \subseteq \Phi(V)$. Hence $\Phi\big(\Phi(V)\big) \cong \Phi(V)$. By definition, $\Phi(V) \cong V$ if and only if $V \in \mathfrak{Rep}(e_x)$. Thus Φ is a retraction from $\mathfrak{Rep}$ onto $\mathfrak{Rep}(e_x)$. An $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$-module homomorphism $V \to W$ between $V, W \in \mathfrak{Rep}$ restricts to a map $\Phi(V) \to \Phi(W)$ because Φ is a functor. If $V \in \mathfrak{Rep}(e_x)$, that is, $V \cong \Phi(V)$, then this means that any $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$-module homomorphism $V \to W$ factors through the embedding $\Phi(W) \to W$, necessarily uniquely. Thus $\operatorname{Hom}(V, W) \cong \operatorname{Hom}\big(V, \Phi(W)\big)$ for all $V \in \mathfrak{Rep}(e_x)$, $W \in \mathfrak{Rep}$. □

We may reformulate the definition of $\mathfrak{Rep}(e_x)$ using a fundamental domain for the $\mathcal{G}_{\mathbb{K}}$-action on $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ$. Recall that $\mathcal{G}_{\mathbb{K}}$ acts transitively on the set of chambers of $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ and that any vertex of $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ is contained in a chamber Δ. Therefore, if Δ is a chamber in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$, then any $\mathcal{G}_{\mathbb{K}}$-orbit on $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ$ contains a vertex of Δ. Since $g e_x g^{-1} = e_{gx}$ for all $g \in \mathcal{G}_{\mathbb{K}}$, we may rewrite

$$\sum_{x \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V) = \sum_{g \in \mathcal{G}_{\mathbb{K}}} \sum_{x \in \Delta^\circ} e_{gx}(V) = \sum_{g \in \mathcal{G}_{\mathbb{K}}} g \cdot \Bigg(\sum_{x \in \Delta^\circ} e_x(V) \Bigg).$$

Thus $V \in \mathfrak{Rep}(e_x)$ if and only if the subspace $\sum_{x \in \Delta^\circ} e_x(V)$ generates V as an $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$-module. If $e_x = \langle K_x \rangle$ for a consistent system of compact open subgroups $(K_x)_{x \in \mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ}$ (see Lemma 2.6), then $e_x(V)$ is the subspace of K_x-invariants in V. Thus $\mathfrak{Rep}(\langle K_x \rangle)$ consists of those representations that are generated by their K_x-invariant vectors for $x \in \Delta^\circ$. This is the situation considered in [3].

If the stabiliser $P_\Delta^\dagger$ operates non-trivially on the vertices of Δ, then we do not need all vertices of Δ to generate representations in $\mathfrak{Rep}(e_x)$: a set of representatives for the orbits of $\mathcal{G}_{\mathbb{K}}$ on Δ suffices. For instance, if $\mathcal{G} = \mathrm{Gl}_d(\mathbb{K})$, then a single vertex suffices (see Section 2.2), and $\mathfrak{Rep}(e_x)$ is the set of all $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$-modules in $\mathcal{C}$ that are generated by the range of $e_{[\mathcal{O}^d]}$, where $[\mathcal{O}^d]$ is the vertex in $\mathcal{BT}$ with stabiliser $\mathrm{Gl}_d(\mathcal{O})$.

Our next goal is to show that $\mathfrak{Rep}(e_x)$ is equivalent to the category of unital $u_\Delta \mathcal{H}(\mathcal{G}_{\mathbb{K}}, R) u_\Delta$-modules for any chamber Δ, where u_Δ denotes the support projection of Δ studied in Section 2.3.

Let $(\Sigma_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite convex subcomplexes of $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ with $\bigcup_{n \in \mathbb{N}} \Sigma_n = \mathcal{BT}(\mathcal{G}_{\mathbb{K}})$, and let $u_n := u_{\Sigma_n}$. Then $u_n \le u_{n+1}$ for all $n \in \mathbb{N}$, that is, $u_n u_{n+1} = u_n = u_{n+1} u_n$. Let

$$\mathcal{H}(e_x) := \bigcup_{n \in \mathbb{N}} u_n \mathcal{H}(\mathcal{G}_{\mathbb{K}}, R) u_n.$$

Since this union is increasing, $\mathcal{H}(e_x)$ is a subalgebra of $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$. By construction, $(u_n)_{n \in \mathbb{N}}$ is an approximate unit of idempotents in $\mathcal{H}(e_x)$. An $\mathcal{H}(e_x)$-module V in $\mathcal{C}$ is called *smooth* if $V = \varinjlim u_n V$, where $u_n V$ denotes the image of u_n as an operator on V.

Proposition 3.3. *The category $\mathfrak{Rep}(e_x)$ is isomorphic to the category of smooth $\mathcal{H}(e_x)$-modules.*

Proof. Any object V of $\mathfrak{Rep}$ is also an $\mathcal{H}(e_x)$-module. By definition, an $\mathcal{H}(e_x)$-module is smooth if and only if $V = \varinjlim u_n V = \varinjlim \sum_{x \in \Sigma_n^\circ} e_x(V)$, that is, if and only if V belongs to $\mathfrak{Rep}(e_x)$. It remains to show that any smooth $\mathcal{H}(e_x)$-module structure extends to an $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$-module structure. If $f \in \mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$ and $g \in \mathcal{H}(e_x)$, then f is supported in some compact subset S of $\mathcal{G}_{\mathbb{K}}$ and $g \in u_n \mathcal{H}(\mathcal{G}_{\mathbb{K}}, R) u_n$ for some $n \in \mathbb{N}$. Choose $N \ge n$ for which Σ_N contains $S \cdot \Sigma_n$ and let λ denote the left regular representation of $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$. Then $\operatorname{im}\lambda(f * g) \subseteq \sum_{x\in\Sigma_N^\circ} \operatorname{im}\lambda(e_x) = u_N\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$, so that $f*g = u_N * f * g = u_N * f * g * u_N$ because $N \geq n$. Thus f is a left multiplier of $\mathcal{H}(e_x)$. Similarly, f is a right multiplier of $\mathcal{H}(e_x)$. Thus any smooth $\mathcal{H}(e_x)$-module is a module over $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$ as well. □

Theorem 3.4. *Let Δ be a chamber of $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ and let $\mathcal{H}(e_x)_\Delta := u_\Delta\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)u_\Delta$. The category $\mathfrak{Rep}(e_x)$ is equivalent to the category of unital $\mathcal{H}(e_x)_\Delta$-modules.*

Proof. This follows from Proposition 3.3 if $\mathcal{H}(e_x)$ is Morita equivalent to $\mathcal{H}(e_x)_\Delta$.

The two-sided ideal in $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$ generated by u_Δ contains $e_{g\sigma} = ge_\sigma g^{-1}$ for all $g \in \mathcal{G}_{\mathbb{K}}$ and $\sigma \in \Delta$ because $e_\sigma \leq u_\Delta$. Hence it contains u_Σ for any finite convex subcomplex Σ of $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ by the formula in Theorem 2.12 for the support projections. Thus the two-sided ideal of $\mathcal{H}(e_x)$ generated by u_Δ contains the approximate unit u_n. This means that the idempotent u_Δ is full in $\mathcal{H}(e_x)$. So by [8, Theorem 2.8] the $\mathcal{H}(e_x)_\Delta$-$\mathcal{H}(e_x)$-bimodules $u_\Delta\mathcal{H}(e_x)$ and $\mathcal{H}(e_x)u_\Delta$ yield a Morita equivalence between $\mathcal{H}(e_x)$ and $\mathcal{H}(e_x)_\Delta$. □

Now we assume that $\mathcal{C}$ has exact countable products in order to study the cohomology of the cellular cochain complex $C^*\bigl(\mathcal{BT}(\mathcal{G}_{\mathbb{K}}), \hat\Gamma(V)\bigr)$ for an $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$-module V in $\mathcal{C}$. Theorem 2.4 yields

$$\mathrm{H}^0\bigl(\mathcal{BT}(\mathcal{G}_{\mathbb{K}}), \hat\Gamma(V)\bigr) \cong \varprojlim u_n(V).$$

Since $u_n(V) = u_n\bigl(\Phi(V)\bigr)$, this implies

$$\mathrm{H}^0\bigl(\mathcal{BT}(\mathcal{G}_{\mathbb{K}}), \hat\Gamma(V)\bigr) \cong \mathrm{H}^0\bigl(\mathcal{BT}(\mathcal{G}_{\mathbb{K}}), \hat\Gamma(\Phi(V))\bigr),$$

so that we may restrict attention to $V \in \mathfrak{Rep}(e_x)$ in the following. Our description of the cohomology is reminiscent of the roughening functor for $\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)$-modules, but the comparison of the two constructions requires an additional assumption:

Lemma 3.5. *Assume that for each compact open subgroup $K \subseteq \mathcal{G}_{\mathbb{K}}$ there is a finite convex subcomplex $\Sigma \subseteq \mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ with $\langle K\rangle\Phi(V) = \sum_{x\in\Sigma^\circ}\langle K\rangle e_x(V)$. Then $\mathrm{H}^0\bigl(\mathcal{BT}(\mathcal{G}_{\mathbb{K}}), \hat\Gamma(V)\bigr)$ is the roughening of $\Phi(V)$ as a representation of $\mathcal{G}_{\mathbb{K}}$. In particular,*

$$\mathrm{S}\bigl(\mathrm{H}^0\bigl(\mathcal{BT}(\mathcal{G}_{\mathbb{K}}), \hat\Gamma(V)\bigr)\bigr) \cong \Phi(V).$$

Proof. Let $(K_n)_{n\in\mathbb{N}}$ be a decreasing sequence of compact open subgroups with $\bigcap K_n = \{1\}$. We may assume that Σ_n is K_0-invariant for all $n \in \mathbb{N}$, so that $u_n := u_{\Sigma_n}$ commutes with K_0. Hence the idempotents u_n and $\langle K_m\rangle$ commute for all $n, m \in \mathbb{N}$. Since u_n is a locally constant function on $\mathcal{G}_{\mathbb{K}}$, it is K_M-invariant for sufficiently large M. This means that there is $M_0 \in \mathbb{N}$ with $\langle K_M\rangle u_n = u_n$ for all $M \geq M_0$. The assumption in the statement means that for each $m \in \mathbb{N}$ there is $N_0 \in \mathbb{N}$ with $\langle K_m\rangle u_N(V) = \langle K_m\rangle u_{N_0}(V)$ for all $N \geq N_0$. It follows that the projective systems $\bigl(\langle K_m\rangle u_n(V)\bigr)_{n,m\in\mathbb{N}}$, $\bigl(u_n(V)\bigr)_{n\in\mathbb{N}}$, and $\bigl(\langle K_m\rangle\Phi(V)\bigr)_{m\in\mathbb{N}}$ are all equivalent, where $\Phi(V) = \sum_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ} e_x(V) = \varinjlim u_n(V)$. This implies the assertion. □

The assumption of the lemma is automatic if V is admissible in the sense that $\langle K\rangle V$ is finitely generated for each compact open subgroup K because the finitely many generators must belong to $\langle K\rangle u_n(V)$ for some $n \in \mathbb{N}$. But this assumption is far from necessary:

Proposition 3.6. *Let R be a field of characteristic not equal to p. The limit $u_\infty := \lim_{n\to\infty} u_n$ exists in the multiplier algebra of $\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$ and is a central idempotent, that is,*

$$u_\infty f = f u_\infty = \lim_{n\to\infty} u_n f = \lim_{n\to\infty} f u_n \qquad \text{for all } f \in \mathcal{H}(\mathcal{G}_\mathbb{K}, R),$$

and the sequences converge in the strong sense of becoming eventually constant.

For any $\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$-module V, we have $u_\infty V = \Phi(V)$ and

$$\mathrm{H}_0\big(\mathcal{BT}(\mathcal{G}_\mathbb{K}), \Gamma(V)\big) \cong u_\infty V, \qquad \mathrm{H}^0\big(\mathcal{BT}(\mathcal{G}_\mathbb{K}), \hat{\Gamma}(V)\big) \cong \mathrm{R}(u_\infty V),$$

where R *denotes the roughening functor.*

Proof. Let $\mathcal{H} := \mathcal{H}(\mathcal{G}_\mathbb{K}, R)$. Since R is a field of characteristic not equal to p, there is a decreasing sequence of compact open subgroups $(K_m)_{m\in\mathbb{N}}$ with $\bigcap K_m = \{1\}$ for which the unital algebras $\langle K_m\rangle\mathcal{H}\langle K_m\rangle$ are Noetherean. For fields of characteristic 0, this is a result of Joseph Bernstein [3]; for fields of finite characteristic not equal to p, this is due to Marie-France Vignéras [16, 2.13].

We fix $m \in \mathbb{N}$ and assume, as we may, that Σ_n is $\langle K_m\rangle$-invariant. Since $\langle K_m\rangle\mathcal{H}\langle K_m\rangle$ is Noetherean, its submodule $\bigcup_{n\in\mathbb{N}}\langle K_m\rangle\mathcal{H}\langle K_m\rangle u_n$ is finitely generated. That is, there exists $n \in \mathbb{N}$ such that $\langle K_m\rangle\mathcal{H}\langle K_m\rangle u_n = \langle K_m\rangle\mathcal{H}\langle K_m\rangle u_N$ for all $N \geq n$. Since $\langle K_m\rangle$, u_n, and u_N are commuting idempotents, this implies $\langle K_m\rangle u_n = \langle K_m\rangle u_N$. Therefore $u_n * f = u_N * f$ and $f * u_n = f * u_N$ for all $N \geq n$ and all $f \in \langle K_m\rangle\mathcal{H}\langle K_m\rangle$. Thus the sequences $(f * u_n)_{n\in\mathbb{N}}$ and $(u_n * f)_{n\in\mathbb{N}}$ eventually become constant. Since m is arbitrary, we get a multiplier $u_\infty := \lim u_n$. It is idempotent because all u_n are idempotent.

Let $f \in \mathcal{H}$ and let $X := \operatorname{supp} f$. For each $n \in \mathbb{N}$, there is $N \geq n$ with $X(\Sigma_n) \subseteq \Sigma_N$. Then $u_N \geq g u_n g^{-1}$ for all $g \in X$ and hence $u_N * f * u_n = f * u_n$. Thus $u_\infty * f * u_\infty = f * u_\infty$. A similar argument yields $u_\infty * f * u_\infty = u_\infty * f$. Thus u_∞ is central.

As already noted in proof of Lemma 3.5, $u_\infty V = \lim_{n\to\infty} u_n V = \Phi(V)$.

Recall that $\langle K_m\rangle u_\infty = \langle K_m\rangle u_n \leq u_n$ for sufficiently large $n \in \mathbb{N}$. Since $u_n \in \mathcal{H}$, we also have $u_n \leq \langle K_M\rangle$ for sufficiently large $M \in \mathbb{N}$, hence $u_n \leq \langle K_M\rangle u_\infty$. As a consequence, the inductive systems $\big(\langle K\rangle_m u_\infty(V)\big)_{m\in\mathbb{N}}$ and $\big(u_n(V)\big)_{n\in\mathbb{N}}$ are equivalent, so that they have isomorphic direct limits. By Theorem 2.4, this yields

$$\mathrm{H}_0\big(\mathcal{BT}(\mathcal{G}_\mathbb{K}), \Gamma(V)\big) \cong \varinjlim u_n(V) \cong \varinjlim \langle K_m\rangle u_\infty(V) \cong \mathrm{S}\big(u_\infty(V)\big) = u_\infty(V)$$

The assertion about H^0 can be proved as in Lemma 3.5. □

Proposition 3.7. *Let R be a field of characteristic not equal to p. Then the subcategory $\mathfrak{Rep}(e_x)$ in the category of* smooth *$\mathcal{H}(\mathcal{G}_\mathbb{K}, R)$-modules in $\mathcal{C}$ is closed under smooth direct product and hence under arbitrary smooth limits. That is, if $(V_i)_{i\in I}$ is a family of objects of $\mathfrak{Rep}(e_x)$, then $\mathrm{S}\big(\prod_{i\in I} V_i\big)$ belongs to $\mathfrak{Rep}(e_x)$ as well.*

Notice that the smoothening of the product is a product in the categorical sense in the subcategory of smooth representations.

Proof. Since (smooth) limits are subobjects of (smooth) products, it suffices to treat products. The assertion is non-trivial because direct products do no commute with arbitrary direct sums, but only with *finite* sums. Let K be a compact open subgroup.

Since $\langle K\rangle\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)\langle K\rangle$ is Noetherean, there exists a finite convex subcomplex $\Sigma\subseteq\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ such that

$$\langle K\rangle\Phi(V)=\langle K\rangle\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)\langle K\rangle\Phi(V) \text{ equals } \sum_{x\in\Sigma^\circ}\langle K\rangle e_x(V)=\langle K\rangle\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)\langle K\rangle u_\Sigma V.$$

for all $V\in\mathfrak{Rep}$. (See the proof of Proposition 3.6.) Since $V_i\cong\Phi(V_i)$ for all $i\in I$ by assumption, we get $\langle K\rangle(V_i)=\sum_{x\in\Sigma^\circ}\langle K\rangle e_x(V_i)$ for all $i\in I$. Therefore

$$\begin{aligned}\langle K\rangle\Bigl(\prod_{i\in I}V_i\Bigr)=\prod_{i\in I}\langle K\rangle V_i=\prod_{i\in I}\sum_{x\in\Sigma^\circ}\langle K\rangle e_x(V_i)\\ =\sum_{x\in\Sigma^\circ}\prod_{i\in I}\langle K\rangle e_x(V_i)=\langle K\rangle\sum_{x\in\Sigma^\circ}e_x\prod_{i\in I}V_i=\langle K\rangle\Phi\prod_{i\in I}V_i.\end{aligned}$$

Thus $\mathrm{S}\bigl(\prod V_i\bigr)=\Phi\bigl(\prod V_i\bigr)=\Phi\circ\mathrm{S}\bigl(\prod V_i\bigr)$, that is, $\mathrm{S}\bigl(\prod V_i\bigr)\in\mathfrak{Rep}(e_x)$. □

4. Towards a Lefschetz character formula

Let R be a field whose characteristic is different from p. Let V be an R-vector space and let $\varrho\colon\mathcal{G}_{\mathbb{K}}\to\operatorname{Aut}(V)$ be a finitely generated, smooth, admissible representation of $\mathcal{G}_{\mathbb{K}}$. That is, any $v\in V$ is K-invariant for some compact open subgroup K, the subspace of K-invariant vectors in V is finite-dimensional for each compact open subgroup K of $\mathcal{G}_{\mathbb{K}}$, and V is finitely generated as a module over $\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)$. This implies that V is generated by its K-invariant vectors for a sufficiently small compact open subgroup $K\subseteq\mathcal{G}_{\mathbb{K}}$. Hence $V\in\mathfrak{Rep}(e_x)$ for a suitable equivariant consistent system of idempotents $e_x\in\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)$ (see [13]). We fix such a system $(e_x)_{x\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})^\circ}$ and consider the associated cosheaf $\Gamma(V)$.

Admissibility implies that $\varrho(f)\in\operatorname{End}(V)$ is a finite rank operator for each $f\in\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)$ and hence has a well-defined trace. This defines an R-linear map $\mathcal{H}(\mathcal{G}_{\mathbb{K}},R)\to R$ called the *character* of ϱ. For $R=\mathbb{C}$ a deep theorem of Harish-Chandra asserts that the character is of the form $f\mapsto\int_{\mathcal{G}_{\mathbb{K}}}f(x)\chi_\varrho(x)\,\mathrm{d}x$ for some locally integrable function χ_ϱ that is locally constant at regular semisimple elements. Thus the character is not just a distribution but a function defined on regular semisimple elements of $\mathcal{G}_{\mathbb{K}}$. The values of this character at regular elliptic elements are computed by Peter Schneider and Ulrich Stuhler in [13], using the resolutions described above. The resulting formula is a Lefschetz fixed point formula for the character because it assembles the character value at a regular elliptic element $g\in\mathcal{G}_{\mathbb{K}}$ from contributions by the fixed points of g in the building $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$.

Jonathan Korman [10] how to extend this computation to general regular compact elements under an additional assumption, which he could verify in the rank-1-case. Theorem 2.4 shows that these assumptions are satisfied in general, so that we can compute the character on all compact elements. The formula we establish here does not yet apply to non-compact regular elements. We plan to discuss more general character formulas elsewhere, using suitable compactifications of the building. Our goal here is more modest.

For each polysimplex $\sigma\in\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$, the cosheaf value $V_\sigma:=e_\sigma(V)$ carries a representation of $P_\sigma^\dagger:=\{g\in\mathcal{G}_{\mathbb{K}}\mid g\sigma=\sigma\}$. We also allow elements of $P_\sigma^\dagger$ to permute the vertices of σ and even to change orientation. The representation of $P_\sigma^\dagger$ is the one that appears in the cellular chain complex and thus involves the orientation character $P_\sigma^\dagger\to\{\pm1\}$ in (1). We let $\chi_\sigma\colon P_\sigma^\dagger\to R$ be the character of the representation of $P_\sigma^\dagger$ on V_σ.

Let K be a compact open subgroup of $\mathcal{G}_{\mathbb{K}}$. We want to compute the restriction of the character χ_ϱ of V to K in terms of the characters χ_σ of the representations V_σ. More precisely, we restrict χ_σ to $K \cap P_\sigma^\dagger$ and then extend it by 0 to K. Summing up these functions over the K-orbit $K\sigma$, we get the character of the K-representation $\operatorname{Ind}_{K\cap P_\sigma^\dagger}^K \operatorname{Res}_{P_\sigma^\dagger}^{K\cap P_\sigma^\dagger} V_\sigma$.

Proposition 4.1. *For each $f \in \mathcal{H}(K)$ there is a finite convex subcomplex Σ_0 in $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$ such that*

$$\chi_\varrho(f) = \int_K f(g) \cdot \sum_{\sigma\in\Sigma} (-1)^{\deg\sigma} \chi_\sigma(g)\,\mathrm{d}g = \int_K f(g) \cdot \sum_{\substack{\sigma\in\Sigma\\ g\sigma=\sigma}} (-1)^{\deg\sigma} \chi_\sigma(g)\,\mathrm{d}g$$

for all K-invariant finite convex subcomplexes $\Sigma \supseteq \Sigma_0$.

Proof. Since $f(V)$ is finite-dimensional, it is contained in $\sum_{x\in\Sigma_0^\circ} e_x(V)$ for some finite convex subcomplex Σ_0. Let Σ be a K-invariant finite convex subcomplex containing Σ_0. Then $C_*\bigl(\Sigma, \Gamma(V)\bigr)$ is a chain complex of K-representations, so that f acts on it by chain maps. Theorem 2.4 implies that $C_*\bigl(\Sigma, \Gamma(V)\bigr)$ is a resolution of $\mathrm{H}_0\bigl(\Sigma, \Gamma(V)\bigr) \cong \sum_{x\in\Sigma^\circ} e_x(V)$, which contains the range of f. Hence the trace of f does not change when we view it as an endomorphism of $\mathrm{H}_0\bigl(\Sigma, \Gamma(V)\bigr)$. The action of f on $C_*\bigl(\Sigma, \Gamma(V)\bigr)$ lifts the action of f on $\sum_{x\in\Sigma^\circ} e_x(V)$.

The vector space $\bigoplus_{n\in\mathbb{N}} C_n\bigl(\Sigma, \Gamma(V)\bigr)$ is finite-dimensional because V is admissible and Σ is finite. Hence the Euler characteristic

$$\sum_{n=0}^\infty (-1)^n \operatorname{tr}\bigl(f|_{C_n(\Sigma,\Gamma(V))}\bigr) = \sum_{n=0}^\infty (-1)^n \int_K f(g) \cdot \operatorname{tr}\bigl(g|_{C_n(\Sigma,\Gamma(V))}\bigr)\,\mathrm{d}g$$

is well-defined and agrees with the trace of f on $\mathrm{H}^0\bigl(\Sigma, \Gamma(V)\bigr)$, which agrees with the trace $\chi_\varrho(f)$ of f on V by our construction of Σ. We rewrite the Euler characteristic above using the decomposition $C_*\bigl(\Sigma, \Gamma(V)\bigr) = \bigoplus_{\sigma\in\Sigma} e_\sigma(V)$:

$$\chi_\varrho(f) = \int_K f(g) \sum_{\substack{\sigma\in\Sigma\\ g\sigma=\sigma}} (-1)^{\deg\sigma} \operatorname{tr}(g|_{e_\sigma(V)})\,\mathrm{d}g = \int_K f(g) \sum_{\substack{\sigma\in\Sigma\\ g\sigma=\sigma}} (-1)^{\deg\sigma} \chi_\sigma(g)\,\mathrm{d}g. \quad \square$$

As a consequence, the restriction of the character to K is the limit of the functions

$$\chi_\Sigma(g) := \sum_{\substack{\sigma\in\Sigma\\ g\sigma=\sigma}} (-1)^{\deg\sigma} \chi_\sigma(g)\,\mathrm{d}g,$$

where Σ runs through the set of finite K-invariant convex subcomplexes of $\mathcal{BT}(\mathcal{G}_{\mathbb{K}})$. This formula is unwieldy because we cannot exchange the limit over Σ and the summation: the cancellation between simplices of different parity is needed for the limit to exist. Recall that the set of simplices σ with $g\sigma = \sigma$ is finite if and only if g is an regular elliptic element of $\mathcal{G}_{\mathbb{K}}$. In this case, the relevant character formula appears already in [13].

5. Conclusion and outlook

An equivariant consistent systems of idempotents in the Hecke algebra of a reductive p-adic group produces a natural cosheaf and a natural sheaf on the building of the group for any representation. The consistency conditions ensure that the homology and cohomology with these coefficients vanishes except in degree zero, where we get a certain subspace and quotient of the representation we started with. The representations for which the zeroth homology of this cosheaf agrees with the given representation form a Serre subcategory. We have used support projections to describe this Serre subcategory as the module category over a suitable corner in the Hecke algebra of the group. These support projections of convex subcomplexes of the building are also a crucial tool for the homology computation. They are described by a surprisingly simple formula, which only defines an idempotent element of the Hecke algebra because of the consistency conditions. Since our homological computations still work for convex subcomplexes of the building, we also get a formula for the values of the character of a representation on regular compact elements, which involves the fixed point subset in the building. But this formula is still unwieldy because the relevant fixed point subsets are infinite for non-elliptic elements, leading to infinite sums that converge only conditionally.

Note added in proof. Jean-Francois Dat kindly pointed out that the proof of Proposition 3.6 is not entirely correct as stated. The problem is that it is not known whether algebras of the form $\langle K\rangle\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)\langle K\rangle$ are always Noetherean when R has nonzero characteristic. This issue is studied (and proven in many cases) in [J.-F. Dat, *Finitude pour les représentations lisses de groupes p-adiques*, J. Inst. Math. Jussieu 8 (2009), no. 2, 261–333].

Therefore the proofs of Propositions 3.6 and 3.7 are only valid under the additional assumption that there exists a decreasing sequence of compact open subgroups $(K_m)_{m\in\mathbb{N}}$ with $\bigcap K_m = \{1\}$, for which the algebras $\langle K_m\rangle\mathcal{H}(\mathcal{G}_{\mathbb{K}}, R)\langle K_m\rangle$ are Noetherean.

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