Periodic cyclic homology of Hecke algebras and their Schwartz completions

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Abstract. We show that the inclusion of an affine Hecke algebra in its Schwartz completion induces an isomorphism on periodic cyclic homology.
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Let $\mathcal{O}(V)$ and $C^\infty(X)$ be the algebras of regular functions on a nonsingular affine complex variety V and of smooth (complex valued) functions on a differentiable manifold X. The Hochschild-Kostant-Rosenberg theorem [HKR] states that there is a natural isomorphism

$$HH_*(\mathcal{O}(V)) \cong \Omega^*(V) \tag{1}$$

between the Hochschild homology of $\mathcal{O}(V)$ and the algebra of differential forms on V, both in the algebraic sense. The smooth analogue of this theorem, due to [Con, §II.6], is

$$HH_*(C^\infty(X)) \cong \Omega^*(X;\mathbb{C}) \tag{2}$$

but now both sides must be interpreted in the topological sense. [1] Moreover the exterior differential d on Ω^* corresponds to the map B on HH_*, which implies that

$$HP_*(\mathcal{O}(V)) \cong H^*_{DR}(V) \tag{3}$$

$$HP_*(C^\infty(X)) \cong H^*_{DR}(X;\mathbb{C}) \tag{4}$$

where the right hand sides are $\mathbb{Z}/2\mathbb{Z}$-graded. However, periodic cyclic homology is much more flexible than Hochschild homology and therefore the conditions on V and X can be relaxed. In particular (3) still holds if V is singular [FT, Theorem 5] and (4) is also valid for orbifolds X [Was, §4].

Suppose now that X is a (smooth) deformation retract of V, endowed with its analytic topology. Because the algebraic and analytic De Rham cohomologies of V are naturally isomorphic [Har, Theorem IV.1.1], the inclusion $X \to V$ induces isomorphisms

$$H^*_{DR}(V) \to H^*_{DR}(X;\mathbb{C}) \tag{5}$$

$$HP_*(\mathcal{O}(V)) \to HP_*(C^\infty(X)) \tag{6}$$

Notice that if X is a compact set of uniqueness for V then $C^\infty(X)$ is a completion of $\mathcal{O}(V)$. This is remarkable since (contrarily to topological K-theory) cyclic homology theories behave badly with respect to completing algebras.

For example consider the C^*-completion $C(X)$ of $C^\infty(X)$. Applying [Joh, Section 1] to [Kam, Corollary 4.9] we see that

$$HH_n(C(X)) = 0 \qquad \text{for } n > 0 \tag{7}$$

[1] As concerns the notation, V is a complex algebraic variety, so functions and differential forms on V automatically have complex values. On the other hand, X is a real manifold, and while it is customary to write $C^\infty(X)$ for $\mathbb{C}$-valued functions, the author believes that it should be mentioned if differential forms (and De Rham cohomology) are considered with complex coefficients.

Hence also $HP_1(C(X)) = 0$ and

$$HP_0(C(X)) = HH_0(C(X)) = C(X) \tag{8}$$

In our main theorem we will show that (6) also holds for a certain class of noncommutative algebras, namely affine Hecke algebras and their Schwartz completions. The reader is referred to the work of Delorme and Opdam [Opd, DO1, DO2] for a precise definition and a thorough study of the representation theory of these algebras. One of the first things to notice is that an affine Hecke algebra is of finite rank over its center, so that we can use the powerful theory of finite type algebras, which was developed by Baum, Kazhdan, Nistor and Schneider [BN, KNS]. The author was particularly inspired by [BN, Theorem 8]:

Theorem 1 *Let $L \to J$ be a spectrum preserving morphism of finite type algebras. Then the induced map $HP_*(L) \to HP_*(J)$ is an isomorphism.*

So, just as for commutative finitely generated algebras, the periodic cyclic homology of a finite type algebra depends only on its spectrum, endowed with Jacobson topology. Unfortunately the spectrum $\hat{\mathcal{H}}$ of an affine Hecke algebra $\mathcal{H}$ is a rather ugly topological space, it is a kind of non-separated scheme over $\mathbb{C}$. Similarly the spectrum $\hat{\mathcal{S}}$ of the associated Schwartz algebra is a non-Hausdorff manifold. Notwithstanding these topological inconveniences, it follows from [DO2] that we can stratify these spectra so that $\hat{\mathcal{S}}$ becomes a deformation retract of $\hat{\mathcal{H}}$. Along these lines we will prove

Theorem 2 *Let $\mathcal{H}$ be an affine Hecke algebra and $\mathcal{S}$ its Schwartz completion. Then the inclusion $\mathcal{H} \to \mathcal{S}$ induces an isomorphism $HP_*(\mathcal{H}) \to HP_*(\mathcal{S})$.*

But first we consider some possible consequences of this theorem. Let F be a nonarchimedean local field, e.g. a p-adic field. Let G be the group of F-rational points of a connected reductive algebraic group, and $\mathcal{B}(G)$ the set of Bernstein components of the smooth dual of G [BD]. The Hecke algebra $\mathcal{H}(G)$ consists of all compactly supported locally constant functions on G, and it decomposes naturally as

$$\mathcal{H}(G) = \bigoplus_{\Omega \in \mathcal{B}(G)} \mathcal{H}(G)_\Omega \tag{9}$$

Similarly $\mathcal{S}(G)$ denotes the Schwartz algebra of all rapidly decreasing locally constant functions on G, which is also an algebraic direct sum

$$\mathcal{S}(G) = \bigoplus_{\Omega \in \mathcal{B}(G)} \mathcal{S}(G)_\Omega \tag{10}$$

It is well known that $\mathcal{H}(G)_\Omega$ tends to be Morita equivalent to the twisted crossed product of a finite group and an affine Hecke algebra, cf. [ABP, Section 5]. In particular it has been proved that, for all Bernstein components of $GL(n, F)$, $\mathcal{H}(G)_\Omega$ is Morita equivalent to a certain affine Hecke algebra $\mathcal{H}_\Omega$ [BK]. Moreover in this case $\mathcal{S}(G)_\Omega$ is Morita equivalent to $\mathcal{S}_\Omega$, so Theorem 2 implies [BHP1, Theorem 1]:

Theorem 3 *The inclusion $\mathcal{H}(GL(n, F)) \to \mathcal{S}(GL(n, F))$ induces an isomorphism on periodic cyclic homology.*

More generally, in [BHP2, Conjecture 8.9] it was conjectured that $HP_*(\mathcal{H}(G)_\Omega) \to HP_*(\mathcal{S}(G)_\Omega)$ is always an isomorphism. Unfortunately we cannot apply the methods in this paper to the aforementioned twisted affine Hecke algebras, because not enough is known about their representation theory. Nevertheless this conjecture might be proved in another way, in connection with the Baum-Connes conjecture for G, see [Laf], [BHP2, Proposition 9.4] and [Sol, Theorem 12].

We recall some of the notations of [Opd]. Let R_0 be a finite, reduced root system with Weyl group W_0 and set of simple roots F_0. Let $\mathcal{R} = (X, Y, R_0, R_0^\vee, F_0)$ be a root datum with affine Weyl group $W = X \rtimes W_0$ and length function $l : W \to \mathbb{N}$. Pick a label function $q : W \to \mathbb{R}^+$, which may take different values on nonconjugate simple reflections. The affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ has a $\mathbb{C}$-basis N_w in bijection with W, and the multiplication is defined by

- $N_v N_w = N_{vw}$ if $l(vw) = l(v) + l(w)$
- $(N_s + q(s)^{-1/2})(N_s - q(s)^{1/2}) = 0$ for a simple reflection $s \in W$

The adjoint of $h = \sum_w c_w N_w$ is $h^* = \sum_w \overline{c_w} N_{w^{-1}}$. The Schwartz algebra $\mathcal{S} = \mathcal{S}(\mathcal{R}, q)$ consists of all (possibly infinite) sums $\sum_{w \in W} c_w N_w$ such that $w \to |c_w|$ is a rapidly decreasing function, with respect to l. It is a nuclear Fréchet *-algebra.

For any $P \subset F_0$ we denote by $\mathcal{H}_P = \mathcal{H}(\mathcal{R}_P, q)$ the affine Hecke algebra with root datum $\mathcal{R}_P = (X_P, Y_P, R_P, R_P^\vee, P)$, where

$$R_P = \mathbb{Q}P \cap R_0 \qquad R_P^\vee = \mathbb{Q}P^\vee \cap R_0^\vee$$
$$X_P = X / X \cap (P^\vee)^\perp \qquad Y_P = Y \cap \mathbb{Q}P^\vee$$

Furthermore we define

$$X^P = X / X \cap \mathbb{Q}P \quad T^P = \mathrm{Hom}(X^P, \mathbb{C}) \quad T_P = \mathrm{Hom}(X_P, \mathbb{C})$$

Recall that T^P (and T_P as well of course) decomposes into a unitary and real split part:

$$T^P = \mathrm{Hom}(X^P, S^1) \times \mathrm{Hom}(X^P, \mathbb{R}^+) = T_u^P \times T_{rs}^P \tag{11}$$

Let Δ_P be the set of isomorphism classes of discrete series of $\mathcal{H}_P$, and write $\Delta = \bigcup_{P \subset F_0} \Delta_P$. Denote by Ξ the analytic variety consisting of all triples $\xi = (P, t, \delta)$ with $P \subset F_0, \delta \in \Delta_P, t \in T^P$, and let Ξ_u be the compact submanifold obtained by restricting to $t \in T_u^P$. For every $\xi \in \Xi$ there exists a so-called generalized minimal principal series representation $\pi(\xi)$ of $\mathcal{H}$. Its underlying vector space $V_\xi = V_{\pi(P,t,\delta)}$ does not depend on t, and we let

$$\mathcal{V}_\Xi = \bigcup_{(P,\delta) \in \Delta} T^P \times V_{\pi(P,t,\delta)}$$

be the corresponding vector bundle over Ξ. Let $\mathcal{W}$ be the groupoid, over the power set of F_0, with $\mathcal{W}_{PQ} = K_Q \times W(P, Q)$, where $K_Q = T^Q \cap T_Q$ and

$$W(P, Q) = \{w \in W_0 : w(P) = Q\}$$

This groupoid acts naturally on Ξ from the left, and for every $g \in \mathcal{W}, \xi = (P, t, \delta) \in \Xi$ there exists an intertwiner

$$\pi(g, \xi) : V_\xi \to V_{g(\xi)}$$

which is rational in t. These intertwiners are unique up to scalars and for any choice there exist numbers $c(\delta, g, g')$ such that

$$\pi(g, g', \xi) = c(\delta, g, g')\pi(g, g'\xi)\pi(g, \xi)$$

In general it is not possible to choose the scalars such that all the $c(\delta, g, g')$ become 1.

The Langlands classification for $\mathcal{H}$ yields [DO2, Corollary 6.19]:

Proposition 4 *For every $\pi \in \hat{\mathcal{H}}$ there exists a unique association class $\mathcal{W}\xi \in \mathcal{W} \setminus \Xi$ such that*

- *π is isomorphic to a subquotient of $\pi(\xi) = \pi(P, t, \delta)$*
- *$|P|$ is maximal with respect to this property*

The resulting map $\hat{\mathcal{H}} \to \mathcal{W} \setminus \Xi$ is surjective and finite to one.

The orbit $\mathcal{W}\xi$ is called the tempered central character of π, and π extends to $\mathcal{S}$ if and only if $\xi \in \Xi_u$. The intertwiners are unitary on Ξ_u, so $\mathcal{W}$ also acts on the sections of the endomorphism bundle of $\mathcal{V}_\Xi$ over Ξ_u by

$$g(f)(\xi) = \pi(g, g^{-1}\xi) f(g^{-1}\xi) \pi(g, g^{-1}\xi)^{-1}$$

Now we can formulate [DO1, Theorem 4.3]:

Theorem 5 *The Fourier transform defines an isomorphism of pre-C^*-algebras*

$$\mathcal{S} \to C^\infty \left(\Xi_u; \mathrm{End}\, \mathcal{V}_\Xi\right)^{\mathcal{W}}$$

At this point the preparations for the proof of Theorem 2 really start. To bring things back to the commutative case we construct stratifications of the spectra of $\mathcal{H}$ and $\mathcal{S}$. Choose an increasing chain

$$\emptyset = \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_n = \Delta$$

of $\mathcal{W}$-invariant subsets of Δ, with the properties

- if $(P, \delta) \in \Delta_i$ and $|Q| > |P|$ then $\Delta_Q \subset \Delta_i$
- the elements of $\Delta_i \setminus \Delta_{i-1}$ form exactly one association class for the action of $\mathcal{W}$

To this correspond two decreasing chains of ideals

$$\begin{aligned}
\mathcal{H} &= I_0 \supset I_1 \supset \cdots \supset I_n = 0 \\
\mathcal{S} &= J_0 \supset J_1 \supset \cdots \supset J_n = 0 \\
I_i &= \{h \in \mathcal{H} : \pi(P, t, \delta)(h) = 0 \text{ if } (P, \delta) \in \Delta_i, t \in T^P\} \\
J_i &= \{h \in \mathcal{S} : \pi(P, t, \delta)(h) = 0 \text{ if } (P, \delta) \in \Delta_i, t \in T_u^P\}
\end{aligned}$$

For every i pick an element $(P_i, \delta_i) \in \Delta_i \setminus \Delta_{i-1}$, let $\mathcal{W}_i$ be the stabilizer of (P_i, δ_i) in $\mathcal{W}$ and write $V_i = V_{\pi(P_i, t, \delta_i)}$. Then an immediate consequence of Theorem 5 is

$$J_{i-1}/J_i \cong C^\infty(T_u^{P_i}; \mathrm{End}\, V_i)^{\mathcal{W}_i} \tag{12}$$

while from Proposition 4 we see that the spectrum of I_j/I_i corresponds to the inverse image of $\Delta_i \setminus \Delta_j$ under the projection $\Xi \to \Delta$. Moreover the induced map $\widehat{I_{i-1}/I_i} \to \mathcal{W}_i \setminus T^{P_i}$ is continuous if we consider $\mathcal{W}_i \setminus T^{P_i}$ as an algebraic variety and endow $\widehat{I_{i-1}/I_i}$ with the Jacobson topology. (In fact it is the central character map for this algebra.)

Recall that the functor HP_* satisfies excision, both in the algebraic [CQ] and the topological [Cun] setting. This means that an extension

$$0 \to I \to A \to A/I \to 0$$

of algebras gives rise to an exact hexagon

$$\begin{array}{ccccc} HP_0(I) & \to & HP_0(A) & \to & HP_0(A/I) \\ \uparrow & & & & \downarrow \\ HP_1(A/I) & \leftarrow & HP_1(A) & \leftarrow & HP_1(I) \end{array}$$

Note however that in the topological case we have to restrict ourselves to admissible extensions, i.e. those admitting a continuous linear splitting.

Together with the five lemma this means that in order to prove Theorem 2 it suffices to show that each inclusion

$$I_{i-1}/I_i \to J_{i-1}/J_i$$

induces an isomorphism on periodic cyclic homology. Therefore we zoom in on J_{i-1}/J_i. By [Opd, Corollary 4.34] we can extend the action of $\mathcal{W}_i$ on $C^\infty(T_u^{P_i};\text{End } V_i)$ to a compact, $\mathcal{W}_i$-invariant tubular neighborhood U of $T_u^{P_i}$ in T^{P_i}. We may assume that U is W_i-equivariantly diffeomorphic to $T_u^{P_i} \times [-1,1]^{\dim T_u^{P_i}}$, and because $[-1,1]$ is compact and contractible we can even arrange things so that the extended intertwiners $\pi(g,\xi)$ are unitary on all of U. It turns out that we can avoid a lot of technical difficulties by replacing J_{i-1}/J_i by $C^\infty(U;\text{End } V_i)^{\mathcal{W}_i}$. This is justified by the following result, which is an application of the techniques developed in [Sol].

Lemma 6 *The inclusion $T_u^{P_i} \to U$ and the Chern character induce isomorphisms*

$$HP_*(J_{i-1}/J_i) \cong HP_*\left(C^\infty(U;\text{End } V_i)^{\mathcal{W}_i}\right) \cong K_*\left(C(U;\text{End } V_i)^{\mathcal{W}_i}\right) \otimes_{\mathbb{Z}} \mathbb{C}$$

Proof. The second isomorphism follows directly from the density theorem for K-theory and [Sol, Theorem 6]. With the help of [Ill] we pick a $\mathcal{W}_i$-equivariant triangulation $\Sigma \to T_u^{P_i}$ and we construct a closed cover

$$\{V_\sigma : \sigma \text{ simplex of } \Sigma\}$$

as on [Sol, p. 9]. Also let D_σ be the subset of V_σ corresponding to the faces of σ. Using the projection $p_u : U \to T_u^{P_i}$ we get a closed cover

$$\{U_\sigma = \sigma \text{ simplex of } \Sigma\}$$

of U, with

$$U_\sigma = p_u^{-1}(V_\sigma) \cong V_\sigma \times [-1,1]^{\dim T_u^{P_i}}$$

According to [Sol, p. 10] it suffices to show that for any simplex σ we have

$$HP_*\left(C_0^\infty(V_\sigma, D_\sigma;\text{End } V_i)^{\mathcal{W}_\sigma}\right) \cong HP_*\left(C_0^\infty(U_\sigma, p_u^{-1}(D_\sigma);\text{End } V_i)^{\mathcal{W}_\sigma}\right) \tag{13}$$

where $\mathcal{W}_\sigma$ is the stabilizer of σ in $\mathcal{W}_i$. Well, $V_\sigma \setminus D_\sigma$ is $\mathcal{W}_\sigma$-equivariantly contractible by construction, and it is an equivariant retract of $U_\sigma \setminus p_u^{-1}(D_\sigma) = p_u^{-1}(V_\sigma \setminus D_\sigma)$, so we can apply [Sol, Lemma 7]. In this context it says that there exists a finite central extension G of $\mathcal{W}_\sigma$ and a linear representation

$$G \ni g \to u_g \in GL(V_i)$$

such that the Fréchet algebras in (13) are isomorphic to

$$C_0^\infty(V_\sigma, D_\sigma;\text{End } V_i)^G \tag{14}$$

$$C_0^\infty(U_\sigma, p_u^{-1}(D_\sigma);\text{End } V_i)^G \tag{15}$$

The G-action is given by

$$g(f)(t) = u_g f(g^{-1}t) u_g^{-1}$$

where we simply lifted the action of $\mathcal{W}_\sigma$ on U_σ to G.

It is clear that the retraction $U_\sigma \to V_\sigma$ induces a diffeotopy equivalence between (14) and (15), so it also induces the desired isomorphism (13). $\Box$

Proof of Theorem 2. Consider the finite collection $\mathcal{L}$ of all irreducible components of $(T^{P_i})^g$, as g runs over $\mathcal{W}_i$. These are all cosets of complex subtori of T^{P_i} and they have nonempty intersections with $T_u^{P_i}$. Extend this to a collection $\{L_j\}_j$ of cosubtori of T^{P_i} by including all irreducible components of intersections of any number of elements of $\mathcal{L}$. Because the action α_i of $\mathcal{W}_i$ on T^{P_i} is algebraic

$$\dim \left(\left(T^{P_i}\right)^g \cap \left(T^{P_i}\right)^w \right) < \max\{\dim \left(T^{P_i}\right)^g , \dim \left(T^{P_i}\right)^w \}$$

if $\alpha_i(w) \neq \alpha_i(g)$. Define $\mathcal{W}_i$-stable submanifolds

$$T_m = \bigcup_{j:\, \dim L_j \leq m} L_j \qquad U_m = T_m \cap U$$

and construct the following ideals

$$\begin{aligned} A_m &= \{h \in I_{i-1}/I_i : \pi(P_i, t, \delta_i)(h) = 0 \text{ if } t \in T_m\} \\ B_m &= C^\infty(U, U_m; \text{End } V_i)^{\mathcal{W}_i} \\ C_m &= C(U, U_m; \text{End } V_i)^{\mathcal{W}_i} \end{aligned}$$

Now we have $A_n = B_n = C_n = 0$ for $n \geq \dim T^{P_i}$ and

$$A_n = I_{i-1}/I_i \quad B_n = C^\infty(U; \text{End } V_i)^{\mathcal{W}_i} \quad C_n = C(U; \text{End } V_i)^{\mathcal{W}_i} \quad \text{for } n < 0$$

Using excision and Lemma 6, it will be sufficient to show that the inclusions

$$A_{m-1}/A_m \to B_{m-1}/B_m$$

induce isomorphisms on HP_*, so let us compute the periodic cyclic homologies of these quotient algebras.

Because T_m is an algebraic subvariety of T^{P_i} the spectrum of A_{m-1}/A_m consists precisely of the irreducible representations of I_{i-1}/I_i with tempered central character in $(P_i, T_m \setminus T_{m-1}, \delta_i)$. We let $r_i(t)$ be the number of $\pi \in \widehat{I_{i-1}/I_i}$ corresponding to (P_i, t, δ_i). From the proof of [DO2, Proposition 6.17] and we see that $r_i(t|t|^s) = r_i(t)\ \forall s > -1$, and $r_i(t|t|^{-1}) = r_i(t)$ if the stabilizers in $\mathcal{W}_i$ of t and $t|t|^{-1}$ are equal. Choose a minimal subset $\{L_{m,k}\}_k$ of $\mathcal{L}$ such that every m-dimensional element of $\mathcal{L}$ is conjugate under $\mathcal{W}_i$ to a $L_{m,k}$. Let $\mathcal{W}_{m,k}$ be the stabilizer of $L_{m,k}$ in $\mathcal{W}_i$ and write $r_k = r_i(t)$ for some $t \in L_{m,k} \setminus T_u^{P_i}$. Then the spectrum of A_{m-1}/A_m is homeomorphic to

$$\begin{aligned} &\bigsqcup_k \bigsqcup_{l=1}^{r_k} (L_{m,k} \setminus T_{m-1}) / \mathcal{W}_{m,k} \qquad = \\ &\bigsqcup_k \bigsqcup_{l=1}^{r_k} (L_{m,k}/\mathcal{W}_{m,k}) \setminus ((L_{m,k} \cap T_{m-1})/\mathcal{W}_{m,k}) \end{aligned}$$

Let us call the left hand side of this expression X_m, and its subset on the right hand side Y_m; these are complex algebraic varieties. Define

$$\mathcal{O}(X_m, Y_m) := \{f \in \mathcal{O}(X_m) : f(Y_m) = 0\}$$

Forgetting about (non)degeneracy, the representation space $V(m,k,l)(t)$ of A_{m-1}/A_m can be chosen independently of $t \in L_{m,k}$. So we can actually embed them all in a single finite dimensional vector space V_m. The resulting maps

$$A_{m-1}/A_m \to \mathcal{O}(X_m, Y_m) \otimes \mathrm{End}(V_m) \leftarrow \mathcal{O}(X_m, Y_m) \tag{16}$$

are spectrum preserving. Thus from [BN, Theorem 8] and [KNS, Theorem 9] we get natural isomorphisms

$$HP_*(A_{m-1}/A_m) \cong HP_*\big(\mathcal{O}(X_m, Y_m)\big) \to \check{H}^*(X_m, Y_m; \mathbb{C}) \tag{17}$$

On the other hand, by [Tou, Théorème IX.4.3] the extension

$$0 \to C^\infty(U, U_m; \mathrm{End}\, V_i) \to C^\infty(U; \mathrm{End}\, V_i) \to C^\infty(U_m; \mathrm{End}\, V_i) \to 0$$

is admissible, and since $\mathcal{W}_i$ is finite the same holds for

$$0 \to B_m \to C^\infty(U; \mathrm{End}\, V_i)^{\mathcal{W}_i} \to C^\infty(U_m; \mathrm{End}\, V_i)^{\mathcal{W}_i} \to 0$$

So by [Sol, Theorem 6] we have isomorphisms

$$HP_*(B_m) \leftarrow K_*(B_m) \otimes_{\mathbb{Z}} \mathbb{C} \to K_*(C_m) \otimes_{\mathbb{Z}} \mathbb{C} \tag{18}$$

By construction the stabilizer in $\mathcal{W}_i$ of $t \in U$ is constant on the connected components of $U_m \setminus U_{m-1}$, and, by the continuity of the intertwiners $\pi(g, P_i, t, \delta_i)$, the same can be said of the type of V_i as a $\langle g \rangle$-representation (on U_m^g). Thus, with $L_{m,k}$ as above, we get

$$\begin{aligned} C_{m-1}/C_m &\cong \bigoplus_k C_0\left((L_{m,k} \cap U_m) \setminus (L_{m,k} \cap U_{m-1})/\mathcal{W}_{m,k}\right) \otimes S_k \\ &= \bigoplus_k C_0\left((L_{m,k} \cap U_m)/\mathcal{W}_{m,k}, (L_{m,k} \cap U_{m-1})/\mathcal{W}_{m,k}\right) \otimes S_k \end{aligned}$$

where the S_k are certain finite dimensional semisimple $\mathbb{C}$-algebras. Because I_{i-1}/I_i is dense in $C^\infty(U; \mathrm{End}\, V_i)^{\mathcal{W}_i}$ we must have $\dim Z(S_k) = r_k$. Consequently

$$\begin{aligned} HP_*(B_{m-1}/B_m) &\cong K_*(C_{m-1}/C_m) \otimes_{\mathbb{Z}} \mathbb{C} \\ &\cong \bigoplus_k \check{H}^*\left((L_{m,k} \cap U_m)/\mathcal{W}_{m,k}, (L_{m,k} \cap U_{m-1})/\mathcal{W}_{m,k}; \mathbb{C}\right)^{r_k} \\ &= \check{H}^*\left(\bigsqcup_k \bigsqcup_{l=1}^{r_k} (L_{m,k} \cap U_m)/\mathcal{W}_{m,k}, \bigsqcup_k \bigsqcup_{l=1}^{r_k} (L_{m,k} \cap U_{m-1})/\mathcal{W}_{m,k}; \mathbb{C} \right) \\ &:= \check{H}^*(X'_m, Y'_m; \mathbb{C}) \end{aligned} \tag{19}$$

where $X'_m := X_m \cap U/\mathcal{W}_i$ and $Y'_m := Y_m \cap U/\mathcal{W}_i$.

It follows from this and the density theorem that the inclusions

$$C_0^\infty(X'_m, Y'_m) \to C_0(X'_m, Y'_m) \cong Z(C_{m-1}/C_m) \to C_{m-1}/C_m \tag{20}$$

induce isomorphisms on K-theory with complex coefficients. From (17) - (20) we construct the commutative diagram

$$\begin{array}{ccccc} HP_*(A_{m-1}/A_m) & \cong & HP_*\big(\mathcal{O}(X_m, Y_m)\big) & \to & \check{H}^*(X_m, Y_m; \mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ HP_*(B_{m-1}/B_m) & \cong & HP_*\big(C_0^\infty(X'_m, Y'_m)\big) & \to & \check{H}^*(X'_m, Y'_m; \mathbb{C}) \end{array} \tag{21}$$

The pair (X'_m, Y'_m) is a deformation retract of (X_m, Y_m), so all maps in this diagram are isomorphisms. Working our way back up, using excision, we find that also

$$HP_*(I_{i-1}/I_i) \to HP_* \left(C^\infty(U; \mathrm{End}\, V_i)^{\mathcal{W}_i} \right) \to HP_*(J_{i-1}/J_i)$$

and finally

$$HP_*(\mathcal{H}) \to HP_*(\mathcal{S})$$

are isomorphisms. $\square$

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