PARAMETERS OF HECKE ALGEBRAS FOR BERNSTEIN COMPONENTS OF p-ADIC GROUPS

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Abstract. Let G be a reductive group over a non-archimedean local field F. Consider an arbitrary Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ in the category of complex smooth G-representations. In earlier work the author showed that there exists an affine Hecke algebra $\mathcal{H}(\mathcal{O}, G)$ whose category of right modules is closely related to $\mathrm{Rep}(G)^{\mathfrak{s}}$. In many cases this is in fact an equivalence of categories, like for Iwahori-spherical representations.

In this paper we study the q-parameters of the affine Hecke algebras $\mathcal{H}(\mathcal{O}, G)$. We compute them in many cases, in particular for principal series representations of quasi-split groups and for classical groups.

Lusztig conjectured that the q-parameters are always integral powers of q_F and that they coincide with the q-parameters coming from some Bernstein block of unipotent representations. We reduce this conjecture to the case of absolutely simple p-adic groups, and we prove it for most of those.

Contents

Date: December 5, 2023.

2010 *Mathematics Subject Classification.* Primary 22E50, Secondary 20G25, 20C08.

Key words and phrases. representation theory, reductive groups, Hecke algebras, non-archimedean local fields.

The author was supported by a Vidi grant "A Hecke algebra approach to the local Langlands correspondence" (nr. 639.032.528) from Nederlands Wetenschappelijk Onderzoek (NWO).

Introduction

It is well-known that affine Hecke algebras play an important role in the representation theory of a reductive group G over a non-archimedean local field F. In many cases a Bernstein block $\mathrm{Rep}(G)^{\mathfrak s}$ in the category of smooth complex G-representations is equivalent with the module category of an affine Hecke algebra (maybe extended with some finite group). This was first shown for Iwahori-spherical representations [IwMa, Bor] and for depth zero representations [Mor]. With the theory of types [BuKu2] such an equivalence of categories was established for representations of $GL_n(F)$, of inner forms of $GL_n(F)$ [SéSt1, SéSt2] and for inner forms of $SL_n(F)$ [ABPS].

An alternative approach goes via the algebra of G-endomorphisms of a progenerator $\Pi^{\mathfrak s}$ of $\mathrm{Rep}(G)^{\mathfrak s}$. The category of right modules of $\mathrm{End}_G(\Pi^{\mathfrak s})$ is naturally equivalent with $\mathrm{Rep}(G)^{\mathfrak s}$. Heiermann [Hei2, Hei3] showed that for symplectic groups, special orthogonal groups, unitary groups and inner forms of $GL_n(F)$, $\mathrm{End}_G(\Pi^{\mathfrak s})$ is always Morita equivalent with an (extended) affine Hecke algebra.

Recently the author generalized this to all Bernstein components of all reductive p-adic groups [Sol4]. In the most general setting some subtleties have to be taken into account: the involved affine Hecke algebra must be extended with the group algebra of a finite group, but that group algebra might be twisted by a 2-cocycle. Also, the resulting equivalence with $\mathrm{Rep}(G)^{\mathfrak s}$ works for finite length representations, but maybe not entirely for representations of infinite length. Nevertheless, the bottom line is that $\mathrm{Rep}(G)^{\mathfrak s}$ is largely governed by an affine Hecke algebra from $\mathrm{End}_G(\Pi^{\mathfrak s})$.

Let M a Levi factor M of a parabolic subgroup P of G such that $\mathrm{Rep}(G)^{\mathfrak s}$ arises by parabolic induction from a supercuspidal representation σ of M. We denote the variety of unramified twists of σ by $\mathcal O \subset \mathrm{Irr}(M)$, and the affine Hecke algebra described above by $\mathcal H(\mathcal O, G)$. If at the same a $\mathfrak s$-type (J, ρ) is known, then the Hecke algebra $\mathcal H(G, J, \rho)$ is Morita equivalent with $\mathrm{End}_G(\Pi^{\mathfrak s})^{op}$. (In fact [BaSa, Appendix A] shows that in most cases $\mathrm{ind}_J^G(\rho)$ is isomorphic with $\Pi^{\mathfrak s}$.) In this setting $\mathcal H(\mathcal O, G)$ can also be constructed from $\mathcal H(G, J, \rho)$.

The next question is of course: what does $\mathcal H(\mathcal O, G)$ look like? Like all affine Hecke algebras, it is determined by a root datum and some q-parameters. The lattice X (from that root datum) can be identified with the character lattice of $\mathcal O$, once the latter has been made into a complex torus by choosing a base point. The root system $\Sigma_{\mathcal O}^\vee$ (also from the root datum) is contained in X and determined by the reducibility points of the family of representations $\{I_P^G(\sigma') : \sigma' \in \mathcal O\}$. Then $\mathcal H(\mathcal O, G)$ contains a maximal commutative subalgebra $\mathbb C[X] \cong \mathbb C[\mathcal O]$ and a finite dimensional Iwahori–Hecke algebra $\mathcal H(W(\Sigma_{\mathcal O}^\vee), q_F^\lambda)$ such that

$$\mathcal H(\mathcal O, G) = \mathbb C[\mathcal O] \otimes_{\mathbb C} \mathcal H(W(\Sigma_{\mathcal O}^\vee), q_F^\lambda) \text{ as vector spaces.}$$

Here q_F denotes the cardinality of the residue field of F, while λ will be defined soon. For every $X_\alpha \in \Sigma_{\mathcal O}^\vee$ there is a $q_\alpha \in \mathbb R_{>1}$ such that

$$I_P^G(\sigma') \text{ is reducible for all } \sigma' \in \mathcal O \text{ with } X_\alpha(\sigma') = q_\alpha .$$

Sometimes there is also a number $q_{\alpha *} \in (1, q_\alpha]$ with the property

$$I_P^G(\sigma') \text{ is reducible for all } \sigma' \in \mathcal O \text{ with } X_\alpha(\sigma') = -q_{\alpha *}.$$

When such a real number does not exist, we put $q_{\alpha *} = 1$. These q-parameters q_α

and $q_{\alpha *}$ appear in the Hecke relations of $\mathcal{H}(W(\Sigma_{\mathcal{O}}^\vee), q_F^\lambda)$:

$$0 = (T_{s_\alpha} + 1)(T_{s_\alpha} - q_F^{\lambda(\alpha)}) \quad \text{with} \quad q_F^{\lambda(\alpha)} = q_\alpha q_{\alpha *} \in \mathbb{R}_{>1}.$$

Further, we define $\lambda^*(\alpha) \in \mathbb{R}_{\geq 0}$ by

$$q_F^{\lambda^*(\alpha)} = q_\alpha q_{\alpha *}^{-1}.$$

Knowing $q_\alpha, q_{\alpha *}$ is also equivalent to knowing the poles of the Harish-Chandra μ-function on $\mathcal{O}$ associated to α. See Section 1 for more details on the above setup.

The representation theory of $\mathcal{H}(\mathcal{O}, G)$ depends in a subtle way on the q-parameters $q_\alpha, q_{\alpha *}$ for $X_\alpha \in \Sigma_{\mathcal{O}}^\vee$, so knowing them helps to understand $\mathrm{Rep}(G)^{\mathfrak{s}}$. That brings us to the main goal of this paper: *determine the q-parameters of $\mathcal{H}(\mathcal{O}, G)$ for as many Bernstein blocks* $\mathrm{Rep}(G)^{\mathfrak{s}}$ *as possible.*

Like for all affine Hecke algebras, there are some constraints on the q_α and $q_{\alpha *}$:

- if $X_\alpha, X_\beta \in \Sigma_{\mathcal{O}}^\vee$ are $W(\Sigma_{\mathcal{O}}^\vee)$-associate, then $q_\alpha = q_\beta$ and $q_{\alpha *} = q_{\beta *}$,
- $q_{\alpha *} > 1$ is only possible if X_α is a short root in a type B_n root system.

Notice that q_α and $q_{\alpha *}$ can be expressed in terms of the "q-base" q_F and the labels $\lambda(\alpha), \lambda^*(\alpha)$. It has turned out [KaLu, Sol1] that the representation theory of an affine Hecke algebra hardly changes if one replaces q_F by another q-base (in $\mathbb{R}_{>1}$) while keeping all labels fixed. If we replace the q-base q_F by q_F^r and $\lambda(\alpha), \lambda^*(\alpha)$ by $\lambda(\alpha)/r, \lambda^*(\alpha)/r$ for some $r \in \mathbb{R}_{>0}$, then q_α and $q_{\alpha *}$ do not change, and in fact $\mathcal{H}(\mathcal{O}, G)$ is not affected at all. In this way one can always scale one of the labels to 1. Hence the representation theory of $\mathcal{H}(\mathcal{O}, G)$ depends mainly on the ratios between the labels $\lambda(\alpha), \lambda^*(\alpha)$ for $X_\alpha \in \Sigma_{\mathcal{O}}^\vee$.

- For irreducible root systems of type A_n, D_n and E_n, $\lambda(\alpha) = \lambda^*(\alpha) = \lambda(\beta)$, for any roots $X_\alpha, X_\beta \in \Sigma_{\mathcal{O}}^\vee$. There is essentially only one label $\lambda(\alpha)$, and it can be scaled to 1 by fixing q_α but replacing q_F by q_α.
- For the irreducible root systems C_n, F_4 and G_2, again $\lambda(\alpha)$ always equals $\lambda^*(\alpha)$. There are two independent labels $\lambda(\alpha)$: one for the short roots and one for the long roots.
- For an irreducible root system of type B_n, $\lambda^*(\alpha)$ need not equal $\lambda(\alpha)$ if X_α is short. Here we have three independent labels: $\lambda(\beta)$ for X_β long, $\lambda(\alpha)$ for X_α short and $\lambda^*(\alpha)$ for X_α short.

Lusztig [Lus5] has conjectured:

Conjecture A. *Let G be a reductive group over a non-archimedean local field, with an arbitrary Bernstein block* $\mathrm{Rep}(G)^{\mathfrak{s}}$. *Let $\Sigma_{\mathcal{O},j}^\vee$ be an irreducible component of the root system $\Sigma_{\mathcal{O}}^\vee$ underlying $\mathcal{H}(\mathcal{O}, G)$. Then:*

(i) *the q-parameters $q_\alpha, q_{\alpha *}$ are powers of q_F, except that for a short root α in a type B_n root system the q-parameters can also be powers of $q_F^{1/2}$ (and then $q_\alpha q_{\alpha *}^{\pm 1}$ is still a power of q_F).*

(ii) *the label functions λ, λ^* on $\Sigma_{\mathcal{O},j}^\vee$ agree with those obtained in the same way from a Bernstein block of unipotent representations of some adjoint simple p-adic group, as in* [Lus3, Lus4].

Conjecture A.(i) is related to a conjecture of Langlands about Harish-Chandra μ-functions [Sha, §2]. For generic representations of quasi-split reductive groups over p-adic fields, [Sha, §3] translates Conjecture A.(i) to a question about poles of adjoint γ-factors. (We do not pursue that special case here.)

Motivation for Conjecture A.(ii) comes from the local Langlands correspondence. It is believed [AMS1] that $\mathrm{Irr}(G) \cap \mathrm{Rep}(G)^{\mathfrak{s}}$ corresponds to a Bernstein component $\Phi_e(G)^{\mathfrak{s}^\vee}$ of enhanced L-parameters for G. To $\Phi_e(G)^{\mathfrak{s}^\vee}$ one can canonically associate an affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$, possibly extended with a twisted group algebra [AMS3, §3.3]. It is expected that the module category of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ is very closely related to $\mathrm{Rep}(G)^{\mathfrak{s}}$, at least the two subcategories of finite length modules should be equivalent.

The nonextended version $\mathcal{H}^\circ(\mathfrak{s}^\vee, q_F^{1/2})$ of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ can be constructed with complex geometry from a connected reductive group $H^\vee$ (the connected centralizer in $G^\vee$ of the image of the inertia group $\mathbf{I}_F$ under the Langlands parameter) and a cuspidal local system ρ on a unipotent orbit for a Levi subgroup $L^\vee$ of $H^\vee$. The exact same data $(H^\vee, L^\vee, \rho)$ also arise from enhanced Langlands parameters (for some reductive p-adic group G') which are trivial on $\mathbf{I}_F$. By the local Langlands correspondence from [Lus3, Lus4, Sol5, Sol6], a Bernstein component of such enhanced L-parameters corresponds to a Bernstein component $\mathrm{Rep}(G')^{\mathfrak{s}'}$ of unipotent G'-representations.

It follows that $\mathcal{H}^\circ(\mathfrak{s}^\vee, q_F^{1/2})$ is isomorphic to $\mathcal{H}^\circ(\mathfrak{s}'^\vee, q_{F'}^{1/2})$. By [Sol5, Theorem 4.4], $\mathcal{H}^\circ(\mathfrak{s}'^\vee, q_{F'}^{1/2})$ is isomorphic to $\mathcal{H}(\mathcal{O}', G')$, which is an affine Hecke algebra associated to a Bernstein block of unipotent representations of G'. If desired one can replace G' by its adjoint group, by [Sol5, Lemma 3.5] that operation changes the affine Hecke algebras a little but preserves the root systems and the q-parameters.

Thus, if there exists a local Langlands correspondence with good properties, Conjecture A is a consequence of what happens on the Galois side of the correspondence. Conversely, new cases of Conjecture A might contribute to new instances of a local Langlands correspondence, via a comparison of possible Hecke algebras on both sides as in [Lus3].

We note that the affine root systems in Lusztig's notation for affine Hecke algebras correspond to affine extensions of our root systems $\Sigma_{\mathcal{O}}$. Now we list all possible label functions from [Lus3, Lus4], for a given irreducible root system (taken a remark at the end of Paragraph 4.6 into account):

Table 1. Labels for affine Hecke algebras from unipotent representations

$\Sigma_{\mathcal{O}}^\vee$	λ(long root)	λ(short root)	λ^*(short root)
A_n, D_n, E_n	–	$\in \mathbb{Z}_{>0}$	$\lambda^* = \lambda$
B_n	1 or 2	$\in \mathbb{Z}_{>0}$	$\in \mathbb{Z}_{\geq 0}$
C_n	$\in \mathbb{Z}_{>0}$	1 or 2	$\lambda^* = \lambda$
F_4	1 or 2	1	1
F_4	1	2	2
F_4	4	1	1
G_2	1 or 3	1	1
G_2	1	3	3
G_2	9	1	1

An important and accessible class of representations is formed by the principal series representations of quasi-split groups G. When G is F-split, the Hecke algebras for Bernstein blocks of such representations were already analysed in [Roc1] via types, under some mild restrictions on the residual characteristic. To every root of a quasi-split group G (relative to a maximal F-split torus) one can associate a splitting field F_α, a finite extension of F.

Theorem B. (see Theorem 4.4 and Corollary 4.5)
Conjecture A holds for all Bernstein blocks in the principal series of a quasi-split connected reductive group over F. *For* $X_\alpha \in \Sigma_{\mathcal{O}}^\vee$ *(with one exception in type* ${}^2 A_{2n}$ *that we analyse as well)* $q_{\alpha *} = 1$ *and* q_α *is the cardinality of the residue field of* F_α.

Theorem B will be employed to establish a canonical local Langlands correspondence for principal series representations of quasi-split F-groups [Sol7].

For parameter computations in Hecke algebras associated to more complicated Bernstein components, we need a reduction strategy. That is the topic of Section 2, which culminates in:

Theorem C. (see Corollary 2.5)
Suppose that Conjecture A holds for the simply connected cover G_{sc} *of* G_{der}. *Then it holds for* G.

This enables us to reduce the verification of Conjecture A to absolutely simple, simply connected groups. For (absolutely) simple groups quite a few results about the parameters of Hecke algebras can be found in the literature, e.g. [BuKu1, Séc, Hei1]. With our current framework we can easily generalize those results, in particular from one group to an isogenous group.

Sécherre and Stevens [Séc, SéSt1, SéSt2] determined the Hecke algebras for all Bernstein blocks for inner forms of $GL_n(F)$. Together with Theorem C that proves Conjecture A for all inner forms of a group of type A.

For classical groups (symplectic, special orthogonal, unitary) we run into the problem that some representation theoretic results have been proven over p-adic fields but not (yet) over local function fields. We overcome this with the method of close fields [Kaz], which Ganapathy recently generalized to arbitrary connected reductive groups [Gan1, Gan2].

Theorem D. (see Corollary 3.7)
Let $\mathrm{Rep}(G)^{\mathfrak{s}}$ *be a Bernstein block for a reductive group* G *over a local function field. Then there exists a Bernstein block* $\mathrm{Rep}(\tilde{G})^{\tilde{\mathfrak{s}}}$ *for a reductive group* $\tilde{G}$ *over a* p*-adic field, such that:*

- G *and* $\tilde{G}$ *come from "the same" algebraic group,*
- $\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{Rep}(\tilde{G})^{\tilde{\mathfrak{s}}}$ *and* $\mathcal{H}(\mathcal{O}, G) \cong \mathcal{H}(\tilde{\mathcal{O}}, \tilde{G})$,
- *the parameters for both these affine Hecke algebras are the same.*

For classical groups over p-adic fields the parameters of the Hecke algebras were determined in [Hei1, Hei3], in terms of Mœglin's classification of discrete series representations [Mœ3]. With a generalization of this method and a closer analysis of the resulting parameters we prove:

Theorem E. (see Paragraph 4.4)
Conjecture A holds for all pure inner forms of quasi-split classical groups, and for all groups isogenous with one of those. This includes all simple groups of type A_n, B_n, C_n, D_n, *except those associated to Hermitian forms on vector spaces over quaternion algebras.*

Theorem E is useful to study Hecke algebras and the local Langlands correspondence for general spin groups [AMS4]. Among classical groups associated to Hermitian forms, Conjecture A only remains open for the non-pure inner forms of quasi-split classical groups. Unfortunately, the current understanding of their representations does not suffice to carry out the strategies we applied to other groups.

Finally, we consider exceptional groups. For most Bernstein components we can reduce the computation of the Hecke algebra parameters to groups of Lie type A_n, B_n, C_n and D_n, but sometimes that does not work. We establish partial results for all simple exceptional groups, most of which can be summarized as follows:

Theorem F. (see Paragraphs 4.5, 4.6 and 4.7)
Conjecture A holds for all simple F-groups of type $G_2, F_4, E_6, {}^2E_6, E_6^{(3)}, {}^3D_4$.

If (for any reductive p-adic group G) $\Sigma_{\mathcal{O}}^\vee$ has an irreducible component $\Sigma_{\mathcal{O},j}^\vee$ of type F_4, then Conjecture A holds for $\Sigma_{\mathcal{O},j}^\vee$.

Our results about F_4 are useful in combination with [Sol3, §6]. There we related the irreducible representations of an affine Hecke algebra with arbitrary positive q-parameters to the irreducible representations of the analogous algebra that has all q-parameters equal to 1. The problem was only that we could not handle certain label functions for type F_4 root systems. Theorem F shows that the label functions which could be handled well in [Sol3, §6] exhaust the label functions that can appear for type F_4 root systems among affine Hecke algebras coming from reductive p-adic groups.

Acknowledgements.
We are grateful to Anne-Marie Aubert, Geo Tam and Stefan Dawydiak for their comments on earlier versions, and in particular for pointing out some problems.

1. Progenerators and endomorphism algebras for Bernstein blocks

We fix some notations and recall relevant material from [Sol4]. Let F be a non-archimedean local field with ring of integers $\mathfrak{o}_F$. Pick a uniformizing element $\varpi_F \in \mathfrak{o}_F$. We denote the cardinality of the residue field $k_F = \mathfrak{o}_F / \varpi \mathfrak{o}_F$ by q_F. Let $|\cdot|_F$ be the norm on F, normalized so that $|\varpi_F|_F = q_F^{-1}$.

Let $\mathcal{G}$ be a connected reductive F-group and let $G = \mathcal{G}(F)$ be its group of F-rational points. We briefly call G a reductive p-adic group. We consider the category $\mathrm{Rep}(G)$ of smooth G-representations on complex vector spaces. Let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible objects in $\mathrm{Rep}(G)$, and $\mathrm{Irr}_{\mathrm{cusp}}(G) \subset \mathrm{Irr}(G)$ the subset of supercuspidal representations.

Let $\mathcal{M}$ be a F-Levi subgroup of $\mathcal{G}$ and write $M = \mathcal{M}(F)$. The group of unramified characters of M is denoted $X_{\mathrm{nr}}(M)$. We fix $(\sigma, E) \in \mathrm{Irr}_{\mathrm{cusp}}(M)$. The set of unramified twists of σ is

$$\mathcal{O} = \{\sigma \otimes \chi : \chi \in X_{\mathrm{nr}}(M)\} \subset \mathrm{Irr}(M).$$

It can be identified with the inertial equivalence class $\mathfrak{s}_M = [M, \sigma]_M$. Let $\mathfrak{s} = [M, \sigma]_G$ be the associated inertial equivalence class for G.

Recall that the supercuspidal support $\mathrm{Sc}(\pi)$ of $\pi \in \mathrm{Irr}(G)$ consists of a Levi subgroup of G and an irreducible supercuspidal representation thereof. Although $\mathrm{Sc}(\pi)$

is only defined up to G-conjugacy, we shall only be interested in supercuspidal supports with Levi subgroup M, and then the supercuspidal representation is uniquely defined up to the natural action of $N_G(M)$ on $\mathrm{Irr}(M)$.

This setup yields a Bernstein component

$$\mathrm{Irr}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Irr}(G) : \mathrm{Sc}(\pi) \in (M, \mathcal{O})\}$$

of $\mathrm{Irr}(G)$. It generates a Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ of $\mathrm{Rep}(G)$, see [BeDe].

Let $M^1 \subset M$ be the group generated by all compact subgroups of M, so that $X_{\mathrm{nr}}(M) = \mathrm{Irr}(M/M^1)$. Then

$$\mathrm{ind}_{M^1}^{M}(\sigma, E) \cong E \otimes_{\mathbb{C}} \mathbb{C}[M/M^1] \cong E \otimes_{\mathbb{C}} \mathbb{C}[X_{\mathrm{nr}}(M)], \tag{1.1}$$

where $\mathbb{C}[M/M^1]$ is the group algebra of the discrete group M/M^1 and $\mathbb{C}[X_{\mathrm{nr}}(M)]$ is the ring of regular functions on the complex torus $X_{\mathrm{nr}}(M)$. Supercuspidality implies that (1.1) is a progenerator of $\mathrm{Rep}(M)^{\mathfrak{s}_M}$. Let $P \subset G$ be a parabolic subgroup with Levi factor M, chosen as prescribed by [Sol4, Lemma 9.1]. Let

$$I_P^G : \mathrm{Rep}(M) \to \mathrm{Rep}(G)$$

be the parabolic induction functor, normalized so that it preserves unitarity. As a consequence of Bernstein's second adjointness theorem [Ren],

$$\Pi^{\mathfrak{s}} := I_P^G(E \otimes \mathbb{C}[X_{\mathrm{nr}}(M)])$$

is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$. That means [Roc2, Theorem 1.8.2.1] that the functor

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \longrightarrow & \mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod} \\ V & \mapsto & \mathrm{Hom}_G(\Pi^{\mathfrak{s}}, V) \end{array}$$

is an equivalence of categories. This motivates the study of the endomorphism algebra $\mathrm{End}_G(\Pi^{\mathfrak{s}})$, which was carried out in [Roc2, Hei2, Sol4]. To describe its structure, we have to recall several objects which lead to the appropriate root datum. The set

$$X_{\mathrm{nr}}(M, \sigma) = \{\chi \in X_{\mathrm{nr}}(M) : \sigma \otimes \chi \cong \chi\}$$

is a finite subgroup of $X_{\mathrm{nr}}(M)$. The map

$$X_{\mathrm{nr}}(M)/X_{\mathrm{nr}}(M, \sigma) \to \mathcal{O} : \chi \mapsto \sigma \otimes \chi$$

is a bijection, and in this way we provide $\mathcal{O}$ with the structure of a complex variety (a torus, but without a canonical base point). The group

$$M_\sigma^2 := \bigcap\nolimits_{\chi \in X_{\mathrm{nr}}(M,\sigma)} \ker \chi$$

has finite index in M, and there are natural isomorphisms

$$\mathrm{Irr}(M_\sigma^2/M^1) \cong X_{\mathrm{nr}}(M)/X_{\mathrm{nr}}(M, \sigma),$$
$$\mathbb{C}[M_\sigma^2/M^1] \cong \mathbb{C}[X_{\mathrm{nr}}(M)/X_{\mathrm{nr}}(M, \sigma)].$$

Here and later on, the notation $\mathbb{C}[?]$ must be interpreted as in (1.1). The group

$$W(G, M) := N_G(M)/M$$

is a Weyl group in most cases (and if it is not, then it is still very close to a Weyl group). The natural action of $N_G(M)$ on $\mathrm{Rep}(M)$ induces an action of $W(G, M)$ on $\mathrm{Irr}(M)$. Let $N_G(M, \mathcal{O})$ be the stabilizer of $\mathcal{O}$ in $N_G(M)$ and write

$$W(M, \mathcal{O}) = N_G(M, \mathcal{O})/M.$$

Thus $W(M,\mathcal{O})$ acts naturally on the complex algebraic variety $\mathcal{O}$. This finite group figures prominently in the Bernstein theory, for instance because the centres of $\mathrm{Rep}(G)^{\mathfrak{s}}$ and of $\mathrm{End}_G(\Pi^{\mathfrak{s}})$ are naturally isomorphic with $\mathbb{C}[\mathcal{O}]^{W(M,\mathcal{O})}$.

Let $\mathcal{A}_M$ be the maximal F-split torus in $Z(\mathcal{M})$, put $A_M = \mathcal{A}_M(F)$ and let $X_*(\mathcal{A}_M) = X_*(A_M)$ be the cocharacter lattice. We write

$$\mathfrak{a}_M = X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad \mathfrak{a}_M^* = X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $\Sigma(G,A_M) \subset X^*(A_M)$ be the set of nonzero weights occurring in the adjoint representation of A_M on the Lie algebra of G, and let $\Sigma_{\mathrm{red}}(A_M)$ be the set of indivisible elements therein.

For every $\alpha \in \Sigma_{\mathrm{red}}(A_M)$ there is a Levi subgroup M_α of G which contains M and the root subgroup U_α, and whose semisimple rank is one higher than that of M. Let $\alpha^\vee \in \mathfrak{a}_M$ be the unique element which is orthogonal to $X^*(A_{M_\alpha})$ and satisfies $\langle \alpha^\vee, \alpha \rangle = 2$.

Recall the Harish-Chandra μ-functions from [Sil2, §1] and [Wal, §V.2]. The restriction of μ^G to $\mathcal{O}$ is a rational, $W(M,\mathcal{O})$-invariant function on $\mathcal{O}$ [Wal, Lemma V.2.1]. It determines a reduced root system [Hei2, Proposition 1.3]

$$\Sigma_{\mathcal{O},\mu} = \{\alpha \in \Sigma_{\mathrm{red}}(A_M) : \mu^{M_\alpha}(\sigma \otimes \chi) \text{ has a zero on } \mathcal{O}\}. \tag{1.2}$$

For $\alpha \in \Sigma_{\mathrm{red}}(A_M)$ the function $\chi \mapsto \mu^{M_\alpha}(\sigma \otimes \chi)$ factors through the quotient map $X_{\mathrm{nr}}(M) \to X_{\mathrm{nr}}(T_\alpha)$, where T_α is the onedimensional subtorus of A_M with Lie algebra spanned by (a multiple of) $\alpha^\vee$. The associated system of coroots is

$$\Sigma_{\mathcal{O},\mu}^\vee = \{\alpha^\vee \in \mathfrak{a}_M : \mu^{M_\alpha}(\sigma \otimes \chi) \text{ has a zero on } \mathcal{O}\}.$$

By the aforementioned $W(M,\mathcal{O})$-invariance of μ^G, $W(M,\mathcal{O})$ acts naturally on $\Sigma_{\mathcal{O},\mu}$ and on $\Sigma_{\mathcal{O},\mu}^\vee$. Let s_α be the unique nontrivial element of $W(M_\alpha, M)$. By [Hei2, Proposition 1.3] the Weyl group $W(\Sigma_{\mathcal{O},\mu})$ can be identified with the subgroup of $W(G,M)$ generated by the reflections s_α with $\alpha \in \Sigma_{\mathcal{O},\mu}$, and as such it is a normal subgroup of $W(M,\mathcal{O})$.

The parabolic subgroup $P = MU$ of G determines a set of positive roots $\Sigma_{\mathcal{O},\mu}^+$ and a basis $\Delta_{\mathcal{O},\mu}$ of $\Sigma_{\mathcal{O},\mu}$. Let $\ell_{\mathcal{O}}$ be the length function on $W(\Sigma_{\mathcal{O},\mu})$ specified by $\Delta_{\mathcal{O},\mu}$. Since $W(M,\mathcal{O})$ acts on $\Sigma_{\mathcal{O},\mu}$, $\ell_{\mathcal{O}}$ extends naturally to $W(M,\mathcal{O})$, by

$$\ell_{\mathcal{O}}(w) = |w(\Sigma_{\mathcal{O},\mu}^+) \cap -\Sigma_{\mathcal{O},\mu}^+|.$$

The set of positive roots also determines a subgroup of $W(M,\mathcal{O})$:

$$\begin{aligned} R(\mathcal{O}) &= \{w \in W(M,\mathcal{O}) : w(\Sigma_{\mathcal{O},\mu}^+) = \Sigma_{\mathcal{O},\mu}^+\} \\ &= \{w \in W(M,\mathcal{O}) : \ell_{\mathcal{O}}(w) = 0\}. \end{aligned} \tag{1.3}$$

The simple transitivity of the action of $W(\Sigma_{\mathcal{O},\mu})$ on the set of positive systems of $\Sigma_{\mathcal{O},\mu}$ [Hum, Theorem 1.8] implies that

$$W(M,\mathcal{O}) = R(\mathcal{O}) \ltimes W(\Sigma_{\mathcal{O},\mu}). \tag{1.4}$$

Recall that $X_{\mathrm{nr}}(M)/X_{\mathrm{nr}}(M,\sigma)$ is isomorphic to the character group of the lattice M_σ^2/M^1. Since M_σ^2 depends only on $\mathcal{O}$, it is normalized by $N_G(M,\mathcal{O})$. In particular the conjugation action of $N_G(M,\mathcal{O})$ on M_σ^2/M^1 induces an action of $W(M,\mathcal{O})$ on M_σ^2/M^1.

Let $h_\alpha^\vee$ be the unique generator of $(M_\sigma^2 \cap M_\alpha^1)/M^1 \cong \mathbb{Z}$ such that $|\alpha(h_\alpha^\vee)|_F > 1$. Recall the injective homomorphism $H_M : M/M^1 \to \mathfrak{a}_M$ defined by

$$q_F^{\langle H_M(m), \gamma \rangle} = |\gamma(m)|_F \qquad \text{for } m \in M, \gamma \in X^*(M).$$

Remark. *This definition is motivated by the correction to* [Sol4]. *In earlier versions we used alternative conventions, which differ from the above by multiplying $h_\alpha^\vee$ and H_M by a factor -1. That does not change the Hecke algebras, it only amounts to a different choice of generators.*

In these terms $H_M(h_\alpha^\vee) \in \mathbb{R}_{>0}\alpha^\vee$. Since M_σ^2 has finite index in M, $H_M(M_\sigma^2/M^1)$ is a lattice of full rank in $\mathfrak{a}_M$. We write

$$(M_\sigma^2/M^1)^\vee = \mathrm{Hom}_{\mathbb{Z}}(M_\sigma^2/M^1, \mathbb{Z}).$$

Composition with H_M and $\mathbb{R}$-linear extension of maps $H_M(M_\sigma^2/M^1) \to \mathbb{Z}$ determines an embedding

$$H_M^\vee : (M_\sigma^2/M^1)^\vee \to \mathfrak{a}_M^*.$$

Then $H_M^\vee(M_\sigma^2/M^1)^\vee$ is a lattice of full rank in $\mathfrak{a}_M^*$.

Proposition 1.1. [Sol4, Proposition 3.5]
Let $\alpha \in \Sigma_{\mathcal{O},\mu}$.

(*a*) *For $w \in W(M, \mathcal{O})$: $w(h_\alpha^\vee) = h_{w(\alpha)}^\vee$.*
(*b*) *There exists a unique $\alpha^\sharp \in (M_\sigma^2/M^1)^\vee$ such that $H_M^\vee(\alpha^\sharp) \in \mathbb{R}\alpha$ and $\langle h_\alpha^\vee, \alpha^\sharp \rangle = 2$.*
(*c*) *Write*

$$\begin{aligned} \Sigma_{\mathcal{O}} &= \{\alpha^\sharp : \alpha \in \Sigma_{\mathcal{O},\mu}\}, \\ \Sigma_{\mathcal{O}}^\vee &= \{h_\alpha^\vee : \alpha \in \Sigma_{\mathcal{O},\mu}\}. \end{aligned}$$

Then $(\Sigma_{\mathcal{O}}^\vee, M_\sigma^2/M^1, \Sigma_{\mathcal{O}}, (M_\sigma^2/M^1)^\vee)$ is a root datum with Weyl group $W(\Sigma_{\mathcal{O},\mu})$.
(*d*) *The group $W(M, \mathcal{O})$ acts naturally on this root datum, and $R(\mathcal{O})$ is the stabilizer of the basis $\Delta_{\mathcal{O}}^\vee$ determined by P.*

We note that $\Sigma_{\mathcal{O}}$ and $\Sigma_{\mathcal{O}}^\vee$ have almost the same type as $\Sigma_{\mathcal{O},\mu}$. Indeed, the roots $H_M^\vee(\alpha^\sharp)$ are scalar multiples of the $\alpha \in \Sigma_{\mathcal{O},\mu}$, so the angles between the elements of $\Sigma_{\mathcal{O}}$ are the same as the angles between the corresponding elements of $\Sigma_{\mathcal{O},\mu}$. It follows that every irreducible component of $\Sigma_{\mathcal{O},\mu}$ has the same type as the corresponding components of $\Sigma_{\mathcal{O}}$ and $\Sigma_{\mathcal{O}}^\vee$, except that type B_n/C_n might be replaced by type C_n/B_n.

For $\alpha \in \Sigma_{\mathrm{red}}(M) \setminus \Sigma_{\mathcal{O},\mu}$, the function μ^{M_α} is constant on $\mathcal{O}$. In contrast, for $\alpha \in \Sigma_{\mathcal{O},\mu}$ it has both zeros and poles on $\mathcal{O}$. By [Sil2, §5.4.2]

$$s_\alpha \cdot \sigma' \cong \sigma' \quad \text{whenever } \mu^{M_\alpha}(\sigma') = 0. \tag{1.5}$$

As $\Delta_{\mathcal{O},\mu}$ is linearly independent in $X^*(A_M)$ and μ^{M_α} factors through A_M/A_{M_α}, there exists a $\tilde{\sigma} \in \mathcal{O}$ such that $\mu^{M_\alpha}(\tilde{\sigma}) = 0$ for all $\alpha \in \Delta_{\mathcal{O},\mu}$. In view of [Sil3, §1] this can even be achieved with a unitary $\tilde{\sigma}$. We replace σ by $\tilde{\sigma}$, which means that from now on we adhere to:

Condition 1.2. $(\sigma, E) \in \mathrm{Irr}(M)$ *is unitary supercuspidal and* $\mu^{M_\alpha}(\sigma) = 0$ *for all* $\alpha \in \Delta_{\mathcal{O},\mu}$.

By (1.5) the entire Weyl group $W(\Sigma_{\mathcal{O},\mu})$ stabilizes the isomorphism class of this σ. However, in general $R(\mathcal{O})$ need not stabilize σ. We identify $X_{\mathrm{nr}}(M)/X_{\mathrm{nr}}(M,\sigma)$ with $\mathcal{O}$ via $\chi \mapsto \sigma \otimes \chi$ and we define

$$X_\alpha \in \mathbb{C}[X_{\mathrm{nr}}(M)/X_{\mathrm{nr}}(M,\sigma)] \text{ by } X_\alpha(\chi) = \chi(h_\alpha^\vee). \tag{1.6}$$

For any $w \in W(M,\mathcal{O})$ which stabilizes σ in $\mathrm{Irr}(M)$, Proposition 1.1.a implies

$$w(X_\alpha) = X_{w(\alpha)} \quad \text{for all } \alpha \in \Sigma_{\mathcal{O},\mu}. \tag{1.7}$$

According to [Sil2, §1] there exist $q_\alpha, q_{\alpha*} \in \mathbb{R}_{\geq 1}$, $c'_{s_\alpha} \in \mathbb{R}_{>0}$ for $\alpha \in \Sigma_{\mathcal{O},\mu}$, such that

$$\mu^{M_\alpha}(\sigma \otimes \cdot) = \frac{c'_{s_\alpha}(1 - X_\alpha)(1 - X_\alpha^{-1})}{(1 - q_\alpha^{-1} X_\alpha)(1 - q_\alpha^{-1} X_\alpha^{-1})} \frac{(1 + X_\alpha)(1 + X_\alpha^{-1})}{(1 + q_{\alpha*}^{-1} X_\alpha)(1 + q_{\alpha*}^{-1} X_\alpha^{-1})} \tag{1.8}$$

as rational functions on $X_{\mathrm{nr}}(M)/X_{\mathrm{nr}}(M,\sigma) \cong \mathcal{O}$. We may modify the choice of σ in Condition 1.2, so that, as in [Hei2, Remark 1.7]:

$$q_\alpha \geq q_{\alpha*} \text{ for all } \alpha \in \Delta_{\mathcal{O},\mu}. \tag{1.9}$$

Then [Sol4, Lemma 3.4] guarantees that the maps $\Sigma_{\mathcal{O},\mu} \to \mathbb{R}_{\geq 0}$ given by q_α and $q_{\alpha*}$ are $W(M,\mathcal{O})$-invariant. Comparing (1.8), Condition 1.2 and (1.9), we see that $q_\alpha > 1$ for all $\alpha \in \Sigma_{\mathcal{O},\mu}$. In particular the zeros of μ^{M_α} occur at

$$\{X_\alpha = 1\} = \{\sigma' \in \mathcal{O} : X_\alpha(\sigma') = 1\}$$

and sometimes at

$$\{X_\alpha = -1\} = \{\sigma' \in \mathcal{O} : X_\alpha(\sigma') = -1\}.$$

When μ^{M_α} has a zero at both $\{X_\alpha = 1\}$ and $\{X_\alpha = -1\}$, the irreducible component of $\Sigma_{\mathcal{O}}^\vee$ containing $h_\alpha^\vee$ has type B_n ($n \geq 1$) and $h_\alpha^\vee$ is a short root [Sol4, Lemma 3.3].

For another characterization of μ_α, we write down an explicit construction. Let $\delta_P : P \to \mathbb{R}_{>0}$ be the modular function. We realize $I_P^G(\sigma \otimes \chi, E)$ on the vector space

$$\big\{ f : G \to E \mid f \text{ is smooth}, f(umg) = \sigma(m)(\chi \delta_P^{1/2})(m) f(g) \ \forall u \in U, m \in M, g \in G \big\},$$

with G acting by right translations. Let $P' = MU'$ be another parabolic subgroup of G with Levi factor M. Following [Wal, §IV.1] we consider the map

$$\begin{array}{llcl} J_{P'|P}(\sigma \otimes \chi) : & I_P^G(\sigma \otimes \chi, E) & \to & I_{P'}^G(\sigma \otimes \chi, E) \\ & f & \mapsto & [g \mapsto \int_{(U \cap U') \backslash U'} f(u'g) \mathrm{d}u']. \end{array} \tag{1.10}$$

Here $\mathrm{d}u'$ denotes a quotient of Haar measures on U' and $U \cap U'$. This integral converges for χ in an open subset of $X_{\mathrm{nr}}(M)$ (independent of f). As such it defines a map

$$\begin{array}{ccc} X_{\mathrm{nr}}(M) \times I_P^G(E) & \to & I_{P'}^G(E), \\ (\chi, f) & \mapsto & J_{P'|P}(\sigma \otimes \chi) f, \end{array}$$

which is rational in χ and linear in f [Wal, Théorème IV.1.1]. Moreover it intertwines the G-representation $I_P^G(\sigma \otimes \chi)$ with $I_{P'}^G(\sigma \otimes \chi)$ whenever it converges. Then

$$J_{P|P'}(\sigma \otimes \chi) J_{P'|P}(\sigma \otimes \chi) \in \mathrm{End}_G(I_P^G(\sigma \otimes \chi, E)) = \mathbb{C}\,\mathrm{id},$$

at least for χ in a Zariski-open subset of $X_{\mathrm{nr}}(M)$. For any $\alpha \in \Sigma_{\mathrm{red}}(M)$ there exists by construction [Wal, §IV.3] a nonzero constant such that

$$J_{M_\alpha \cap P | s_\alpha(M_\alpha \cap P)}(\sigma \otimes \chi) J_{s_\alpha(M_\alpha \cap P) | M_\alpha \cap P}(\sigma \otimes \chi) = \frac{\text{constant}}{\mu^{M_\alpha}(\sigma \otimes \chi)}, \tag{1.11}$$

as rational functions of $\chi \in X_{\mathrm{nr}}(M)$. We note that

$$(U \cap s_\alpha(U)) \backslash s_\alpha(U) = U_{-\alpha} \quad \text{and} \quad (U \cap s_\alpha(U)) \backslash U = U_\alpha,$$

where $U_{\pm\alpha}$ denotes a root subgroup with respect to A_M. That allows us to simplify (1.11) to

$$\begin{aligned} J_{s_\alpha(M_\alpha \cap P)|M_\alpha \cap P}(\sigma \otimes \chi) f &= [g \mapsto \textstyle\int_{U_{-\alpha}} f(u_- g) \mathrm{d}u_-], \\ J_{M_\alpha \cap P|s_\alpha(M_\alpha \cap P)}(\sigma \otimes \chi) f &= [g \mapsto \textstyle\int_{U_{\alpha}} f(u_+ g) \mathrm{d}u_+], \end{aligned} \tag{1.12}$$

where $\mathrm{d}u_\pm$ is a Haar measure on $U_{\pm\alpha}$. The numbers q_α, q_α^{-1} (and $q_{\alpha*}, q_{\alpha*}^{-1}$ when $q_{\alpha*} \neq 1$) are precisely the values of $X_\alpha(\chi) = X_\alpha(\sigma \otimes \chi)$ at which $\mu^{M_\alpha}(\sigma \otimes \chi)$ has a pole, and in view of (1.11) these are also given by the χ for which

$$J_{M_\alpha \cap P|s_\alpha(M_\alpha \cap P)}(\sigma \otimes \chi) J_{s_\alpha(M_\alpha \cap P)|M_\alpha \cap P}(\sigma \otimes \chi) = 0.$$

For other non-unitary $\sigma \otimes \chi \in \mathcal{O}$ the operators (1.12) are invertible, and by the Langlands classfication [Ren, Théorème VII.4.2] $I_{P \cap M_\alpha}^{M_\alpha}(\sigma \otimes \chi)$ is irreducible.

Corollary 1.3. *The poles of* μ^{M_α} *are precisely the non-unitary* $\sigma \otimes \chi \in \mathcal{O}$ *for which* $I_{P \cap M_\alpha}^{M_\alpha}(\sigma \otimes \chi)$ *is reducible.*

We endow the based root datum

$$\big(\Sigma_{\mathcal{O}}^\vee, M_\sigma^2/M^1, \Sigma_{\mathcal{O}}, (M_\sigma^2/M^1)^\vee, \Delta_{\mathcal{O}}^\vee\big)$$

with the parameter q_F and the labels

$$\lambda(\alpha) = \log(q_\alpha q_{\alpha*}) / \log(q_F), \quad \lambda^*(\alpha) = \log(q_\alpha q_{\alpha*}^{-1}) / \log(q_F).$$

To avoid ambiguous terminology, we will call the q_α and $q_{\alpha*}$ q-parameters and refer to q_F as the q-base. Replacing the q-base by another real number > 1 hardly changes the representation theory of Hecke algebras.

To these data we associate the affine Hecke algebra

$$\mathcal{H}(\mathcal{O}, G) = \mathcal{H}\big(\Sigma_{\mathcal{O}}^\vee, M_\sigma^2/M^1, \Sigma_{\mathcal{O}}, (M_\sigma^2/M^1)^\vee, \lambda, \lambda^*, q_F\big).$$

By definition it is the vector space

$$\mathbb{C}[M_\sigma^2/M^1] \otimes_{\mathbb{C}} \mathbb{C}[W(\Sigma_{\mathcal{O},\mu})]$$

with multiplication given by the following rules:

- $\mathbb{C}[M_\sigma^2/M^1] \cong \mathbb{C}[\mathcal{O}]$ is embedded as subalgebra,
- $\mathbb{C}[W(\Sigma_{\mathcal{O},\mu})] = \mathrm{span}\{T_w : w \in W(\Sigma_{\mathcal{O},\mu})\}$ is embedded as the Iwahori–Hecke algebra $\mathcal{H}(W(\Sigma_{\mathcal{O},\mu}), q_F^\lambda)$, that is,

  $$T_w T_v = T_{wv} \quad \text{if } \ell_{\mathcal{O}}(w) + \ell_{\mathcal{O}}(v) = \ell_{\mathcal{O}}(wv),$$

  $$(T_{s_\alpha} + 1)(T_{s_\alpha} - q_F^{\lambda(\alpha)}) = (T_{s_\alpha} + 1)(T_{s_\alpha} - q_\alpha q_{\alpha*}) = 0 \quad \text{if } \alpha \in \Delta_{\mathcal{O},\mu},$$

- for $\alpha \in \Delta_{\mathcal{O},\mu}$ and $m \in M_\sigma^2/M^1$ (corresponding to $X_m \in \mathbb{C}[M_\sigma^2/M^1]$):

  $$X_m T_{s_\alpha} - T_{s_\alpha} X_{s_\alpha(m)} = \big(q_\alpha q_{\alpha*} - 1 + X_\alpha^{-1}(q_\alpha - q_{\alpha*})\big) \frac{X_m - X_{s_\alpha(m)}}{1 - X_\alpha^{-2}}.$$

This affine Hecke algebra is related to $\mathrm{End}_G(\Pi^{\mathfrak{s}})$ in the following way. Let $\mathrm{End}_G^\circ(\Pi^{\mathfrak{s}})$ be the subalgebra of $\mathrm{End}_G(\Pi^{\mathfrak{s}})$ built, as in [Sol4, §5.2], using only $\mathbb{C}[X_{\mathrm{nr}}(M)]$,

$X_{\mathrm{nr}}(M,\sigma)$ and $W(\Sigma_{\mathcal{O},\mu})$ – so omitting $R(\mathcal{O})$. By [Sol4, Corollary 5.8] there exist elements $\mathcal{T}_r \in \mathrm{End}_G(\Pi^{\mathfrak{s}})^\times$ for $r \in R(\mathcal{O})$, such that

$$\mathrm{End}_G(\Pi^{\mathfrak{s}}) = \bigoplus\nolimits_{r \in R(\mathcal{O})} \mathrm{End}^\circ_G(\Pi^{\mathfrak{s}}) \mathcal{T}_r. \tag{1.13}$$

The calculations in [Sol4, §6–8] apply also to $\mathrm{End}^\circ_G(\Pi^{\mathfrak{s}})$ and they imply, as in [Sol4, Corollary 9.4], an equivalence of categories

$$\mathrm{End}^\circ_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod}_{\mathrm{f}} \longleftrightarrow \mathcal{H}(\mathcal{O},G) - \mathrm{Mod}_{\mathrm{f}}. \tag{1.14}$$

Here $-\mathrm{Mod}_{\mathrm{f}}$ denotes the category of finite length right modules. To go from $\mathrm{End}^\circ_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod}_{\mathrm{f}}$ to $\mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod}_{\mathrm{f}}$ is basically an instance of Clifford theory for a finite group acting on an algebra. In reality it is more complicated [Sol4, §9], but still relatively easy. Consequently the essence of the representation theory of $\mathrm{End}_G(\Pi^{\mathfrak{s}})$ (and thus of $\mathrm{Rep}(G)^{\mathfrak{s}}$) is contained in the affine Hecke algebra $\mathcal{H}(\mathcal{O},G)$.

Slightly better results can be obtained if we assume that the restriction of (σ,E) to M^1 decomposes without multiplicities bigger than one – which by [Roc1, Remark 1.6.1.3] holds for very large classes of reductive p-adic groups. Assuming it for (σ,E), [Sol4, Theorem 10.9] says that there exist:

- a smaller progenerator $(\Pi^{\mathfrak{s}})^{X_{\mathrm{nr}}(M,\sigma)}$ of $\mathrm{Rep}(G)^{\mathfrak{s}}$,
- a Morita equivalent subalgebra $\mathrm{End}_G\big((\Pi^{\mathfrak{s}})^{X_{\mathrm{nr}}(M,\sigma)}\big)$ of $\mathrm{End}_G(\Pi^{\mathfrak{s}})$,
- a subalgebra $\mathrm{End}^\circ_G\big((\Pi^{\mathfrak{s}})^{X_{\mathrm{nr}}(M,\sigma)}\big)$ of $\mathrm{End}_G\big((\Pi^{\mathfrak{s}})^{X_{\mathrm{nr}}(M,\sigma)}\big)$, which is canonically isomorphic with $\mathcal{H}(\mathcal{O},G)$,
- elements $J_r \in \mathrm{End}_G\big((\Pi^{\mathfrak{s}})^{X_{\mathrm{nr}}(M,\sigma)}\big)^\times$ for $r \in R(\mathcal{O})$, such that

$$\mathrm{End}_G\big((\Pi^{\mathfrak{s}})^{X_{\mathrm{nr}}(M,\sigma)}\big) = \bigoplus\nolimits_{r \in R(\mathcal{O})} \mathrm{End}^\circ_G\big((\Pi^{\mathfrak{s}})^{X_{\mathrm{nr}}(M,\sigma)}\big) J_r.$$

As announced in the introduction, we want to determine the parameters $q_\alpha, q_{\alpha *}$ for $\alpha \in \Delta_{\mathcal{O},\mu}$, or equivalently the label functions $\lambda, \lambda^* : \Sigma_{\mathcal{O},\mu} \to \mathbb{R}_{\geq 0}$ of $\mathcal{H}(\mathcal{O},G)$.

When $\Sigma_{\mathcal{O},\mu}$ is empty, $\mathcal{H}(\mathcal{O},G) \cong \mathbb{C}[\mathcal{O}]$ and it does not have parameters or labels. When $\Sigma_{\mathcal{O},\mu} = \{\alpha,-\alpha\}$, it can already be quite difficult to identify q_α and $q_{\alpha *}$. For instance, when G is split of type G_2 and M has semisimple rank one, we did not manage to compute q_α and $q_{\alpha *}$ for all supercuspidal representations of M. (This was achieved recently in [AuXu].)

Yet, for $\mathcal{H}(\mathcal{O},G)$ this is hardly troublesome. Namely, any affine Hecke algebra $\mathcal{H}$ with $\Sigma_{\mathcal{O},\mu} = \{\alpha,-\alpha\}$ and $q_\alpha, q_{\alpha *} \in \mathbb{C} \setminus \{0,-1\}$ can be analysed very well. Firstly, one can determine all its irreducible representations directly, as done in [Sol3, §2.2]. Secondly, with [Lus2] the representation theory of $\mathcal{H}$ can be reduced to that of two graded Hecke algebras $\mathbb{H}_k$ with root system of rank ≤ 1. One of them has label $k_\alpha = \log(q_\alpha)/\log(q_F)$ and underlying vector space $T_1(\mathcal{O})$, the other has label $k_{\alpha *} = \log(q_{\alpha *})/\log(q_F)$ and underlying vector space $T_{\chi_-}(\mathcal{O})$ (for some $\chi_- \in \mathcal{O}$ with $X_\alpha(\chi_-) = -1$).

For graded Hecke algebras with root system $\{\alpha,-\alpha\}$ and a fixed underlying vector space, there are just two isomorphism classes: one with label $k \neq 0$ and one with label $k = 0$. For both there is a nice geometric construction of the irreducible representations of $\mathbb{H}_k$, see [Lus1] and [AMS2, Theorem 3.11]. This is an instance of a construction that underlies the representation theory of affine Hecke algebras associated to unipotent representations of p-adic groups [Lus3, Lus4]. Let us summarise that:

(1.15) for $\mathcal{H}(\mathcal{O},G)$ with $\mathrm{rk}(\Sigma_{\mathcal{O},\mu}) = 1$,
Conjecture A holds on the level of the underlying graded Hecke algebras.

While this does not settle Conjecture A for all affine Hecke algebras in the rank one cases, it looks like a satisfactory outcome.

2. Reduction to simply connected groups

In this section we reduce the analysis of the parameters of $\mathcal{H}(\mathcal{O},G)$ to the case where $\mathcal{G}$ is absolutely simple and simply connected. Consider a homomorphism between connected reductive F-groups $\eta : \tilde{\mathcal{G}} \to \mathcal{G}$ such that:

- the kernel of $\mathrm{d}\eta : \mathrm{Lie}(\tilde{\mathcal{G}}) \to \mathrm{Lie}(\mathcal{G})$ is central,
- the cokernel of η is a commutative F-group.

These properties imply [Sol2, Lemma 5.1] that on the derived groups η restricts to

$$\text{a central isogeny } \eta_{\mathrm{der}} : \tilde{\mathcal{G}}_{\mathrm{der}} \to \mathcal{G}_{\mathrm{der}} \tag{2.1}$$

Such a map induces a homomorphism on F-rational points

$$\eta : \tilde{G} = \tilde{\mathcal{G}}(F) \to \mathcal{G}(F) = G$$

and a pullback functor $\eta^* : \mathrm{Rep}(G) \to \mathrm{Rep}(\tilde{G})$.

Lemma 2.1. *Let $\pi \in \mathrm{Irr}(G)$. Then $\eta^*(\pi)$ is a finite direct sum of irreducible $\tilde{G}$-representations.*

Proof. According to [Tad, Lemma 2.1] this holds for the inclusion of $\mathcal{G}_{\mathrm{der}}$ in $\mathcal{G}$. Taking that into account, [Sil1] says that pullback along $\eta_{\mathrm{der}} : \tilde{G}_{\mathrm{der}} \to G_{\mathrm{der}}$ has the desired property. This shows that $\mathrm{Res}^{\tilde{G}}_{\tilde{G}_{\mathrm{der}}} \eta^*(\pi)$ is a finite direct sum of irreducible $\tilde{G}_{\mathrm{der}}$-representations. As in the proof of [Tad, Lemma 2.1], that implies the same property for $\eta^*(\pi)$. $\square$

By (2.1), η induces a bijection

$$\begin{array}{ccc} \{\text{Levi subgroups of } G\} & \to & \{\text{Levi subgroups of } \tilde{G}\} \\ M & \mapsto & \tilde{M} = \eta^{-1}(M) \end{array} .$$

One also sees from (2.1) that η induces a bijection

$$\begin{array}{ccc} \Sigma(G, A_M) & \to & \Sigma(\tilde{G}, A_{\tilde{M}}) \\ \alpha & \mapsto & \tilde{\alpha} = \alpha \circ \eta \end{array} .$$

For each $\alpha \in \Sigma_{\mathrm{red}}(A_M)$ this yields an isomorphism of F-groups

$$\eta_\alpha : \mathcal{U}_{\tilde{\alpha}} \to \mathcal{U}_\alpha .$$

This implies that η^* preserves cuspidality [Sil1, Lemma 1]. Further, pullback along η restricts to an algebraic group homomorphism $\eta^* : X_{\mathrm{nr}}(M) \to X_{\mathrm{nr}}(\tilde{M})$.

Proposition 2.2. *Let $(\sigma, E) \in \mathrm{Irr}_{\mathrm{cusp}}(M)$ and let $\tilde{\sigma} \in \mathrm{Irr}_{\mathrm{cusp}}(\tilde{M})$ be a constituent of $\eta^*(\sigma)$. For $\alpha \in \Sigma_{\mathrm{red}}(A_M)$ there exists $\tilde{c}_\alpha \in \mathbb{C}^\times$ such that*

$$\mu^{M_\alpha}(\sigma \otimes \chi) = \tilde{c}_\alpha \mu^{\tilde{M}_{\tilde{\alpha}}}(\tilde{\sigma} \otimes \eta^*(\chi))$$

as rational functions of $\chi \in X_{\mathrm{nr}}(M)$.

Proof. In view of the explicit shape (1.8), it suffices to show that the two rational functions have precisely the same poles. Using the relation (1.11), it suffices to show that

$$J_{M_\alpha \cap P | s_\alpha (M_\alpha \cap P)} (\sigma \otimes \chi) J_{s_\alpha (M_\alpha \cap P) | M_\alpha \cap P} (\sigma \otimes \chi) = 0 \quad \Longleftrightarrow \tag{2.2}$$
$$J_{\eta^{-1}(M_\alpha \cap P) | \eta^{-1}(s_\alpha (M_\alpha \cap P))} (\tilde\sigma \otimes \eta^* (\chi)) J_{\eta^{-1}(s_\alpha (M_\alpha \cap P)) | \eta^{-1}(M_\alpha \cap P)} (\tilde\sigma \otimes \eta^* (\chi)) = 0.$$

Since $\eta_\alpha : U_{\tilde\alpha} \to U_\alpha$ is an isomorphism, we may choose Haar measures on U_α and $U_{\tilde\alpha}$ such that the latter is pullback along η_α of the former. Then (1.12) shows that the J-operators on both lines of (2.2) do the same thing, namely

$$f \mapsto \big[g \mapsto \int_U f(ug) \mathrm{d}u \big],$$

where U stands for U_α or $U_{-\alpha}$. The only real difference between the two lines of (2.2) lies in their domain. Since $\tilde\sigma \otimes \eta^*(\chi)$ is a subrepresentation of $\eta^*(\sigma \otimes \chi)$, it is clear that the implication $\Rightarrow$ holds.

Conversely, suppose that the second line of (2.2) is 0, for a particular χ. Let $\tilde E \subset E$ be the subspace on which $\tilde\sigma$ is defined, so that $I_{\eta^{-1}P}^{\tilde G}(\tilde E) \cong I_P^G (\tilde E)$ is the vector space underlying $I_{\eta^{-1}P}^{\tilde G}(\tilde\sigma \otimes \eta^*(\chi))$. It is a linear subspace of $I_P^G(E)$, on which $I_P^G(\sigma \otimes \chi)$ is defined. Then

$$J_{M_\alpha \cap P | s_\alpha (M_\alpha \cap P)} (\sigma \otimes \chi) J_{s_\alpha (M_\alpha \cap P) | M_\alpha \cap P} (\sigma \otimes \chi) \tag{2.3}$$

coincides on $I_P^G(\tilde E)$ with

$$J_{\eta^{-1}(M_\alpha \cap P) | \eta^{-1}(s_\alpha (M_\alpha \cap P))} (\tilde\sigma \otimes \eta^* (\chi)) J_{\eta^{-1}(s_\alpha (M_\alpha \cap P)) | \eta^{-1}(M_\alpha \cap P)} (\tilde\sigma \otimes \eta^* (\chi)),$$

so annihilates $I_P^G(\tilde E)$. But by (1.11) the operator (2.3) is a scalar on $I_P^G(E)$, so it annihilates that entire space. ☐

From Proposition 2.2 and (1.2) we deduce:

Corollary 2.3. *In the setting of Proposition 2.2, write* $\tilde{\mathcal O} = X_{\mathrm{nr}}(\tilde M)\tilde\sigma$. *Then* $\Sigma_{\tilde{\mathcal O},\mu}$ *equals*

$$\eta^*(\Sigma_{\mathcal O,\mu}) = \{\tilde\alpha = \alpha \circ \eta : \alpha \in \Sigma_{\mathcal O,\mu}\}.$$

We warn that Proposition 2.2 and Corollary 2.3 do not imply that $q_\alpha = q_{\tilde\alpha}$. The problem is that X_α need not equal $X_{\tilde\alpha} \circ \eta^*$. To make the relation precise, we have to consider $h_\alpha^\vee$, $h_{\tilde\alpha}^\vee$ and their images (via H_M and $H_{\tilde M}$) in $\mathfrak a_M$ and $\mathfrak a_{\tilde M}$. We note that $\mathrm d\eta : \mathrm{Lie}(A_{\tilde M}) \to \mathrm{Lie}(A_M)$ induces a linear map $\mathfrak a_\eta : \mathfrak a_{\tilde M} \to \mathfrak a_M$. Further, η induces a group homomorphism

$$\eta : (\tilde M \cap \tilde M^1_{\tilde\alpha}) / \tilde M^1 \to (M \cap M^1_\alpha)/M^1. \tag{2.4}$$

Both the source and the target of (2.4) are isomorphic to $\mathbb Z$, so the map is injective.

Proposition 2.4. (*a*) *For* $\alpha \in \Sigma_{\mathcal O,\mu}$, *there exists a* $N_\alpha \in \{1/2, 1, 2\}$ *such that*

$$H_M(h_\alpha^\vee) = N_\alpha \mathfrak a_\eta \big(H_{\tilde M}(h_{\tilde\alpha}^\vee) \big).$$

(*b*) *If (2.4) is bijective, then* $N_\alpha \in \{1,2\}$. *This happens for instance when* η *restricts to an isomorphism between the almost direct* F*-simple factors of* $\tilde{\mathcal G}$ *and* $\mathcal G$ *corresponding to* $\tilde\alpha$ *and* α,

(*c*) *If* $\eta^*(\sigma)$ *is irreducible, then* $N_\alpha \in \{1/2, 1\}$.

(*d*) *Let* $\Sigma^\vee_{\mathcal{O},j}$ *be an irreducible component of* $\Sigma^\vee_{\mathcal{O}}$, *and regard it as a subset of* $\mathfrak{a}_M$ *via* H_M. *Consider the irreducible component*

$$\Sigma^\vee_{\tilde{\mathcal{O}},j} = \{h^\vee_{\tilde\alpha} : h^\vee_\alpha \in \Sigma^\vee_{\mathcal{O},j}\}$$

of $\Sigma^\vee_{\tilde{\mathcal{O}}}$. *There are three possibilities:*

- (*i*) $N_\alpha = 1$ *for all* $h^\vee_\alpha \in \Sigma^\vee_{\mathcal{O},j}$.
- (*ii*) $\Sigma^\vee_{\mathcal{O},j} \cong B_n$, $\Sigma^\vee_{\tilde{\mathcal{O}},j} \cong C_n$, $N_\alpha = 1$ *for* $h^\vee_\alpha \in \Sigma^\vee_{\mathcal{O},j}$ *long and* $N_\beta = 1/2$ *for* $h^\vee_\beta \in \Sigma^\vee_{\mathcal{O},j}$ *short. Then*
$$q_{\tilde\beta *} = 1,\ q_\beta = q_{\beta *} = q_{\tilde\beta}^{1/2},\ \lambda^*(\beta) = 0 \text{ and } \lambda(\beta) = \lambda(\tilde\beta) = \lambda^*(\tilde\beta).$$
- (*iii*) $\Sigma^\vee_{\mathcal{O},j} \cong C_n$, $\Sigma^\vee_{\tilde{\mathcal{O}},j} \cong B_n$, $N_\alpha = 1$ *for* $h^\vee_\alpha \in \Sigma^\vee_{\mathcal{O},j}$ *short and* $N_\beta = 2$ *for* $h^\vee_\beta \in \Sigma^\vee_{\mathcal{O},j}$ *long. Then*
$$q_{\beta *} = 1,\ q_{\tilde\beta}^2 = q_{\tilde\beta *}^2 = q_\beta,\ \lambda^*(\tilde\beta) = 0 \text{ and } \lambda(\tilde\beta) = \lambda(\beta) = \lambda^*(\beta).$$

(*e*) *The modifications of the labels in part (d) preserve the class of labels in Table 1.*

Proof. (a) As both sides of (2.4) are isomorphic to $\mathbb{Z}$, the definition of $h^\vee_\alpha$ implies that the statement holds for some $N_\alpha \in \mathbb{Q}_{>0}$. Then $\eta(X_{\tilde\alpha}) = X_\alpha^{1/N_\alpha}$, and this is a well-defined function on $X_{\mathrm{nr}}(M)$ because it equals evaluation at $\eta(h^\vee_{\tilde\alpha})$. We plug this into the equality of μ-functions from Proposition 2.2, and we use the formula (1.8) both for M and for $\tilde M$. That yields an equality of two rational functions on $X_{\mathrm{nr}}(M)$, one built from X_α and one built from X_α^{1/N_α}. The equality of the numerators of these two functions reads

$$\begin{aligned} c'_{s_\alpha}(1 - X_\alpha)(1 - X_\alpha^{-1}) \big[(1 + X_\alpha)(1 + X_\alpha^{-1})\big] = \\ \tilde c_\alpha c'_{s_{\tilde\alpha}}(1 - X_\alpha^{1/N_\alpha})(1 - X_\alpha^{-1/N_\alpha}) \big[(1 + X_\alpha^{1/N_\alpha})(1 + X_\alpha^{-1/N_\alpha})\big]. \end{aligned} \tag{2.5}$$

Here the term $(1 + X_\alpha)(1 + X_\alpha^{-1})$ must be omitted when $q_{\alpha *} = 1$. On the other hand $q_\alpha > 1$ because $\alpha \in \Sigma_{\mathcal{O},\mu}$, so the zeros at $X_\alpha = 1$ do not cancel against something in the denominator of (1.8). Analogous considerations apply to the second line of (2.5). Now we see that there are only three values of N_α for which (2.5) is possible: $N_\alpha = 1$, $N_\alpha = 1/2$ (when the factor $(1 + X_\alpha^{1/N_\alpha})(1 + X_\alpha^{-1/N_\alpha})$ is not there) and $N_\alpha = 2$ (when $(1 + X_\alpha)(1 + X_\alpha^{-1})$ is omitted).

(b) By (2.1) η induces an isomorphism between the respective adjoint groups. From $G \to G_{\mathrm{ad}} \to \tilde G_{\mathrm{ad}}$ we get an action of G on $\tilde G$, by "conjugation". All the $\tilde M$-constituents of $\eta^*(\sigma)$ are associated (up to isomorphism) by elements of M. For $m \in M$, $\mathrm{Ad}(m) : \tilde M \to \tilde M$ does not affect unramified characters of $\tilde M$. It follows that any $\chi \in X_{\mathrm{nr}}(\tilde M)$ which stabilizes $\tilde\sigma$, also stabilizes $\eta^*(\sigma)$. That implies

$$\eta^{-1}\big((M^2_\sigma \cap M^1_\alpha)/M^1\big) \subset (\tilde M^2_{\tilde\sigma} \cap \tilde M^1_{\tilde\alpha})/\tilde M^1.$$

That and the assumed bijectivity show that

$$h^\vee_\alpha \in \eta(\tilde M^2_{\tilde\sigma} \cap \tilde M^1_{\tilde\alpha})/\tilde M^1.$$

By definition $h^\vee_{\tilde\alpha}$ generates $(\tilde M^2_{\tilde\sigma} \cap \tilde M^1_{\tilde\alpha})/\tilde M^1$, so an integer multiple of its image under η equals $h^\vee_\alpha$. By part (a) the multiplication factor is at most 2.

(c) If $\chi \in X_{\mathrm{nr}}(M, \sigma)$, then $\eta^*(\sigma) \otimes \eta^*(\chi) = \eta^*(\sigma \otimes \chi)$ is isomorphic with $\eta^*(\sigma)$. Hence

$$\eta^*(X_{\mathrm{nr}}(M, \sigma)) \subset X_{\mathrm{nr}}(\tilde M, \eta^*(\sigma)),$$

which implies that $\eta(\tilde M^2_{\eta^*(\sigma)}) \subset M^2_\sigma$. As $h_\alpha^\vee$ generates $(M^2_\sigma \cap M^1_\alpha)/M^1$ and $\eta(h_{\tilde\alpha})$ lies in that group, $\eta(h_{\tilde\alpha})$ is a multiple of $h_\alpha^\vee$. Combine that with part (a).
(d) In case $N_\alpha = 2$, $2\eta(X_{\tilde\alpha}) = X_\alpha$. Then Proposition 2.2 and (1.8) entail $q_\alpha = 1$ and $q_{\tilde\alpha} = q_{\tilde\alpha *} = q_\alpha^{1/2}$. Notice that this is only possible when $\Sigma^\vee_{\tilde{\mathcal O},j} \cong C_n$.

When $N_\alpha = 1/2$, we have $\eta(X_{\tilde\alpha}) = 2X_\alpha$. For the same reasons as above, $q_{\tilde\alpha} = 1$ and $q_\alpha = q_{\alpha *} = q_{\tilde\alpha}^{1/2}$. By [Sol4, Lemma 3.3] this is only possible if $\Sigma^\vee_{\mathcal O,j}$ has type B_n.
(e) Parts (d,ii) and (d,iii) just switch the second line (with $\lambda^* = 0$) and the third line of Table 1. $\square$

We remark that examples of case (ii) are easy to find, it already occurs for $SL_2(F) \to PGL_2(F)$ and the unramified principal series (as worked out in Paragraph 4.1). For an instance of case (iii) see Example 4.8.

We can apply Propositions 2.2 and 2.4 in particular with $\tilde{\mathcal G}$ equal to the simply connected cover $\mathcal G_{\rm sc}$ of $\mathcal G_{\rm der}$, that yields:

Corollary 2.5. *Suppose that Conjecture A holds for* $\tilde G = G_{\rm sc}$ *and* $[\tilde M, \tilde\sigma]_{G_{\rm sc}}$. *Then it holds for* G *and* $[M,\sigma]_G$.

Every simply connected F-group is a direct product of F-simple simply connected groups, say

$$\mathcal G_{\rm sc} = \prod\nolimits_i \mathcal G_{\rm sc}^{(i)} .$$

Everything described in Section 1 decomposes accordingly, for instance any $\tilde\sigma \in {\rm Irr}_{\rm cusp}(G_{\rm sc})$ can be factorized as

$$\tilde\sigma = \boxtimes_i \sigma^{(i)} \quad \text{with} \quad \sigma^{(i)} \in {\rm Irr}_{\rm cusp}(G_{\rm sc}^{(i)}).$$

For every F-simple simply connected F-group $\mathcal G_{\rm sc}^{(i)}$ there exists a finite separable field extension F'/F and an absolutely simple, simply connected F'-group $\mathcal G_{\rm sc}^{\prime (i)}$, such that $\mathcal G_{\rm sc}^{(i)}$ is the restriction of scalars from F' to F of $\mathcal G_{\rm sc}^{\prime (i)}$. Then

$$G_{\rm sc}^{(i)} = \mathcal G_{\rm sc}^{(i)}(F) = \mathcal G_{\rm sc}^{\prime (i)}(F') = G_{\rm sc}^{\prime (i)},$$

so $\sigma^{(i)}$ can be regarded as a supercuspidal representation of $G_{\rm sc}^{\prime (i)}$. Of course that last step does not change the parameters q_α and $q_{\alpha *}$ associated to $\sigma^{(i)}$. On the other hand, that step does replace q_F by $q_{F'}$ and changes the labels $\lambda(\alpha)$ and $\lambda^*(\alpha)$ by a factor $\log(q_F)/\log(q_{F'})$. As this is the same scalar factor for all $\alpha \in \Sigma_{\tilde{\mathcal O}^{(i)},\mu}$, it is innocent. With these steps we reduced the computation of the parameters $q_\alpha, q_{\alpha *}, \lambda(\alpha)$ and $\lambda^*(\alpha)$ to the case where $\mathcal G$ is absolutely simple and simply connected.

Sometime it is more convenient to study, instead of a simply connected simple group, a reductive group with that as derived group. For instance, the groups $GL_n, U_n, {\rm GSpin}_n$ are often easier than, respectively, $SL_n, SU_n, {\rm Spin}_n$. In such situations, the following result comes in handy.

Proposition 2.6. [Tad, Propositions 2.2 and 2.7]
Suppose that $\tilde{\mathcal G}$ *is a connected reductive* F*-subgroup of* $\mathcal G$ *that contains* $\mathcal G_{\rm der}$. *For every* $\tilde\pi \in {\rm Irr}(\tilde G)$ *there exists a* $\pi \in {\rm Irr}(G)$ *such that* ${\rm Res}^G_{\tilde G}(\pi)$ *contains* $\tilde\pi$. *Moreover* $\tilde\pi$ *is supercuspidal if and only if* π *is supercuspidal.*

We note that in this setting the inclusion $\imath : \tilde{\mathcal G} \to \mathcal G$ satisfies the conditions stated at the start of the paragraph.

Corollary 2.7. *Let $\mathcal{G}, \tilde{\mathcal{G}}$ be as in Proposition 2.6. Then Conjecture A holds for G if and only if it holds for $\tilde{G}$.*

Proof. Let M be a Levi subgroup of G and let $\tilde{\pi} \in \mathrm{Irr}_{\mathrm{cusp}}(M \cap \tilde{G})$. An appropriate π is obtained from Proposition 2.6 applied to $\imath : \mathcal{M} \cap \tilde{\mathcal{G}} \to \mathcal{M}$. Then $\tilde{\pi}$ is a constituent of $\imath^*(\pi)$. This also works the other way round: if we start with $\pi \in \mathrm{Irr}_{\mathrm{cusp}}(M)$ we can choose as $\tilde{\pi}$ any constituent of $\imath^*(\pi)$. Now we can apply Proposition 2.4, which says that the Hecke algebras $\mathcal{H}(X_{\mathrm{nr}}(M \cap \tilde{G})\tilde{\pi}, \tilde{G})$ and $\mathcal{H}(X_{\mathrm{nr}}(M)\pi, G)$ have root systems and parameters related as in cases (i) or (iii) of Proposition 2.4.d. $\square$

3. Reduction to characteristic zero

For several classes of reductive groups, stronger results are available over p-adic fields than over local function fields. With the method of close local fields [Kaz, Gan2], will show that all relevant results about affine Hecke algebras associated to Bernstein components can be transferred from characteristic zero to positive characteristic.

We start with an arbitrary local field of characteristic p. Choose a p-adic field $\tilde{F}$ which is ℓ-close to F, that is

$$\mathfrak{o}_F / \varpi_F^\ell \mathfrak{o}_F \cong \mathfrak{o}_{\tilde{F}} / \varpi_{\tilde{F}}^\ell \mathfrak{o}_{\tilde{F}} \qquad \text{as rings.} \tag{3.1}$$

As remarked in [Del], such a field $\tilde{F}$ exists for every given $\ell \in \mathbb{Z}_{>0}$. If (3.1) holds, then it is also valid for every $m < \ell$, and in particular the residue fields $\mathfrak{o}_F / \varpi_F \mathfrak{o}_F$ and $\mathfrak{o}_{\tilde{F}} / \varpi_{\tilde{F}} \mathfrak{o}_{\tilde{F}}$ are isomorphic. We note that

$$F^\times / (1 + \varpi_F^\ell \mathfrak{o}_F) \cong \mathbb{Z} \times \mathfrak{o}_F^\times / (1 + \varpi_F^\ell \mathfrak{o}_F) \cong \mathbb{Z} \times \mathfrak{o}_{\tilde{F}}^\times / (1 + \varpi_{\tilde{F}}^\ell \mathfrak{o}_{\tilde{F}}) = \tilde{F}^\times / (1 + \varpi_{\tilde{F}}^\ell \mathfrak{o}_{\tilde{F}}). \tag{3.2}$$

Let $\mathbf{I}_F^\ell$ be the ℓ-th ramification subgroup of $\mathrm{Gal}(F_s/F)$. By [Del, (3.5.1)] there is a group isomorphism (unique up to conjugation)

$$\mathrm{Gal}(F_s/F)/\mathbf{I}_F^\ell \cong \mathrm{Gal}(\tilde{F}_s/\tilde{F})/\mathbf{I}_{\tilde{F}}^\ell, \tag{3.3}$$

and similarly with Weil groups. According to [Del, Proposition 3.6.1], for $m < \ell$ this isomorphism is compatible with the Artin reciprocity map

$$\mathbf{W}_F / \mathbf{I}_F^\ell \to F^\times / (1 + \varpi_F^m \mathfrak{o}_F).$$

Let $\mathcal{G}$ be a connected reductive F-group. We want to exhibit "the same" group over a p-adic field. The quasi-split inner form $\mathcal{G}^*$ of $\mathcal{G}$ is determined by the action of $\mathrm{Gal}(F_s/F)$ on the based absolute root datum of $\mathcal{G}$. That action factors through a finite quotient of $\mathrm{Gal}(F_s/F)$, so there exists a $\ell \in \mathbb{Z}_{>0}$ such that $\mathbf{I}_F^\ell$ acts trivially. The group $\mathcal{G}$ is an inner twist of $\mathcal{G}^*$, and the inner twists of $\mathcal{G}^*$ are parametrized naturally by

$$H^1(F, \mathcal{G}^*_{\mathrm{ad}}) \cong \mathrm{Irr}\big(Z(G^{*\vee}_{\mathrm{sc}})^{\mathbf{W}_F}\big). \tag{3.4}$$

Now we pick a p-adic field $\tilde{F}$ which is ℓ-close to F, and we define $\tilde{\mathcal{G}}^*$ to be the quasi-split $\tilde{F}$-group with the same based root absolute root datum as $\mathcal{G}^*$ and Galois action transferred from that of $\mathcal{G}^*$ via (3.3). Then $\mathcal{G}^*$ and $\tilde{\mathcal{G}}^*$ have the same Langlands dual group (in a form where $\mathbf{I}_F^\ell$ has been divided out) and hence

$$Z(\tilde{G}^{*\vee}_{\mathrm{sc}})^{\mathbf{W}_{\tilde{F}}} \cong Z(G^{*\vee}_{\mathrm{sc}})^{\mathbf{W}_F}. \tag{3.5}$$

We define $\tilde{\mathcal{G}}$ to be the inner twist of $\tilde{\mathcal{G}}^*$ parametrized by the character of $Z(\tilde{G}^{*\vee}_{\mathrm{sc}})^{\mathbf{W}_{\tilde{F}}}$ that is transformed by (3.5) into the character of $Z(G^{*\vee}_{\mathrm{sc}})^{\mathbf{W}_F}$ that parametrizes $\mathcal{G}$.

The following descriptions are based on recent work of Ganapathy [Gan1, Gan2]. It applies when F and $\tilde{F}$ are ℓ-close with ℓ large enough. The relation between $\mathcal{G}$ and $\tilde{\mathcal{G}}$ is the same as in these papers, although over there it is reached in a slightly different way, without (3.4). Let $\mathcal{T} \subset \mathcal{G}$ be the maximal F-torus from which the root datum is built, and let $\mathcal{S} \subset \mathcal{T}$ be the maximal F-split subtorus. In the Bruhat–Tits building $\mathcal{B}(\mathcal{G}, F)$, $S = \mathcal{S}(F)$ determines an apartment $\mathbb{A}_S$.

The same constructions can be performed for $\tilde{\mathcal{G}}$. Then $X_*(\mathcal{S}) \cong X_*(\tilde{\mathcal{S}})$ extends to an isomorphism of polysimplicial complexes $\mathbb{A}_S \cong \mathbb{A}_{\tilde{S}}$. For every special vertex $x \in \mathbb{A}_S$, we get a special vertex $\tilde{x} \in \mathbb{A}_{\tilde{S}}$. For $m \in \mathbb{Z}_{\geq 0}$, there is a refined version of the Moy–Prasad group $G_{x,m}$, see [Gan1]. It is a compact open normal subgroup of G_x, the G-stabilizer of x. More precisely, there is an $\mathfrak{o}_F$-group scheme $\mathcal{G}_x$ (a slightly improved version of the parahoric group schemes constructed in [BrTi]), such that

$$G_{x,m} = \mathcal{G}_x(\varpi_F^m \mathfrak{o}_F) \qquad \forall m \in \mathbb{Z}_{\geq 0}.$$

By construction [Gan1, §2.D.3], $G_{x,m}$ is totally decomposed in the sense of [Bus, §1]. This means that, for any ordering of the root system $\Sigma(G, S)$, the product map

$$(G_{x,m} \cap Z_G(S)) \times \prod\nolimits_{\alpha \in \Sigma(G,S)} (G_{x,m} \cap U_\alpha) \longrightarrow G_{x,m}$$

is a bijection. Here U_α is the root subgroup of G with respect to $\alpha \in \Sigma(G, S)$ (to be distinguished from the earlier U_α when M is not a minimal Levi subgroup of G).

All the above applies to $\tilde{\mathcal{G}}$ as well. The following results generalize [Kaz] to non-split groups.

Theorem 3.1. [Gan1, Corollary 6.3]
Fix $m \in \mathbb{Z}_{>0}$ and let $\ell \in \mathbb{Z}_{>0}$ be large enough. The isomorphisms (3.1) *induce an isomorphism of group schemes*

$$\mathcal{G}_x \times_{\mathfrak{o}_F} \mathfrak{o}_F / \varpi_F^m \mathfrak{o}_F \cong \mathcal{G}_{\tilde{x}} \times_{\mathfrak{o}_{\tilde{F}}} \mathfrak{o}_{\tilde{F}} / \varpi_{\tilde{F}}^m \mathfrak{o}_{\tilde{F}}$$

and group isomorphisms

$$G_{x,0}/G_{x,m} = \mathcal{G}_x(\mathfrak{o}_F / \varpi_F^m \mathfrak{o}_F) \cong \mathcal{G}_{\tilde{x}}(\mathfrak{o}_{\tilde{F}} / \varpi_{\tilde{F}}^m \mathfrak{o}_{\tilde{F}}) = \tilde{G}_{\tilde{x},0}/\tilde{G}_{\tilde{x},m}.$$

We endow G with the Haar measure that gives the parahoric subgroup $G_{x,0}$ volume 1. The vector space $C_c(G_{x,m} \backslash G / G_{x,m})$ with the convolution product is an associative algebra, denoted $\mathcal{H}(G, G_{x,m})$.

Theorem 3.2. [Gan2, Theorem 4.1]
Fix $m \in \mathbb{Z}_{>0}$ and let $\ell \in \mathbb{Z}_{>0}$ be large enough. The isomorphisms from Theorem 3.1 and the Cartan decomposition give rise to a bijection

$$\zeta_m : G_{x,m} \backslash G / G_{x,m} \to \tilde{G}_{\tilde{x},m} \backslash \tilde{G} / \tilde{G}_{\tilde{x},m}.$$

This map extends to an algebra isomorphism

$$\zeta_m^G : \mathcal{H}(G, G_{x,m}) \to \mathcal{H}(\tilde{G}, \tilde{G}_{\tilde{x},m}).$$

In particular ζ_m induces a group isomorphism $G/G^1 \to \tilde{G}/\tilde{G}^1$, and hence a group isomorphism

$$\overline{\zeta_m^G} : X_{\mathrm{nr}}(G) = \mathrm{Irr}(G/G^1) \to \mathrm{Irr}(\tilde{G}/\tilde{G}^1) = \mathrm{Irr}(\tilde{G}). \tag{3.6}$$

Let $\mathrm{Rep}(G, G_{x,m})$ be the category of smooth G-representations that are generated by their $G_{x,m}$-fixed vectors. Recall that $G_{x,m}$ is a totally decomposed open normal subgroup of the good maximal compact subgroup G_x of G. From [BeDe, §3.7–3.9] we know that there is an equivalence of categories

$$\begin{array}{ccc} \mathrm{Rep}(G, G_{x,m}) & \longrightarrow & \mathrm{Mod}(\mathcal{H}(G, G_{x,m})) \\ V & \mapsto & V^{G_{x,m}} \end{array} . \tag{3.7}$$

From (3.7) and Theorem 3.2 one obtains equivalences of categories

$$\begin{array}{cclcl} (\zeta_m^G)_* & : & \mathrm{Mod}(\mathcal{H}(G, G_{x,m})) & \longrightarrow & \mathrm{Mod}(\mathcal{H}(\tilde{G}, \tilde{G}_{\tilde{x},m})) \\ \overline{\zeta_m^G} & : & \mathrm{Rep}(G, G_{x,m}) & \longrightarrow & \mathrm{Rep}(\tilde{G}, \tilde{G}_{\tilde{x},m}) \end{array} , \tag{3.8}$$

which constitute the core of the method of close local fields. We will need many properties of these equivalences, starting with two easy ones about characters.

Lemma 3.3. (*a*) *Via* (3.6), *the equivalence of categories* $\overline{\zeta_m^G}$ *preserves twists by unramified characters.*

(*b*) ζ_m *induces an isomorphism*

$$A_G/(A_G \cap G_{x,m}) \longrightarrow A_{\tilde{G}}/(A_{\tilde{G}} \cap \tilde{G}_{\tilde{x},m}).$$

The effect of $\overline{\zeta_m^G}$ *on* A_G*-characters of* G*-representations is push-forward along this isomorphism.*

Proof. (a) This is clear from (3.6).

(b) Notice that

$$A_G/A_G \cap G_{x,m} \cong X_*(\mathcal{A}_G) \otimes_{\mathbb{Z}} (F^\times/1 + \varpi_F^m \mathfrak{o}_F), \tag{3.9}$$

and similarly for $\tilde{G}$. Since ζ_m comes from the isomorphism $X_*(S) \cong X_*(\tilde{S})$, it induces a linear bijection $X_*(A_G) \to X_*(A_{\tilde{G}})$, and hence an isomorphism from (3.9) to its counterpart for $\tilde{G}$. The A_G-characters of representations in $\mathrm{Rep}(G, G_{x,m})$ are precisely the characters of (3.9), and $\overline{\zeta_m^G}$ pushes them forward along ζ_m. □

Let P be a parabolic subgroup of G with a Levi factor M, which contains S. By [Bus, §1.6] the normalized parabolic functor I_P^G sends $\mathrm{Rep}(M, M_{x,m})$ to $\mathrm{Rep}(G, G_{x,m})$. We will exploit an expression for this functor [Bus] in terms that can be transferred to $\tilde{G}$ with Theorems 3.1 and 3.2.

Let P^{op} be the parabolic subgroup of G that is opposite to P with respect to M. Let $M_{x,m} = G_{x,m} \cap M$ be the version of $G_{x,m}$ for M. Recall that an element $g \in M$ is called $(P, G_{x,m})$-positive if

$$g(G_{x,m} \cap P)g^{-1} \subset G_{x,m} \cap P \quad \text{and} \quad g(G_{x,m} \cap P^{op})g^{-1} \supset G_{x,m} \cap P^{op}.$$

Let $\mathcal{H}^+(M, M_{x,m})$ be the subalgebra of $\mathcal{H}(M, M_{x,m})$ consisting of functions that are supported on $(P, G_{x,m})$-positive elements. In [Bus, §3.3], which is based on [BuKu2], a canonical injective algebra homomorphism

$$j_P : \mathcal{H}^+(M, M_{x,m}) \to \mathcal{H}(G, G_{x,m})$$

is given. Let $\tilde{P}$ and $\tilde{M}$ be the subgroups of $\tilde{G}$ corresponding to P and M via the equality of based root data. All the above constructions also work in $\tilde{G}$, and we endow the resulting objects with tildes.

Lemma 3.4. (*a*) ζ_m^M *restricts to an algebra isomorphism from* $\mathcal{H}^+(M, M_{x,m})$ *to* $\mathcal{H}^+(\tilde{M}, \tilde{M}_{\tilde{x},m})$.

(*b*) *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{H}(G, G_{x,m}) \arrow[r, "\zeta_m^G"] & \mathcal{H}(\tilde{G}, \tilde{G}_{\tilde{x},m}) \\
\mathcal{H}^+(M, M_{x,m}) \arrow[u, "j_P"] \arrow[r, "\zeta_m^M"] & \mathcal{H}^+(\tilde{M}, \tilde{M}_{\tilde{x},m}) \arrow[u, "j_{\tilde{P}}"]
\end{tikzcd} .$$

Proof. (a) The property "$(P, G_{x,m})$-positive" can be expressed in terms of the Cartan decomposition of M. Namely, the elements of a double coset $M_{x,0} g M_{x,0}$ with $g \in Z_M(S)$ are $(P, G_{x,m})$-positive if and only if

$$|\alpha(g)|_F \leq 1 \text{ for all } \alpha \in \Sigma(G, S) \text{ that appear in } \mathrm{Lie}(P). \tag{3.10}$$

(Notice that $|\alpha|_F$ extends naturally to a character of $Z_G(S)$ because S is cocompact in $Z_G(S)$.) The map ζ_m from Theorem 3.2 for M preserve the property (3.10), because it comes from the isomorphism $X_*(S) \cong X_*(\tilde{S})$, which preserves positivity of roots. Thus ζ_m maps $(P, G_{x,m})$-positive elements to $(\tilde{P}, \tilde{G}_{\tilde{x},m})$-positive elements, and then Theorem 3.2 provides the desired isomorphism.
(b) We endow M (resp. $\tilde{M}$) with the Haar measure that gives $M_{x,0}$ (resp. $\tilde{M}_{\tilde{x},0}$) volume 1. Suppose that $f \in \mathcal{H}^+(M, M_{x,m})$ has support $M_{x,m} g M_{x,m}$ with $g \in M$. The map j_P is characterized by: $j_P f$ has support $G_{x,m} g G_{x,m}$ and

$$j_P f(g) = f(g) \delta_P(g) \mu_M(M_{x,m}) \mu_G(G_{x,m})^{-1}. \tag{3.11}$$

By Theorem 3.1

$$\mu_G(G_{x,m}) = [G_{x,0} : G_{x,m}]^{-1} = [\tilde{G}_{\tilde{x},0} : \tilde{G}_{\tilde{x},m}]^{-1} = \mu_{\tilde{G}}(\tilde{G}_{\tilde{x},m}),$$

and similarly for $\mu_M(M_{x,m})$. It is well-known that $\delta_P(g)$ is the product, over all $\alpha \in \Sigma(G, S)$ that appear in $\mathrm{Lie}(P)$, of the factors $|\alpha(g)|_F^{\dim U_\alpha / U_{2\alpha}}$. The root subgroup U_α contains the root subgroup $U_{2\alpha}$ if 2α is also a root, and otherwise $U_{2\alpha} = \{1\}$ by definition. See [Ren, Lemme V.5.4] for a proof (although there a different convention is used, which results in replacing g by g^{-1}). By Theorem 3.1 $\dim U_\alpha$ equals $\dim U_{\tilde{\alpha}}$, where $\tilde{\alpha} \in \Sigma(\tilde{G}, \tilde{S})$ corresponds to α. Furthermore δ_P is trivial on compact subgroups, so $\delta_P(g)$ depends only on $M_{x,m} g M_{x,m}$. It follows that

$$\delta_P(M_{x,m} g M_{x,m}) = \delta_{\tilde{P}}(\zeta_m(M_{x,m} g M_{x,m})). \tag{3.12}$$

Knowing that, we take another look at (3.11) and we see that $\zeta_m^G \circ j_P = j_{\tilde{P}} \circ \zeta_m^M$. $\square$

Let $\mathcal{I}_{P,m} : \mathrm{Mod}(\mathcal{H}(M, M_{x,m})) \to \mathrm{Mod}(\mathcal{H}(G, G_{x,m}))$ be the composition of $\mathrm{Res}^{\mathcal{H}(M,M_{x,m})}_{\mathcal{H}^+(M,M_{x,m})}$ and

$$\begin{array}{ccc}
\mathrm{Mod}(\mathcal{H}^+(M, M_{x,m})) & \to & \mathrm{Mod}(\mathcal{H}(G, G_{x,m})) \\
V & \mapsto & \mathrm{Hom}_{\mathcal{H}^+(M,M_{x,m}))}(\mathcal{H}(G, G_{x,m}), V)
\end{array},$$

where $\mathcal{H}(G, G_{x,m})$ is regarded as a left $\mathcal{H}^+(M, M_{x,m})$-module via j_P.

Theorem 3.5. (*a*) *The equivalences of categories* (3.8) *are compatible with normalized parabolic induction, in the sense that the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Rep}(G, G_{x,m}) \arrow[r, "\overline{\zeta_m^G}"] & \mathrm{Rep}(\tilde{G}, \tilde{G}_{\tilde{x},m}) \\
\mathrm{Rep}(M, M_{x,m}) \arrow[u, "I_P^G"] \arrow[r, "\overline{\zeta_m^M}"] & \mathrm{Rep}(\tilde{M}, \tilde{M}_{\tilde{x},m}) \arrow[u, "I_{\tilde{P}}^{\tilde{G}}"]
\end{tikzcd} .$$

(*b*) *The equivalences of categories* (3.8) *are compatible with normalized Jacquet restriction, in the sense that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Rep}(G, G_{x,m}) \arrow[r, "\overline{\zeta^G_m}"] \arrow[d, "J^G_P"] & \mathrm{Rep}(\tilde G, \tilde G_{\tilde x,m}) \arrow[d, "J^{\tilde G}_{\tilde P}"] \\ \mathrm{Rep}(M, M_{x,m}) \arrow[r, "\overline{\zeta^M_m}"] & \mathrm{Rep}(\tilde M, \tilde M_{\tilde x,m}) \end{tikzcd} .$$

(*c*) $\overline{\zeta^G_m}$ *and its inverse send supercuspidal representations to supercuspidal representation. The same holds for unitary supercuspidal representations.*

(*d*) $\overline{\zeta^G_m}$ *and its inverse preserve temperedness and essential square-integrability.*

Proof. (a) Lemma 3.4 ensures that the diagram

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{H}(G, G_{x,m})) \arrow[r, "(\zeta^G_m)_*"] & \mathrm{Mod}(\mathcal{H}(\tilde G, \tilde G_{\tilde x,m})) \\ \mathrm{Mod}(\mathcal{H}(M, M_{x,m})) \arrow[r, "(\zeta^M_m)_*"] \arrow[u, "\mathcal{I}_{P,m}"] & \mathrm{Mod}(\mathcal{H}(\tilde M, \tilde M_{\tilde x,m})) \arrow[u, "\mathcal{I}_{\tilde P,m}"] \end{tikzcd}$$

commutes. According to [Bus, §4.1], the unnormalized parabolic induction functor Ind^G_P fits in a commutative diagram

$$\begin{tikzcd} \mathrm{Rep}(G, G_{x,m}) \arrow[r] & \mathrm{Mod}(\mathcal{H}(G, G_{x,m})) \\ \mathrm{Rep}(M, M_{x,m}) \arrow[r] \arrow[u, "\mathrm{Ind}^G_P"] & \mathrm{Mod}(\mathcal{H}(M, M_{x,m})) \arrow[u, "\mathcal{I}_{P,m}"] \end{tikzcd} ,$$

where the horizontal arrows are the equivalences of categories from (3.7). Of course the same holds for $\tilde G$. These two commutative diagrams entail that

$$\overline{\zeta^G_m} \circ \mathrm{Ind}^G_P = \mathrm{Ind}^{\tilde G}_{\tilde P} \circ \overline{\zeta^M_m} .$$

In view of (3.12), if we twist this equality on the left hand side by $\delta_P^{1/2}$ and on the right hand side by $\delta_{\tilde P}^{1/2}$, it remains valid. That yields exactly the desired relation with normalized parabolic induction.

(b) By Frobenius reciprocity J^G_P is left adjoint to I^G_P, so by part (a) $\overline{\zeta^M_m} \circ J^G_P \circ \overline{\zeta^G_m}^{-1}$ is left adjoint to $I^{\tilde G}_{\tilde P}$. Now we use the uniqueness of adjoints.

(c) The first claim follows from part (a), or alternatively from part (b). For the second claim, we note that a supercuspidal G-representation is unitary if and only if its central character is unitary. As A_G is cocompact in $Z(G)$, that is equivalent to: the A_G-character is unitary. By Lemma 3.3.b, $\overline{\zeta^G_m}$ preserve the latter property.

(d) For the property "square integrable modulo centre" one can follow the proof of [Badu, Théorème 2.17.b], reformulated in the setting of [Gan2]. Combining that with Lemma 3.3.a, we find that $\overline{\zeta^G_m}$ also preserves essential square-integrability.

By [Wal, Proposition III.4.1], every irreducible tempered representation $\tau \in \mathrm{Rep}(G, G_{x,m})$ is a direct summand of a completely reducible representation of the form $I^G_P(\pi)$, where $\pi \in \mathrm{Rep}(M, M_{x,m})$ is square-integrable modulo centre. By the above and part (a),

$$\overline{\zeta^G_m}\big(I^G_P(\pi)\big) \cong I^{\tilde G}_{\tilde P}\big(\overline{\zeta^M_m}(\pi)\big) \tag{3.13}$$

is also a direct sum of irreducible tempered representations. As $\overline{\zeta^G_m}(\tau)$ is a direct summand of (3.13), it is tempered. □

Consider an inertial equivalence class $\mathfrak{s} = [M, \sigma]_G$, where $S \subset M$. Choose $m \in \mathbb{Z}_{>0}$ such that $\mathrm{Rep}(G)^{\mathfrak{s}} \subset \mathrm{Rep}(G, G_{x,m})$, and similarly for all Levi subgroups of G containing M. This is easy for supercuspidal Bernstein components and possible in general because parabolic induction preserves depths [MoPr, Theorem 5.2]. Fix $\ell \in \mathbb{Z}_{>m}$ so that Theorems 3.1, 3.2 and 3.5 apply. We may and will assume that σ fulfills Condition 1.2. By Theorem 3.5.d the $\tilde{M}$-representation $\tilde{\sigma} = \overline{\zeta_m^M}(\sigma)$ is unitary and supercuspidal. We write $\tilde{\mathfrak{s}}, \tilde{\mathcal{O}}$ etc. for objects constructed from $\tilde{\sigma}$.

Proposition 3.6. (*a*) *The bijection* $\Sigma(G, A_M) \to \Sigma(\tilde{G}, A_{\tilde{M}})$*, induced by the equality of the root data of* $\mathcal{G}$ *and* $\tilde{\mathcal{G}}$*, sends* $\Sigma_{\mathcal{O},\mu}$ *onto* $\Sigma_{\tilde{\mathcal{O}},\mu}$*.*

(*b*) *Let* $\alpha \in \Sigma_{\mathcal{O},\mu}$ *with image* $\tilde{\alpha} \in \Sigma_{\tilde{\mathcal{O}},\mu}$*. The pullback of* $X_{\tilde{\alpha}}$ *along* (3.6) *is* X_α *and* $q_\alpha = q_{\tilde{\alpha}}, q_{\alpha *} = q_{\tilde{\alpha} *}$*.*

Proof. Let $\alpha \in \Sigma_{\mathrm{red}}(A_M)$, with image $\tilde{\alpha} \in \Sigma_{\mathrm{red}}(A_{\tilde{M}})$. The groups $\mathcal{M}_\alpha$ and $\tilde{\mathcal{M}}_{\tilde{\alpha}}$ correspond via the equality of root data of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. For $\chi \in X_{\mathrm{nr}}(M_\alpha)$, Theorem 3.5.a implies that

$$\overline{\zeta_m^M}(\sigma \otimes \chi) = \tilde{\sigma} \otimes \overline{\zeta_m^M}(\chi).$$

By (3.8), $I^{M_\alpha}_{P \cap M_\alpha}(\sigma \otimes \chi)$ is reducible if and only if $I^{\tilde{M}_{\tilde{\alpha}}}_{\tilde{P} \cap \tilde{M}_{\tilde{\alpha}}}\big(\tilde{\sigma} \otimes \overline{\zeta_m^M}(\chi)\big)$ is reducible. If $\alpha \notin \Sigma_{\mathcal{O},\mu}$, then $I^{M_\alpha}_{P \cap M_\alpha}(\sigma \otimes \chi)$ is irreducible for all non-unitary $\chi \in X_{\mathrm{nr}}(M_\alpha)$. It follows that $I^{\tilde{M}_{\tilde{\alpha}}}_{\tilde{P} \cap \tilde{M}_{\tilde{\alpha}}}\big(\tilde{\sigma} \otimes \tilde{\chi}\big)$ is irreducible for all non-unitary $\tilde{\chi} \in X_{\mathrm{nr}}(\tilde{M}_{\tilde{\alpha}})$, and by Corollary 1.3 $\tilde{\alpha} \notin \Sigma_{\tilde{\mathcal{O}},\mu}$.

On the other hand, suppose that $\alpha \in \Sigma_{\mathcal{O},\mu}$. Then $I^{M_\alpha}_{P \cap M_\alpha}(\sigma \otimes \chi)$ is reducible for a $\chi \in X_{\mathrm{nr}}(M_\alpha)$ with $X_\alpha(\chi) = q_\alpha > 1$. It is clear from the construction of X_α in (1.6) that $X_{\tilde{\alpha}} \circ \overline{\zeta_m^{M_\alpha}}$ is a multiple of X_α. Consequently $I^{\tilde{M}_{\tilde{\alpha}}}_{\tilde{P} \cap \tilde{M}_{\tilde{\alpha}}}\big(\tilde{\sigma} \otimes \overline{\zeta_m^M}(\chi)\big)$ is reducible and $X_{\tilde{\alpha}}(\overline{\zeta_m^M}(\chi)) \in \mathbb{R}_{>0} \setminus \{1\}$. With Corollary 1.3 we conclude that $\tilde{\alpha} \in \Sigma_{\tilde{\mathcal{O}},\mu}$.
(b) By (3.8) and Theorem 3.2, the bijection ζ_m induces a bijection

$$(M_\sigma^2 \cap M_\alpha^1)/M^1 \longrightarrow (\tilde{M}_{\tilde{\sigma}}^2 \cap \tilde{M}_{\tilde{\alpha}}^1)/\tilde{M}^1 \cong \mathbb{Z}.$$

The element $h_\alpha^\vee$ generates $(M_\sigma^2 \cap M_\alpha^1)/M^1$, while $h_{\tilde{\alpha}}^\vee$ generates $(\tilde{M}_{\tilde{\sigma}}^2 \cap \tilde{M}_{\tilde{\alpha}}^1)/\tilde{M}^1$. These generators are determined by conditions $\nu_F(\alpha(h_\alpha^\vee)) > 0$ and $\nu_{\tilde{F}}(\tilde{\alpha}(h_{\tilde{\alpha}}^\vee)) > 0$, respectively. As $\nu_{\tilde{F}} \circ \tilde{\alpha} \circ \zeta_m = \nu_F \circ \alpha$, we can conclude that

$$\zeta_m(h_\alpha^\vee) = h_{\tilde{\alpha}}^\vee \quad \text{and} \quad X_\alpha = X_{\tilde{\alpha}} \circ \overline{\zeta_m^{M_\alpha}}. \tag{3.14}$$

Then $X_{\tilde{\alpha}}(\overline{\zeta_m^{M_\alpha}}(\chi)) = q_\alpha$ and $I^{\tilde{M}_{\tilde{\alpha}}}_{\tilde{P} \cap \tilde{M}_{\tilde{\alpha}}}\big(\tilde{\sigma} \otimes \overline{\zeta_m^M}(\chi)\big)$ is reducible, so $q_\alpha = q_{\tilde{\alpha}}$.

If $q_{\alpha *} > 1$, then $I^M_{P \cap M_\alpha}(\sigma \otimes \chi')$ is reducible for a $\chi' \in X_{\mathrm{nr}}(M_\alpha)$ with $X_\alpha(\chi') = -q_{\alpha *}$. In that case $I^{\tilde{M}_{\tilde{\alpha}}}_{\tilde{P} \cap \tilde{M}_{\tilde{\alpha}}}\big(\tilde{\sigma} \otimes \overline{\zeta_m^M}(\chi')\big)$ is also reducible and $X_{\tilde{\alpha}}(\overline{\zeta_m^{M_\alpha}}(\chi')) = -q_{\alpha *}$, so by Corollary 1.3 $q_{\tilde{\alpha} *} = q_{\alpha *}$. When $q_{\alpha *} = 1$, $I^M_{P \cap M_\alpha}(\sigma \otimes \chi')$ is irreducible for all $\chi' \in X_{\mathrm{nr}}(M_\alpha)$ with $X_\alpha(\chi') \in \mathbb{R}_{<-1}$. That translates to $\tilde{M}_{\tilde{\alpha}}$, and then Corollary 1.3 implies that $q_{\tilde{\alpha} *} = 1$. $\square$

We summarise the conclusions of this sections:

Corollary 3.7. *Let* $\mathrm{Rep}(G)^{\mathfrak{s}}$ *be an arbitrary Bernstein block for a connected reductive group* $\mathcal{G}$ *over a local function field* F*. There exist:*

- *a* p*-adic field* $\tilde{F}$*, sufficiently close to* F*,*
- *a connected reductive* $\tilde{F}$*-group* $\tilde{\mathcal{G}}$ *with the same based root datum as* $\mathcal{G}$*,*

- *a Bernstein block* $\mathrm{Rep}(\tilde{G})^{\tilde{\mathfrak{s}}}$ *for* $\tilde{G}$*,*

such that:

- $\mathrm{Rep}(G)^{\mathfrak{s}}$ *is equivalent with* $\mathrm{Rep}(\tilde{G})^{\tilde{\mathfrak{s}}}$*,*
- $\mathcal{H}(\mathcal{O}, G)$ *is isomorphic with* $\mathcal{H}(\tilde{\mathcal{O}}, \tilde{G})$*,*
- *whenever* $\alpha \in \Sigma_{\mathcal{O},\mu}$ *and* $\tilde{\alpha} \in \Sigma_{\tilde{\mathcal{O}},\mu}$ *correspond (via Proposition 3.6),* $\lambda(\alpha) = \lambda(\tilde{\alpha})$ *and* $\lambda^*(\alpha) = \lambda^*(\tilde{\alpha})$*.*

Proof. It only remains to establish the isomorphism of affine Hecke algebras. From (3.8) and Theorem 3.2 we get the bijection $M^2_\sigma / M^1 \to \tilde{M}^2_{\tilde{\sigma}} / \tilde{M}^1$. From (3.14) we obtain the bijection $\Sigma^\vee_{\mathcal{O}} \to \Sigma^\vee_{\tilde{\mathcal{O}}}$. Dualizing these two bijections, we obtain an isomorphism from the root datum underlying $\mathcal{H}(\mathcal{O}, G)$ to the root datum underlying $\mathcal{H}(\tilde{\mathcal{O}}, \tilde{G})$. It respects the bases because $\mathcal{G}$ and $\tilde{\mathcal{G}}$ have the same based root datum. By Proposition 3.6.b the parameters $q_\alpha, q_{\alpha *}$ are the same on both sides. As

$$q_F = [\mathfrak{o}_F : \varpi_F \mathfrak{o}_F] = [\mathfrak{o}_{\tilde{F}} : \varpi_{\tilde{F}} \mathfrak{o}_{\tilde{F}}] = q_{\tilde{F}},$$

also the label functions λ, λ^* on both sides correspond via $\alpha \mapsto \tilde{\alpha}$. $\square$

4. HECKE ALGEBRA PARAMETERS FOR SIMPLE GROUPS

4.1. Principal series of split groups.

The affine Hecke algebras for Bernstein blocks in the principal series of split groups were worked out in [Roc1], under some mild assumptions on the residual characteristic of F. In particular, for roots $\alpha \in \Sigma_{\mathcal{O},\mu}$ one finds $q_\alpha = q_F$ and $q_{\alpha *} = 1$. We will derive the same conclusion in a different way, which avoids any restrictions on the residual characteristic. Using a little input from [BeDe] we will evaluate the intertwining operators (1.10) directly, which is instructive but unfortunately seems infeasible outside the principal series. While the results in this paragraph are not original and the kind of calculation is also not new, we have been unable to locate such computations in the literature in the generality that is required for [Sol7]. The closest we found is [Cas, §3], which however applies only when the underlying characters of tori are unramified.

Let $\mathcal{G}$ be a split connected reductive F-group. We may assume that $\mathcal{G}$ is a Chevalley group, so defined over $\mathbb{Z}$. Let $\mathcal{T}$ be a maximal F-split torus of $\mathcal{G}$ and write $T = \mathcal{T}(F)$. We consider an inertial equivalence class $\mathfrak{s} = [T, \sigma]_G$, where σ is a character of T that fulfills Condition 1.2.

For $\alpha \in \Sigma(\mathcal{G}, \mathcal{T})$ the group $\mathcal{M}_\alpha$ is generated by $\mathcal{T}$ and the root subgroups $\mathcal{U}_\alpha, \mathcal{U}_{-\alpha}$. It has root system $\Sigma(\mathcal{M}_\alpha, \mathcal{T}) = \{\alpha, -\alpha\}$ and parabolic subgroups $\mathcal{P}_\alpha = \langle \mathcal{T}, \mathcal{U}_\alpha \rangle$, $\mathcal{P}_{-\alpha} = \langle \mathcal{T}, \mathcal{U}_{-\alpha} \rangle$. Let $u_\alpha : F \to U_\alpha$ and $u_{-\alpha} : F \to U_{-\alpha}$ be the coordinates coming from the Chevalley model.

We assume that $s_\alpha \cdot \sigma = \sigma$, a condition which by (1.5) is necessary for $\sigma \in \Sigma_{\mathcal{O},\mu}$. Then $\sigma \circ \alpha^\vee = (\sigma \circ \alpha^\vee)^{-1}$, so $\sigma \circ \alpha^\vee$ has order ≤ 2 in $\mathrm{Irr}(F^\times)$. When the residual characteristic of F is not 2, this implies that $\sigma \circ \alpha^\vee$ has depth zero. Of course the cases with $\sigma \circ \alpha^\vee$ of positive depth are more involved.

We start the search for q_α with elements of $I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)$ that are as close as possible to fixed by the Iwahori subgroup

$$I = u_\alpha(\mathfrak{o}_F) \mathcal{T}(\mathfrak{o}_F) u_{-\alpha}(\varpi_F \mathfrak{o}_F).$$

For $x \in F^\times$ we write

$$s_\alpha(x) = u_\alpha(-x^{-1}) u_{-\alpha}(x) u_\alpha(-x^{-1}) \in N_{M_\alpha}(T).$$

It follows quickly from the Iwasawa decomposition of M_α that

$$M_\alpha = P_\alpha I \sqcup P_\alpha s_\alpha I, \quad \text{where } s_\alpha = s_\alpha(1).$$

Consider the elements $f_1, f_s \in I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)$ defined by

$$\begin{array}{lllll} \mathrm{supp}(f_1) = P_\alpha I & f_1(u_\alpha(x) t u_{-\alpha}(y)) & = & (\sigma\chi\delta_{P_\alpha}^{1/2})(t) & x \in F, t \in T, y \in \varpi_F \mathfrak{o}_F, \\ \mathrm{supp}(f_s) = P_\alpha s_\alpha I & f_s(u_\alpha(x) t u_{-\alpha}(y) s_\alpha) & = & (\sigma\chi\delta_{P_\alpha}^{1/2})(t) & x \in F, t \in T, y \in \mathfrak{o}_F. \end{array}$$

We endow F with the Haar measure that gives $\mathfrak{o}_F$ volume 1. We compute

$$J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_1(1_G) = \int_F f_1(u_{-\alpha}(x)) \mathrm{d}x = \mathrm{vol}(\varpi_F \mathfrak{o}_F) = q_F^{-1}, \tag{4.1}$$

$$\begin{aligned} & J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_1(s_\alpha) \; = \; \int_F f_s\big(u_{-\alpha}(x) s_\alpha\big) \mathrm{d}x \; = \\ & \int_{F^\times} f_s\big(u_\alpha(-x^{-1}) u_{-\alpha}(x) u_\alpha(-x^{-1}) u_\alpha(x^{-1}) s_\alpha\big) \mathrm{d}x \; = \\ & \int_{F^\times} f_s\big(s_\alpha(x) u_\alpha(x^{-1}) s_\alpha\big) \mathrm{d}x \; = \; \int_{F^\times} f_s\big(s_\alpha(x) s_\alpha u_{-\alpha}(-x^{-1})\big) \mathrm{d}x \; = \\ & \int_{F^\times} f_s\big(\alpha^\vee(-x^{-1}) u_{-\alpha}(-x^{-1})\big) \mathrm{d}x \; = \\ & \int_{F^\times} (\sigma\chi\delta_{P_\alpha}^{1/2}) \circ \alpha^\vee(-x^{-1}) f_s\big(u_{-\alpha}(-x^{-1})\big) \mathrm{d}x \; = \\ & \sum_{n=1}^{\infty} \int_{\varpi_F^{-n} \mathfrak{o}_F^\times} (\sigma\chi\delta_{P_\alpha}^{1/2}) \circ \alpha^\vee(-x^{-1}) \mathrm{d}x. \end{aligned} \tag{4.2}$$

As $\chi\delta_{P_\alpha}^{1/2}$ is unramified and $\sigma \circ \alpha^\vee$ is quadratic,

$$(\sigma\chi\delta_P^{1/2}) \circ \alpha^\vee|_{\mathfrak{o}_F^\times} = \sigma \circ \alpha^\vee|_{\mathfrak{o}_F^\times} \quad \text{is quadratic.} \tag{4.3}$$

If (4.3) is nontrivial, then

$$\int_{\varpi_F^{-n} \mathfrak{o}_F^\times} (\sigma\chi\delta_{P_\alpha}^{1/2}) \circ \alpha^\vee(-x^{-1}) \mathrm{d}x = (\sigma\chi\delta_{P_\alpha}^{1/2})(\varpi_F^n) \int_{\mathfrak{o}_F^\times} \sigma \circ \alpha^\vee(-x^{-1}) \mathrm{d}x = 0. \tag{4.4}$$

In that case $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_1(s_\alpha) = 0$. On the other hand, when (4.3) is trivial:

$$\begin{aligned} J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_1(s_\alpha) &= \sum_{n=1}^{\infty} \int_{\varpi_F^{-n} \mathfrak{o}_F^\times} (\sigma\chi)(\alpha^\vee(\varpi_F^n)) \left|\alpha(\alpha^\vee(\varpi_F^n))\right|^{1/2} \mathrm{d}x \\ &= \sum_{n=1}^{\infty} (\sigma\chi)(\alpha^\vee(\varpi_F))^n \mathrm{vol}(\varpi_F^{-n} \mathfrak{o}_F^\times) \, |\varpi_F^n| \\ &= \sum_{n=1}^{\infty} (\sigma\chi)(\alpha^\vee(\varpi_F))^n (1 - q_F^{-1}) \; = \; \frac{(1 - q_F^{-1})(\sigma\chi)(\alpha^\vee(\varpi_F))}{1 - (\sigma\chi)(\alpha^\vee(\varpi_F))}. \end{aligned}$$

Similar calculations show that

$$\begin{aligned} &J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_s(s_\alpha) = 1, \\ &J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_s(1_G) = 0 \quad \text{if } \sigma \circ \alpha^\vee |_{\mathfrak{o}_F^\times} \neq 1, \\ &J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_s(1_G) = \frac{1 - q_F^{-1}}{1 - (\sigma\chi)(\alpha^\vee(\varpi_F))} \quad \text{if } \sigma \circ \alpha^\vee |_{\mathfrak{o}_F^\times} = 1. \end{aligned}$$

Case I: $\sigma \circ \alpha^\vee$ is unramified

Here $\mathrm{Rep}(M_\alpha)^{\mathfrak{s}}$ is isomorphic with the Iwahori-spherical Bernstein block and $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)$ restricts to a $\mathcal{H}(M_\alpha, I)$-homomorphism

$$I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)^I \to I_{P_{-\alpha}}^{M_\alpha}(\sigma \otimes \chi)^I. \tag{4.5}$$

The space $I_{P_{-\alpha}}^{M_\alpha}(\sigma \otimes \chi)^I$ has a basis f'_1, f'_s where $\mathrm{supp}(f'_1) = P_{-\alpha} I$ and $\mathrm{supp}(f'_s) = P_{-\alpha} s_\alpha I$. Abbreviating $z_\alpha = (\sigma \otimes \chi) \circ \alpha^\vee(\varpi_F)$, the above calculations entail that the matrix of (4.5) respect to the given bases is

$$\begin{pmatrix} q_F^{-1} & \frac{1 - q_F^{-1}}{1 - z_\alpha} \\ \frac{1 - q_F^{-1}}{z_\alpha^{-1} - 1} & 1 \end{pmatrix}.$$

An equivalent result was obtained in [Cas, Theorem 3.4]. Similarly one checks that $J_{P_\alpha|P_{-\alpha}}(\sigma \otimes \chi)$ restricts to

$$\begin{pmatrix} 1 & \frac{1 - q_F^{-1}}{z_\alpha - 1} \\ \frac{1 - q_F^{-1}}{1 - z_\alpha^{-1}} & q_F^{-1} \end{pmatrix} : I_{P_{-\alpha}}^{M_\alpha}(\sigma \otimes \chi)^I \to I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)^I.$$

We find that $J_{P_\alpha|P_{-\alpha}}(\sigma \otimes \chi) J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)$ restricts to

$$\Big(q_F^{-1} + \frac{(1 - q_F^{-1})^2}{(1 - z_\alpha)(1 - z_\alpha^{-1})} \Big) \mathrm{id} : I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)^I \to I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)^I. \tag{4.6}$$

We already know that $J_{P_\alpha|P_{-\alpha}}(\sigma \otimes \chi) J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)$ is a scalar multiple of the identity on $I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)$, so (4.6) gives that scalar. We note that (4.6) has a pole at $z_\alpha = 1$ and that (4.6) is zero if and only if $z_\alpha = q_F$ or $z_\alpha = q_F^{-1}$. As σ is unitary and $\chi \in \mathrm{Hom}(M_\alpha, \mathbb{R}_{>0})$, this is equivalent to

$$\sigma \circ \alpha^\vee = 1 \quad \text{and} \quad \chi \circ \alpha^\vee(\varpi_F) \in \{q_F, q_F^{-1}\}. \tag{4.7}$$

Since $M_\sigma^2 = T$, $h_\alpha^\vee$ generates T/T^1. If $\alpha^\vee(\varpi_F^{-1}) = h_\alpha^\vee$, (4.7) says that $q_\alpha = q_F$ and $q_{\alpha*} = 1$. If $\alpha^\vee(\varpi_F^{-1}) = 2h_\alpha^\vee$, then (4.7) means $q_\alpha = q_F^{1/2} = q_{\alpha*}$. But in that case we can also define $X_\alpha(\chi) = \chi(\alpha^\vee(\varpi_F^{-1}))$ instead of $X_\alpha(\chi) = \chi(h_\alpha^\vee)$. These new X_α also form a root system, which embeds naturally in $R(\mathcal{G}, \mathcal{T})^\vee$. From the presentation after Corollary 1.3 one sees that this redefinition does not change the affine Hecke algebra. Hence we can achieve $q_\alpha = q_F, q_{\alpha*} = 1$ in all these cases.

Case II: $\sigma \circ \alpha^\vee$ is ramified

For $r \in \mathbb{Z}_{>0}$, M_α has compact open subgroups

$$\begin{aligned} J_r &= x_\alpha(\varpi_F^r \mathfrak{o}_F) \mathcal{T}(\varpi_F^r \mathfrak{o}_F) x_{-\alpha}(\varpi_F^r \mathfrak{o}_F), \\ H_r &= x_\alpha(\varpi_F^{2r-1} \mathfrak{o}_F) \mathcal{T}(\varpi_F^r \mathfrak{o}_F) x_{-\alpha}(\varpi_F \mathfrak{o}_F). \end{aligned}$$

Here $\mathcal{T}(\varpi_F^r \mathfrak{o}_F)$ is a shorthand for the kernel of $\mathcal{T}(\mathfrak{o}_F) \to \mathcal{T}(\mathfrak{o}_F / \varpi_F^r \mathfrak{o}_F)$.

Lemma 4.1. *There exists* $r \in \mathbb{Z}_{>0}$ *such that* $\mathcal{T}(\varpi_F^r \mathfrak{o}_F) \subset \ker(\sigma)$ *and* $\mathrm{Rep}(M_\alpha)^{\mathfrak{s}}$ *is a direct factor of*

$$\mathrm{Rep}(M_\alpha, H_r) \cong \mathrm{Mod}(\mathcal{H}(M_\alpha, H_r)).$$

Proof. Choose an odd $r \in \mathbb{Z}_{>0}$ such that $\mathcal{T}(\varpi_F^r \mathfrak{o}_F) \subset \ker(\sigma)$ and $I_{P_\alpha}^{M_\alpha}(\sigma)^{J_r} \neq 0$. Then $I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)^{J_r} \neq 0$ for any $\chi \in X_{\mathrm{nr}}(T)$ because J_r is compact. Hence

$$\mathrm{Rep}(M_\alpha)^{\mathfrak{s}} \subset \mathrm{Rep}(M_\alpha, J_r).$$

We note that J_r is a normal subgroup of the hyperspecial parahoric subgroup $\mathcal{M}_\alpha(\mathfrak{o}_F)$ of M_α. It is known from [BeDe] that $\mathrm{Rep}(M_\alpha, J_r)$ is a direct product of finitely many Bernstein blocks of $\mathrm{Rep}(M_\alpha)$, and that

$$\mathrm{Rep}(M_\alpha, J_r) \to \mathrm{Mod}(\mathcal{H}(M_\alpha, J_r)) : V \mapsto V^{J_r} \tag{4.8}$$

is an equivalence of categories. Consider conjugation by $\alpha^\vee(\varpi_F^{(r-1)/2})$. This sends J_r to H_r and induces equivalences of categories

$$\mathrm{Rep}(M_\alpha, J_r) \cong \mathrm{Rep}(M_\alpha, H_r), \quad \mathrm{Mod}(\mathcal{H}(M_\alpha, J_r)) \cong \mathrm{Mod}(\mathcal{H}(M_\alpha, H_r)). \qquad \square$$

Lemma 4.1 tells us that most aspects of $I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)$ can already be detected on $I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)^{H_r}$.

Lemma 4.2. *The double cosets in* $P_\alpha \backslash M_\alpha / H_r$ *can be represented by*

$$\{1_G\} \cup \{(u_{-\alpha}(z) s_\alpha : z \in \mathfrak{o}_F / \varpi_F^{2r-1} \mathfrak{o}_F\}.$$

Similarly $P_{-\alpha} \backslash M_\alpha / H_r$ *can be represented by* $\{s_\alpha\} \cup \{u_\alpha(\mathfrak{o}_F)/u_\alpha(\varpi_F^{2r-1}\mathfrak{o}_F)\}$.

Proof. From the Iwasawa decomposition $M_\alpha = P_\alpha \mathcal{M}_\alpha(\mathfrak{o}_F)$ we get

$$P_\alpha \backslash M_\alpha / H_r \cong (P_\alpha \cap \mathcal{M}_\alpha(\mathfrak{o}_F)) \backslash \mathcal{M}_\alpha(\mathfrak{o}_F) / H_r. \tag{4.9}$$

Recall that by the Bruhat decomposition of $\mathcal{M}_\alpha(k_F)$:

$$\mathcal{M}_\alpha(\mathfrak{o}_F) = I \sqcup I s_\alpha I = I \sqcup u_\alpha(\mathfrak{o}_F) \mathcal{T}(\mathfrak{o}_F) u_{-\alpha}(\mathfrak{o}_F) s_\alpha. \tag{4.10}$$

Furthermore, we note that $(P_\alpha \cap \mathcal{M}_\alpha(\mathfrak{o}_F)) H_r = I$ and

$$(P_\alpha \cap \mathcal{M}_\alpha(\mathfrak{o}_F)) u_{-\alpha}(z) s_\alpha H_r = (P_\alpha \cap \mathcal{M}_\alpha(\mathfrak{o}_F)) u_{-\alpha}(z + \varpi_F^{2r-1}\mathfrak{o}_F) s_\alpha \qquad z \in \mathfrak{o}_F.$$

In combination with (4.9) and (4.10) that yields the desired representatives for (4.9).

The representatives for the second double coset space are found in analogous fashion, now using

$$\mathcal{M}_\alpha(\mathfrak{o}_F) = s_\alpha I \sqcup s_\alpha I s_\alpha I = u_{-\alpha}(\mathfrak{o}_F) \mathcal{T}(\mathfrak{o}_F) u_\alpha(\varpi_F \mathfrak{o}_F) s_\alpha \sqcup u_{-\alpha}(\mathfrak{o}_F) \mathcal{T}(\mathfrak{o}_F) u_\alpha(\mathfrak{o}_F)$$

instead of (4.10). $\square$

It follows from Lemma 4.2 that $I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)^{H_r}$ has a basis $\{f_1\} \cup \{f_{zs} : z \in \mathfrak{o}_F / \varpi_F^{2r-1} \mathfrak{o}_F\}$. Here $\mathrm{supp}(f_1) = P_\alpha H_r = P_\alpha I$ as before and

$$\mathrm{supp}(f_{zs}) = P_\alpha u_{-\alpha}(z) H_r s_\alpha = P_\alpha x_{-\alpha}(z + \varpi_F^{2r-1}\mathfrak{o}_F) s_\alpha,$$
$$f_{zs}(u_\alpha(x) t u_{-\alpha}(y) s_\alpha) = (\sigma \chi \delta_{P_\alpha}^{1/2})(t) \qquad x \in F, y \in z + \varpi_F^{2r-1}\mathfrak{o}_F, t \in T.$$

The next result can be deduced from [Roc1, Theorem 6.3] when the characteristic of F is not 2.

Proposition 4.3. *Recall that* $\sigma \circ \alpha^\vee$ *is ramified and* $s_\alpha \cdot \sigma = \sigma$.

(a) *The functions* $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_1$ *and* $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_{zs}$ *(with* $z \in \mathfrak{o}_F / \varpi_F^{2r-1} \mathfrak{o}_F$*) of* $\chi \in X_{\mathrm{nr}}(M_\alpha)$ *do not have any poles.*

(b) $\alpha \notin \Sigma_{\mathcal{O},\mu}$.

Proof. (a) Note that $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)$ preserves the H_r-invariance of an element f of the given basis. By Lemma 4.2 it suffices to check the values of $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f$ at $\{s_\alpha\} \cup u_\alpha(\mathfrak{o}_F)$. From the earlier computations (4.1) and (4.2) we know that $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_1$ does not have poles at 1_G or at s_α. For $y \in \mathfrak{o}_F \setminus \varpi_F^{2r-1} \mathfrak{o}_F$ the multiplication rules in $SL_2(F)$ (which surjects on $M_{\alpha,\mathrm{der}}$) enable us to compute

$$
\begin{aligned}
J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_1(u_\alpha(y)) &= \int_F f_1\big(u_{-\alpha}(x) u_\alpha(y)\big) \mathrm{d}x \\
&= \int_F f_1\big(u_\alpha(\frac{y}{1+xy}) \alpha^\vee(\frac{1}{1+xy}) u_{-\alpha}(\frac{x}{1+xy})\big) \mathrm{d}x \qquad (4.11) \\
&= \int_F \big(\sigma \chi \delta_{P_\alpha}^{1/2}\big)\big(\alpha^\vee(\frac{1}{1+xy})\big) f_1\big(u_{-\alpha}(\frac{x}{1+xy})\big) \mathrm{d}x.
\end{aligned}
$$

In terms of the new variable $x' := 1 + xy$ this becomes

$$
|y|^{-1} \int_F \big(\sigma \chi \delta_{P_\alpha}^{1/2}\big)\big(\alpha^\vee(x')^{-1}\big) f_1\big(u_{-\alpha}(\frac{x'-1}{yx'})\big) \mathrm{d}x'
$$

The integrand is nonzero if and only if $\frac{x'-1}{yx'} \in \varpi_F \mathfrak{o}_F$, which is equivalent to

$$
(x'-1)/x' \in y \varpi_F \mathfrak{o}_F \subset \varpi_F \mathfrak{o}_F .
$$

That is only possible when $|x'| = 1$, so (4.11) becomes an integral of a continuous function over the compact set $\mathfrak{o}_F^\times$. In particular it converges and $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_1$ does not have any poles.

With calculations as in (4.2) we check the other basis elements f_{zs}:

$$
\begin{aligned}
J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_{zs}(s_\alpha) &= \int_F f_{zs}\big(u_{-\alpha}(x) s_\alpha\big) \mathrm{d}x = \mathrm{vol}(z + \varpi_F^{2r-1} \mathfrak{o}_F) = q_F^{1-2r}, \\
J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_{zs}(u_\alpha(y)) &= \int_F f_{zs}\big(u_{-\alpha}(x) u_\alpha(y)\big) \mathrm{d}x \\
&= \int_{F^\times} f_{zs}\big(u_\alpha(-x^{-1}) u_{-\alpha}(x) u_\alpha(-x^{-1}) u_\alpha(y + x^{-1})\big) \mathrm{d}x \\
&= \int_{F^\times} f_{zs}\big(s_\alpha(x) u_\alpha(y + x^{-1})\big) \mathrm{d}x \qquad (4.12) \\
&= \int_{F^\times} \big(\sigma \chi \delta_{P_\alpha}^{1/2}\big)\big(\alpha^\vee(x^{-1})\big) f_{zs}\big(s_\alpha u_\alpha(y + x^{-1})\big) \mathrm{d}x \\
&= \int_{F^\times} \big(\sigma \chi \delta_{P_\alpha}^{1/2}\big)\big(\alpha^\vee(x^{-1})\big) f_{zs}\big(s_\alpha(x) u_\alpha(y + x^{-1})\big) \mathrm{d}x \\
&= \int_{F^\times} \big(\sigma \chi \delta_{P_\alpha}^{1/2}\big)\big(\alpha^\vee(x^{-1})\big) f_{zs}\big(u_{-\alpha}(-y - x^{-1}) s_\alpha\big) \mathrm{d}x.
\end{aligned}
$$

When $-y \notin z + \varpi_F^{2r-1} \mathfrak{o}_F$, this integral is supported on a compact subset of F, and it converges. When $-y \in z + \varpi_F^{2r-1} \mathfrak{o}_F$, the support condition on x becomes

$|x| \geq q_F^{2r-1}$, and the integral reduces to

$$\sum_{n=2r-1}^{\infty} \int_{\varpi_F^{-n} \mathfrak{o}_F^\times} (\sigma \chi \delta_{P_\alpha}^{1/2})(\alpha^\vee (x^{-1})) \mathrm{d}x.$$

Since $\sigma \circ \alpha^\vee$ is ramified and quadratic, it is nontrivial on $\mathfrak{o}_F^\times$. Then (4.3) and (4.4) show that every term of the above sum is zero. We conclude that $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi) f_{zs}$ also does not have any poles.
(b) Part (a) and Lemma 4.2 show that $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)$ does not have any poles on $I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)^{H_r}$. Similar computations (which we omit) show that $J_{P_\alpha|P_{-\alpha}}(\sigma \otimes \chi)$ does not have any poles on $I_{P_{-\alpha}}^{M_\alpha}(\sigma \otimes \chi)^{H_r}$. By Lemma 4.1 they neither have poles on, respectively, $I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)$ and $I_{P_{-\alpha}}^{M_\alpha}(\sigma \otimes \chi)$. Then (1.11) says that $\mu^{M_\alpha}(\sigma \otimes \chi)$ is nonzero for $\chi \in X_{\mathrm{nr}}(T)$, which by definition means $\alpha \notin \Sigma_{\mathcal{O},\mu}$. $\square$

Let us combine the conclusions for all possible $\sigma \circ \alpha^\vee$:

Theorem 4.4. *Suppose that $\alpha \in \Sigma_{\mathcal{O},\mu}$, for a principal series Bernstein component of a F-split group G. Define $X_\alpha(\chi) = \chi(\alpha^\vee(\varpi_F^{-1}))$. Then $\sigma \circ \alpha^\vee = 1$ and $q_\alpha = q_F, q_{\alpha*} = 1$.*

4.2. **Principal series of quasi-split groups.**

We consider a quasi-split non-split connected reductive F-group $\mathcal{G}$. By Section 2 we may suppose that $\mathcal{G}$ is absolutely simple. Then it is an outer form of Lie type A_n, D_n or E_6.

Let $\mathcal{T}$ be the centralizer of a maximal F-split torus $\mathcal{S}$ in $\mathcal{G}$, and let σ be a character of T satisfying Condition 1.2. Let $\mathrm{Gal}(F_s/\tilde{F})$ be the normal subgroup of $\mathrm{Gal}(F_s/F)$ that acts trivially on $X^*(\mathcal{T})$, so that $\tilde{F}/F$ is a minimal Galois extension splitting $\mathcal{T}$.

Consider a root $\alpha \in \Sigma_{\mathcal{O},\mu}$. By a suitable choice of a basis of $\Sigma(G, S) \subset \Sigma(G, A_T)$, we may assume that α is simple. It corresponds to a unique Galois orbit $\mathbf{W}_F \alpha_T$ in $\Sigma(\mathcal{G}, \mathcal{T})$. Then

$$U_\alpha(F) = \Big(\prod_{\beta_T \in \mathbf{W}_F \alpha_T} U_{\beta_T}(F_s) \Big)^{\mathbf{W}_F} \cong U_{\alpha_T}(F_s)^{\mathbf{W}_{F,\alpha_T}} \cong F_s^{\mathbf{W}_{F,\alpha_T}} =: F_\alpha. \tag{4.13}$$

The field F_α does not depend on the choice of α_T (up to isomorphism) and is known as a splitting field for α.

By construction the numbers $q_\alpha, q_{\alpha*}$ depend only on the group M_α. Parts (b–c) of Proposition 2.4 apply, so we may even replace $\mathcal{M}_\alpha$ by its derived subgroup $\mathcal{M}_{\alpha,\mathrm{der}}$.

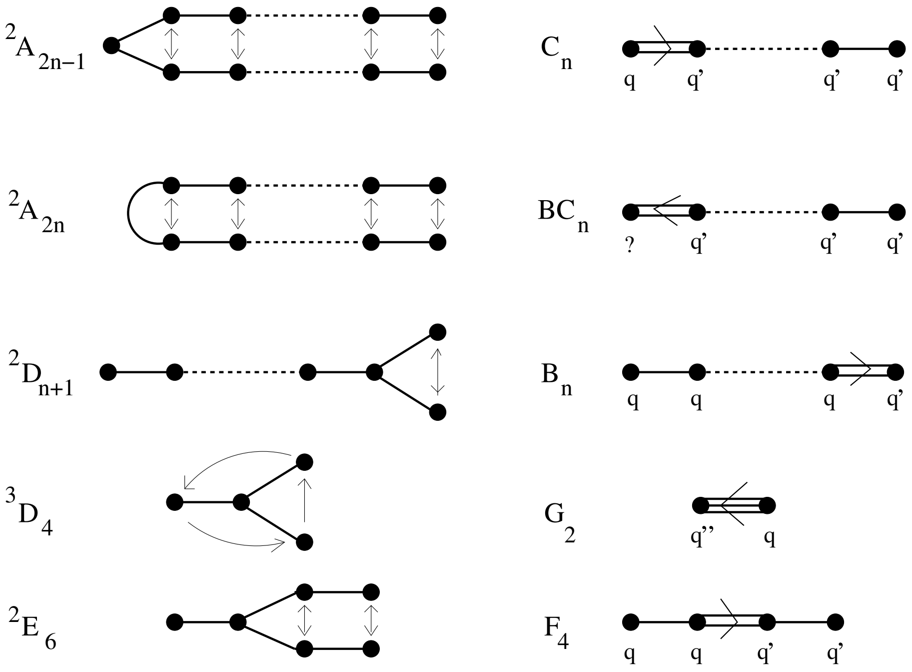
Suppose for the moment that the elements of $\mathbf{W}_F \alpha_T \subset \Sigma(\mathcal{G}, \mathcal{T})$ are mutually orthogonal. Then $\mathcal{M}_{\alpha,\mathrm{der}}$ is isomorphic to the restriction of scalars, from F_α to F, of SL_2 or PGL_2. Now q_α and $q_{\alpha*}$ can be computed in $SL_2(F_\alpha)$ or $PGL_2(F_\alpha)$, as in Paragraph 4.1. (Recall that even for PGL_2 we insisted that X_α is based on $\alpha^\vee$ rather than on $h_\alpha^\vee$.) By Theorem 4.4 $\sigma \circ \alpha^\vee = 1$, $q_{\alpha*} = 1$ and $q_\alpha = q_{F_\alpha}$ is the cardinality of the residue field of F_{α_T}. From Galois theory for local fields [Ser] it is known that

$$\begin{aligned} |\mathbf{W}_F \alpha_T| = [\mathbf{W}_F : \mathbf{W}_{F,\alpha_T}] = e_{F_\alpha/F} f_{F_\alpha/F} = \\ [\mathbf{I}_F : \mathbf{I}_F \cap \mathbf{W}_{F,\alpha_T}] \cdot [\mathbf{W}_F/\mathbf{I}_F : \mathbf{W}_{F,\alpha_T}\mathbf{I}_F/\mathbf{I}_F] = |\mathbf{I}_F \alpha_T| \cdot f_{F_\alpha/F}. \end{aligned} \tag{4.14}$$

Since $\mathbf{I}_F$ is normal in $\mathbf{W}_F$, the number

$$q_{F_\alpha} = q_F^{f_{F_\alpha/F}} = q_F^{|\mathbf{W}_F \alpha_T|/|\mathbf{I}_F \alpha_T|} \tag{4.15}$$

Table 2. Dynkin diagrams and parameters for quasi-split groups



depends only on α, and not on the choice of α_T. This leads to the possibilities for the Dynkin diagrams (with Galois action indicated by arrows), the relative root systems and the q_α in Table 2. We stress that the parameters q_α only come into play when $\alpha \in \Sigma_{\mathcal{O},\mu}$, for $\alpha \in \Sigma(A_T) \setminus \Sigma_{\mathcal{O},\mu}$ they are not defined. Recall that the root system underlying $\mathcal{H}(\mathcal{O}, G)$ is $\Sigma_{\mathcal{O}}^\vee$, which is a rescaled version of $\Sigma_{\mathcal{O},\mu}^\vee$, so obtained from the dual of the root system on the right hand side of the table.

In Table 2 $q = q_F$ and $q' \in \{q_F, q_F^2\}$, according to (4.15). For a F-group of type $^3D_4$, $[\tilde{F} : F]$ can be of degree 3 or 6. In both cases $[F_\alpha : F] = 3$ for the roots α not fixed by $\mathbf{W}_F$, so $q'' \in \{q_F, q_F^3\}$. Thus Conjecture A holds in all these cases.

It remains to consider the case where the elements of $\mathbf{W}_F \alpha_T$ are not orthogonal. From the above diagrams we see that this happens only once (up to Weyl group conjugacy) for absolutely simple groups, namely for certain pairs of roots in type $^2A_{2n}$. With Proposition 2.4 we can transfer the determination of q_α and $q_{\alpha*}$ (which no longer needs to be 1) to the simply connected cover of $\mathcal{M}_{\alpha,\mathrm{der}}$, which is isomorphic to SU_3. This does not change the q-parameters, by Proposition 2.4.(b–c). Because we cannot reduce the issue to SL_2 or PGL_2, the necessary computations are more involved.

With Section 2 we can further transfer these computations to the F-group U_3, which is a little easier. Indeed, for that group all the Hecke algebras were computed by means of types by the author's PhD student Badea [Bade]. In particular, it was shown in [Bade, §2.7 and §5.2.1] that only the following possibilities can arise:

(i) $q_\alpha = q_{F_\alpha} = q_F, q_{\alpha*} = 1$,
(ii) $q_\alpha = q_F, q_{F_\alpha} = q_F^2, q_{\alpha*} = 1$,
(iii) $q_\alpha = q_{F_\alpha} = q_F^2, q_{\alpha*} = q_F$.

The option (i) leads to an affine Hecke algebra with all q_α for $\alpha \in \Sigma_{\mathcal{O},\mu}$ equal, which occurs in Table 1. In case (ii) the connected component $\Sigma^\vee_{\mathcal{O},j}$ of $\Sigma^\vee_{\mathcal{O}}$ containing $h^\vee_\alpha$ has type B_m (for some $m \leq n$) and $q_\beta = q_F^2$ for all other simple roots in $\Sigma^\vee_{\mathcal{O},j}$. The possibility (iii) arises only from the Iwahori-spherical principal series. The latter consists of unipotent representations, so that Conjecture A is automatic.

We have to be a little careful, because it is assumed in [Bade] that the residual characteristic of F is not 2. For the Iwahori-spherical principal series that is not a problem, those affine Hecke algebras are known from [Bor] regardless of the residual characteristic. For ramified characters of $T \subset U_3(F)$ it is troublesome, because some computations in [Bade] change substantially in residual characteristic 2. To be sure in those cases as well we refer to Theorem 4.9, where all the q-parameters for $U_n(F)$ are computed in a different way (for arbitrary F but with much heavier machinery). From Lemma 4.11 one sees are the only options for $U_3(F)$ in residual characteristic 2 are still (i), (ii) and (iii).

Let us state the above conclusions concisely:

Corollary 4.5. *Conjecture A holds for all Bernstein blocks in the principal series of a quasi-split connected reductive group G over F. When we base X_α on $\alpha^\vee$, $q_{\alpha*} = 1$ and $q_\alpha = q_{F_\alpha}$ (except for one root in type ${}^2A_{2n}$).*

4.3. Inner forms of Lie type A_n.

We consider simple F-groups $\mathcal{G}$ that are inner forms of a split group of type A_{n-1}. The simply connected cover of $\mathcal{G}$ is an inner form of SL_n, so isomorphic to the derived subgroup of an inner form of GL_n. In view of Section 2 it suffices to consider the latter case, so with G isomorphic to $GL_m(D)$ for a division algebra D with centre F and $\dim_F(D) = (n/m)^2$.

For every Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ there exists a type (J,ρ) [SéSt2]. We can write $\mathfrak{s} = [M,\sigma]_G$ in the form

$$M = \prod\nolimits_i GL_{m_i}(D)^{e_i}, \sigma = \boxtimes_i \sigma_i^{\otimes e_i},$$

where the various σ_i differ by more than an unramified character. The associated Hecke algebra $\mathcal{H}(G,J,\rho)$ is a tensor product of affine Hecke algebras of type GL_{e_i} [SéSt1], so the underlying root system has irreducible components of type A_{e_i-1}, for suitable $e_i \leq n$. The same result was obtained around the same time in [Hei2], using $\Pi^{\mathfrak{s}}$. The parameters of such a type GL_{e_i} affine Hecke algebra were determined explicitly in [Séc, Théorème 4.6], they are of the form $q_\alpha = q_F^f, q_{\alpha*} = 1$ for a specific positive integer f. Thus λ and λ^* are constant and equal to f on the underlying root system A_{e_i-1}. From [Hei2, 1.13–1.15] or [SéSt2] we also see that

$$W(M,\mathcal{O}) = W(\Sigma_{\mathcal{O},\mu}) \cong \prod\nolimits_{e_i} S_{e_i} \tag{4.16}$$

and $R(\mathcal{O}) = \{1\}$. From that, (1.14) and

$$\mathrm{Mod} - \mathcal{H}(G,J,\rho) \cong \mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod}$$

we deduce that $\mathcal{H}(G,J,\rho)$ is Morita equivalent with $\mathcal{H}(\mathcal{O},G)^{op}$. These are both affine Hecke algebras, and then Morita equivalence implies that $\mathcal{H}(G,J,\rho)$ and $\mathcal{H}(\mathcal{O},G)$ and $\mathcal{H}(\mathcal{O},G)^{op}$ are isomorphic. We summarise:

Theorem 4.6. [Heiermann, Sécherre–Stevens]
Let $\mathcal{G}$ be an inner form of a simple F-split group of type A_{n-1}, and let $\mathfrak{s}$ be an inertial

equivalence class for G. Then the root system underlying $\mathcal{H}(\mathcal{O}, G)$ has irreducible components of type A_{e-1} with $e \leq n$. The label functions λ, λ^* are constant on A_{e-1}, and equal to an integer f.

We note that such parameters already occur for Iwahori-spherical representations. Namely, consider $GL_m(D)$ where $\dim_F(D) = f^2$. Its Iwahori–Hecke algebra is isomorphic with an affine Hecke algebra of type GL_e with parameters q_F^f.

More explicit information about f comes from [SéSt2, Introduction]. Every type GL_e affine Hecke algebra as above comes from a supercuspidal representation $\pi^{\boxtimes e}$ of $GL_{m'/e}(D)$ for some $m' \leq m$. Then f equals the torsion number

$$t_\pi = |X_{\mathrm{nr}}(GL_{m'/e}(D), \pi)|$$

times the reducibility number s_π. With the Jacquet–Langlands correspondence [Badu, DKV] one can relate the torsion and reducibility numbers of π to the same numbers for a specific discrete series representation $JL(\pi)$ of $GL_{nm'/em}(F)$. More information about those numbers is already known from [BeZe, BuKu1]. From that or from a comparison with Langlands parameters as in [AMS3, p. 57], one sees that s_π divides $\frac{nm'}{me}$ and that t_π divides $\frac{nm'}{mes_\pi}$. Therefore

$$f = s_\pi t_\pi \quad \text{divides} \quad \frac{nm'}{me} \leq \frac{n}{e}. \tag{4.17}$$

We note that in all these cases $\alpha^\vee$ generates $H_M(M^1_\alpha / M^1)$, because the derived groups are simply connected. The torsion number t_π says precisely that

$$H_M(M^2_\sigma \cap M^1_\alpha / M^1) = \mathbb{Z} t_\pi \alpha^\vee.$$

Consider a F-split connected reductive group $\mathcal{M}_\alpha$ with root system of type A_{n+m-1}. Let $\mathcal{M}$ be the standard F-Levi subgroup of $\mathcal{M}_\alpha$ obtained by omitting a simple root α, with root system of type $A_{n-1} \times A_{m-1}$. Then the simply connected cover of M_{der} is isomorphic to $SL_n(F) \times SL_m(F)$.

Put $\mathfrak{s} = [M, \sigma]_{M_\alpha}$ for some $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(M)$. The inflation of $\sigma|_{M_{\mathrm{der}}}$ to the simply connected cover M_{sc} of M_{der} can be written as a finite direct sum

$$\bigoplus\nolimits_i \sigma_i \boxtimes \sigma'_i \text{ with } \sigma_i \in \mathrm{Irr}_{\mathrm{cusp}}(SL_n(F)), \sigma'_i \in \mathrm{Irr}_{\mathrm{cusp}}(SL_m(F)).$$

From Theorem 4.6, (4.17) and Section 2 we obtain the following criterion for Hecke algebra parameters in split type A groups:

Corollary 4.7. *Let M_α, M and σ be as above.*

- (*a*) *If $n \neq m$, then s_α does not give rise to an element of $N_{M_\alpha}(M)/M$.*
- (*b*) *Suppose that $n = m$ and that, for any i, σ_i and σ'_i are not isomorphic. Then s_α does not give rise to an element of $W(M, \mathcal{O})$.*
- (*c*) *Suppose that $n = m$ and that, for at least one i, σ_i and σ' are isomorphic. Then $\Sigma_{\mathcal{O},\mu} = \{\alpha, -\alpha\}$ and s_α gives rise to an element of $W(M, \mathcal{O})$ that exchanges the two almost direct simple factors of M_{der}.*

  When $M_\alpha = GL_{2n}(F)$, the q-parameters for $\mathcal{H}(\mathcal{O}, M_\alpha)$ are $q_{\alpha *} = 1$ and $q_\alpha = q_F^f$. Here f is the torsion number $t_{\sigma_i} \in \mathbb{Z}_{>0}$, which divides n.

Proof. (a) This is clear, because such an element would have to exchange the two almost direct simple factors of M_{der}.
(b) Now s_α does give an element of $N_{M_\alpha}(M)/M$, which exchanges the two almost direct simple factors of M_{der}. By Proposition 2.2 we may lift to the simply connected

cover M_{sc}, picking one irreducible constituent $\sigma_i \otimes \sigma_i'$ of the inflation of $\sigma|_{M_{\mathrm{der}}}$. As M_{sc} does not have nontrivial unramified characters, stabilizing $\mathfrak{s}$ has become stabilizing $\sigma_i \otimes \sigma_i'$. Clearly s_α does that if and only if σ_i and σ_i' are isomorphic.
(c) This follows from Theorem 4.6. $\Box$

In part (c) for $M_\alpha \neq GL_{2n}(F)$, it may still be necessary to apply Proposition 2.4.d to obtain the precise parameters.

Example 4.8. *Consider the inclusion* $\eta : SL_4(F) \to GL_4(F)$ *and the Levi subgroups* $M = GL_2(F)^2$ *and* $\tilde M = S(GL_2(F)^2)$. *Let* $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(GL_2(F))$ *with*

$$X_{\mathrm{nr}}(GL_2(F), \sigma) = \{1, \chi_-\}.$$

We may assume that $\sigma|_{SL_2(F)}$ *decomposes as a direct sum of two irreducible representations, both stable under diag*$(a, b) \in GL_2(F)$ *for all* $a, b \in \mathfrak{o}_F^\times$. *Then* $\sigma \otimes \sigma \in \mathrm{Irr}(M)$ *and* $\eta^*(\sigma \otimes \sigma)$ *is a direct sum of two irreducible* $\tilde M$*-representations* $\tilde\sigma_1, \tilde\sigma_2$, *permuted by diag*$(\varpi_F, 1) \in M$. *Here*

$$\eta^*(X_{\mathrm{nr}}(M, \sigma \otimes \sigma)) = \{1, \chi_- \otimes 1\}$$

but tensoring by $\chi_- \otimes 1$ *exchanges* $\tilde\sigma_1$ *and* $\tilde\sigma_2$. *It follows that*

$$X_{\mathrm{nr}}(\tilde M, \tilde\sigma_1) = X_{\mathrm{nr}}(\tilde M, \tilde\sigma_2) = \{1\}.$$

The root systems of the Hecke algebras are $\{\alpha, -\alpha\}$ *and* $\{\tilde\alpha, -\tilde\alpha\}$, *while* $h_\alpha^\vee = \eta(h_{\tilde\alpha}^\vee)^2 \in M/M^1$. *So this is an instance of Proposition 2.4.d.(iii).*

4.4. **Classical groups.**

We look at classical groups associated to Hermitian forms on F-vector spaces. Let $\mathcal{G}^*$ be a symplectic group or a special orthogonal group (not necessarily split). It was shown in [Hei2] that $\mathrm{End}_G(\Pi^{\mathfrak{s}})$ is Morita equivalent with the crossed product of $\mathcal{H}(\mathcal{O}, G)$ and $R(\mathcal{O})$, where $\mathcal{H}(\mathcal{O}, G)$ is a tensor product of affine Hecke algebras with lattice $\mathbb{Z}^e$ and root system A_{e-1}, B_e, C_e or D_e. When $\mathcal{G}^*$ is F-split, the parameters are computed in [Hei1], relying on [Mœ2]. Later the (quasi-)split assumption in [Mœ2] was lifted in [MoRe], which means that [Hei1] also applies to pure inner forms of quasi-split groups.

We also allow $\mathcal{G}^*$ to be a special unitary group. With Section 2 we reduce that to U_n, a unitary group U_n which splits over a separable quadratic extension $\tilde F/F$. According to [Hei3, Theorem 1.8 and §C.5], the above description of $\mathcal{H}(\mathcal{O}, G)$ is also valid for U_n. Unfortunately there is no real proof of these claims in [Hei3], but it is similar to [Hei2] and in fact an instance of the more general results of [Sol4]. Also according to [Hei3, §C], the parameters of these affine Hecke algebras can be computed as in [Hei1]. This uses the results of [Mœ1, Mœ2, Mœ3].

Recall that every F-Levi subgroup of G^* is of the form

$$\mathcal{M}^*(F) \cong \prod\nolimits_i GL_{n_i}(F') \times \mathcal{H}^*(F), \tag{4.18}$$

where $\mathcal{H}^*$ is of the same type as $\mathcal{G}^*$, but of smaller rank. Here $F' = \tilde F$ for (special) unitary groups and $F' = F$ otherwise. Let f be residue degree of F'/F, so 2 for unramified (special) unitary groups and 1 otherwise.

Let $\mathcal{G}$ be a group isogenous to $\mathcal{G}^*$ and let $\mathcal{M}$ be a F-Levi subgroup of $\mathcal{G}$.

Theorem 4.9. *Let* G *be isogenous to* G^* *as above, and consider an inertial equivalence class* $\mathfrak{s} = [M, \sigma]_G$. *Fix an irreducible component* $\Sigma_{\mathcal{O},j}^\vee$ *of* $\Sigma_{\mathcal{O}}^\vee$ *as in Section*

2. Associate a supercuspidal representation ρ of $GL_{d_\rho}(F')$ to $\Sigma^\vee_{\mathcal{O},j}$ as in [Hei1, §2.1 and §3]*, and let $t = t_\rho \in \mathbb{Z}_{>0}$ be its torsion number, a divisor of d_ρ.*

(a) When $\Sigma^\vee_{\mathcal{O},j} \cong C_e$ and $h^\vee_\alpha$ is a long root, there exists an integer $a_+ \in \mathbb{Z}_{>0}$ such that $q_\alpha = q_{F'}^{ta_+}, q_{\alpha*} = 1$ and $\lambda(\alpha) = \lambda^*(\alpha) = fta_+$.

(b) When $\Sigma^\vee_{\mathcal{O},j} \cong B_e$ and $h^\vee_\alpha$ is a short root, there exist integers $a \geq a_- \geq -1$ such that $q_\alpha = q_{F'}^{t(a+1)/2}, q_{\alpha*} = q_{F'}^{t(a_-+1)/2}$ and $\lambda(\alpha) = ft(a + a_- + 2)/2$, $\lambda^*(\alpha) = ft(a - a_-)/2$.

(c) For all other $h^\vee_\alpha \in \Sigma^\vee_{\mathcal{O},j}$, $q_\alpha = q^t_{F'}, q_{\alpha*} = 1$ and $\lambda(\alpha) = \lambda^*(\alpha) = ft$.

Suppose that $\mathcal{M}$ is isogenous to (4.18)*, and that the complex dual group of $\mathcal{H}^*$ consists of matrices of size $N^\vee$. Then*

$$\Big\lfloor (\frac{a+1}{2})^2 \Big\rfloor + \Big\lfloor (\frac{a_-+1}{2})^2 \Big\rfloor \leq N^\vee d_\rho^{-1}$$

in case (b) and $a_+^2 \leq 2N^\vee d_\rho^{-1} + 1$ in case (a).

We note that for a maximal Levi subgroup $M^* = GL_{d_\rho}(F') \times H^*$ of G^*, the root system $\Sigma^\vee_{\mathcal{O}}$ has type B_1 or is empty.

Proof. By Corollary 3.7 we may assume that F has characteristic zero.

For G^* the claims (b) and (c) follow from [Hei1, Proposition 3.4] and [Hei3, §C.5]. The role of the torsion number t is to replace the sublattice $\mathbb{Z}^e$ of M/M^1 corresponding to $\Sigma^\vee_{\mathcal{O},j}$ by $(t\mathbb{Z})^e$, which is a direct summand of M^2_σ/M^1. In this process, all the labels $\lambda(\alpha)$ and $\lambda^*(\alpha)$ are multiplied by t.

The numbers a, a_- come from [Mœ1, Proposition 4], [Mœ2, §1.3–1.4] and [Mœ3, Théorème 3.1], where they are computed in terms of reducibility of the parabolic induction of a supercuspidal representation $\rho \otimes \pi$ of $GL_{d_\rho}(F') \times H^*$. This shows that in general we have to use F' instead of F. For (special) unitary groups, the factors $GL_{d_\rho}(\tilde{F})$ in (4.18) cause another factor f in all the parameters, as explained in [Hei3, §C].

Recall that the Jordan block of $\pi \in \mathrm{Irr}(H^*)$ is built from the pairs (ρ, a) that we consider (but those with $a \leq 0$ omitted), by adding new pairs according to the rule

$$\text{if } (\rho, a) \in \mathrm{Jord}(\pi) \text{ and } a > 2 \text{ then } (\rho, a-2) \in \mathrm{Jord}(\pi).$$

It was shown in [Mœ2, §1.4] and [Mœ1, Proposition 4] that

$$\sum\nolimits_{(\rho,a)\in \mathrm{Jord}(\pi)} a d_\rho = N^\vee. \tag{4.19}$$

We fix a ρ and let ρ_- be the unramified twist of ρ from which a_- is determined. Isolating the terms with ρ and ρ_- in (4.19), we obtain

$$\begin{aligned} N^\vee \geq \sum\nolimits_{a':(\rho,a')\in \mathrm{Jord}(\pi)} a' d_\rho + \sum\nolimits_{a':(\rho_-,a')\in \mathrm{Jord}(\pi)} a' d_{\rho_-} \\ = \Big\lfloor (\frac{a+1}{2})^2 \Big\rfloor d_\rho + \Big\lfloor (\frac{a_-+1}{2})^2 \Big\rfloor d_\rho. \end{aligned} \tag{4.20}$$

Case (a) for $\mathcal{G}^*$ is not mentioned explicitly in [Hei1], it is an instance of case (b) when we focus on the Weyl group (not on the root system). As the lattice containing $\Sigma^\vee_{\mathcal{O},j}$ is isomorphic to $\mathbb{Z}^e$, the construction of $h^\vee_\alpha$ entails that $h^\vee_\alpha \notin 2\mathbb{Z}^e$, so that $\Sigma^\vee_{\mathcal{O},j}$ does not have type C_e. Still, this root system occurs if $\Sigma^\vee_{\mathcal{O},j} \cong B_e$ and $q_\alpha = q_{\alpha*}$.

Then we can replace $h_\alpha^\vee$ by $h_{\alpha/2}^\vee = 2h_\alpha^\vee$, X_α by X_α^2, B_e by C_e, q_α by $q_\alpha^2 = q_\alpha q_{\alpha *}$ and $q_{\alpha *}$ by 1, without changing the Hecke algebra $\mathcal{H}(\mathcal{O}, G)$. We find

$$\lambda(\alpha/2) = \lambda^*(\alpha/2) = \lambda(\alpha) + \lambda^*(\alpha) = tf(a+1),$$

so the a_+ for $2h_\alpha^\vee$ is 1 plus the a from B_e. When a_+ is odd, the previously established bound on $a = a_-$ directly yields the new bound on a_+. When a_+ is even, (4.20) says

$$2\Big(\big(\frac{a+1}{2}\big)^2 - \frac{1}{4} \Big) \leq N^\vee d_\rho^{-1}, \text{ so } a_+^2 = (a+1)^2 \leq 2N^\vee d_\rho^{-1} + 1.$$

When $\mathcal{G}$ is a quotient of $\mathcal{G}^*$, Section 2 enables us to reduce to G^*. According to Proposition 2.4, in the process the labels for α must be multiplied by some $N_\alpha \in \{1/2, 1, 2\}$. But for type A roots nothing really changes along $G^* \to G$ (the computations can be placed entirely in a general linear group) so $N_\alpha = 1$. For other roots β either $N_\beta = 1$ or (if $\Sigma^\vee_{\mathcal{O},j} \cong C_e$) $N_\beta = 2$ or (if $\Sigma^\vee_{\mathcal{O},j} \cong B_e$) $N_\beta = 1/2$. In the last two cases types C_e and B_e are exchanged, and the relations between the parameters are the same as between cases (a) and (b) of the theorem.

In the remaining cases $\mathcal{G}$ is a spin group (or a half-spin group, but by passing to the simply connected cover, as allowed by Corollary 2.5, we reduce that case to a spin group). Then $\mathcal{G}$ can be embedded in a general spin group GSpin_n – not necessarily split, but at least a pure inner form of a quasi-split group. The Levi subgroups of GSpin_n follow the same pattern as for SO_n, and their discrete series representations can be classified as for special orthogonal groups, see [Mœ2, §1.3, §1.5] and [Mœ3, §4.4, §6.3]. Consequently the results of [Hei1, Hei2] also hold for GSpin_n, with the only difference that the lattices in the Hecke algebras have rank one higher than for SO_n. Thus Theorem 4.9 holds for GSpin_n, possibly with correction factors $N_\beta \in \{1/2, 1, 2\}$ as above for quotients of G^*. By Corollary 2.7 the theorem also holds for the derived group $\mathcal{G}$ of GSpin_n. $\square$

In the generality of Theorem 4.9 it is hard to make the integers a_+, a and a_- more explicit, since they depend in a very subtle way on the involved supercuspidal representations. If one restricts to specific classes of Bernstein components, more can be said about the Hecke algebra parameters. In particular, for the principal series representations of quasi-split classical groups the method in Paragraphs 4.1 and 4.2 yields the concrete q-parameters.

To check the integrality of the label functions λ, λ^*, we analyse the parity of a and a_-. As explained in [Hei1, §1], this boils down to comparing the Langlands parameter ρ of a supercuspidal representation of $GL_k(F_\alpha)$ with the Langlands parameter of a supercuspidal representation of H (a classical group of the same type as G).

For G of Lie type B_n, C_n, D_n or ${}^2D_n$, ρ is self-dual. Then a is odd if and only if ρ and the complex dual group $H^\vee$ of H have the same type (orthogonal or symplectic).

For G of Lie type ${}^2A_{n-1}$, ρ is conjugate-dual, that is, the contragredient $\rho^\vee$ is isomorphic to $s \cdot \rho$ for $s \in \mathbf{W}_F \setminus \mathbf{W}_{\tilde{F}}$. From [GGP, Theorem 8.1] we see that the standard representation of $H^\vee$ is conjugate-orthogonal or conjugate-symplectic, depending on the size of $H^\vee$. Just as above, a is odd if and only if ρ and this standard representation have the same type.

The parity of a_- is determined by analogous considerations, starting from a different self-dual or conjugate-dual representation $\rho \otimes \chi$ with χ an unramified character. More specifically, let t_ρ be the torsion number of $\rho \in \mathrm{Irr}(\mathbf{W}_F)$, that is, the number of unramified characters χ such that $\rho \cong \rho \otimes \chi$. Then there exist precisely $2t_\tau$ unramified characters χ of $\mathbf{W}_F$ such that $\rho \otimes \chi$ is self-dual, namely those with $\chi^{2t_\rho} = 1$. When moreover $\chi^{t_\rho} \neq 1$, $\rho \otimes \chi \not\cong \rho$. That provides a unique (up to isomorphism) $\rho_- = \rho \otimes \chi$ which is self-dual and not isomorphic to ρ. The number a_- is computed from this ρ_-. The same applies to conjugate-dual representations of $\mathbf{W}_{\tilde F}$.

The following result extends [Hei1, Proposition 1.3].

Proposition 4.10. (*a*) *Let $\rho \in \mathrm{Irr}(\mathbf{W}_F)$ be a self-dual and let ρ_- be as above.*

- *If t_ρ is odd, then ρ and ρ_- have the same type (orthogonal or symplectic).*
- *If t_ρ is even, then ρ and ρ_- can have the same or opposite type.*

(*b*) *Let $\tilde\rho \in \mathrm{Irr}(\mathbf{W}_{\tilde F})$ be a conjugate-dual and let $\tilde\rho_-$ be the unique (up to isomorphism) conjugate-dual twist of $\tilde\rho$ by an unramified character.*

- *If $\tilde F/F$ is ramified and $t_{\tilde\rho}$ is odd, then $\tilde\rho$ and $\tilde\rho_-$ have the same type (conjugate-orthogonal or conjugate-symplectic).*
- *If $\tilde F/F$ is ramified and $t_{\tilde\rho}$ is even, then $\tilde\rho$ and $\tilde\rho_-$ can have the same or opposite type.*
- *If $\tilde F/F$ is unramified, then $\tilde\rho$ and $\tilde\rho_-$ have opposite type.*

Proof. (a) Since the inertia group $\mathbf{I}_F$ is normal and $\mathbf{W}_F/\mathbf{I}_F \cong \mathbb{Z}$, ρ can be analysed well by restriction to $\mathbf{I}_F$. Clifford theory tells us that there exist mutually inequivalent irreducible $\mathbf{I}_F$-representations $\rho_1, \ldots, \rho_t$ such that

$$\mathrm{Res}^{\mathbf{W}_F}_{\mathbf{I}_F} \rho \cong \rho_1 \oplus \cdots \oplus \rho_t \tag{4.21}$$

and a Frobenius element Frob of $\mathbf{W}_F$ permutes the ρ_i cyclically. The unramified characters χ that stabilize ρ are precisely those for which $\chi(\mathrm{Frob}^t)$ acts trivially on ρ_1, so t equals the torsion number t_ρ.

If ρ_1 is self-dual, then so are all the ρ_i, and the $\mathbf{W}_F$-invariant bilinear form on ρ is a direct sum of $\mathbf{I}_F$-invariant bilinear forms on the ρ_i. Then the type of ρ is the same as the type of ρ_1, which depends only on $\mathbf{I}_F$ and is not affected by twisting with unramified characters. This can happen for even t and for odd t.

If ρ_1 is not-self dual, then none of the ρ_i is self-dual. In that case t is even and the dual of (ρ_i, V_i) is isomorphic to $\rho_{i^\vee}$ for a unique integer $i^\vee$. Further the $\mathbf{W}_F$-invariant bilinear form on ρ restricts on $\rho_i \times \rho_{i^\vee}$ to z times the canonical pairing, for some $z \in \mathbb{C}^\times$. Similarly it restricts on $\rho_{i^\vee} \times \rho_i$ to $z^\vee$ times the canonical pairing. It is easy to check that the representation $\rho_{i^\vee} \oplus \rho_i$ of $\mathbf{I}_F \rtimes \langle \mathrm{Frob}^{t/2} \rangle$ is self-dual and $z^\vee = \pm z$ where $\pm$ indicates the type of the representation. Then the type of ρ is the same as the type of $\rho_{i^\vee} \oplus \rho_i$.

By self-duality of ρ and $\rho_- = \rho \otimes \chi$ and $\rho \not\cong \rho_-$, we must have $\chi(\mathrm{Frob}^t) = -1$ and $\chi(\mathrm{Frob}^{t/2}) = \pm i$. In particular the representation $(\rho_{i^\vee} \oplus \rho_i) \otimes \chi$ of $\mathbf{I}_F \rtimes \langle \mathrm{Frob}^{t/2} \rangle$ is not self-dual with respect to the same bilinear form as $\rho_{i^\vee} \oplus \rho_i$. To make $(\rho_{i^\vee} \oplus \rho_i) \otimes \chi$ self-dual, we can take the bilinear form where in the above description $z^\vee$ is replaced by $-z^\vee$. This changes the sign of the bilinear form, so ρ and ρ_- have opposite type.
(b) When $\tilde F/F$ is ramified, we can pick a representative for $\mathbf{W}_F/\mathbf{W}_{\tilde F}$ in $\mathbf{I}_F$. Then the notions conjugate-dual, conjugate-orthogonal and conjugate-symplectic can be defined in the same way for $\mathbf{I}_{\tilde F}$-representations. The proof of part (a) applies to $\tilde\rho \in \mathrm{Irr}(\mathbf{W}_{\tilde F})$, when we replace self-dual by conjugate-dual. The conclusion is that $\tilde\rho$ and $\tilde\rho_-$ have the same type.

When $\tilde F/F$ is unramified, we pick a representative s for $\mathbf{W}_F/\mathbf{W}_{\tilde F}$ so that s^2 is a Frobenius element of $\mathbf{W}_{\tilde F}$. Conjugate-duality is still defined for $\mathbf{I}_{\tilde F}$-representations (because $\mathbf{I}_{\tilde F}$ is normal in $\mathbf{W}_F$), but the type of such a representation is not (because $s^2 \notin \mathbf{I}_{\tilde F}$). Nevertheless, we can still decompose $\tilde\rho \in \mathrm{Irr}(\mathbf{W}_{\tilde F})$ as $\mathbf{I}_{\tilde F}$-representation like in (4.21). We see that the $\mathbf{W}_{\tilde F}$-invariant bilinear pairing between $\tilde\rho$ and $s \cdot \tilde\rho$ restricts to a pairing between $(\tilde\rho_i, V_i)$ and $(\tilde\rho_{i^\vee}, V_{i^\vee})$ for a unique $i^\vee$. By definition [GGP, §3], the type of $\tilde\rho$ is given by the sign ± in

$$\langle v, v' \rangle = \pm \langle v', \tilde\rho(s^2) v \rangle \qquad \forall v, v' \in V_{\tilde\rho}. \tag{4.22}$$

We consider v in V_1 and $v' \in V_{1^\vee}$ such that the pairing (4.22) is nonzero. Then $\tilde\rho(s^2)v$ must also belong to V_1, but at the same time $\tilde\rho(s^2)$ permutes the $\tilde\rho_i$ cyclically. That renders (4.22) impossible, unless $t_{\tilde\rho} = 1$. But then $\tilde\rho|_{\mathbf{I}_{\tilde F}}$ is irreducible and isomorphic to $s \cdot \tilde\rho^\vee|_{\mathbf{I}_{\tilde F}}$. In this situation the bilinear pairing between $\tilde\rho$ and $s \cdot \tilde\rho$ is already determined by their structure as $\mathbf{I}_{\tilde F}$-representations.

The same applies to $\tilde\rho_- = \tilde\rho \otimes \tilde\chi$. Then the conjugate-duality of $\tilde\rho$ and $\tilde\rho_-$ implies that $\tilde\chi$ is quadratic. It cannot be trivial because $\tilde\rho$ and $\tilde\rho_-$ are not isomorphic, so $\tilde\chi$ is the unique unramified character of $\mathbf{W}_{\tilde F}$ of order two. By [GGP, Lemma 3.4] $\tilde\chi$ is conjugate-symplectic, and by [GGP, Lemma 3.5.ii] $\tilde\rho$ and $\tilde\rho_-$ have opposite type. □

Proposition 4.10 is the key to the following result.

Lemma 4.11. *We assume the setting of Theorem 4.9.b.*

(*a*) *When $\mathcal{G}^*$ is a special unitary group which splits over an unramified extension $\tilde F/F$, a and a_- have different parity.*

(*b*) *For all other $\mathcal{G}^*$ eligible in Theorem 4.9: if t_ρ is odd, then a and a_- have the same parity.*

(*c*) *All the labels $\lambda(\alpha), \lambda^*(\alpha)$ in Theorem 4.9 are integers.*

Proof. (b) Assume first that $\mathcal{G}$ does not have Lie type ${}^2A_{n-1}$. In the proof of Theorem 4.9 we saw how the issue can be reduced from $\mathcal{G}$ to $\mathcal{G}^*$ or GSpin_n. To $\mathcal{G}^*$ and GSpin_n we apply Proposition 4.10.a and the remarks above it.
(a) When $\mathcal{G}$ does have Lie type ${}^2A_{n-1}$, Section 2 allows to reduce to $\mathcal{G}^* = SU_n$, and then to U_n. Now we apply Proposition 4.10.b and the remarks above it.
(c) It is clear that the labels in parts (a) and (c) of Theorem 4.9 are integers. We recall that the labels in Theorem 4.9.b are

$$\lambda(\alpha) = t_\rho f(a + a_- + 2)/2 \quad \text{and} \quad \lambda^*(\alpha) = t_\rho f(a - a_-)/2.$$

These are integers, except possibly when a and a_- have different parity. In the cases where $\mathcal{G}^*$ is an unramified special unitary group, $f = 2$ and again the labels are integers. In the other cases with a and a_- of different parity, part (b) of the current lemma tells us that t_ρ is even, which makes the labels integral. □

Having checked Conjecture A.(i), we turn to Conjecture A.(ii).

Lemma 4.12. *Consider a root system of type A_{e-1}, B_e, C_e or D_e, with label functions λ, λ^* as in Theorem 4.9. There exist:*

- *a simple group $\mathcal{G}$ over a nonarchimedean local field $\tilde F$,*
- *a Bernstein block $\mathrm{Rep}(\tilde G)^{\mathfrak{s}}$, which consists of unipotent representations of $\tilde G = \mathcal{G}(\tilde F)$,*
- *a $\mathfrak{s}$-type (J, ρ),*

such that $\mathcal{H}(\tilde G, J, \rho)$ is an affine Hecke algebra with the given root system and the given label functions.

Proof. When the root system has type A_{e-1} (resp. D_e) we take $G = GL_e$ (resp. SO_{2e}). Choose a non-archimedean local field $\tilde{F}$ with residue field of order $q_{\tilde{F}} = q_F^{ft}$. We take the Iwahori-spherical Bernstein block and let J be an Iwahori subgroup of $\tilde{G}$. Then (J, triv) is a type and $\mathcal{H}(\tilde{G}, J, \mathrm{triv})$ is an affine Hecke algebra with parameters $q_{\tilde{F}}$. We obtain labels $\lambda(\alpha) = \lambda^*(\alpha) = ft$.

Suppose that the root system is C_e (or B_e with $a_- = -1$, that boils down to the same thing). We choose the q-base q_F^{ft}, which can be achieved by considering $\tilde{F}$-groups. Thus reduce to the situation where the short root α has label 1 and the long root β has label $a^+ \in \mathbb{Z}_{>0}$. Now see [Lus3, 7.40–7.42] when a_+ is even and [Lus3, 7.56] when a_+ is odd. In each case, a type for the associated Bernstein component is produced in [Lus3, §1].

Suppose that the root system is B_e and that $a_- \geq 0$. Let β be a short root and α a long root. When $a + a_-$ is even we take the q-base q_F^{ft} and we reduce to the labels

$$\lambda(\alpha) = 1, \quad \lambda(\beta) = (a + a_- + 2)/2, \quad \lambda^*(\beta) = (a - a_-)/2.$$

Depending on the parities of $\lambda(\beta)$ and $\lambda^*(\beta)$, see [Lus3, 7.38–7.39] (both even) or [Lus3, 7.48–7.49] (both odd) or [Lus3, 7.54–7.55] (one even, one odd).

When $a + a_-$ is odd we take the q-base $q_F^{ft/2}$ (a power of q_F by Lemma 4.11) and we reduce to the labels

$$\lambda(\alpha) = 2, \quad \lambda(\beta) = a + a_- + 2, \quad \lambda^*(\beta) = a - a_-.$$

See [Lus4, 11.2–11.3] for an appropriate Bernstein component consisting of unipotent representations. □

We covered all simple groups of type $A_n, {}^2A_n$ or B_n, but some simple groups of Lie type C_n, D_n or ${}^2D_n$ remain. With the classification of inner twists via Galois cohomology and the Kottwitz isomorphism [Kot] we can count them, and realizations of those groups can be found in [Spr, §17.2–17.3]:

- the non-split (non-pure) inner twist of a symplectic group,
- the two non-pure inner twists of a split even special orthogonal group,
- the non-pure inner twist of a quasi-split even special orthogonal group,
- groups isogenous to one of the above.

We note that (apart from the last entry) this list consists of classical groups associated to Hermitian forms on vector spaces over quaternionic division algebras. As far as we are aware, much less is known about the representation theory of these groups. They are ruled out in [Mœ1, Mœ2, Mœ3], so it is not clear which Hecke algebra labels can arise.

For unipotent representations, this is known completely [Lus3, Lus4, Sol5, Sol6], and that indicates that Theorem 4.9 might hold for these groups. The relevant label functions λ, λ^*, in the tables [Lus3, 7.44–7.46 and 7.51–7.53], occur also in Theorem 4.9 (with $a - a_-$ odd, like for unitary groups).

4.5. **Groups of Lie type G_2.**

Up to isogeny, there are three absolutely simple F-groups whose relative root system has type G_2:

- the split group G_2,
- the quasi-split group ${}^3D_4$, which splits over a Galois extension $\tilde{F}/F$ of degree 3 or 6,

- the non-split inner forms $E_6^{(3)}$, which split over the cubic unramified extension $F^{(3)}/F$.

Let $G = \mathcal{G}(F)$ denote the rational points of one of these groups. Let M be a Levi subgroup of G and write $\mathfrak{s} = [M, \sigma]_G, \mathcal{O} = X_{\mathrm{nr}}(M)\sigma$. When the semisimple rank of M is ≥ 1, $\Sigma_{\mathcal{O},\mu}$ has rank ≤ 1. For those cases we refer to (1.15).

Otherwise $\mathcal{M}$ is a minimal F-Levi subgroup of $\mathcal{G}$. For $G = G_2(F)$, $\mathrm{Rep}(G)^{\mathfrak{s}}$ consists of principal series representations. In Theorem 4.4 we proved that $q_\alpha = q_F$ and $q_{\alpha *} = 1$ for $\alpha \in \Sigma_{\mathcal{O},\mu}$. For $G = {}^3D_4(F)$ $\mathrm{Rep}(G)^{\mathfrak{s}}$ also belongs to the principal series. We showed in (4.15) that $q_{\alpha *} = 1, q_\alpha = q_F$ for long roots $\alpha \in \Sigma_{\mathcal{O},\mu}$ and $q_{\beta *} = 1, q_\beta \in \{q_F, q_F^3\}$ for short roots $\beta \in \Sigma_{\mathcal{O},\mu}$. Notice that in $\Sigma_{\mathcal{O}}^\vee$ the lengths of the roots are reversed.

The group $G = E_6^{(3)}$ involves a central simple F-algebra D of dimension $3^2 = 9$. We assume for the moment that $\mathcal{G}$ is simply connected, so that we can apply some reduction steps from Section 2 more easily. For a short root $\alpha \in \Sigma_{\mathcal{O},\mu}$, the inclusion $M \to M_\alpha$ is isogenous to

$$S(GL_1(D)^2) \times GL_1(F) \to S(GL_1(D)^2) \times SL_2(F).$$

In particular the coroot $\alpha^\vee$ is orthogonal to M_{der} and the restriction of σ to the image of $\alpha^\vee$ is a direct sum of finitely many characters. Hence the same computations as in Paragraph 4.1 apply here, with M instead of T. Thus $q_\alpha = q_F$ and $q_{\alpha *} = 1$. On the other hand, for a long root $\beta \in \Sigma_{\mathcal{O},\mu}$ the inclusion $M \to M_\beta$ is isogenous to

$$S(GL_1(D)^2) \times GL_1(F) \to SL_2(D) \times GL_1(F).$$

Again with Section 2 the computation of the parameters can be transferred to $GL_1(D)^2 \to GL_2(D)$, which is discussed in Paragraph 4.3. Then Theorem 4.6 and (4.17) show that $q_{\beta *} = 1$ and $q_\beta = q_F^f$ where f divides 3. All this based on an X_α defined as evaluation at $\alpha^\vee(\varpi_F^{-1})$. We still have to take the effect of the isogenies

$$M_\beta \leftarrow SL_2(D) \times GL_1(F) \to GL_2(D) \tag{4.23}$$

into account. As worked out in Proposition 2.4, this goes via changing $h_\alpha^\vee$. Since the derived groups are simply connected, no $\alpha^\vee/2$ can be involved, and this effect comes only from changes in the torsion number $|X_{\mathrm{nr}}(M, \sigma)|$. That boils down to the torsion number of a representation of $GL_1(D)$, so it can only be 1 or 3. In terms of cocharacter lattices both maps in (4.23) are index 2 inclusions, and 2 is coprime to 3, so actually the torsion numbers do not change along these inclusions. We conclude that the labels are $\lambda(\alpha) = 1$ and $\lambda(\beta) \in \{1, 3\}$ (and the same for λ^*).

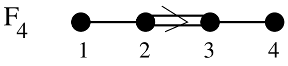
When $\mathcal{G}$ is not simply connected, we can apply Proposition 2.2 to compare with its simply connected cover. If $\Sigma_{\mathcal{O},\mu}$ has rank > 1, then it is isomorphic to $A_1 \times A_1$, A_2 or G_2. In the latter two cases we are not in the instances (ii) or (iii) of Proposition 2.4.d, so Proposition 2.4.d.(i) tells us that the parameters do not change when we pass from $\mathcal{G}$ to its simply connected cover. In the first case there could be a change as in Proposition 2.4.d.(ii) when we go to a cover of $\mathcal{G}$, but that does not bother us because we already understand affine Hecke algebras of type A_1 completely – see the discussion before (1.15).

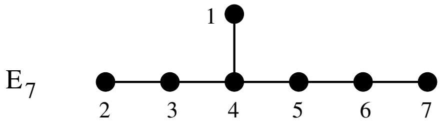
4.6. **Groups of Lie type F_4.**

Just as for G_2 we will analyse all possibilities for the parameters, by reduction to earlier cases. Up to isogeny there are three absolutely simple F-groups with relative root system of type F_4:

- the split group F_4,
- the quasi-split group ${}^2E_6$, split over a separable quadratic extension F'/F,
- the non-split inner form $E_7^{(2)}$, split over the unramified quadratic extension $F^{(2)}/F$.

Supported by Section 2, we only consider the simply connected version of these groups. We number the bases of F_4 and E_7 as follows:





Let D be a central simple F-algebra of dimension $2^2 = 4$. The anisotropic kernel of $E_7^{(2)}(F)$ corresponds to the labels $1, 5, 7$ and is isomorphic to $SL_1(D)^3$.

Let $G = \mathcal{G}(F)$ denote the rational points of one of the above groups. Fix a maximal F-split torus $S = \mathcal{S}(F)$ and let Δ be a basis of $\Sigma(\mathcal{G}, \mathcal{S})$. Let $M_J = \mathcal{M}_J(F)$ be the standard Levi subgroup associated to $J \subset \Delta$. Write $\mathfrak{s} = [M_J, \sigma]_G$ and $\mathcal{O} = X_{\mathrm{nr}}(M_J)\sigma$. We will verify Conjecture A for G, M_J, except that in a few cases for $\mathcal{G} = E_7^{(3)}$ we cannot work it out.

Recall from [Hei2, Proposition 1.3] that $\alpha \in \Sigma_{\mathcal{O},\mu}$ implies $s_\alpha \in W(G, M)$. Let $\Sigma_W(A_{M_J})$ be the set of $\alpha \in \Sigma_{\mathrm{red}}(A_{M_J})$ for which $W(G, M)$ contains s_α. Such s_α belong to the normalizer of W_J in $W(F_4)$, which links our setup to [How]. It is shown in [How, Theorem 6], that $\Sigma_W(A_{M_J})$ is a root system. The type of $\Sigma_W(A_{M_J})$, as well as a lot of other useful data, are collected in [How, p. 74].

J is empty, $\mathbf{\Sigma_W(A_{M_J}) \cong F_4}$

For F_4 and ${}^2E_6$, $\mathrm{Rep}(G)^{\mathfrak{s}}$ consists of principal series representations. For $G = F_4(F)$ we proved in Theorem 4.4 that $q_\alpha = q_F$ and $q_{\alpha *} = 1$ for all $\alpha \in \Sigma_{\mathcal{O},\mu}$.

For $G = {}^2E_6(F)$, we showed in (4.15) that $q_{\alpha *} = 1, q_\alpha = q_F$ for long roots $\alpha \in \Sigma_{\mathcal{O},\mu}$ and $q_{\beta *} = 1, q_\beta \in \{q_F, q_F^2\}$ for short roots $\beta \in \Sigma_{\mathcal{O},\mu}$. Notice that in $\Sigma_{\mathcal{O}}^\vee$ the lengths of the roots are reversed.

For $G = E_7^{(2)}(F)$ and $\alpha \in \{\alpha_1, \alpha_2\}$, the inclusion $M_\emptyset \to M_{\{\alpha\}}$ is isogenous to

$$GL_1(F)^2 \times S(GL_1(D)^3) \longrightarrow GL_2(F) \times S(GL_1(D)^3).$$

The direct factors $S(GL_1(D)^3)$ do not influence the rest, so can be ignored for the computation of the parameters. It follows that $q_\alpha = q_F, q_{\alpha *} = 1$.

For $\alpha \in \{\alpha_3, \alpha_4\}$, we can instead consider the inclusion

$$GL_1(F)^2 \times S(GL_1(D)^3) \longrightarrow GL_1(F) \times S(GL_2(D) \times GL_1(D)).$$

With Section 2 we reduce this to $GL_1(D)^2 \to GL_2(D)$, and then Paragraph 4.3 tells us that $q_{\alpha *} = 1$ and $q_\alpha \in \{q_F, q_F^2\}$.

$\mathbf{J = \{\alpha_3\}}$ or $\mathbf{J = \{\alpha_4\}, \Sigma_W(A_{M_J}) \cong B_3}$

These two J's are $W(F_4)$-conjugate, so it suffices to consider $J = \{\alpha_4\}$. The parameters for α_1 and α_2 are the same as when J is empty, so $q_{\alpha_1} = q_{\alpha_2} = q_F$ and $q_{\alpha_1 *} = q_{\alpha_2 *} = 1$.

The short simple root of $\Sigma_W(A_{M_J})$ is $\beta = \alpha_2 + 2\alpha_3 + \alpha_4$, which is orthogonal to α_2 and α_4. The inclusion $M_J \to M_{J \cup \{\beta\}}$ is isogenous to, depending on the type of

$\mathcal{G}$:

$$\begin{array}{ccccc} F_4 & GL_1(F)^3 \times SL_2(F) & \to & GL_1(F)^2 \times SL_2(F)^2 \\ {}^2E_6 & GL_1(F)^2 \times GL_1(F') \times SL_2(F') & \to & GL_1(F)^2 \times SL_2(F')^2 \\ E_7^{(2)} & GL_1(F) \times GL_1(D) \times GL_2(D) & \to & GL_1(F) \times SL_2(D) \times GL_2(D) \end{array}$$

Again the determination of the q-parameters can be simplified with Section 2. Then we see from Paragraph 4.3 that $q_{\beta *} = 1$ and $q_\beta \in \{q_F, q_F^2\}$.

$\mathbf{J} = \{\alpha_\mathbf{1}\}$ or $\mathbf{J} = \{\alpha_\mathbf{2}\}, \mathbf{\Sigma_W(A_{M_J})} \cong \mathbf{C_3}$
These two J's are $W(F_4)$-conjugate, so it suffices to consider $J = \{\alpha_1\}$. The parameters for α_3 and α_4 are the same as when J is empty.

The long simple root of $\Sigma_W(A_{M_J})$ is $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3$, which is orthogonal to α_1 and α_4. The inclusion $M_J \to M_{J \cup \{\beta\}}$ is isogenous to:

$$\begin{array}{cccc} F_4 & SL_2(F) \times GL_1(F)^3 & \to & SL_2(F)^2 \times GL_1(F)^2 \\ {}^2E_6 & SL_2(F) \times GL_1(F) \times GL_1(F')^2 & \to & SL_2(F)^2 \times GL_1(F')^2 \\ E_7^{(2)} & SL_2(F) \times GL_1(F) \times S(GL_1(D)^3) & \to & SL_2(F)^2 \times S(GL_1(D)^3) \end{array}$$

In each of the three cases, this reduces to $GL_1(F) \to SL_2(F)$. Hence $q_\beta = q_F$ and $q_{\beta *} = 1$.

$\mathbf{J}$ equals $\{\alpha_\mathbf{1}, \alpha_\mathbf{4}\}$ or $\{\alpha_\mathbf{1}, \alpha_\mathbf{3}\}$ or $\{\alpha_\mathbf{2}, \alpha_\mathbf{4}\}, \mathbf{\Sigma_W(A_{M_J})} \cong \mathbf{A_1} \times \mathbf{A_1}$
These three subsets of Δ are associate under the Weyl group $W(F_4)$, so it suffices to consider $J = \{\alpha_1, \alpha_3\}$. Up to a sign there are just two possibilities for $\alpha \in \Sigma_{\mathcal{O},\mu} \cong A_1 \times A_1$. These can be represented by α_2 and $\beta = \alpha_2 + 2\alpha_3 + 2\alpha_4$. We note that β is orthogonal to α_2 and α_3, but not to α_1. The inclusion $M_J \to M_{J \cup \{\beta\}}$ is isogenous to:

$$\begin{array}{cccc} F_4 & GL_2(F) \times GL_2(F) & \to & SL_3(F) \times GL_2(F) \\ {}^2E_6 & GL_2(F) \times GL_2(F') & \to & SL_3(F) \times GL_2(F') \\ E_7^{(2)} & GL_2(F) \times S(GL_2(D) \times GL_1(D)) & \to & SL_3(F) \times S(GL_2(D) \times GL_1(D)) \end{array}$$

In all three cases this boils down to $GL_2(F) \to SL_3(F)$, so $q_\beta = q_F$ and $q_{\beta *} = 1$.

We also list inclusions isogenous to $M_J \to M_{J \cup \{\alpha_2\}}$:

$$\begin{array}{cccc} F_4 & GL_2(F) \times SO_3(F) \times GL_1(F) & \to & SO_7(F) \times GL_1(F) \\ {}^2E_6 & GL_2(F) \times SO_4^*(F) \times GL_1(F) & \to & SO_8^*(F) \times GL_1(F) \\ E_7^{(2)} & GL_2(F) \times SO_6'(F) \times GL_1(D) & \to & SO_{10}'(F) \times GL_1(D) \end{array}$$

Here SO_{2n}^* denotes a quasi-split special orthogonal group, while SO_{2n}' stands for a non-split inner form of SO_{2n}. For the parameter computations, the direct factors $GL_1(F)$ and $GL_1(D)$ can be ignored. In all three cases Theorem 4.9.b shows that $q_{\alpha_2} = q_F^{t(a+1)/2}$ and $q_{\alpha_2 *} = q_F^{t(a_- +1)/2}$, where $t \in \{1, 2\}$. A small correction might still come from the involved isogenies via Proposition 2.4.

$\mathbf{J} = \{\alpha_\mathbf{1}, \alpha_\mathbf{2}\}, \mathbf{\Sigma_W(A_{M_J})} \cong \mathbf{G_2}$
Now α_3 gives rise to a short root of $\Sigma_W(A_{M_J})$, and to a long root of $\Sigma_{\mathcal{O}}^\vee$. Analysing

the inclusion $M_J \to M_{J\cup\{\alpha_4\}}$ up to isogeny, we obtain:

group	inclusion	q_{α_4}
F_4	$SL_3(F) \times GL_1(F)^2 \to SL_3(F) \times GL_2(F)$	q_F
${}^2E_6$	$SL_3(F) \times GL_1(F')^2 \to SL_3(F) \times GL_2(F')$	$q_{F'}$
$E_7^{(2)}$	$SL_3(F) \times SL_1(D) \times GL_1(D)^2 \to SL_3(F) \times SL_1(D) \times GL_2(D)$	q_F^f

where $f \in \{1,2\}$. In all three cases $q_{\alpha_4 *} = 1$ by Theorem 4.6. Since $\alpha_4^\vee$ is orthogonal to $M_{J,\mathrm{der}}$, the computations behind these q-parameters work equally well in $M_{J\cup\{\alpha_4\}}$, no corrections from isogenies are needed.

For α_3 we find

group	inclusion	q_{α_3}
F_4	$GL_3(F) \times GL_1(F) \to SO_6(F) \times GL_1(F)$	$q_F^{t(a+1)/2}$
${}^2E_6$	$GL_3(F) \times SO_2^*(F) \times GL_1(F') \to SO_8^*(F) \times GL_1(F')$	$q_F^{t(a+1)/2}$
$E_7^{(2)}$	$GL_3(F) \times SO'_4(F) \times GL_1(D) \to SO'_{10}(F) \times GL_1(D)$	$q_F^{t(a+1)/2}$

where $t \in \{1,3\}$. The parameter $q_{\alpha_3 *}$ equals $q_F^{t(a_- +1)/2}$. Recall the bound on a and a_- from Theorem 4.9.

When $\Sigma_{\mathcal{O},\mu}$ has type G_2, [Sol4, Lemma 3.3] says that $q_{\alpha_3 *} = 1$. Then $a_- = -1$ and Lemma 4.11 tells us that a is odd. For F_4 and $E_6^{(2)}$ that means $a = -1$ and $q_{\alpha_3} = 1$, so that actually $\Sigma_{\mathcal{O},\mu}$ does not have type G_2. For $E_7^{(2)}$ it would still be possible that $a = 1$, so that $q_{\alpha_3} = q_F^t$. But then the Langlands parameter of a representation of $SO'_4(F)$ would be the sum of a three-dimensional and a one-dimensional representation of $\mathbf{W}_F$ which is not compatible with the isogeny to $SL_1(D)^2$. Hence this case does not arise, and we conclude that for $J = \{\alpha_1, \alpha_2\}$ the root system $\Sigma_{\mathcal{O},\mu}$ has rank ≤ 1.

$\mathbf{J = \{\alpha_3, \alpha_4\}, \Sigma_W(A_{M_J}) \cong G_2}$

Now a long root of $\Sigma_W(A_{M_J})$ comes from α_1, and in $\Sigma_{\mathcal{O}}^\vee$ a short root comes from α_1. The inclusion $M_J \to M_{J\cup\{\alpha_1\}}$ is isogenous to:

$$\begin{array}{cccc} F_4 & GL_1(F)^2 \times SL_3(F) & \to & GL_2(F) \times SL_3(F) \\ {}^2E_6 & GL_1(F)^2 \times SL_3(F') & \to & GL_2(F) \times SL_3(F') \\ E_7^{(2)} & GL_1(F)^2 \times SL_3(D) & \to & GL_2(F) \times SL_3(D) \end{array}$$

In each case the parameters can be analysed already with $GL_1(F)^2 \to GL_2(F)$, and Theorem 4.6 tells us that $q_{\alpha_1} = q_F, q_{\alpha_1 *} = 1$.

Let us also consider the inclusion $M_J \to M_{J\cup\{\alpha_2\}}$ up to isogenies:

group	inclusion	q_{α_2}	$q_{\alpha_2 *}$
F_4	$GL_1(F) \times GL_3(F) \to GL_1(F) \times SO_7(F)$	$q_F^{t(a+1)/2}$	$q_F^{t(a_- +1)/2}$
${}^2E_6$	$GL_1(F) \times GL_3(F') \to GL_1(F) \times U_6(F)$	$q_{F'}^{t(a+1)/2}$	$q_F^{t(a_- +1)/2}$
$E_7^{(2)}$	$GL_1(F) \times GL_3(D) \to GL_1(F) \times SO_6(D)$	?	?

Here $t \in \{1,3\}$ and by Theorem 4.9 $0 \geq a \geq a_- \geq -1$. When $\Sigma_{\mathcal{O},\mu} \cong G_2$, we know from [Sol4, Lemma 3.3] that $q_{\alpha_2 *} = 1$. With Lemma 4.11 that implies $q_{\alpha_2} = 1$ for F_4 and for ${}^2E_6$ if F'/F is ramified. For ${}^2E_6$ with F'/F unramified, it is still possible that $a = 0$, so that $q_{\alpha_2} = q_{F'}^{t/2} = q_F^t$. For the same reasons as after (4.23), no corrections from isogenies are needed. For $E_7^{(2)}$ the analysis involves quaternionic special orthogonal groups, a case which remains open.

$\mathbf{J} = \{\alpha_2, \alpha_3\}, \boldsymbol{\Sigma}_{\mathrm{red}}(\mathbf{A}_{\mathbf{M_J}}) \cong \mathbf{B_2}$

Here α_1 gives rise to a long root and α_4 to a short root of $\Sigma_{\mathrm{red}}(A_{M_J})$. We assume that $\alpha_1, \alpha_4 \in \Sigma_{\mathcal{O},\mu}$, otherwise $\Sigma_{\mathcal{O},\mu}$ is isomorphic to a root subsystem of $A_1 \times A_1$ and the situation is simpler. We would like to say that in $\Sigma_{\mathcal{O}}^\vee$ the relation between the lengths of the roots is reversed, but that is not so obvious because $h_{\alpha_i}^\vee$ need not be exactly $\alpha_i^\vee(\varpi_F^{-1})$, maybe it has to scaled.

Up to isogenies, the inclusions $M_J \to M_{J \cup \{\alpha_1\}}$ are:

$$\begin{array}{cccc} F_4 & GL_1(F) \times SO_5(F) \times GL_1(F) & \to & SO_7(F) \times GL_1(F) \\ {}^2E_6 & GL_1(F) \times SO_6^*(F) \times GL_1(F') & \to & SO_8^*(F) \times GL_1(F') \\ E_7^{(2)} & GL_1(F) \times SO_8'(F) \times GL_1(D) & \to & SO_{10}'(F) \times GL_1(D) \end{array} \tag{4.24}$$

Each of the involved isogenies is a twofold cover of the groups listed above, and on the left hand side that covering does not involve the direct factor $GL_1(F)$. Hence passing to that cover does not change $X_{\mathrm{nr}}(M,\sigma) \cap X_{\mathrm{nr}}(GL_1(F))$. For (4.24) this intersection is trivial, so also for the analogous setting inside G. This shows that

$$h_{\alpha_1}^\vee = \alpha_1^\vee(\varpi_F^{-1}) \in M/M^1.$$

Now the parameters associated with α_1 are given by Theorem 4.9.b, namely

$$\lambda(\alpha_1) = (a + a_- + 2)/2 \quad \text{and} \quad \lambda^*(\alpha_1) = (a - a_-)/2.$$

Here $a \geq a_- \geq -1$ and $a \leq N^\vee$ with $N^\vee \in \{4,6,8\}$ depending on $\mathcal{G}$.

Up to isogenies, the inclusion $M_J \to M_{J \cup \{\alpha_4\}}$ is:

$$\begin{array}{cccc} F_4 & GL_1(F) \times Sp_4(F) \times GL_1(F) & \to & GL_1(F) \times Sp_6(F) \\ {}^2E_6 & GL_1(F) \times U_4(F) \times GL_1(F') & \to & GL_1(F) \times U_6(F) \\ E_7^{(2)} & GL_1(F) \times SO_4(D) \times GL_1(D) & \to & GL_1(F) \times SO_6(D) \end{array} \tag{4.25}$$

The same argument as for α_1 shows that $h_{\alpha_4}^\vee = \alpha_4^\vee(\varpi_F^{-1})$. In the root system $\Sigma_{\mathcal{O}}^\vee$ we now have the short simple root $h_{\alpha_1}^\vee$ and the long simple root $h_{\alpha_4}^\vee$. We recall from [Sol4, Lemma 3.3] that $q_{\alpha_4 *} = 1$ and $\lambda(\alpha_4) = \lambda^*(\alpha_4)$. From (4.25) and Theorem 4.9 we deduce that $a_- = -1$ and $q_{\alpha_4} = q_F^a$, at least for F_4 and ${}^2E_6$. For $E_7^{(2)}$ this involves quaternionic special orthogonal groups, which we could not handle in Theorem 4.9. As explained before Proposition 4.10, a is an odd integer. Moreover, Theorem 4.9 tells us that $(a+1)^2/4 \leq N^\vee \in \{5,4\}$. It follows that $a \leq 3$, and then

$$\lambda(\alpha_4) = \lambda^*(\alpha_4) = (a+1)/2 \in \{1,2\}.$$

We take this opportunity to point out a typo in [Lus3] relevant to us. Namely, when we run the above arguments with σ the unique supercuspidal unipotent representation of $M_J \subset G = F_4(F)$, we obtain the parameters $\lambda(\alpha_1) = 2, \lambda^*(\alpha_1) = 1, \lambda(\alpha_4) = 2$. In [Lus3, §7.31] these are given as $\lambda(\alpha_1) = 3, \lambda^*(\alpha_1) = 1, \lambda(\alpha_4) = 3$. We already took this into account by not including labels (3,3,1) for B_2 in Table 1.

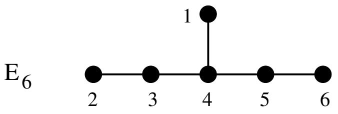
$|\mathbf{J}| = \mathbf{3}$ or $|\mathbf{J}| = \mathbf{4}$

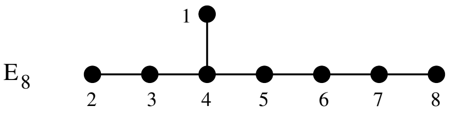
In these cases $\Sigma_{\mathcal{O},\mu}$ has rank ≤ 1, and we refer to (1.15).

Summarising: we checked our main conjecture for absolutely simple groups with relative root system of type F_4, except that for the group $E_7^{(2)}$ we are not sure when $J = \{\alpha_3, \alpha_4\}$ or $J = \{\alpha_2, \alpha_3\}$. These cases can be settled once we understand symplectic and special orthogonal groups of quaternionic type better.

4.7. **Groups of Lie type E_6, E_7, E_8.**

We consider simply connected F-split groups of type E_n. We number E_6 and E_8 (or rather their bases Δ) as





and E_7 similarly (as on page 38). The number of inequivalent Levi subgroups is quite large, which renders a case-by-case analysis as for G_2 and F_4 elaborate. An advantage is of course that all these Levi subgroups are simply connected and F-split, so the analysis of Hecke algebra parameters for E_n consists of the principal series (dealt with in Paragraph 4.1) and contributions from split groups of lower rank. For Levi subgroups of semisimple rank $n-1$ the root system $\Sigma_{\mathcal{O},\mu}$ has rank ≤ 1, and before (1.15) we discussed all such cases.

For E_6 and Levi subgroups of semisimple rank at most 4, the q-parameters can be computed via inclusions $M \to M_\alpha$ where M_α has semisimple rank at most 5. These $\mathcal{M}_\alpha$ are not exceptional, so the q-parameters can be found from Paragraphs 4.3 and 4.4. For the irreducible components of $\Sigma_{\mathcal{O},\mu}$ of type A, Conjecture A just says that every parameter q_α is a power of q_F. That is readily verified in each case. Therefore we focus on the subsets $J \subset \Delta$ such that $\Sigma_W(A_{M_J})$ has a component of type B_n, C_n, F_4 or G_2. The possible J can be found by inspecting the tables on [How, p.75–77].

$\mathbf{J = \{\alpha_1, \alpha_3\}, \Sigma_W(A_{M_J}) \cong B_3}$

The long simple roots $\alpha \in \Sigma_W(A_{M_J})$ correspond to $\alpha_5, \alpha_6 \in \Delta$, which are orthogonal to $M_{J,\mathrm{der}}$. Hence the computations reduce to those in Paragraph 4.1, and yield $q_{\alpha*} = 1, q_\alpha = q_F$ (or $\alpha \notin \Sigma_{\mathcal{O},\mu}$).

The short simple root β of $\Sigma_W(A_{M_J})$ comes from $\alpha_4 \in \Delta$. Here $M_{\beta,\mathrm{der}} \cong SL_4(F)$ and $M \cap M_{\beta,\mathrm{der}} \cong S(GL_1(F)^2)$. If $\beta \in \Sigma_{\mathcal{O},\mu}$, then Corollary 4.7 associates to $GL_1(F)^2 \to GL_2(F)$ the parameters $q_{\beta*} = 1$ and $q_\beta = q_F^f$ with $f \in \{1, 2\}$. Under the isogenies that transfer back to $M \to M_\beta$, $h_\beta^\vee$ remains equal to $\beta^\vee$, so the q-parameters do not change.

$\mathbf{J = \{\alpha_1, \alpha_3, \alpha_4\}, \Sigma_W(A_{M_J}) \cong B_2}$

The long simple root α of $\Sigma_W(A_{M_J})$ comes from $\alpha_6 \in \Delta$, which is orthogonal to $M_{J,\mathrm{der}}$. Hence $q_\alpha = q_F$ and $q_{\alpha*} = 1$.

The short simple root $\beta \in \Sigma_W(A_{M_J})$ comes from $\alpha_5 \in \Delta$. Here $M_{\beta,\mathrm{der}} \cong \mathrm{Spin}_8(F)$ and $M \cap M_{\beta,\mathrm{der}}$ is a twofold cover of $SO_6(F) \times GL_1(F)$. The q-parameters for this setting are known from Theorem 4.9:

$$q_\beta = q_F^{(a+1)/2}, q_{\beta*} = q_F^{(a_-+1)/2} \text{ where } \Big\lfloor (\frac{a+1}{2})^2 \Big\rfloor + \Big\lfloor (\frac{a_-+1}{2})^2 \Big\rfloor \leq 6,$$

so $a \leq 4$. When we apply Proposition 2.4 to $M_{\beta,\mathrm{der}} \to M_\beta$, the parameters stay the same or (only when $a = a_-$) Proposition 2.4.d.(iii) applies.

$\mathbf{J = \{\alpha_2, \alpha_3, \alpha_5, \alpha_6\}, \Sigma_W(A_{M_J}) \cong G_2}$
The long simple root $\alpha \in \Sigma_W(A_{M_J})$ comes from $\alpha_1 \in \Delta$. That one is orthogonal to $M_{J,\mathrm{der}}$, so by Paragraph 4.1 $q_\alpha = q_F$ and $q_{\alpha *} = 1$.

The short simple root $\beta \in \Sigma_W(A_{M_J})$ comes from $\alpha_4 \in \Delta$. Now $M_{\beta,\mathrm{der}} \cong SL_6(F)$ and $M \cap M_{\beta,\mathrm{der}} \cong S(GL_3(F)^2)$. The same arguments as above for $J = \{\alpha_1, \alpha_3\}$ shows that here (if $\beta \in \Sigma_{\mathcal{O},\mu}$) $q_{\beta *} = 1$ and $q_\alpha = q_F^f$ with $f \in \{1, 3\}$.

Having checked Conjecture A for E_6, we turn to the simply connected split F-groups of type E_7 and E_8. For most $J \subset \Delta$, the q-parameters of $\mathcal{H}(\mathcal{O}, G)$ can be analysed as before. However, some J behave like $\{\alpha_2, \alpha_3\}$ for F_4, where we found it hard to relate the parameters of the two simple roots to each other. For other J (only in E_8) the computation of the q-parameters can only be reduced to inclusions of Lie type $A_2 \times A_1 \times A_2 \to E_6$ or $D_6 \to E_7$ or $E_6 \to E_7$, and we do not know an effective method in these cases. Therefore we settle for a modest goal:

Lemma 4.13. *For groups of Lie type E_6, E_7 or E_8, Conjecture A holds whenever the root system $\Sigma_{\mathcal{O},\mu}$ has a component of type F_4.*

Proof. From [How, p.75–79] one sees that in only very few cases $\Sigma_{\mathcal{O},\mu}$ has a component of type F_4. For any root α in a type F_4 root system, [Sol4, Lemma 3.3] shows that $q_{\alpha *} = 1$, and then Proposition 2.4 entails that no involved isogeny can change the parameters.

For $G = E_7(F)$ there is only one J with $\Sigma_W(A_{M_J}) \cong F_4$, namely $J = \{\alpha_1, \alpha_5, \alpha_7\}$. The q-parameters can be obtained in the same way as for $E_7^{(2)}(F)$ and $J = \emptyset$, as treated in Paragraph 4.6. The only difference is that an inclusion $S(GL_1(D)^2) \to SL_2(D)$ must be replaced by an inclusion $S(GL_2(F)^2) \to SL_4(F)$, but from Paragraph 4.3 we know that exactly the same q-parameters can occur for both these inclusions. Thus $q_\alpha = q_F, q_{\alpha *} = 1$ for any long root $\alpha \in \Sigma_{\mathcal{O},\mu} \cong F_4$ and $q_{\beta *} = 1, q_\beta \in \{q_F, q_F^2\}$ for any short root $\beta \in \Sigma_{\mathcal{O},\mu}$.

For $E_8(F)$ and $J = \{\alpha_1, \alpha_5, \alpha_7\}$ we also have $\Sigma_W(A_{M_J}) \cong F_4$. This case can be handled just as for E_7, and leads to the same q-parameters.

For $E_8(F)$ and $J = \{\alpha_1, \alpha_3, \alpha_5\}$ we have $\Sigma_W(A_{M_J}) \cong F_4 \times A_1$. The long simple roots of F_4 come from $\alpha_7, \alpha_8 \in \Delta$. These are orthogonal to M_{der}, so $q_\alpha = q_F$ and $q_{\alpha *} = 1$. According to [How, p. 75] the short simple roots β of F_4 are associated to an inclusion $S(GL_2(F)^2) \to SL_4(F)$. We can use the same X_β as for $GL_2(F)^2 \to GL_4(F)$, for which Corollary 4.7 shows that $q_{\beta *} = 1$ and $q_\beta \in \{q_F, q_F^2\}$.

The only remaining case with $\Sigma_W(A_{M_J}) \cong F$ is $J = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$. Like in the previous case $q_\alpha = q_F, q_{\alpha *} = 1$ for any long simple root $\alpha \in \Sigma_{\mathcal{O},\mu}$. Both short simple roots β of F_4 come from a non-simple root in E_8, for which $M \cap M_{\beta,\mathrm{der}} \to M_{\beta,\mathrm{der}}$ is isomorphic to the inclusion of a double cover of $SO_8(F) \times GL_1(F)$ in $\mathrm{Spin}_{10}(F)$. According to Theorem 4.9 the resulting q-parameters are

$$q_\beta = q_F^{(a+1)/2} \text{ and } q_{\beta *} = q_F^{(a_- +1)/2}, \text{ where } \left\lfloor (\frac{a+1}{2})^2 \right\rfloor + \left\lfloor (\frac{a+1}{2})^2 \right\rfloor \leq 8.$$

Since $\Sigma_{\mathcal{O},\mu}$ has type F_4, $q_{\beta *} = 1$ and $a_- = -1$. From Lemma 4.11 we know that a and a_- have the same parity, so a is odd. The estimate shows that $a < 5$, so $a \in \{1, 3\}$ and $q_\beta \in \{q_F, q_F^2\}$ as desired. □

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