DEPTH AND THE LOCAL LANGLANDS CORRESPONDENCE

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ABSTRACT. Let G be an inner form of a general linear group over a non-archimedean local field. We prove that the local Langlands correspondence for G preserves depths. We also show that the local Langlands correspondence for inner forms of special linear groups preserves the depths of essentially tame Langlands parameters.

CONTENTS

1. INTRODUCTION

Let F be a non-archimedean local field and let G be a connected reductive group over F. Let $\Phi(G)$ denote the collection of equivalence classes of Langlands parameters for G, and $\mathrm{Irr}(G)$ the set of (isomorphism classes of) irreducible smooth G-representations. A central role in the representation theory of such groups is played by the local Langlands correspondence (LLC). It is supposed to be a map

$$\mathrm{Irr}(G) \to \Phi(G)$$

that enjoys several naturality properties [Bor, Vog]. The LLC should preserve interesting arithmetic information, like local L-functions and ϵ-factors. A lesser-known invariant that makes sense on both sides of the LLC is *depth*.

The depth of a Langlands parameter ϕ is easy to define. For $r \in \mathbb{R}_{\geq 0}$ let $\mathrm{Gal}(F_s/F)^r$ be the r-th ramification subgroup of the absolute Galois group of F. Then the depth of ϕ is the smallest number $d(\phi) \geq 0$ such that ϕ is trivial on $\mathrm{Gal}(F_s/F)^r$ for all $r > d(\phi)$.

The depth $d(\pi)$ of an irreducible G-representation π was defined by Moy and Prasad [MoPr1, MoPr2], in terms of filtrations $P_{x,r}(r \in \mathbb{R}_{\geq 0})$ of the parahoric subgroups $P_x \subset G$. On the basis of several examples (see below) it is reasonable to expect that for many Langlands parameters $\phi \in \Phi(G)$ with L-packet $\Pi_\phi(G) \subset \mathrm{Irr}(G)$ one has

$$(1) \qquad\qquad d(\phi) = d(\pi) \quad \text{ for all } \pi \in \Pi_\phi(G).$$

This relation would be useful for several reasons. Firstly, it allows one to employ some counting arguments in the local Langlands correspondence, because (up to unramified twists) there are only finitely many irreducible representations and Langlands parameters whose depth is at most a specified upper bound.

Secondly, it would be a step towards a more explicit LLC. One can try to determine the groups $P_{x,r}/P_{x,r+\epsilon}$ ($\epsilon > 0$ small) and their representations explicitly, and to match them with representations of $\mathrm{Gal}(F_s/F)/\mathrm{Gal}(F_s/F)^{r+\epsilon}$.

Thirdly, one can use (1) as a working hypothesis when trying to establish a local Langlands correspondence, to determine whether or not two irreducible representations stand a chance of belonging to the same L-packet.

The most basic case of depth preservation concerns Langlands parameters $\phi \in \Phi(G)$ that are trivial on both the inertia group $\mathbf{I}_F$ and on $\mathrm{SL}_2(\mathbb{C})$. These can be regarded as Langlands parameters of negative depth. Such a ϕ is only relevant for G if G is quasi-split and splits over an unramified extension of F. In that case one can say that an irreducible G-representation has negative depth if it possesses a nonzero vector fixed by a hyperspecial maximal compact subgroup. The Satake isomorphism shows how to each such representation one can associate (in a natural way) a Langlands parameter of the above kind.

The G-representations of depth zero have been subjected to ample study, see for example [GSZ, Mor, DBRe, Mœ]. According to Moy–Prasad, an irreducible representation has depth zero if and only if it has nonzero vectors fixed by the prounipotent radical of some parahoric subgroup of G. This includes Iwahori-spherical representations and Lusztig's unipotent representations [Lus1, Lus2]. A Langlands parameter has depth zero if and only if it is trivial on the wild inertia subgroup of the absolute Galois group of F. For depth zero the equality (1) is conjectured, and proven in certain cases, in [DBRe]. It fits very well with the aforementioned work of Lusztig.

In positive depth there is the result of Yu [Yu2, §7.10], who proved (1) for unramified tori. For $\mathrm{GL}_n(F)$, (1) was claimed in [Yu1, §2.3.6] and proved in [ABPS2, Proposition 4.5]. For $\mathrm{GSp}_4(F)$, (1) is proved in [Gan, § 10]. We refer to [GrRe, Ree, ReYu] for some interesting examples of positive depth Langlands parameters and supercuspidal representations. Most of these examples satisfy (1), but in [ReYu, §7.4–7.5] some particular cases are mentioned in which (1) does not hold. All these counterexamples appear in small residual characteristics. So it remains to be seen in which generality the local Langlands correspondence will preserve depths.

In this paper we will prove that the local Langlands correspondence preserves depth for the inner forms of $\mathrm{GL}_n(F)$. In a few non-split cases, this was done before in [LaRa]. For inner forms of $\mathrm{SL}_n(F)$, we will prove an inequality between depths, which becomes an equality if the Langlands parameter is essentially tame in the sense that it maps the wild inertia group to a maximal torus of $\mathrm{PGL}_n(\mathbb{C})$. Every Langlands parameter for an inner form of $\mathrm{SL}_n(F)$ is essentially tame if the residual characteristic of F does not divide n.

Let D be a division algebra with centre F, of dimension d^2 over F. Then $G = \mathrm{GL}_m(D)$ is an inner form of $\mathrm{GL}_n(F)$ with $n = dm$. There is a reduced norm map $\mathrm{Nrd}: \mathrm{GL}_m(D) \to F^\times$ and the derived group $G_{\mathrm{der}} := \ker(\mathrm{Nrd} : G \to F^\times)$ is an inner form of $\mathrm{SL}_n(F)$. Every inner form of $\mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$ is isomorphic to one of this kind.

The main steps in the proof of our depth-preservation theorem are:

- With the Langlands classification one reduces the problem to essentially square-integrable representations and elliptic Langlands parameters.
- Express the depth in terms of ϵ-factors and conductors. This is a technical step which involves detailed knowledge of the representation theory of G. Here it is convenient to use an alternative but equivalent version of depth, the normalized level of an irreducible G-representation.
- Show that the Jacquet–Langlands correspondence for $G = \mathrm{GL}_m(D)$ preserves ϵ-factors. Since the LLC for $\mathrm{GL}_m(D)$ is defined as a composition of the Jacquet–Langlands correspondence with the LLC for $\mathrm{GL}_n(F)$ and the latter is known to preserve ϵ-factors, this proves depth-preservation for inner forms of $\mathrm{GL}_n(F)$.
- Relate the depth for G_{der} to depth for G. For irreducible representations nothing changes, but for Langlands parameters the depth can decrease if one replaces the dual group $\mathrm{GL}_n(\mathbb{C})$ by $\mathrm{PGL}_n(\mathbb{C})$. Using several properties of the Artin reciprocity map, we show that such a decrease in depth cannot occur if the Langlands parameter is essentially tame.

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2. The local Langlands correspondence for inner forms of $\mathrm{GL}_n(F)$

2.1. **The statement of the correspondence.**

The local Langlands correspondence for supercuspidal representations of $\mathrm{GL}_n(F)$ was established in the important papers [LRS, HaTa, Hen2]. Together with the Jacquet–Langlands correspondence this provides the LLC for essentially square-integrable representations of inner forms $G = \mathrm{GL}_m(D)$ of $\mathrm{GL}_n(F)$. It is extended to all irreducible G-representations via the Zelevinsky classification [Zel, DKV], see [HiSa, ABPS1]. For these groups every L-packet is a singleton and the LLC is a

canonical bijective map

$$(2) \qquad \operatorname{rec}_{D,m} : \operatorname{Irr}(\operatorname{GL}_m(D)) \to \Phi(\operatorname{GL}_m(D)).$$

A remarkable aspect of Langlands' conjectures [Vog] is that it is better to consider not just one reductive group at a time, but all inner forms of a given group simultaneously. Inner forms share the same Langlands dual group, so in (2) the right hand side is the same for all inner forms G of the given group. Then one can turn (2) into a bijection by defining a suitable equivalence relation on the set of inner forms and taking the corresponding union of the sets $\operatorname{Irr}(G)$ on the left hand side (see Theorem 2.1 below).

We define the equivalence classes of such inner forms to be in bijection with the isomorphism classes of central simple F-algebras of dimension n^2 via $\mathrm{M}_m(D) \mapsto \operatorname{GL}_m(D)$, respectively $\mathrm{M}_m(D) \mapsto \operatorname{GL}_m(D)_{\mathrm{der}}$.

As Langlands dual group of G we take $\operatorname{GL}_n(\mathbb{C})$. To deal with inner forms it is advantageous to consider the conjugation action of $\operatorname{SL}_n(\mathbb{C})$ on these two groups. It induces a natural action of $\operatorname{SL}_n(\mathbb{C})$ on the collection of Langlands parameters for $\operatorname{GL}_n(F)$. For any such parameter ϕ we can define

$$(3) \qquad C(\phi) = Z_{\operatorname{SL}_n(\mathbb{C})}(\operatorname{im} \phi), \quad \text{and} \quad \mathcal{S}_\phi = C(\phi)/C(\phi)^\circ.$$

Notice that the centralizers are taken in $\operatorname{SL}_n(\mathbb{C})$ and not in the Langlands dual group.

Via the Langlands correspondence the non-trivial irreducible representations of $\mathcal{S}_\phi$ are associated to irreducible representations of non-split inner forms of $\operatorname{GL}_n(F)$. For example, consider a Langlands parameter ϕ for $\operatorname{GL}_2(F)$ which is elliptic, that is, whose image is not contained in any torus of $\operatorname{GL}_2(\mathbb{C})$. Then $\mathcal{S}_\phi = Z(\operatorname{SL}_2(\mathbb{C})) \cong \{\pm 1\}$. The pair $(\phi, \operatorname{triv}_{\mathcal{S}_\phi})$ parametrizes an essentially square-integrable representation of $\operatorname{GL}_2(F)$ and $(\phi, \operatorname{sgn}_{\mathcal{S}_\phi})$ parametrizes an irreducible representation of the inner form $D^\times$, where D denotes a noncommutative division algebra of dimension 4 over F.

The enhanced version of the local Langlands correspondence for all inner forms of general linear groups over nonarchimedean local fields says:

Theorem 2.1. [ABPS2, Theorem 1.1]
There is a canonical bijection between:

- *pairs (G, π) with $\pi \in \operatorname{Irr}(G)$ and G an inner form of $\operatorname{GL}_n(F)$, considered up to equivalence;*
- *$\operatorname{GL}_n(\mathbb{C})$-conjugacy classes of pairs (ϕ, ρ) with $\phi \in \Phi(\operatorname{GL}_n(F))$ and $\rho \in \operatorname{Irr}(\mathcal{S}_\phi)$.*

Via the Kottwitz isomorphism [Kot, Proposition 6.4] every character of $Z(\operatorname{SL}_n(\mathbb{C}))$ determines a central simple F-algebra $\mathrm{M}_m(D)$. As $Z(\operatorname{SL}_n(\mathbb{C})) \subset C(\phi)$, for any Langlands parameter as above a character of $\mathcal{S}_\phi$ determines an inner form $\operatorname{GL}_m(D)$ of $\operatorname{GL}_n(F)$. In contrast with the usual LLC, our L-packets for inner forms of general linear groups need not be singletons. To be precise, the packet Π_ϕ contains the unique representation $\operatorname{rec}_{D,m}^{-1}(\phi)$ of $G = \operatorname{GL}_m(D)$ if ϕ is relevant for G, and no G-representations otherwise.

2.2. The Jacquet–Langlands correspondence.

A representation π of G is called essentially square-integrable if $\pi|_{G_{\mathrm{der}}}$ is square-integrable and there exists an unramified character χ of G such that $\pi \otimes \chi$ is unitary. We denote the set of (equivalence classes of) irreducible essentially square-integrable G-representations by $\mathrm{Irr}_{\mathrm{ess}L^2}(G)$. There is a natural bijection between $\mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_n(F))$ and $\mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_m(D))$, discovered first for $\mathrm{GL}_2(F)$ by Jacquet and Langlands [JaLa]. The local Langlands correspondence for $\mathrm{GL}_m(D)$ is constructed with the help of the Jacquet–Langlands correspondence. Here we recall some useful properties of the latter correspondence.

Theorem 2.2. *Let $\mathrm{GL}_m(D)$ be an inner form of $\mathrm{GL}_n(F)$. There exists a canonical bijection*

$$\mathrm{JL} : \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_n(F)) \to \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_m(D))$$

with the following properties:

(a) *There is a canonical identification of the semisimple elliptic conjugacy classes in $\mathrm{GL}_n(F)$ with those in $\mathrm{GL}_m(D)$. Let $g \in \mathrm{GL}_n(F)$ and $g' \in \mathrm{GL}_m(D)$ be semisimple elliptic elements in the same conjugacy class and let θ_π be the character of $\pi \in \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_n(F))$. Then*

$$(-1)^n \theta_\pi(g) = (-1)^m \theta_{\mathrm{JL}(\pi)}(g').$$

(b) *JL preserves twists with characters of $F^\times$:*

$$\mathrm{JL}(\pi \otimes \chi \circ \det) = \mathrm{JL}(\pi) \otimes \chi \circ \mathrm{Nrd}.$$

(c) *JL respects contragredients: $\mathrm{JL}(\pi^\vee) = \mathrm{JL}(\pi)^\vee$.*

(d) *Let P' be a standard parabolic subgroup of $\mathrm{GL}_m(D)$, with Levi factor $M' = \prod_i \mathrm{GL}_{m_i}(D)$. Let P be the corresponding standard parabolic subgroup of $\mathrm{GL}_n(F)$, with Levi factor $M = \prod_i \mathrm{GL}_{dm_i}(F)$. Then the Jacquet modules $r_P^{\mathrm{GL}_n(F)}(\pi)$ and $r_{P'}^{\mathrm{GL}_m(D)}(\mathrm{JL}(\pi))$ are either both zero or both irreducible and essentially square-integrable. In the latter case*

$$\mathrm{JL}\big(r_P^{\mathrm{GL}_n(F)}(\pi)\big) = r_{P'}^{\mathrm{GL}_m(D)}(\mathrm{JL}(\pi)).$$

In other words, JL and its version for M and M' respect Jacquet restriction.

(e) *JL preserves supercuspidality.*

(f) *$\mathrm{JL}(\mathrm{St}_{\mathrm{GL}_n(F)}) = \mathrm{St}_{\mathrm{GL}_m(D)}$, where St_G denotes the Steinberg representation of G.*

(g) *JL preserves γ-factors:*

$$\gamma(s, \mathrm{JL}(\pi), \psi) = \gamma(s, \pi, \psi) \quad \text{for any nontrivial character } \psi \text{ of } F.$$

(h) *JL preserves L-functions: $L(s, \mathrm{JL}(\pi)) = L(s, \pi)$.*

(i) *JL preserves ϵ-factors: $\epsilon(s, \mathrm{JL}(\pi), \psi) = \epsilon(s, \pi, \psi)$.*

Proof. The correspondence, which is in fact characterized by property (a), is proven over p-adic fields in [DKV] and over local fields of positive characteristic in [Bad]. Properties (b) and (c) are obvious in view of (a). The same goes for property (f) in the case $m = 1$, because then $\mathrm{St}_{\mathrm{GL}_m(D)}$ is just the trivial representation of $D^\times$. For (d) see [Bad, §5], in particular Proposition B. Obviously (d) implies (e). Property (f) for $m > 1$ follows from (f) for $m = 1$ and property (d). Property (g) was proven over local function fields in [Bad, p. 741], with an argument that also works over p-adic fields.

Properties (h) and (i) were claimed in [DKV], with the difference that the ϵ-factors of π and $\mathrm{JL}(\pi)$ are said to agree only up to a sign $(-1)^{n+m}$. This sign is due to a convention that does not agree with [GoJa], which we use for the definition of ϵ-factors. Unfortunately the argument for (h) and (i) given in [DKV, §B.1] is incorrect. Instead, we will establish (h) by direct calculation.

Let ν_D denote the unramified character $g' \mapsto \|\mathrm{Nrd}\, g'\|_F$ of $\mathrm{GL}_m(D)$. Consider any $\pi' \in \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_m(D))$. By [DKV, §B.2] or [Tad, §2] there exist:

- integers a, b, s_σ such that $ab = m$ and s_σ divides ad;
- an irreducible supercuspidal representation σ of $\mathrm{GL}_a(D)$,

such that π' is a consituent of the parabolically induced representation

$$(4) \qquad \Pi' := I_{\mathrm{GL}_a(D)^b}^{\mathrm{GL}_m(D)}\big(\nu_D^{s_\sigma \frac{1-b}{2}}\sigma \otimes \nu_D^{s_\sigma \frac{3-b}{2}}\sigma \otimes \cdots \otimes \nu_D^{s_\sigma \frac{b-1}{2}}\sigma\big).$$

By [Jac, Proposition 2.3] the L-function of (4) is the product of L-functions of the inducing representations:

$$(5) \qquad L(s, \Pi') = \prod_{k=1}^{b} L(s, \nu_D^{s_\sigma(k-(1+b)/2)}\sigma) = \prod_{k=1}^{b} L(s + s_\sigma(k - (1+b)/2), \sigma).$$

By definition $L(s, \pi')^{-1}$ is a monic polynomial in q^{-s}, and by [Jac, 2.7.4] it is a factor of the monic polynomial $L(s, \Pi')^{-1}$. Now there are two cases to be distinguished, depending on whether σ is an unramified representation of $D^\times$ or not.

Case 1: $a = 1, b = m$ and σ is unramified.
There exists an unramified character χ of $F^\times$ such that $\sigma = \chi \circ \mathrm{Nrd}$. By [DKV, §B.2] or [Tad, §2] (4) only has an essentially square-integrable subquotient if $s_\sigma = d$. Then $\pi' \cong \mathrm{St}_{\mathrm{GL}_m(D)} \otimes \chi \circ \mathrm{Nrd}$, so $\mathrm{JL}^{-1}(\pi') \cong St_{\mathrm{GL}_n(F)} \otimes \chi \circ \det$. With property (f) this enables us to compute the γ-factor. Let ω_F be a uniformizer of F, $\mathfrak{o}_F$ the ring of integers and $\mathfrak{p}_F$ its maximal ideal. Assume that ψ is trivial on $\mathfrak{p}_F$ but not on $\mathfrak{o}_F$. Then

$$(6) \quad \gamma(s, \pi', \psi) = \gamma(s, \mathrm{St}_{\mathrm{GL}_n(F)} \otimes \chi \circ \det, \psi) = (-1)^n q^{n/2} \frac{1 - q^{-s+(1-n)/2}\chi(\omega_F)}{1 - q^{-s+(1+n)/2}\chi(\omega_F)}.$$

By [GoJa, Proposition 4.4], (5) becomes

$$(7) \quad \prod_{k=1}^{m} L(s + d(k - (1+m)/2), \chi \circ \mathrm{Nrd}) = \prod_{k=1}^{m} L(s + d(k - (1+m)/2) + (d-1)/2, \chi).$$

Now we apply [GoJa, Theorem 7.11.4]. It is stated only for $\mathrm{GL}_n(F)$, but the proof with zeros and poles of L-functions goes through because we know $\gamma(s, \pi', \psi)$. We find that for the L-function of π' we need only the factor $k = m$ of (7):

$$L(s, \pi', \psi) = L(s + (n-1)/2, \chi) = (1 - q^{-s+(1-n)/2}\chi(\omega_F))^{-1}.$$

In particular the whole calculation works with $d = 1$, so

$$(8) \quad L(s, \mathrm{St}_{\mathrm{GL}_m(D)} \otimes \chi \circ \mathrm{Nrd}) = L(s, \mathrm{St}_{\mathrm{GL}_n(F)} \otimes \chi \circ \det) = L(s + (n-1)/2, \chi).$$

Case 2: all other σ.
Then [GoJa, Proposition 5.11] says that $L(s, \sigma \otimes \chi) = 1$ for every unramified character χ of $\mathrm{GL}_a(D)$. Hence $L(s, \Pi') = 1$ by (5). We observed above that $L(s, \pi')^{-1}$ is a factor of $L(s, \Pi')$, so $L(s, \pi') = 1$. Because JL is bijective, $\mathrm{JL}^{-1}(\pi')$ is not an

unramified twist of the Steinberg representation, so $L(s, \mathrm{JL}^{-1}(\pi')) = 1$ as well. This proves property (h).

In view of the relation

$$\epsilon(s, \pi, \psi) = \gamma(s, \pi, \psi) L(s, \pi) L(1 - s, \pi^\vee), \tag{9}$$

(i) follows directy from (c), (g) and (h). $\qquad\square$

We record a particular consequence of equations (6), (8) and (9):

$$\epsilon(s, \mathrm{St}_{\mathrm{GL}_m(D)} \otimes \chi \circ \mathrm{Nrd}, \psi) = (-1)^{n-1} \epsilon(s, \chi, \psi) = (-1)^{n-1} q^{s-1/2} \chi(\omega_F^{-1}) \tag{10}$$

for any character ψ of F which is trivial on $\mathfrak{p}_F$ but not on $\mathfrak{o}_F$.

2.3. Depth for Langlands parameters.

Let F_s be a separable closure of F and let $\mathrm{Gal}(F_s/F)$ be the absolute Galois group of F. We recall some properties of its ramification groups (with respect to the upper numbering), as defined in [Ser, Remark IV.3.1]:

- $\mathrm{Gal}(F_s/F)^{-1} = \mathrm{Gal}(F_s/F)$ and $\mathrm{Gal}(F_s/F)^0 = \mathbf{I}_F$, the inertia group.
- For every $l \in \mathbb{R}_{\geq 0}$, $\mathrm{Gal}(F_s/F)^l$ is a compact subgroup of $\mathbf{I}_F$. It consists of all $\gamma \in \mathrm{Gal}(F_s/\bar{F})$ which, for every finite Galois extension E of F contained in F_s, act trivially on the ring $\mathfrak{o}_E/\mathfrak{p}_E^{i(l,E)}$ (where $i(l, E) \in \mathbb{Z}_{\geq 0}$ can be found with [Ser, §IV.3]).
- $l \in \mathbb{R}_{\geq 0}$ is called a jump of the filtration if

$$\mathrm{Gal}(F_s/F)^{l+} := \bigcap_{l' > l} \mathrm{Gal}(F_s/F)^{l'}$$

does not equal $\mathrm{Gal}(F_s/F)^l$. The set of jumps of the filtration is countably infinite and need not consist of integers.

Recall [Bor] that a Langlands parameter for $\mathrm{GL}_m(D)$ is a continuous homomorphism

$$\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$$

such that:

- $\phi(\mathbf{W}_F)$ consists of semisimple elements;
- $\phi|_{\mathrm{SL}_2(\mathbb{C})} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ is a morphism of complex algebraic groups;
- ϕ is relevant for $\mathrm{GL}_m(D)$. This means that the conjugacy class of a Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$ minimally containing $\mathrm{im}(\phi)$ should correspond to a conjugacy of class of Levi subgroups of $\mathrm{GL}_m(D)$.

We define the depth of such a Langlands parameter as

$$d(\phi) := \inf\{l \geq 0 \mid \mathrm{Gal}(F_s/F)^{l+} \subset \ker\phi\}.$$

We say that $\phi \in \Phi(\mathrm{GL}_n(F))$ is *elliptic* if its image is not contained in any proper Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$.

Let ψ be a nontrivial character of F and let $c(\psi)$ be the largest integer c such that ψ is trivial on $\mathfrak{p}_F^{-c}$. The ϵ factor of ϕ (and ψ) was defined in [Tat]. It takes the form

$$\epsilon(s, \phi, \psi) = \epsilon(0, \phi, \psi) q^{-s(a(\phi) + nc(\psi))} \text{ with } \epsilon(0, \phi, \psi) \in \mathbb{C}^\times. \tag{11}$$

Here $a(\phi) \in \mathbb{Z}_{\geq 0}$ is the Artin conductor of ϕ (called $f(\phi)$ in [Ser, §VI.2]). To study $a(\phi)$ it is convenient to rewrite ϕ in terms of the Weil–Deligne group. For $\gamma \in \mathbf{W}_F$

we write $\|\gamma\| = q$ if γ induces the automorphism $x \mapsto x^q$ on the residue field of F_s. Define

$$(12) \qquad \phi_0(\gamma) = \phi(\gamma, 1)\phi\left(1, \begin{pmatrix} \|\gamma\|^{1/2} & 0 \\ 0 & \|\gamma\|^{-1/2} \end{pmatrix}\right),$$

so ϕ_0 is a representation of $\mathbf{W}_F$ which agrees with ϕ on $\mathbf{I}_F$. Define $N \in \mathfrak{gl}_n(\mathbb{C})$ as the nilpotent element $\log \phi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$. Then (ϕ_0, N) is the Weil–Deligne representation of $\mathbf{W}_F \ltimes \mathbb{C}$ corresponding to ϕ.

Denote the vector space $\mathbb{C}^n$ endowed with the representation ϕ by V, and write $V_N = \ker(N : V \to V)$. By definition [Tat, §4.1.6]

$$(13) \qquad a(\phi) = a(\phi_0) + \dim(V^{\mathbf{I}_F}/V_N^{\mathbf{I}_F}),$$

$$(14) \qquad \epsilon(s, \phi, \psi) = \epsilon(0, \phi_0, \psi) \det\left(-\operatorname{Frob}|_{V^{\mathbf{I}_F}/V_N^{\mathbf{I}_F}}\right) q^{-s(a(\phi)+nc(\psi))},$$

where Frob denotes a geometric Frobenius element of $\mathbf{W}_F$.

Lemma 2.3. *For any elliptic $\phi \in \Phi(GL_n(F))$*

$$(15) \qquad d(\phi) := \begin{cases} 0 & \text{if } \mathbf{I}_F \subset \ker(\phi), \\ \dfrac{a(\phi)}{n} - 1 & \text{otherwise,} \end{cases}$$

Proof. This was proved in [ABPS2, Lemma 4.4] under the additional assumption $\mathrm{SL}_2(\mathbb{C}) \subset \ker \phi$. We will reduce to that special case.

Since ϕ is elliptic, it defines an irreducible n-dimensional representation V of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$. Hence there are irreducible representations (ϕ_1, V_1) of $\mathbf{W}_F$ and (ϕ_2, V_2) of $\mathrm{SL}_2(\mathbb{C})$ such that

$$(16) \qquad (\phi, V) = (\phi_1, V_1) \otimes (\phi_2, V_2).$$

In particular $V^{\mathbf{I}_F} = V_1^{\mathbf{I}_F} \otimes V_2$. Suppose first that $V_1^{\mathbf{I}_F} = V_1$. Then $\mathbf{I}_F \subset \ker \phi$, so $d(\phi) = 0$ by definition. Now suppose $V_1^{\mathbf{I}_F} \neq V_1$. As (ϕ_1, V_1) is irreducible and $\mathbf{I}_F$ is normal in $\mathbf{W}_F$, we must have $V_1^{\mathbf{I}_F} = 0$. Hence $V^{\mathbf{I}_F} = 0$, which by (12) and (13) implies $a(\phi) = a(\phi_0)$. By [Ser, Corollary VI.2.1'] $a(\phi_0)$ is additive in V and depends only on

$$\phi_0|_{\mathbf{I}_F} = \phi|_{\mathbf{I}_F} = \phi_1|_{\mathbf{I}_F} \otimes \operatorname{id}_{V_2}.$$

Now it follows from (16) that

$$(17) \qquad a(\phi) = a(\phi_1) \dim V_2 = n a(\phi_1)/\dim V_1.$$

As $\ker \phi_1 \supset \mathrm{SL}_2(\mathbb{C})$ we may apply [ABPS2, Lemma 4.4], which together with (17) gives

$$d(\phi_1) = \frac{a(\phi_1)}{\dim V_1} - 1 = \frac{a(\phi)}{n} - 1.$$

To conclude, we note that $d(\phi_1) = d(\phi)$ by (16). $\square$

2.4. The depth of representations of $\mathrm{GL}_m(D)$.

Let $k_D = \mathfrak{o}_D/\mathfrak{p}_D$ be the residual field of D. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in $\mathrm{M}_m(D)$. The Jacobson radical of $\mathfrak{A}$ will be denoted by $\mathfrak{P}$. Let $r = e_D(\mathfrak{A})$ and $e = e_F(\mathfrak{A})$ denote the integers defined by $\mathfrak{p}_D\mathfrak{A} = \mathfrak{P}^r$ and $\mathfrak{p}_F\mathfrak{A} = \mathfrak{P}^e$, respectively. We have

$$(18) \qquad e_F(\mathfrak{A}) = d\, e_D(\mathfrak{A}).$$

The normalizer in G of $\mathfrak{A}^{\times}$ will be denoted by

$$\mathfrak{K}(\mathfrak{A}) := \left\{ g \in G \,:\, g^{-1}\mathfrak{A}^{\times}g = \mathfrak{A}^{\times} \right\}.$$

Define a sequence of compact open subgroups of $G = \mathrm{GL}_m(D)$ by

$$U^0(\mathfrak{A}) := \mathfrak{A}^{\times}, \quad \text{and} \quad U^j(\mathfrak{A}) := 1 + \mathfrak{P}^j, \;\; j \geq 1.$$

Then $\mathfrak{A}^{\times}$ is a parahoric subgroup of G and $U^1(\mathfrak{A})$ is its pro-unipotent radical. We define the *normalized level* of an irreducible representation π of G to be

$$(19) \qquad\qquad d(\pi) := \min\left\{ j/e_F(\mathfrak{A}) \right\},$$

where $(j, \mathfrak{A})$ ranges over all pairs consisting of an integer $j \geq 0$ and a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{M}_m(D)$ such that π contains the trivial character of $U^{j+1}(\mathfrak{A})$.

Remark 2.4. When π is a representation of $\mathrm{GL}_n(F)$, our notion of normalized level coincides with that of [BuHe2, § 12.6]. However when π is a representation of $D^{\times}$ (or more generally of $\mathrm{GL}_m(D)$), the normalized level of π as defined above is not equal to the level $\ell_D(\pi)$ defined in [BuHe2, § 54.1] (resp. $\ell(\pi)$ defined by Broussous in [BaBr, Théorème A.1.2]): we have

$$d(\pi) = \frac{1}{d}\,\ell_D(\pi) \quad (\text{resp.} \quad d(\pi) = \frac{1}{d}\,\ell(\pi).$$

This reflects the fact that we have divided by $e_F(\mathfrak{A})$ instead of $e_D(\mathfrak{A})$ in Eqn. (19).

The following proposition will allow to use both results that were written in the setting of the normalized level, as general results on the depth in the sense of Moy and Prasad.

Proposition 2.5. *The normalized level of $\pi \in \mathrm{Irr}(G)$ equals its Moy–Prasad depth.*

Proof. Let us denote the Moy–Prasad depth of (π, V_π) by $d_{\mathrm{MP}}(\pi)$ for the duration of this proof. For any point x of the Bruhat–Tits building $\mathcal{B}(G)$ of G, consider the Moy–Prasad filtrations $P_{x,r}, P_{x,r+}$ $(r \in \mathbb{R}_{\geq 0})$ of the parahoric subgroup $P_x \subset G$ [MoPr1, §2]. We normalize these filtrations by using the valuation on F_s which maps $F^{\times}$ onto $\mathbb{Z}$. Then $d_{\mathrm{MP}}(\pi)$ is the minimal $r \in \mathbb{R}_{\geq 0}$ such that $V_\pi^{P_{x,r+}} \neq 0$ for some $x \in \mathcal{B}(G)$, see [MoPr2, §3.4].

Any hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{M}_m(D)$ is associated to a unique facet $\mathcal{F}(\mathfrak{A})$ of $\mathcal{B}(G)$. The filtration $\{U^j(\mathfrak{A}) \mid j \in \mathbb{Z}_{\geq 0}\}$ was compared with the Moy–Prasad groups for $x \in \mathcal{F}(\mathfrak{A})$ by Broussous and Lemaire. Let $x_{\mathfrak{A}}$ be the barycenter of $\mathcal{F}(\mathfrak{A})$. From [BrLe, Proposition 4.2 and Appendix A] and the definition of $e_F(\mathfrak{A})$ we see that

$$U^j(\mathfrak{A}) = P_{x_{\mathfrak{A}}, j/e_F(\mathfrak{A})} \text{ for all } j \in \mathbb{Z}_{\geq 0}.$$

Hence the definitions of the normalized level and the Moy–Prasad depth are almost equivalent, the only difference being that for $d_{\mathrm{MP}}(\pi)$ we must consider all points of $\mathcal{B}(G)$, whereas for $d(\pi)$ we may only use barycenters of facets of $\mathcal{B}(G)$. Thus it remains to check the following claim: there exists a facet $\mathcal{F}$ of $\mathcal{B}(G)$ with barycenter $x_{\mathcal{F}}$, such that V_π has nonzero $P_{x_{\mathcal{F}}, d_{\mathrm{MP}}(\pi)+}$-invariant vectors.

This is easy to see with the explicit constructions of the groups $P_{x,r}$ at hand, but we prefer not to delve into those details here. In fact, since every chamber of $\mathcal{B}(G)$ intersects every G-orbit in $\mathcal{B}(G)$, it suffices to consider facets contained in the closure of a fixed "standard" chamber. Then the claim becomes equivalent to saying that $x_{\mathcal{F}}$ is an "optimal point" in the sense of [MoPr1, §6.1]. That is assured by [MoPr1, Remark 6.1], which is applicable because the root system of G is of type A_{m-1}. $\square$

2.5. Conductors of representations of $\mathrm{GL}_m(D)$.

Let $\epsilon(s, \pi, \psi)$ denote the Godement–Jacquet local constant [GoJa]. It takes the form

(20) $$\epsilon(s, \pi, \psi) = \epsilon(0, \pi, \psi)\, q^{-f(\pi,\psi)s}, \quad \text{where } \epsilon(0, \pi, \psi) \in \mathbb{C}^\times.$$

Recall that $c(\psi)$ is the largest integer c such that $\mathfrak{p}_F^{-c} \subset \ker \psi$. In the previous section we had $c(\psi) = -1$.

A representation of $D^\times$ is called *unramified* if it is trivial on $\mathfrak{o}_D^\times$. An unramified representation of $D^\times$ is a character and has depth zero.

Proposition 2.6. *Let π be a supercuspidal irreducible representation of G. We have*

(21) $$f(\pi, \psi) = \begin{cases} n\,(c(\psi)+1) - 1 & \text{if } m = 1 \text{ and } \pi \text{ is unramified,} \\ n\,(d(\pi) + 1 + c(\psi)) & \text{otherwise.} \end{cases}$$

Proof. We suppose first that $m = 1$ (so $d = n$) and π is unramified. The required formula can be read off from (10) if $c(\psi) = -1$. For general ψ, applying [BuFr1, Theorem 3.2.11] and taking in account [BuFr1, (1.2.7), (1.2.8), (1.2.10)], we obtain

$$f(\pi, \psi) = (d(1 - d - dc(\psi)) \cdot (-\tfrac{1}{d}) = d + dc(\psi) - 1.$$

Hence the first case of Eqn. (21) holds.

From now on, we will assume that $m \geq 2$ or π is ramified. Then by combining [BaBr, Théorème A.2.1] with the fact that the Godement–Jacquet L-function $L(s, \pi)$ is 1, we see that π satisfies the conditions of Theorem 3.3.8 of [BuFr2]. Choose $(j, \mathfrak{A})$ as in (19), so $d(\pi) = j/e_F(\mathfrak{A})$ and π contains the trivial representation of $U^{j+1}(\mathfrak{A})$. Recall that $n = md$. By applying the formula of [BuFr2, Theorem 3.3.8 (iv)], we obtain

$$q^{f(\pi,\psi)} = \left[\mathfrak{A} : \mathfrak{p}_F^{c(\psi)+1}\mathfrak{P}^j\right]^{1/n}.$$

On the other hand, the $\mathfrak{o}_F$-order $\mathfrak{A}$ is G-conjugate to the standard principal $\mathfrak{o}_F$-order of $\mathrm{M}_m(D)$ defined by the partition $(t, \ldots, t)$ (r-times) of m, where $m = rt$ and $r = e_D(\mathfrak{A})$. Hence we have $\mathfrak{A}/\mathfrak{P} \simeq (\mathrm{M}_t(k_D))^r$. It follows that

$$[\mathfrak{A} : \mathfrak{P}] = (q^d)^{t^2 r} = q^{drt^2}.$$

Hence we get

$$f(\pi, \psi) = \frac{drt^2(j + e + ec(\psi))}{n} = n\left(\frac{j}{e} + 1 + c(\psi)\right),$$

since $drt^2 = nt = n^2/e$.

On the other hand it follows from [SéSt1, Corollaire 5.22] that there exists a maximal simple type (J, λ) in G, and an extension Λ of λ to the normalizer $\bar{J} = \mathrm{N}_G(\lambda)$ of λ, such that

$$\pi = \mathrm{c-Ind}_{\bar{J}}^G \Lambda.$$

By the construction of the type (J, λ), we have $d(\pi) \leq j/e$. Conversely, let $[\mathfrak{A}', j', j'-1, \beta']$ be a stratum contained in π. Then if $[\mathfrak{A}', j', j'-1, \beta']$ is such that its normalized level j'/e' is minimal among the normalized levels of all the strata contained in π, it is necessarily fundamental [Bro, Theorem 1.2.1. (ii)]. Since all the fundamental strata contained in π have the same normalized level [BaBr, Théorème A.1.2], we get $j/e = d(\pi)$. $\qquad\square$

Theorem 2.7 below proves the validity of Conjecture 4.3 of [LaRa]. In the case when F has characteristic 0, it is due to Lansky and Raghuram for the groups $\mathrm{GL}_n(F)$ and $D^\times$, [LaRa, Theorem 3.1], and for certain representations of $\mathrm{GL}_2(D)$, [LaRa, Theorem 4.1]. Our proof is inspired by those of these results.

Theorem 2.7. *The depth $d(\pi)$ and the conductor $f(\pi) := f(\pi, \psi) - nc(\psi)$ of each essentially square-integrable irreducible representation π of $\mathrm{GL}_m(D)$ are linked by the following relation:*

$$
(22) \qquad d(\pi) = \begin{cases} 0 & \text{if } \pi \text{ is an unramified twist of } \mathrm{St}_{\mathrm{GL}_m(D)} \\ \dfrac{f(\pi) - n}{n} & \text{otherwise.} \end{cases},
$$

In particular

$$
(23) \qquad d(\pi) = \max\left\{ \frac{f(\pi) - n}{n}, 0 \right\}.
$$

Proof. Let $\pi \in \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_m(D))$. We use the same notation as for π' in the proof of Theorem 2.2.h, so π is consituent of

$$
I^{\mathrm{GL}_m(D)}_{\mathrm{GL}_a(D)^b}\left(\nu_D^{s_\sigma \frac{(1-b)}{2}} \sigma \otimes \nu_D^{s_\sigma \frac{(3-b)}{2}} \sigma \otimes \cdots \otimes \nu_D^{s_\sigma \frac{(b-1)}{2}} \sigma \right),
$$

where $\sigma \in \mathrm{Irr}(\mathrm{GL}_a(D))$ is supercuspidal. Since the depth is preserved by parabolic induction [MoPr2, Theorem 5.2], we get

$$
d(\pi) = d\left(\nu_D^{s_\sigma \frac{(1-b)}{2}} \sigma \otimes \nu_D^{s_\sigma \frac{(3-b)}{2}} \sigma \otimes \cdots \otimes \nu_D^{s_\sigma \frac{(b-1)}{2}} \sigma \right).
$$

It follows that

$$
(24) \qquad d(\pi) = d(\sigma).
$$

We will apply Proposition 2.6 to the supercuspidal representation σ of $\mathrm{GL}_a(D)$. In the special case σ is an unramified representation of $D^\times$ (hence $a = 1$ in this case), Eqn. (21) gives

$$
f(\sigma, \psi) = d\left(c(\psi) + 1 \right) - 1,
$$

that is, $f(\sigma) = d - 1$. Hence we get

$$
\frac{f(\sigma) - d}{d} = -\frac{1}{d}.
$$

Then it implies that

$$
\max\left\{ \frac{f(\sigma) - d}{d}, 0 \right\} = \max\left\{ -\frac{1}{d}, 0 \right\} = 0 = d(\sigma),
$$

in other words, Eqn. (23) holds for the unramified representations of $D^\times$.

In the other cases (that is, $a \neq 1$ or σ is ramified), (21) gives $f(\sigma) = ad(d(\sigma)+1)$, that is,

$$
(25) \qquad \frac{f(\sigma)}{ad} = d(\sigma) + 1.
$$

Since $d(\sigma) \geq 0$ (by definition of the depth), we obtain that

$$
(26) \qquad d(\sigma) = \max\left\{ \frac{f(\sigma) - ad}{ad}, 0 \right\}.
$$

Hence (22) holds for every supercuspidal irreducible representation of $\mathrm{GL}_a(D)$, with $a \geq 1$ an arbitrary integer.

Recall that s_σ is an integer dividing ad, say $ad = a^* s_\sigma$ with $a^* \in \mathbb{Z}$. The image $\mathrm{JL}^{-1}(\sigma)$ of σ under the Jacquet-Langlands correspondence is equivalent to the Langlands quotient of the parabolically induced representation

$$I_{\mathrm{GL}_{a^*}(F)^{s_\sigma}}^{\mathrm{GL}_{a^* s_\sigma}(F)} (\nu_F^{\frac{(1-s_\sigma)}{2}} \sigma^* \otimes \nu_F^{\frac{(3-s_\sigma)}{2}} \sigma^* \otimes \cdots \otimes \nu_F^{\frac{(s_\sigma-1)}{2}} \sigma^*),$$

where σ^* is a unitary supercuspidal irreducible representation of $\mathrm{GL}_{a^*}(F)$ and $\nu_F(g^*) = |\det(g^*)|_F$.

The representation $\mathrm{JL}^{-1}(\pi)$ is equivalent to a constituent of the parabolically induced representation

$$I_{\mathrm{GL}_{a^*}(F)^{bs_\sigma}}^{\mathrm{GL}_{adb}(F)} (\nu_F^{\frac{(1-bs_\sigma)}{2}} \sigma^* \otimes \nu_F^{\frac{(3-bs_\sigma)}{2}} \sigma^* \otimes \cdots \otimes \nu_F^{\frac{(bs_\sigma-1)}{2}} \sigma^*).$$

We recall from [Hen, § 2.6] the formula describing the epsilon factor of $\mathrm{JL}^{-1}(\pi)$ in terms of the local factors of σ^*:

$$(27) \qquad \epsilon(s, \mathrm{JL}^{-1}(\pi), \psi) = \prod_{i=0}^{bs_\sigma-1} \epsilon(s+i, \sigma^*, \psi) \prod_{j=0}^{bs_\sigma-2} \frac{L(-s-j, \check{\sigma}^*)}{L(s+j, \sigma^*)}.$$

Since the Jacquet–Langlands correspondence preserves the ϵ-factors (see Theorem 2.2 i) we have

$$\epsilon(s, \mathrm{JL}^{-1}(\pi), \psi) = \epsilon(s, \pi, \psi).$$

Thus we have obtained the following formula

$$(28) \qquad \epsilon(s, \pi, \psi) = \prod_{i=0}^{bs_\sigma-1} \epsilon(s+i, \sigma^*, \psi) \prod_{j=0}^{bs_\sigma-2} \frac{L(-s-j, \check{\sigma}^*)}{L(s+j, \sigma^*)}.$$

If $\pi = \mathrm{St}_{\mathrm{GL}_m(D)} \otimes \chi$ for some unramified character χ of $D^\times$, it follows from (10) that $f(\pi, \psi) = -1$ in the case where $c(\psi) = -1$, hence we obtain

$$(29) \qquad\qquad\qquad f(\pi) = n - 1.$$

From now on we assume π is not equivalent to a representation of the form $\mathrm{St}_{\mathrm{GL}_m(D)} \otimes \chi$, with χ an unramified character of $D^\times$ (that is, we have $m \neq 1$ or σ ramified). Then Theorem 2.2 b and f implies that similarly $\mathrm{JL}^{-1}(\pi)$ is not a twist of $\mathrm{St}_{\mathrm{GL}_n(F)}$ by an unramified character of $F^\times$. Thus we have $a^* \neq 1$ or σ^* ramified. It follows that $L(-s-j, \check{\sigma}^*) = L(s+j, \sigma^*) = 1$, and we obtain from (28) that

$$(30) \qquad\qquad\qquad f(\pi) = bs_\sigma \, f(\sigma^*).$$

In the special case when $b = 1$ the equation (30) gives

$$(31) \qquad\qquad\qquad f(\sigma) = s_\sigma \, f(\sigma^*).$$

Then using (24) and (26) we get

$$(32) \qquad d(\pi) = d(\sigma) = \max\left\{ \frac{bs_\sigma f(\sigma^*) - bad}{bad}, 0 \right\} = \max\left\{ \frac{f(\pi) - n}{n}, 0 \right\}. \qquad\qquad \square$$

2.6. **Depth preservation.**

Corollary 2.8. *The Jacquet–Langlands correspondence preserves the depth of essentially square-integrable representations of* $\mathrm{GL}_m(D)$.

Proof. Theorem 2.2.i shows in particular that the Jacquet–Langlands correspondence preserves conductors. Now Theorem 2.7 shows that it preserves depths as well. □

Theorems 2.7 and 2.2 are also the crucial steps to show that the local Langlands correspondence for inner forms of $\mathrm{GL}_m(D)$ preserves depths. With similar considerations we show that it also preserves L-functions, ϵ-factors and γ-factors. We abbreviate these three to "local factors". For the basic properties of the local factors of Langlands parameters we refer to [Tat].

Theorem 2.9. *The local Langlands correspondence for representations of* $\mathrm{GL}_m(D)$ *preserves L-functions, ϵ-factors, γ-factors and depths. In other words, for every irreducible smooth representation π of* $\mathrm{GL}_m(D)$:

$$
\begin{aligned}
L(s,\pi) &= L(s,\mathrm{rec}_{D,m}(\pi)), \\
\epsilon(s,\pi,\psi) &= \epsilon(s,\mathrm{rec}_{D,m}(\pi),\psi), \\
\gamma(s,\pi,\psi) &= \gamma(s,\mathrm{rec}_{D,m}(\pi),\psi) \\
d(\pi) &= d(\mathrm{rec}_{D,m}(\pi)).
\end{aligned}
$$

Proof. It is well-known that the local Langlands correspondence for $\mathrm{GL}_n(F)$ preserves local factors, see the introduction of [HaTa].

Assume first that π is essentially square-integrable. Recall the notations of the ϵ factors of π and of $\phi := \mathrm{rec}_{D,m}(\pi) \in \Phi(\mathrm{GL}_m(D))$ from (11) and (20). By definition

$$\mathrm{rec}_{D,m}(\pi) = \mathrm{rec}_{F,n}(\mathrm{JL}^{-1}(\pi)),$$

so by Theorem 2.2 $\mathrm{rec}_{D,m}$ preserves the ϵ-factors of π:

(33) $\qquad \epsilon(0,\phi,\psi)\, q^{-s(a(\phi)+nc(\psi))} = \epsilon(s,\phi,\psi) = \epsilon(s,\pi,\psi) = \epsilon(0,\pi,\psi)q^{-sf(\pi,\psi)}.$

Hence, with the notation from Theorem 2.7:

(34) $\qquad\qquad\qquad f(\pi) = f(\pi,\psi) - nc(\psi) = a(\phi).$

The properties of $\mathrm{rec}_{F,n}$ imply that ϕ is elliptic. By combining Lemma 2.3 with Theorem 2.7 and (34), we obtain that $d(\phi) = d(\pi)$ whenever π is essentially square-integrable.

Now let π be any irreducible representation of $\mathrm{GL}_m(D)$. By the Langlands classification, there exist a parabolic subgroup $P \subset \mathrm{GL}_m(D)$ with Levi factor M and an irreducible essentially square-integrable representation ω of M, such that π is a quotient of $I_P^{\mathrm{GL}_m(D)}(\omega)$. Moy and Prasad proved in [MoPr2, Theorem 5.2] that π and ω have the same depth. By [Jac, Theorem 3.4] π and ω have the same L-functions and ϵ-factors and by [Jac, (2.3) and (2.7.3)] they also have the same γ-factors.

On the other hand, M is isomorphic to a product of groups of the form $\mathrm{GL}_{m_i}(D)$, so the local Langlands correspondence for M is simply the product of that for the $\mathrm{GL}_{m_i}(D)$. The Langlands parameters $\mathrm{rec}_{D,m}(\pi)$ and $\mathrm{rec}_M(\omega)$ are related via an inclusion of the complex dual groups $\prod_i \mathrm{GL}_{dm_i}(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$. Hence these two Langlands parameters also have the same depth and local factors.

Because we already proved that the LLC preserves depths for essentially square-integrable representations of $\mathrm{GL}_m(D)$ or M, we can conclude that

$$d(\pi) = d(\omega) = d(\mathrm{rec}_M(\omega)) = d(\mathrm{rec}_{D,m}(\pi)),$$

and similarly for the local factors. □

3. The local Langlands correspondence for inner forms of $\mathrm{SL}_n(F)$

3.1. The statement of the correspondence.

Recall that F is a non-archimedean local field and that the equivalence classes of inner forms of $\mathrm{SL}_n(F)$ are in bijection with the isomorphism classes of central simple F-algebras of dimension n^2, via $\mathrm{M}_m(D) \mapsto \mathrm{GL}_m(D)_{\mathrm{der}}$. As mentioned after Theorem 2.1, every character of $Z(\mathrm{SL}_n(\mathbb{C}))$ gives rise to such an algebra via the Kottwitz isomorphism.

The local Langlands correspondence for $\mathrm{GL}_m(D)_{\mathrm{der}}$ is implied by that for $\mathrm{GL}_m(D)$, in the following way. A Langlands parameter

$$\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$$

which is relevant for $\mathrm{GL}_m(D)_{\mathrm{der}}$ can be lifted it to a Langlands parameter

$$\overline{\phi} : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$$

which is relevant for $\mathrm{GL}_m(D)$, by [Wei]. Then $\mathrm{rec}^{-1}_{m,D}(\overline{\phi})$ is an irreducible representation of $\mathrm{GL}_m(D)$ which, upon restriction to $\mathrm{GL}_m(D)_{\mathrm{der}}$, decomposes as a finite direct sum of irreducible representations. The packet $\Pi_\phi(\mathrm{GL}_m(D)_{\mathrm{der}})$ is defined as the set of irreducible constituents of $\mathrm{Res}^{\mathrm{GL}_m(D)}_{\mathrm{GL}_m(D)_{\mathrm{der}}} \mathrm{rec}^{-1}_{m,D}(\overline{\phi})$.

For these groups it is more interesting to consider the enhanced Langlands correspondence, where ϕ is supplemented with an irreducible representation of a finite group. In addition to the groups defined in (3), we write

$$(35) \qquad \mathcal{Z}_\phi = Z(\mathrm{SL}_n(\mathbb{C}))/Z(\mathrm{SL}_n(\mathbb{C})) \cap C(\phi)^\circ \cong Z(\mathrm{SL}_n(\mathbb{C}))C(\phi)^\circ/C(\phi)^\circ.$$

Any character of $\mathcal{Z}_\phi$ determines a character of $Z(\mathrm{SL}_n(\mathbb{C}))$, and hence an inner form of $\mathrm{SL}_n(F)$. An enhanced Langlands parameter is a pair (ϕ, ρ) with $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$. The groups in (3), (35) are related to the more usual component group

$$S_\phi := Z_{\mathrm{PGL}_n(\mathbb{C})}(\mathrm{im}\ \phi)/Z_{\mathrm{PGL}_n(\mathbb{C})}(\mathrm{im}\ \phi)^\circ$$

by the short exact sequence

$$1 \to \mathcal{Z}_\phi \to \mathcal{S}_\phi \to S_\phi \to 1.$$

Hence $\mathcal{S}_\phi$ has more irreducible representations than S_ϕ. Via the enhanced Langlands correspondence the additional ones are associated to irreducible representations of non-split inner forms of $\mathrm{SL}_n(F)$. The following result is due to Hiraga and Saito [HiSa, Theorem 12.7] for generic representations of $\mathrm{GL}_m(D)$ when char $F = 0$.

Theorem 3.1. [ABPS2, Theorem 1.2]
There exists a bijective correspondence between:

- *pairs $(\mathrm{GL}_m(D)_{\mathrm{der}}, \pi)$ with $\pi \in \mathrm{Irr}(\mathrm{GL}_m(D)_{\mathrm{der}})$ and $\mathrm{GL}_m(D)_{\mathrm{der}}$ an inner form of $\mathrm{SL}_n(F)$, considered up to equivalence;*
- *$\mathrm{SL}_n(\mathbb{C})$-conjugacy classes of pairs (ϕ, ρ) with $\phi \in \Phi(\mathrm{SL}_n(F))$ and $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$.*

Here the group $\mathrm{GL}_m(D)_{\mathrm{der}}$ *determines* $\rho|_{\mathcal{Z}_\phi}$ *and conversely. The correspondence satisfies the desired properties from* [Bor, §10.3], *with respect to restriction from inner forms of* $\mathrm{GL}_n(F)$, *temperedness and essential square-integrability of representations.*

We remark that the above bijection need not be canonical if $\Pi_\phi(\mathrm{GL}_m(D)_{\mathrm{der}})$ has more than one element.

3.2. The depth of representations of $\mathrm{GL}_m(D)_{\mathrm{der}}$.

For the depth of an irreducible representation of $\mathrm{GL}_m(D)_{\mathrm{der}}$ there are two candidates. Besides the Moy–Prasad depth one can define the normalized level, just as in (19). This was done for representations of $\mathrm{SL}_n(F)$ in [BuKu]. However, Proposition 2.5 quickly reveals that these two notions agree:

Proposition 3.2. *The Moy–Prasad depth of an irreducible representation of* $\mathrm{GL}_m(D)_{\mathrm{der}}$ *equals its normalized level.*

Proof. Let us compare the descriptions of the two kinds of depth with those given in the proof of Proposition 2.5. By definition $\mathrm{GL}_m(D)$ and $\mathrm{GL}_m(D)_{\mathrm{der}}$ have the same Bruhat–Tits building. The Moy–Prasad depth is defined in terms of the groups

$$(36) \qquad P'_{x,r} = P_{x,r} \cap \mathrm{GL}_m(D)_{der} \quad \text{with} \quad x \in \mathcal{B}(GL_m(D)).$$

The normalized level is expressed with the groups

$$U^j(\mathfrak{A})' = U^j(\mathfrak{A}) \cap \mathrm{GL}_m(D)_{der},$$

where $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in $M_m(D)$. With these groups instead of $P_{x,r}$ and $U^j(\mathfrak{A})$ the entire proof of Proposition 2.5 carries over to $\mathrm{GL}_m(D)_{der}$. $\square$

It turns out that the depth of an irreducible $\mathrm{GL}_m(D)_{\mathrm{der}}$-representation π behaves nicely with respect to restriction from $\mathrm{GL}_m(D)$. To be precise, equals the minimum of the depths of the irreducible $\mathrm{GL}_m(D)$-representations that contain π. (Notice that this minimum is always attained because all depths for inner forms of $\mathrm{GL}_n(F)$ lie in $\frac{1}{n}\mathbb{Z}$.)

Proposition 3.3. *Let* $\pi \in \mathrm{Irr}(\mathrm{GL}_m(D)_{\mathrm{der}})$ *and let* $\overline{\pi} \in \mathrm{Irr}(\mathrm{GL}_m(D))$ *be such that*

- π *is a direct summand of* $\mathrm{Res}^{\mathrm{GL}_m(D)}_{\mathrm{GL}_m(D)_{\mathrm{der}}}(\overline{\pi})$;
- $d(\overline{\pi}) \leq d(\overline{\pi} \otimes \chi \circ \mathrm{Nrd})$ *for every character* χ *of* $F^\times$.

Then $d(\pi) = d(\overline{\pi})$.

Proof. In the case $G = \mathrm{GL}_n(F)$, this is guaranteed by Proposition 3.2 and [BuKu, Proposition 1.7.iii]. The same proof works for $\mathrm{GL}_m(D)$ but this would be cumbersome, one would have to check that everything in [BuKu, p.265–268] also works with a division algebra instead of a field.

Instead, we select some parts of [BuKu, §1] to provide a shorter proof. Pick a $x \in \mathcal{B}(G)$ such that $(\overline{\pi}, V)$ has nonzero vector fixed by $P_{x,d(\overline{\pi})+}$. Then

$$V^{P'_{x,d(\overline{\pi})+}} \supset V^{P_{x,d(\overline{\pi})+}} \neq 0,$$

so there is an irreducible $\mathrm{GL}_m(D)_{\mathrm{der}}$-subrepresentation (π_1, V_1) of $\overline{\pi}$ with

$$V_1^{P'_{x,d(\overline{\pi})+}} \neq 0 \quad \text{and} \quad d(\pi_1) \leq d(\overline{\pi}).$$

Since $\overline{\pi}$ is irreducible, π_1 is isomorphic to a $\mathrm{GL}_m(D)$-conjugate of π. Conjugation by $g \in \mathrm{GL}_m(D)$ sends any Moy–Prasad group $P_{y,r}$ to $P_{g(y),r}$. So this operation preserves depths and

$$(37) \qquad\qquad d(\pi) = d(\pi_1) \le d(\overline{\pi}).$$

Suppose now that $d(\pi) < d(\overline{\pi})$. Take a nonzero $v \in V^{P'_{x,d(\pi)+}}$ and consider

$$V_v := \mathrm{span}\{\overline{\pi}(g)v \mid g \in P_{x,d(\pi)+}\}.$$

As $P'_{x,d(\pi)+}$ is normal in $P_{x,d(\pi)+}$, it acts trivially on V_v, and V_v can be regarded as representation of

$$P_{x,d(\pi)+}/P'_{x,d(\pi)+} \cong \mathrm{Nrd}(P_{x,d(\pi)+}) \subset F^\times.$$

Hence there is a character χ of $F^\times$ such that $\chi^{-1} \circ \mathrm{Nrd}$ appears in the action of $P_{x,d(\pi)+}$ on V_v. Then the irreducible $\mathrm{GL}_m(D)$-representation $\overline{\pi} \otimes \chi \circ \mathrm{Nrd}$ has a nonzero vector fixed by $P_{x,d(\pi)+}$, so

$$d(\overline{\pi} \otimes \chi \circ \mathrm{Nrd}) \le d(\pi) < d(\overline{\pi}).$$

This contradicts the assumptions of proposition, so (37) must be an equality. $\square$

3.3. The depth of Langlands parameters for $\mathrm{GL}_m(D)_{\mathrm{der}}$.

The depth of a Langlands parameter $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$ for an inner form of $\mathrm{SL}_n(F)$ is defined as in Section 2.3:

$$d(\phi) = \inf\{l \in \mathbb{R}_{\ge 0} \mid \mathrm{Gal}(F_s/F)^{l+} \subset \ker \phi\}.$$

The following result may be considered as the non-archimedean analogue of [ChKa, Theorem 1] in the case of the geometric local Langlands correspondence.

Corollary 3.4. *Let* $\pi \in \mathrm{Irr}(\mathrm{GL}_m(D)_{\mathrm{der}})$ *with Langlands parameter* $\phi \in \Phi(\mathrm{SL}_n(F))$. *Then* $d(\pi) \ge d(\phi)$.

Proof. Let $\overline{\pi}$ be as in Proposition 3.3, so $d(\overline{\pi}) = d(\pi)$. Put $\overline{\phi} = \mathrm{rec}_{D,m}(\overline{\pi})$, this is a lift of ϕ to $\mathrm{GL}_n(\mathbb{C})$ and Theorem 2.9 says that $d(\overline{\phi}) = d(\overline{\pi})$.

We remark that by the compatibility of the LLC with character twists

$$(38) \qquad\qquad d(\overline{\phi}) \le d(\overline{\phi} \otimes \gamma) \text{ for every character } \gamma \text{ of } \mathbf{W}_F.$$

For any lift $\overline{\phi} \in \Phi(\mathrm{GL}_n(F))$ of ϕ we have $\ker \overline{\phi} \subset \ker \phi$, so $d(\overline{\phi}) \ge d(\phi)$. $\square$

It is possible that the inequality in Corollary 3.4 is strict. The following example was pointed out to the authors by Mark Reeder.

Example 3.5. Take $F = \mathbb{Q}_2$ and a Langlands parameter $\phi : \mathbf{W}_{\mathbb{Q}_2} \to \mathrm{PGL}_2(\mathbb{C})$ which is trivial on $\mathrm{SL}_2(\mathbb{C})$ and has image isomorphic to the symmetric group S_4. (Such a L-parameter exists, see for example [Wei].) We claim that $d(\phi) = 1/3$.
Let Ad denote the adjoint representation of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathfrak{sl}_2(\mathbb{C}) = \mathrm{Lie}(\mathrm{PGL}_2(\mathbb{C}))$. Then $\mathrm{Ad}\circ\phi$ is an irreducible 3-dimensional representation of $\mathbf{W}_{\mathbb{Q}_2}$. Since $\mathrm{PGL}_2(\mathbb{C})$ is the adjoint group of $\mathfrak{sl}_2(\mathbb{C})$, $\mathrm{Ad}\circ\phi$ has the same kernel and hence the same depth as ϕ. One can check that $\mathrm{Ad}(\phi(\mathbf{I}_F)) \cong A_4$ and that the image of the wild inertia subgroup $\mathbf{P}_F$ is isomorphic to the Klein four group. With the formula [GrRe, (1)] for the Artin conductor we find that $a(\mathrm{Ad} \circ \phi) = 4$. By Lemma 2.3 (with $n = 3$) $d(\mathrm{Ad} \circ \phi) = 1/3$.

Let $\overline{\phi} : \mathbf{W}_{\mathbb{Q}_2} \to \mathrm{GL}_2(\mathbb{C})$ be a lift of ϕ. This is an irreducible 2-dimensional representation. We claim that $d(\overline{\phi}) \geq 1/2$.

With a suitable basis transformation we can achieve that

$$\phi(\mathbf{P}_{\mathbb{Q}_2}) = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\} \subset \mathrm{PGL}_2(\mathbb{C}).$$

Let $1, w_2, w_3, w_4 \in \mathbf{P}_{\mathbb{Q}_2}$ be preimages of these elements. Irrespective of the choice of the lift of ϕ we have

$$[\overline{\phi}(w_3), \overline{\phi}(w_4)] = [\left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)] = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C}).$$

Now we see the problem: since $\phi|_{\mathbf{P}_{\mathbb{Q}_2}}$ is not a sum of characters, commutators pop up in $\overline{\phi}|_{\mathbf{P}_{\mathbb{Q}_2}}$, and these will cause more ramification.

Put $E = F_s^{\ker \overline{\phi}}$ and endow $\mathrm{Gal}(E/\mathbb{Q}_2) \cong \overline{\phi}(\mathbf{W}_{\mathbb{Q}_2})$ with the lower numbered filtration. The image of $\mathbf{P}_{\mathbb{Q}_2}$ is $\mathrm{Gal}(E/\mathbb{Q}_2)_1$ and $[w_3, w_4] \in [\mathrm{Gal}(E/\mathbb{Q}_2)_1, \mathrm{Gal}(E/\mathbb{Q}_2)_1]$. Then $[w_3, w_4] \in \mathrm{Gal}(E/\mathbb{Q}_2)_3$ by [Ser, Proposition IV.2.10], so $\overline{\phi}$ is nontrivial on this ramification group. If we lift ϕ with as little ramification as possible, $\overline{\phi}(\mathbf{W}_{\mathbb{Q}_2})$ is an index 2 central extension of S_4. Writing $d_j = |\overline{\phi}(\mathrm{Gal}(E/\mathbb{Q}_2)_j)|$, we have

$$d_0 = 24, d_1 = 8, d_2 = d_3 = 2 \text{ and } d_j = 1 \text{ for } j > 3.$$

The formula [GrRe, (1)] gives

$$a(\overline{\phi}) = \frac{\dim(\overline{\phi})}{d_0} \sum_{j \geq 0 : d_j > 1} d_j = \frac{2}{24}(24 + 8 + 2 + 2) = 3.$$

Now Lemma 2.3 says that $d(\overline{\phi}) = 1/2$.

To show that Corollary 3.4 is in many cases an equality, we will make use of several well-known properties of the Artin reciprocity map $\mathbf{a}_F : \mathbf{W}_F \to F^\times$. In particular:

Theorem 3.6. $\mathbf{a}_F(\mathrm{Gal}(F_s/F)^l) = U_F^{[l]}$ for all $l \in \mathbb{R}_{\geq 0}$.

Proof. For any finite abelian extension E/F, [Ser, Corollary 3 to Theorem XV.2.1] says that the Artin reciprocity map gives an isomorphism

(39) $$\mathbf{a}_F : \mathrm{Gal}(E/F)^l \to U_F^{[l]}/(\mathrm{N}_{E/F}(E^\times) \cap U_F^{[l]}).$$

Let F_s^{ab} be the maximal abelian extension of F contained in F_s. Taking the projective limit of (39) over all finite subextensions of F_s^{ab}/F, we obtain an isomorphism

$$\mathbf{a}_F : \mathrm{Gal}(F_s^{\mathrm{ab}}/F)^l \to U_F^{[l]}.$$

We note that $\mathrm{Gal}(F_s^{\mathrm{ab}}/F)$ is the quotient of $\mathrm{Gal}(F_s/F)$ modulo the closure of its commutator subgroup. Hence $\mathbf{a}_F : \mathbf{W}_F \to F^\times$ factors via $\mathrm{Gal}(F_s^{\mathrm{ab}}/F)$. $\square$

Recall from [BuHe1] that a Langlands parameter for $\mathrm{GL}_n(F)$ is *essentially tame* if its restriction to the wild inertia subgroup $\mathbf{P}_F$ of $\mathbf{W}_F$ is a direct sum of characters. Clearly $\overline{\phi}$ is essentially tame if and only if $\overline{\phi}(\mathbf{P}_F)$ lies in a maximal torus of $\mathrm{GL}_n(\mathbb{C})$, which in turn is equivalent to $\phi(\mathbf{P}_F)$ lying in a maximal torus of $\mathrm{PGL}_n(\mathbb{C})$.

Definition 3.7. *A Langlands parameter ϕ for an inner form of $\mathrm{SL}_n(F)$ is essentially tame if $\phi(\mathbf{P}_F)$ lies in a maximal torus of $\mathrm{PGL}_n(\mathbb{C})$.*

An important difference with Example 3.5 is that such a projective representation of $\mathbf{P}_F$ can be lifted to a n-dimensional representation of $\mathbf{P}_F$ with the same depth. By [BuHe1, Corollary A.4] any L-parameter for (an inner form of) $\mathrm{GL}_n(F)$ is essentially tame if the residual characteristic of F does not divide n. Our definition is such that the same holds for Langlands parameters for (inner forms of) $\mathrm{SL}_n(F)$.

For such L-parameters the LLC does preserve depths:

Theorem 3.8. *Let $\phi \in \Phi(\mathrm{SL}_n(F))$ be essentially tame and relevant for $\mathrm{GL}_m(D)_{\mathrm{der}}$. Then $d(\pi) = d(\phi)$ for every $\pi \in \Pi_\phi(\mathrm{GL}_m(D)_{\mathrm{der}})$.*

Proof. Let $\overline{\phi}$ be as in (38), so $d(\overline{\phi}) = d(\pi)$.

First we consider the case where $\overline{\phi}$ is an irreducible n-dimensional representation of $\mathbf{W}_F$. As $\overline{\phi}$ is essentially tame, [BuHe1, Theorem A.3] shows that there exist a finite, tamely ramified Galois extension E/F and a smooth character $\xi : \mathbf{W}_E \to \mathbb{C}^\times$ such that $\overline{\phi} = \mathrm{ind}_{\mathbf{W}_E}^{\mathbf{W}_F} \xi$. We may and will assume that E is contained in our chosen separable closure F_s of F. By Mackey's induction–restriction formula

$$\mathrm{Res}_{\mathbf{W}_E}^{\mathbf{W}_F}(\overline{\phi}) = \bigoplus\nolimits_{s \in \mathbf{W}_F/\mathbf{W}_E} \xi^s \ , \quad \text{where} \quad \xi^s(w) = \xi(s^{-1}ws).$$

The elements of $\mathbf{W}_F \setminus \mathbf{W}_E$ permute the $\mathbf{W}_E$-subrepresentations ξ^s nontrivially, so they cannot lie in the kernel of $\overline{\phi}$:

$$\ker \overline{\phi} = \mathbf{W}_E \cap \ker \overline{\phi} = \{w \in \mathbf{W}_E : \xi^s(w) = 1 \ \forall s \in \mathbf{W}_F\}.$$

Let $\mathrm{pr} \colon \mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$ be the canonical projection. Then $\phi = \mathrm{pr} \circ \overline{\phi}$ and

$$(40) \qquad \ker \phi = \overline{\phi}^{-1}(Z(\mathrm{GL}_n(\mathbb{C}))) = \{w \in \mathbf{W}_E : \xi^s(w) = \xi(w) \ \forall s \in \mathbf{W}_F\}.$$

Suppose that $d(\overline{\phi}) > d(\phi)$. In view of the definition of $d(\phi)$,

$$(41) \quad \ker \overline{\phi} \supset \mathbf{W}_E \cap \mathrm{Gal}(F_s/F)^{d(\overline{\phi})+}, \text{ but } \ker \overline{\phi} \not\supset \mathbf{W}_E \cap \mathrm{Gal}(F_s/F)^{d(\overline{\phi})} \subset \ker \phi.$$

The relation between the upper and the lower numbering of the filtration subgroups of $\mathbf{W}_F$ [Ser, §IV.3], combined with the compatibility of the lower numbering with subgroups [Ser, Proposition IV.1.2], provides a $l \in \mathbb{R}_{\geq 0}$ such that

$$(42) \qquad\qquad\qquad \mathbf{W}_E \cap \mathrm{Gal}(F_s/F)^{d(\overline{\phi})} = \mathrm{Gal}(F_s/E)^l.$$

In fact $l > 0$ because $d(\overline{\phi}) > d(\phi) \geq 0$. Since $\mathrm{Res}_{\mathbf{W}_E}^{\mathbf{W}_F} \overline{\phi}$ is a direct sum of characters, it factors through the Artin reciprocity map $\mathbf{a}_E : \mathbf{W}_E \to E^\times$. With (41) we see that

$$\mathbf{a}_E(\mathrm{Gal}(F_s/E)^l) \neq \mathbf{a}_E(\mathrm{Gal}(F_s/E)^{l+}).$$

By Theorem 3.6 applied to F_s/E, l must be a positive integer. When we transfer the conjugation action of $\mathbf{W}_F$ on $\mathbf{W}_E$ to $E^\times$ via Artin reciprocity, it becomes the standard action of $\mathrm{Gal}(E/F) \cong \mathbf{W}_F/\mathbf{W}_E$ on $E^\times$. Now (40) says that ξ is a $\mathrm{Gal}(E/F)$-invariant character of U_E^l. Since $l \in \mathbb{Z}_{>0}$ and E/F is tamely ramified, U_E^l is a cohomologically trivial $\mathrm{Gal}(E/F)$-module. According to [BuHe1, Lemma A.1], these properties imply that ξ factors through the norm map $\mathrm{N}_{E/F}$, and there is a unique smooth character

$$\xi' \text{ of } U_E^l \cap F^\times = \mathrm{N}_{E/F}(U_E^l) \text{ such that } \xi = \xi' \circ \mathrm{N}_{E/F} \text{ on } U_E^l.$$

Since $F^\times / \ker(\xi')$ is a finitely generated abelian group and $\mathbb{C}^\times$ is divisible, we can extend ξ' to a smooth character of $F^\times$. Via Artin reciprocity this yields a character ξ_F of $\mathbf{W}_F$. From (42) and the commutative diagram [Ser, §XI.3]

$$
\begin{array}{ccc}
\mathbf{W}_E & \longrightarrow & \mathbf{W}_F \\
\mathbf{a}_E \downarrow & & \downarrow \mathbf{a}_F \\
E^\times & \xrightarrow{\ N_{E/F}\ } & F^\times
\end{array}
$$

we see that $\xi_F = \xi$ on $\mathrm{Gal}(F_s/E)^{d(\overline{\phi})} \cap \mathbf{W}_E$. Then $\overline{\phi} \otimes \xi_F^{-1}$ is another lift of ϕ to $\Phi(\mathrm{GL}_n(F))$, and $\ker \overline{\phi} \otimes \xi_F^{-1}$ contains $\mathrm{Gal}(F_s/E)^{d(\overline{\phi})}$. Thus $d(\overline{\phi} \otimes \xi_F^{-1}) < d(\overline{\phi})$, which contradicts the definition of $\overline{\phi}$. We have shown that $d(\overline{\phi}) = d(\phi)$ if $\overline{\phi}|_{\mathbf{W}_F}$ is irreducible.

For a general essentially tame parameter $\overline{\phi}$ for $\mathrm{GL}_n(F)$, $\overline{\phi}|_{\mathbf{W}_F}$ is a direct sum of irreducible essentially tame parameters ψ_i for $\mathrm{GL}_{n_i}(F)$, with $n_i \le n$. Writing $\psi_i = \mathrm{ind}_{\mathbf{W}_{E_i}}^{\mathbf{W}_F} \xi_i$ as above, we obtain from (40) that

$$(43) \qquad \ker \phi = \{w \in \cap_i \mathbf{W}_{E_i} : \xi_i^s(w) = \xi_j(w) \text{ for all } i,j \text{ and all } s \in \mathbf{W}_F\}.$$

In general this is smaller than

$$\ker(\oplus_i \mathrm{pr}_i \circ \psi_i) = \{w \in \cap_i \mathbf{W}_{E_i} : \xi_i^s(w) = \xi_i(w) \text{ for all } i \text{ and all } s \in \mathbf{W}_F\},$$

where $\mathrm{pr}_i : \mathrm{GL}_{n_i}(\mathbb{C}) \to \mathrm{PGL}_{n_i}(\mathbb{C})$ denotes the canonical projection. Comparing all these kernels we deduce that

$$(44) \qquad \max_i d(\psi_i) = d(\overline{\phi}) \ge d(\phi) \ge d(\oplus_i \mathrm{pr}_i \circ \psi_i) = \max_i d(\mathrm{pr}_i \circ \psi_i).$$

However, we cannot just twist $\overline{\phi}$ with a character of $\mathbf{W}_F$ derived from the most ramified of the ψ_i as in the irreducible case, because that could make the depth of another ψ_j much larger.

We suppose once again that $d(\overline{\phi}) > d(\phi)$. Then

$$\ker \overline{\phi} \supset \mathrm{Gal}(F_s/F)^{d(\overline{\phi})+}, \text{ but } \ker \overline{\phi} \not\supset \mathrm{Gal}(F_s/F)^{d(\overline{\phi})} \subset \ker \phi.$$

By (43) all the ξ_i agree on $\mathrm{Gal}(F_s/F)^{d(\overline{\phi})}$. The above method produces characters ξ_i' of $U_{E_i}^{d_i} \cap F^\times$, which agree on $\mathbf{a}_F(\mathbf{W}_F^{d(\overline{\phi})})$. Put $\xi' = \xi_i'\big|_{\mathbf{a}_F(\mathbf{W}_F^{d(\overline{\phi})})}$. Now the same argument as in the irreducible case leads to a contradiction with (38). $\qquad\square$

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