

PSEUDO-REDUCTIVE AND QUASI-REDUCTIVE GROUPS OVER NON-ARCHIMEDEAN LOCAL FIELDS

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ABSTRACT. Among connected linear algebraic groups, quasi-reductive groups generalize pseudo-reductive groups, which in turn form a useful relaxation of the notion of reductivity. We study quasi-reductive groups over non-archimedean local fields, focusing on aspects involving their locally compact topology.

For such groups we construct valuated root data (in the sense of Bruhat–Tits) and we make them act nicely on affine buildings. We prove that they admit Iwasawa and Cartan decompositions, and we construct small compact open subgroups with an Iwahori decomposition.

We also initiate the smooth representation theory of quasi-reductive groups. Among others, we show that their irreducible smooth representations are uniformly admissible, and that all these groups are of type I.

Finally we discuss how much of these results remains valid if we omit the connectedness assumption on our linear algebraic groups.

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INTRODUCTION

Pseudo-reductive and quasi-reductive groups generalize connected reductive groups. These classes of linear algebraic groups were already known to Tits [BoTi] and Springer [Spr], who developed some basic theory. The recent work of Conrad, Gabber and Prasad [CGP, CoPr1, CoPr2] has revived the interest in these groups.

These sources are mainly concerned with the structure of pseudo-reductive groups over arbitrary or separably closed fields. Properties which involve the locally compact topology of pseudo-reductive or quasi-reductive groups over local fields have been investigated far less (apart from reductive groups of course). With this paper we try to narrow that gap.

Let \mathcal{G} be a connected linear algebraic group defined over a field F . The F -unipotent radical $\mathcal{R}_{u,F}(\mathcal{G})$ of \mathcal{G} is the largest connected normal unipotent F -subgroup of \mathcal{G} (so it is contained in the usual unipotent radical of \mathcal{G}). By definition, \mathcal{G} is pseudo-reductive as F -group if $\mathcal{R}_{u,F}(\mathcal{G}) = 1$.

Over a field of characteristic zero, every connected linear algebraic group \mathcal{G} admits a Levi decomposition, that is, it can be written as the semidirect product of its unipotent radical and a reductive subgroup. But this is not always the case over fields F of positive characteristic p . Firstly, suitable Levi factors need not exist, even if F is algebraically closed [CGP, §A.6]. Secondly, the unipotent radical need not be defined over F . For example, suppose that F'/F is an inseparable field extension of degree p and that \mathcal{G}' is a nontrivial connected reductive F' -group. Then restriction of scalars yields a pseudo-reductive F -group $R_{F'/F}(\mathcal{G}')$ which is not reductive [CGP, Proposition 1.1.10].

Nevertheless, \mathcal{G} is always an extension of a pseudo-reductive F -group by a unipotent F -group. Namely, in the short exact sequence

$$(1) \quad 1 \rightarrow \mathcal{R}_{u,F}(\mathcal{G}) \rightarrow \mathcal{G} \rightarrow \mathcal{G}/\mathcal{R}_{u,F}(\mathcal{G}) \rightarrow 1$$

the quotient group $\mathcal{G}/\mathcal{R}_{u,F}(\mathcal{G})$ is easily seen to be pseudo-reductive over F . Thus one can try to understand connected linear algebraic F -groups in terms of pseudo-reductive F -groups and unipotent F -groups. This is considerably easier over perfect fields, for then every pseudo-reductive F -group is in fact reductive and every connected unipotent group is F -split.

Over a general field F , every connected unipotent group is an extension of a F -wound unipotent group by a F -split unipotent group. Such F -wound unipotent groups were studied deeply by Tits (see [Oes] and [CGP, Appendix B]). Although their structure is described quite well, it is safe to say that they are much more complicated than split unipotent groups.

Every connected linear algebraic F -group \mathcal{G} has a maximal normal F -split unipotent subgroup, its F -split unipotent radical $\mathcal{R}_{us,F}(\mathcal{G})$. As in [BrTi2, §1.1.12], we say that \mathcal{G} is quasi-reductive over F if $\mathcal{R}_{us,F}(\mathcal{G}) = 1$. Since $\mathcal{R}_{u,F}(\mathcal{G}) \supset \mathcal{R}_{us,F}(\mathcal{G})$, every pseudo-reductive F -group is also quasi-reductive. If F is perfect, then every connected unipotent F -group is F -split, so every quasi-reductive F -group is also pseudo-reductive and in fact reductive. When F is not perfect, there do exist quasi-reductive F -groups that are not pseudo-reductive, for instance nontrivial F -wound unipotent groups.

In the short exact sequence

$$(2) \quad 1 \rightarrow \mathcal{R}_{us,F}(\mathcal{G}) \rightarrow \mathcal{G} \rightarrow \mathcal{G}/\mathcal{R}_{us,F}(\mathcal{G}) \rightarrow 1$$

the quotient $\mathcal{G}/\mathcal{R}_{us,F}(\mathcal{G})$ is always quasi-reductive [CGP, Corollary B.3.5]. Thus, as an alternative to (1), one can try to analyse connected linear algebraic groups in terms of quasi-reductive groups and split unipotent groups. An advantage of doing this with quasi-reductive groups (as compared to pseudo-reductive groups) is that split unipotent groups are much easier than non-split unipotent groups, and that (2) always splits as a sequence of F -varieties [Spr, Theorem 14.2.6]. For every quasi-reductive F -group \mathcal{G} , the F -unipotent radical $\mathcal{R}_{u,F}(\mathcal{G})$ is F -wound. Thus (1) shows that every quasi-reductive F -group is an extension of a pseudo-reductive F -group by a F -wound unipotent group.

Now we consider a local field F and a quasi-reductive F -group \mathcal{G} . If F is archimedean, then $\mathcal{G}(F)$ is a real reductive group – an object which we do not investigate in this paper. When F is non-archimedean and has characteristic zero, $\mathcal{G}(F)$ is a reductive p -adic group. The structure of such groups, and more generally of reductive groups over discretely valued fields, was studied deeply by Bruhat and Tits [BrTi1, BrTi2, Tit].

The most interesting case for us arises when F is a local field of positive characteristic and \mathcal{G} is quasi-reductive but not reductive. From the work of Borel and Tits (see [BoTi] and [CGP, Appendix C.2]) it is known that \mathcal{G} shares many properties with reductive groups. Summarizing: there exists a maximal F -split torus in \mathcal{G} , an associated root system, a Weyl group and root subgroups, which are F -split unipotent.

Since F is a local field, we can also consider $G = \mathcal{G}(F)$ with the topology coming from the metric on F . Then it becomes a locally compact, totally disconnected, unimodular group (Lemma 1.3). We investigate several properties of G involving this locally compact topology.

Theorem 1. (see Theorem 3.4 and Propostion 3.6)

The group G has a generating root datum with a prolonged valuation in the sense of [BrTi1].

In fact it was already shown in [CGP, §C.2.28] that G has a generating root datum, we provide a prolonged valuation thereof.

Theorem 2. (see Theorem 3.8)

To $\mathcal{G}(F)$ we can associate two affine buildings: a thick building $\mathcal{BT}(\mathcal{G}, F)$ coming from a double Tits system in $\mathcal{G}(F)$ and an extended building $\mathcal{B}(\mathcal{G}, F)$, which is the direct product of $\mathcal{BT}(\mathcal{G}, F)$ and a vector space. Moreover $\mathcal{B}(\mathcal{G}, F)$ is a universal space for proper $\mathcal{G}(F)$ -actions.

Theorem 3. (see Theorem 3.11)

The group G has arbitrarily small compact open subgroups K that admit an Iwahori decomposition with respect to a given maximal F -split torus S in G .

More explicitly, let $P = MU, \bar{P} = M\bar{U}$ be a pair of opposite pseudo-parabolic F -subgroups of G containing $Z_G(S)$. Then the multiplication map

$$(K \cap U) \times (K \cap M) \times (K \cap \bar{U}) \rightarrow K$$

is bijective.

Theorem 4. (see Theorem 3.13)

The group G admits Iwasawa decompositions. More precisely, let x be a special vertex of $\mathcal{B}(\mathcal{G}, F)$. Then the isotropy group G_x is a maximal compact subgroup of G and $G = PG_x$ for every pseudo-parabolic F -subgroup P of G .

Theorem 5. (see Theorem 3.14)

The group G admits Cartan decompositions. More precisely, let S be a maximal F -split torus in G and let x be a special vertex of the apartment of $\mathcal{B}(\mathcal{G}, F)$ associated to S . Then there exists a finitely generated semigroup $A \subset Z_G(S)$ such that $G = \bigsqcup_{a \in A} G_x a G_x$.

In the course of proving the Cartan decomposition, we also establish an Iwahori–Bruhat decomposition. That is, we show that the double cosets with respect to a kind of Iwahori subgroup of G are in bijection with a suitable (extended) affine Weyl group.

The upshot of all the above geometric results is that quasi-reductive F -groups are actually not so different from reductive groups. The main difference lies in the structure of Cartan subgroups and “ F -Cartan subgroups”, i.e. the centralizers of maximal F -split tori in \mathcal{G} . Upon dividing out its center, such a F -Cartan subgroup becomes F -anisotropic (see Lemma 3.3). By [Con, Proposition A.5.7] the F -rational points of any F -anisotropic group form a compact group, and conversely. Thus the F -rational points of any F -Cartan subgroup of \mathcal{G} form a group which is compact modulo its center. But, whereas the structure of F -tori and of anisotropic reductive F -groups is understood very well, this is much less the case for commutative or anisotropic pseudo- or quasi-reductive F -groups.

With this in mind we start to investigate smooth complex representations of quasi-reductive groups. In the representation theory of reductive p -adic groups F -Cartan subgroups are often treated as black boxes, in the sense that one only uses that they are compact modulo a central torus. Therefore most of the elementary representation theory of reductive F -groups should remain valid for quasi-reductive groups. In particular, almost everything in the influential preprint [Cas] should hold for quasi-reductive groups. Using Renard’s treatment [Ren] of Bernstein’s work as main reference, we check that several well-known results involving parabolic induction generalize to quasi-reductive groups. Our main result about G -representations is uniform admissibility:

Theorem 6. (see Theorem 4.9)

Let K be a compact open subgroup of G . There exists a bound $N(G, K) \in \mathbb{N}$ such that $\dim_{\mathbb{C}}(V^K) \leq N(G, K)$ for every irreducible smooth G -representation V . The same holds when V is a topologically irreducible unitary G -representation.

As a consequence, every quasi-reductive group over a non-archimedean local field has type I (Corollary 4.10).

Finally, we check in Section 5 which of the above results remain valid for disconnected linear algebraic F -groups with a quasi-reductive connected component. This turns out to be a nontrivial issue, because such groups do not always possess good maximal compact subgroups. But under mild conditions almost everything we did generalizes.

Several of our proofs are much easier for pseudo-reductive groups than for quasi-reductive groups. In the pseudo-reductive case one can show Theorems 2 – 5 without using Theorem 1. The main technique, relying heavily on [CGP], is reduction to the cases of semisimple groups and of commutative pseudo-reductive groups. We put our results and arguments for pseudo-reductive groups in Section 2. Let us note here that our results about parabolic induction and restriction for pseudo-reductive groups (see Paragraph 4.1) rely only on Section 2.

Only in Section 3 we turn to quasi-reductive groups. A difficulty in the generalization from pseudo-reductive to quasi-reductive arises from taking F -rational points in (1). The resulting sequence

$$1 \rightarrow \mathcal{R}_{u,F}(\mathcal{G})(F) \rightarrow \mathcal{G}(F) \rightarrow (\mathcal{G}/\mathcal{R}_{u,F}(\mathcal{G}))(F)$$

need not be exact at the right hand side, even when \mathcal{G} is quasi-reductive. We analyse the failure of surjectivity in Theorem 3.2, showing that the image of the quasi-reductive group $\mathcal{G}(F)$ in the pseudo-reductive group $(\mathcal{G}/\mathcal{R}_{u,F}(\mathcal{G}))(F)$ has finite index and contains several important subgroups. But still, the non-surjectivity makes it hard to derive certain results for quasi-reductive groups from those for pseudo-reductive groups. To overcome this, we feel forced to appeal to Bruhat–Tits theory. Obviously this approach is rather technical, but once we have established the setup with valuated root data, we get many beautiful results quite easily. Moreover, even for pseudo-reductive groups we can produce more precise results than before.

One motivation for this paper was our desire to prove the Baum–Connes conjecture [BCH] for pseudo-reductive and quasi-reductive groups over local function fields. We prepare specifically for that in Paragraph 5.2. In joint work with K. Li (Münster) we intend to use results from [Laf] and from this paper to verify that conjecture for all linear algebraic groups over non-archimedean local fields, thus complementing the results of [CEN].

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1. NOTATIONS AND PRELIMINARIES

Our default field is called F . Our algebraic groups are assumed to be smooth and affine, unless explicitly stated otherwise. When \mathcal{G} is an algebraic group defined over F , we will often denote its group of F -rational points $\mathcal{G}(F)$ by G (and similarly $\mathcal{T}(F) = T$ etcetera). When F'/F is a finite field extension, we denote the Weil restriction functor by $R_{F'/F}$. Thus $R_{F'/F}(\mathcal{G})$ is a linear algebraic F -group with $\mathcal{R}_{F'/F}(\mathcal{G})(F) = \mathcal{G}(F')$.

We denote the derived group of \mathcal{G} by $\mathcal{D}(\mathcal{G})$. The algebraic groups $\mathcal{R}_{u,F}(\mathcal{G})$ and $\mathcal{R}_{us,F}(\mathcal{G})$ are by definition the F -unipotent radical and the F -split unipotent radical of \mathcal{G} . We say that \mathcal{G} is F -anisotropic if it contains neither the additive group \mathcal{G}_a nor the multiplicative group \mathcal{G}_m as an F -subgroup.

Starting from Paragraph 2.1, F will be local and non-archimedean. Then the group $\mathcal{G}(F)$ is endowed with the topology coming from the metric on F . A phrase like “ S is a maximal F -split torus of \mathcal{G} ” means that $S = \mathcal{S}(F)$, where \mathcal{S} is a maximal F -split torus in \mathcal{G} . By $Z(\mathcal{G})$ we mean the scheme-theoretic center of \mathcal{G} , in contrast to $Z(G)$, which is the center of G as an abstract group. Similarly $Z_{\mathcal{G}}(X)$ will be the scheme-theoretic centralizer of X in \mathcal{G} , whereas $Z_G(X)$ denotes the ordinary centralizer of X in G .

By an affine building we mean those buildings that appear in Bruhat–Tits theory for reductive groups over discretely valued fields [Tit]. In other words, every affine building is a direct product of Euclidean/affine buildings that are simplicial complexes. Typical instances of the latter are the real line and the building coming from an irreducible affine Tits system [BrTi1, §2]. All actions of topological groups on topological spaces (e.g. buildings) are assumed to be continuous.

Let us recall here a few results which do not have a natural place elsewhere in the paper, as they are valid in far greater generality. We will use them repeatedly in Paragraph 2.1.

Proposition 1.1. [Con, Proposition A.5.7]

Let \mathcal{G} be a connected linear algebraic group defined over a local field F . Then \mathcal{G} is F -anisotropic if and only if $\mathcal{G}(F)$ is compact.

Proposition 1.2. *Let \mathcal{G} be a connected linear algebraic group defined over a local field F , and let \mathcal{N} be a normal F -subgroup.*

- (a) *The map $\mathcal{G}(F) \rightarrow (\mathcal{G}/\mathcal{N})(F)$ is a submersion of F -analytic manifolds. In particular this map is open and smooth.*
- (b) *The map $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{N}$ sends maximal F -split tori onto maximal F -split tori.*
- (c) *Suppose moreover that \mathcal{G}/\mathcal{N} is quasi-reductive over F (e.g. pseudo-reductive over F). Then the image of $\mathcal{G}(F)$ in $(\mathcal{G}/\mathcal{N})(F)$ is an open subgroup of finite index.*

Proof. (a) According to [Spr, Proposition 5.5.10 and Corollary 12.2.2], \mathcal{G}/\mathcal{N} is again a connected linear algebraic F -group. By [Spr, Corollary 5.5.4] the map $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{N}$ is a separable morphism of F -varieties, and by [Spr, Theorem 4.3.7] its differential at any point of \mathcal{G} is surjective. Hence it is an F -analytic submersion.

(b) Let \mathcal{S} be a maximal F -split torus in \mathcal{G} . Its image \mathcal{S}_N in \mathcal{G}/\mathcal{N} is a maximal F -split torus over there [Bor, Theorem 22.6.ii]. We note that by part (a) this result works when F is not perfect – see also [CGP, Lemma C.2.31].

(c) By part (a) the image of $\mathcal{G}(F)$ in $(\mathcal{G}/\mathcal{N})(F)$ is open. Let \mathcal{S} and \mathcal{S}_N be as above. By [Con, Lemma 4.1.2.i] the image of the map $\mathcal{S}(F) \rightarrow \mathcal{S}_N(F)$ is open and has finite index. These facts allow us to conclude with [Con, Proposition 4.1.9]. \square

For any smooth linear algebraic group \mathcal{G} , the modulus character $\delta_{\mathcal{G}} : \mathcal{G} \rightarrow GL_1$ is defined as the determinant of the adjoint representation of \mathcal{G} on its Lie algebra. Equivalently, $\delta_{\mathcal{G}}$ describes the action of \mathcal{G} on the one-dimensional space of top-degree left-invariant differential forms on \mathcal{G} .

The following proof was kindly communicated to the author by Gopal Prasad.

Lemma 1.3. *The modulus character of any quasi-reductive F -group is trivial.*

Proof. Let F_s be a separable closure of F . By [CGP, Proposition 1.1.9]

$$\mathcal{R}_{u, F_s}(F_s \otimes_F \mathcal{G}) = F_s \otimes_F \mathcal{R}_{u, F}(\mathcal{G}),$$

and by [CGP, Proposition B.3.2] right hand side is F_s -wound. Hence $F_s \otimes_F \mathcal{G}$ is quasi-reductive, and we may assume that F is separably closed.

Let \mathcal{T} be any maximal F -torus in \mathcal{G} , and let $\mathcal{C} = Z_{\mathcal{G}}(\mathcal{T})$ be the associated Cartan F -subgroup of \mathcal{G} . By [CGP, Proposition 1.2.6] $\mathcal{G} = \mathcal{CD}(\mathcal{G})$. Clearly the algebraic character $\delta_{\mathcal{G}}$ is trivial on the derived group $\mathcal{D}(\mathcal{G})$, so it is enough to prove that $\delta_{\mathcal{G}}$ is trivial on \mathcal{C} .

Since \mathcal{G} is quasi-reductive, the decomposition of its Lie algebra as \mathcal{T} -representation yields a root system Φ [CGP, Theorem C.2.15]. For any root $\alpha \in \Phi$, it is shown in [CGP, pp. 601–602] that $N_{\mathcal{G}}(\mathcal{T})(F)$ contains an element n_{α} which acts on $\Phi(\mathcal{G}, \mathcal{T})$ as the reflection associated to α . In particular conjugation by n_{α} interchanges the root subgroups \mathcal{U}_{α} and $\mathcal{U}_{-\alpha}$, which therefore have the same dimension. For $t \in \mathcal{T}(F)$ one computes

$$\begin{aligned} \delta_{\mathcal{G}}(t) &= \det(\text{Ad}(t) : \text{Lie}(\mathcal{G}) \rightarrow \text{Lie}(\mathcal{G})) \\ &= \prod_{\alpha \in \Phi} \det(\text{Ad}(t) : \text{Lie}(\mathcal{U}_{\alpha}) \rightarrow \text{Lie}(\mathcal{U}_{\alpha})) = \prod_{\alpha \in \Phi} \alpha(t)^{\dim U_{\alpha}} \\ &= \prod_{\alpha \in \Phi/\{\pm 1\}} \alpha(t)^{\dim U_{\alpha}} (-\alpha)(t)^{\dim U_{-\alpha}} \\ &= \prod_{\alpha \in \Phi/\{\pm 1\}} \alpha(t)^{\dim U_{\alpha}} \alpha(t)^{-\dim U_{\alpha}} = 1 \end{aligned}$$

Thus $\delta_{\mathcal{G}}$ restricted to \mathcal{T} is trivial and $\delta_{\mathcal{G}}$ restricted to \mathcal{C} factors through the group \mathcal{C}/\mathcal{T} . By the maximality of \mathcal{T} , \mathcal{C}/\mathcal{T} has no nontrivial tori, which means that it is unipotent. But a unipotent group does not admit nontrivial algebraic characters, so $\delta_{\mathcal{G}}$ is trivial on \mathcal{C} . \square

2. PSEUDO-REDUCTIVE GROUPS

Based on the work of Tits, Conrad, Gabber and Prasad, we give a rough description of the structure of pseudo-reductive groups. In [CGP] and [CoPr1] pseudo-reductive groups are classified in terms of reductive groups and commutative pseudo-reductive groups. Among them is a class of “standard pseudo-reductive” groups, which turns out to exhaust all possibilities if $\text{char}(F) \notin \{2, 3\}$. We now recall the standard construction.

Let F_i be a finite extension of a field F . (The interesting case here is when F_i/F is inseparable.) Let \mathcal{G}_i be an absolutely simple and simply connected F_i -group, and let \mathcal{T}_i be a maximal F_i -torus of \mathcal{G}_i . Then $R_{F_i/F}(\mathcal{G}_i)$ is a pseudo-reductive F -group [CGP, Proposition 1.1.10] and by [CGP, Proposition A.5.15]

$$(3) \quad R_{F_i/F}(\mathcal{T}_i) \text{ is the centralizer of a maximal } F\text{-torus in } R_{F_i/F}(\mathcal{G}_i).$$

Furthermore $R_{F_i/F}(\mathcal{G}_i)$ is reductive if and only if F_i/F is separable. Let i run through a finite index set I_s and put

$$(4) \quad \mathcal{G}' = \prod_{i \in I_s} R_{F_i/F}(\mathcal{G}_i), \quad \mathcal{T}' = \prod_{i \in I_s} R_{F_i/F}(\mathcal{T}_i).$$

Let \mathcal{C} be a commutative pseudo-reductive F -group, endowed with F -homomorphisms

$$(5) \quad \mathcal{T}' \xrightarrow{\phi_{\mathcal{C}}} \mathcal{C} \xrightarrow{\psi_{\mathcal{C}}} \prod_{i \in I_s} R_{F_i/F}(\mathcal{T}_i/Z(\mathcal{G}_i))$$

such that $\psi_{\mathcal{C}} \circ \phi_{\mathcal{C}} : \mathcal{T}' \rightarrow \prod_{i \in I_s} R_{F_i/F}(\mathcal{T}_i/Z(\mathcal{G}_i))$ is the canonical map. (We warn that $\psi_{\mathcal{C}} \circ \phi_{\mathcal{C}}$ need not be surjective.) The group $\prod_{i \in I_s} R_{F_i/F}(\mathcal{T}_i/Z(\mathcal{G}_i))$ acts on \mathcal{G}' , by $R_{F_i/F}$ applied to the conjugation action of $\mathcal{T}_i/Z(\mathcal{G}_i)$ on \mathcal{G}_i . Hence we can use $\psi_{\mathcal{C}}$ to build a semi-direct product $\mathcal{G}' \rtimes \mathcal{C}$. The map

$$\begin{aligned} \alpha : \mathcal{T}' &\rightarrow \mathcal{G}' \rtimes \mathcal{C}, \\ t &\mapsto (t^{-1}, \phi_{\mathcal{C}}(t)) \end{aligned}$$

provides an embedding of \mathcal{T}' as a central F -subgroup $\mathcal{G}' \rtimes \mathcal{C}$. Then

$$(6) \quad \mathcal{G} := (\mathcal{G}' \rtimes \mathcal{C})/\alpha(\mathcal{T}')$$

is a pseudo-reductive F -group [CGP, Proposition 1.4.3]. Every standard pseudo-reductive group arises in this way from data $(\mathcal{G}', \prod_{i \in I_s} F_i/F, \mathcal{T}', \mathcal{C})$, which are essentially unique [CGP, §4.2].

From now on, \mathcal{G} is any pseudo-reductive F -group and we assume that $[F : F^2] \leq 2$ if $\text{char}(F) = 2$. According to [CGP, Theorem 10.2.1], \mathcal{G} factors as

$$(7) \quad \mathcal{G} = \mathcal{G}_{nr} \times \mathcal{G}_r,$$

where \mathcal{G}_{nr} is “totally non-reduced” and \mathcal{G}_r has a reduced root system (when computed over a separable closure of F).

By [CGP, Theorem 10.2.1.(2)] \mathcal{G}_r admits a generalized standard presentation, similar to (6). First let \mathcal{G}_i, F_i and \mathcal{T}_i be as above, so with \mathcal{G}_i an absolutely simple and simply connected F_i -group and \mathcal{T}_i a maximal F_i -torus in \mathcal{G}_i . Next we allow more possible \mathcal{G}_i , indexed by a new set I_e . Namely, they may also be “basic exotic pseudo-reductive”, as in [CGP, Definition 7.2.6]. The group $R_{F_i/F}(\mathcal{G}_i)$ is then called “exotic pseudo-reductive”. Such groups exist only if the characteristic p of F is 2 or 3. For every such index $i \in I_e$, let \mathcal{T}_i be the centralizer of a maximal F_i -torus in \mathcal{G}_i . The construction in [CGP, §7.2] entails that \mathcal{T}_i is commutative. Moreover, by [CGP, Propositions 7.1.5 and 7.3.3] there exist a semisimple simply connected F_i -group $\overline{\mathcal{G}}_i$ and a canonical homomorphism of F -groups

$$(8) \quad (\mathcal{R}_{F_i/F} \mathcal{G}_i)(F) = \mathcal{G}_i(F_i) \longrightarrow \overline{\mathcal{G}}_i(F_i) = (\mathcal{R}_{F_i/F} \overline{\mathcal{G}}_i)(F),$$

which is a homeomorphism if F_i is a local field. In that case (8) provides a bijection between the collections of maximal F -split tori on both sides [CGP, Corollary 7.3.4]. Since every pseudo-parabolic F -subgroup is determined by a cocharacter with values in a maximal F -split torus, (8) also induces a bijection between the pseudo-parabolic F -subgroups on both sides.

Using all these objects we put

$$(9) \quad \mathcal{T}' = \prod_{i \in I_s \cup I_e} R_{F_i/F}(\mathcal{T}_i), \quad \mathcal{G}'_r = \prod_{i \in I_s \cup I_e} R_{F_i/F}(\mathcal{G}_i).$$

Here \mathcal{T}' is a Cartan subgroup of \mathcal{G}'_r . Although this looks the same as in (4), there are more possibilities when $\text{char}(F) \in \{2, 3\}$. The remainder of the construction of reduced pseudo-reductive group is the same as before, we write it down for reference. We need a commutative pseudo-reductive F -group \mathcal{C} and F -homomorphisms

$$(10) \quad \mathcal{T}' \xrightarrow{\phi_{\mathcal{C}}} \mathcal{C} \xrightarrow{\psi_{\mathcal{C}}} \prod_{i \in I_s \cup I_e} R_{F_i/F}(Z_{\mathcal{G}_i, \mathcal{T}_i}),$$

where $Z_{\mathcal{G}_i, \mathcal{T}_i}$ is an algebraic group of automorphisms of \mathcal{G}_i which restrict to the identity on \mathcal{T}_i [CGP, p. 427]. When F_i is local and $\overline{\mathcal{T}}_i$ denotes the image of \mathcal{T}_i in $\overline{\mathcal{G}}_i$,

(8) implies that

$$Z_{\mathcal{G}_i, \mathcal{T}_i}(F_i) \cong (\overline{\mathcal{T}}_i / Z(\overline{\mathcal{G}}_i))(F_i).$$

With (10) we build the semidirect product $\mathcal{G}'_r \rtimes \mathcal{C}$. Define

$$\alpha : \mathcal{T}' \rightarrow \mathcal{T}' \times \mathcal{C}, \quad t \mapsto (t^{-1}, \phi_{\mathcal{C}}(t)).$$

Then $\mathcal{T}' \times \mathcal{C} = \alpha(\mathcal{T}') \times \mathcal{C}$ is the centralizer of \mathcal{T}' in $\mathcal{G}'_r \rtimes \mathcal{C}$. Now we define

$$(11) \quad \mathcal{G}_r = (\mathcal{G}'_r \rtimes \mathcal{C}) / \alpha(\mathcal{T}').$$

Here the image of \mathcal{G}'_r in \mathcal{G}_r is the derived group $\mathcal{D}(\mathcal{G}_r)$ [CGP, Proposition 4.1.4.(1) and Remark 10.1.11]. Furthermore \mathcal{C} embeds naturally in \mathcal{G}_r as a Cartan subgroup, namely the centralizer of a maximal F -torus in \mathcal{C} .

Totally non-reduced pseudo-reductive groups exist only when $\text{char}(F) = 2$. Their classification is much easier if $[F : F^2] = 2$, so we assume that now. By [CGP, Proposition 10.1.4]

$$(12) \quad \mathcal{G}_{nr} = \prod_{i \in I_{nr}} R_{F'_i/F}(\mathcal{G}'_{nr,i}),$$

where each $\mathcal{G}'_{nr,i}$ is a ‘‘basic non-reduced pseudo-simple’’ group over a finite extension F'_i of F . By [CGP, 10.1.2 and Theorem 9.9.3] there exist $n_i \in \mathbb{N}$ and quadratic inseparable extensions F''_i/F'_i such that the (solvable) radical $\mathcal{R}(\mathcal{G}'_{nr,i})$ is defined over F''_i and

$$\mathcal{G}'_{nr,i} / \mathcal{R}(\mathcal{G}'_{nr,i}) \cong Sp_{2n_i} \text{ as } F''_i\text{-groups.}$$

This gives rise to a canonical homomorphism of F'_i -groups

$$(13) \quad \mathcal{G}'_{nr,i} \rightarrow R_{F''_i/F'_i}(Sp_{2n_i}).$$

If $[F'_i : (F'_i)^2] = 2$ then by [CGP, Proposition 9.9.2] the F'_i -group homomorphism

$$(14) \quad \mathcal{G}'_{nr,i}(F'_i) \rightarrow Sp_{2n_i}(F''_i) \text{ is bijective.}$$

Proposition 1.2 implies that (14) induces a bijection between the maximal F -split tori on both sides. Furthermore [CGP, Proposition 11.4.4] says that (13) provides a bijection between the pseudo-parabolic F'_i -subgroups of $\mathcal{G}'_{nr,i}$ and the pseudo-parabolic F''_i -subgroups of Sp_{2n_i} .

The shape of a general pseudo-reductive F -group becomes [CGP, Theorem 10.2.1]

$$(15) \quad \mathcal{G} = \mathcal{G}_{nr} \times (\mathcal{G}'_r \rtimes \mathcal{C}) / \alpha(\mathcal{T}') = (\mathcal{G}_{nr} \times \mathcal{G}'_r \rtimes \mathcal{C}) / \alpha(\mathcal{T}').$$

Since \mathcal{G}_{nr} is perfect [CGP, Definition 10.1.1], the image of $\mathcal{G}_{nr} \times \mathcal{G}'_r$ in \mathcal{G} equals the derived group $\mathcal{D}(\mathcal{G})$.

2.1. Structure over non-archimedean local fields.

The above holds over any field, from now on we specialize to a non-archimedean local field F . Like every variety over such a field, $\mathcal{G}(F)$ is locally compact and totally disconnected. In particular it has a modular function $\delta_{\mathcal{G}(F)}$. We recall from [PIRa, p. 167–168] that the modular function of any linear algebraic F -group $\mathcal{H}(F)$ can be computed as the norm of its modulus character:

$$(16) \quad \delta_{\mathcal{H}(F)}(h) = \left\| \det(\text{Ad}(h) : \text{Lie}(\mathcal{H}) \rightarrow \text{Lie}(\mathcal{H})) \right\|_F.$$

In view of Lemma 1.3, this means that $\mathcal{G}(F)$ is unimodular.

We write $\mathcal{G}' = \mathcal{G}_{nr} \times \mathcal{G}'_r$ and

$$(17) \quad \mathcal{G}' \rtimes \mathcal{C} = \mathcal{G}'(F) \rtimes \mathcal{C}(F) = \prod_{i \in I_s \cup I_e} \mathcal{G}'_i(F_i) \rtimes \mathcal{C}(F).$$

By [CGP, Propositions 4.1.4.(3) and 10.2.2.(3)] all maximal F -tori of \mathcal{G}'_r are equally good for the standard presentation, which means that we may choose any maximal F_i -torus \mathcal{T}_i in \mathcal{G}_i . Thus we may and will assume that every \mathcal{T}_i contains a maximal F_i -split torus of \mathcal{G}_i .

Let \mathcal{S} be the unique maximal F -split torus in \mathcal{C} . The image of $\mathcal{S}(F)$ in $\mathcal{G}_r(F)$ contains the image of $\mathcal{T}_i(F)$ in $\mathcal{G}_r(F)$, for each $i \in I_s \cup I_e$. By the assumption on \mathcal{T}_i and by [CGP, Proposition A.5.15], the image of \mathcal{S} in \mathcal{G}_r is a maximal F -split torus in there. Let \mathcal{S}_\emptyset be a maximal F -split torus in \mathcal{G} containing \mathcal{S} . By (15) it factorizes as

$$(18) \quad \mathcal{S}_\emptyset = (\mathcal{S}_\emptyset \cap \mathcal{G}_{nr}) \times \mathcal{S}.$$

By [CGP, Proposition 2.2.12] the preimage of any maximal F -torus of \mathcal{G}_r under $\mathcal{G}'_r \rtimes \mathcal{C} \rightarrow \mathcal{G}_r$ contains a unique maximal F -torus of $\mathcal{G}'_r \rtimes \mathcal{C}$. Hence the preimage of $\mathcal{S} \subset \mathcal{C} \subset \mathcal{G}_r$ in $\mathcal{G}'_r \rtimes \mathcal{C}$ contains a unique maximal F -split torus, say $\mathcal{S}' \times \mathcal{S}$. Then

$$(19) \quad \mathcal{S}' \subset \mathcal{G}'_r \quad \text{and} \quad (\mathcal{S}_\emptyset \cap \mathcal{G}_{nr}) \times (\mathcal{S}' \times \mathcal{S}) \subset \mathcal{G}_{nr} \times \mathcal{G}'_r \rtimes \mathcal{C}$$

are maximal F -split tori.

Recall that $[F : F^p] = p$ for every non-archimedean local field of residual characteristic p . For all our purposes, i.e. for groups over local fields, (8) shows that we may replace every exotic pseudo-reductive group \mathcal{G}_i by $\overline{\mathcal{G}}_i$ and \mathcal{T}_i by its image $\overline{\mathcal{T}}_i$ in $\overline{\mathcal{G}}_i$. Then [CGP, Theorem 1.3.9] allows us to replace $Z_{\mathcal{G}_i, \mathcal{T}_i}$ in (10) by $\overline{\mathcal{T}}_i/Z(\overline{\mathcal{G}}_i)$, like in (5). Similarly (14) shows that we may replace every non-reduced pseudo-simple group $\mathcal{G}'_{nr, i}$ by $R_{F''_i/F'_i}(Sp_{2n_i})$. In particular we obtain

$$(20) \quad G' = \mathcal{G}_{nr}(F) \times \mathcal{G}'_r(F) = \prod_{i \in I_{nr}} Sp_{2n_i}(F''_i) \times \prod_{i \in I_s \cup I_e} \mathcal{G}_i(F_i),$$

where we may read $\overline{\mathcal{G}}_i(F_i)$ for an exotic pseudo-reductive group $\mathcal{G}_i(F_i)$. By the remarks after (8) and (14), these replacements preserve the notions of maximal F -split torus and of pseudo-parabolic F -subgroup.

Lemma 2.1. (a) \mathcal{C}/\mathcal{S} is F -anisotropic and $\mathcal{C}(F)/\mathcal{S}(F)$ is compact.

(b) $Z_{\mathcal{G}}(\mathcal{S}_\emptyset)/\mathcal{S}_\emptyset$ is F -anisotropic and $Z_{\mathcal{G}}(\mathcal{S}_\emptyset)(F)/\mathcal{S}_\emptyset(F)$ is compact.

Proof. (a) Let $\mathcal{T}_{\mathcal{C}}$ be the maximal F -torus in \mathcal{C} . Then $\mathcal{C}/\mathcal{T}_{\mathcal{C}}$ is a commutative F -group without nontrivial F -tori, so by [Spr, Theorem 13.3.6] it has no nontrivial tori, which means that it is unipotent. If $\mathcal{C}/\mathcal{T}_{\mathcal{C}}$ would contain any F -split unipotent subgroup, then by [Con, Lemma 4.1.4] so would \mathcal{C} . That would contradict the pseudo-reductivity of \mathcal{C} , so $\mathcal{C}/\mathcal{T}_{\mathcal{C}}$ is a F -wound unipotent group. Since \mathcal{S} is the maximal F -split subtorus of $\mathcal{T}_{\mathcal{C}}$, $\mathcal{T}_{\mathcal{C}}/\mathcal{S}$ and \mathcal{C}/\mathcal{S} are F -anisotropic. Now Proposition 1.1 tells us that $(\mathcal{C}/\mathcal{S})(F)$ is compact. Since \mathcal{S} is F -split, its first Galois cohomology group is trivial and $\mathcal{C}(F)/\mathcal{S}(F)$ is isomorphic to $(\mathcal{C}/\mathcal{S})(F)$ as topological groups.

(b) In view of (15), we first ignore \mathcal{G}_{nr} and focus on \mathcal{G}_r . The preimage of $Z_{\mathcal{G}_r}(\mathcal{S})$ is $Z_{\mathcal{G}'_r}(\mathcal{S}') \rtimes \mathcal{C}$. By [CGP, Proposition A.5.15.ii] there are (unique) maximal F_i -split tori \mathcal{S}_i in \mathcal{G}_i that \mathcal{S}' is contained in $\prod_i R_{F_i/F}(\mathcal{S}_i)$. Since F^\times is Zariski-dense and cocompact in F_i^\times , $\mathcal{S}'(F)$ is Zariski-dense and cocompact in $\mathcal{S}_i(F_i)$, and $R_{F_i/F}(\mathcal{S}_i)/\mathcal{S}'$ is F -anisotropic. Hence

$$(21) \quad Z_{\mathcal{G}'_r}(\mathcal{S}')(F) = \prod_{i \in I_s \cup I_e} R_{F_i/F}(Z_{\mathcal{G}_i}(\mathcal{S}_i))(F).$$

For the standard factors of \mathcal{G}'_r , the reductivity of \mathcal{G}_i and the maximality of \mathcal{S}_i imply that each $Z_{\mathcal{G}_i}(\mathcal{S}_i)/\mathcal{S}_i$ is F_i -anisotropic. For the exotic factors \mathcal{G}_i one first observes

that \mathcal{S}_i is mapped to a maximal F_i -split torus $\overline{\mathcal{S}}_i$ of $\overline{\mathcal{G}}_i$ by (8). Then $Z_{\overline{\mathcal{G}}_i}(\overline{\mathcal{S}}_i)/\overline{\mathcal{S}}_i$ is F_i -anisotropic because $\overline{\mathcal{G}}_i$ is reductive. Proposition 1.1 says that $(Z_{\overline{\mathcal{G}}_i}(\overline{\mathcal{S}}_i)/\overline{\mathcal{S}}_i)(F_i)$ is compact. Since $\overline{\mathcal{S}}_i$ is F_i -split this is homeomorphic to $Z_{\overline{\mathcal{G}}_i}(\overline{\mathcal{S}}_i)(F_i)/\overline{\mathcal{S}}_i(F_i)$, which by (8) is homeomorphic to $Z_{\mathcal{G}_i}(\mathcal{S}_i)(F_i)/\mathcal{S}_i(F_i)$ as topological groups. In particular these groups are compact. By Proposition 1.2.c also $(Z_{\mathcal{G}_i}(\mathcal{S}_i)/\mathcal{S}_i)(F_i)$ is compact, so by Proposition 1.1 $Z_{\mathcal{G}_i}(\mathcal{S}_i)/\mathcal{S}_i$ is F_i -anisotropic – just as for the standard factors.

From the short exact sequence

$$\begin{aligned} 1 \rightarrow \prod_{i \in I_s \cup I_e} R_{F_i/F}(\mathcal{S}_i)/\mathcal{S}' &\rightarrow \prod_{i \in I_s \cup I_e} R_{F_i/F}(Z_{\mathcal{G}_i}(\mathcal{S}_i))/\mathcal{S}' \\ &\rightarrow \prod_{i \in I_s \cup I_e} R_{F_i/F}(Z_{\mathcal{G}_i}(\mathcal{S}_i))/R_{F_i/F}(\mathcal{S}_i) \rightarrow 1 \end{aligned}$$

we see that the middle term is F -anisotropic. By (21) so is $Z_{\mathcal{G}'_r}(\mathcal{S}')/\mathcal{S}'$. Then

$$1 \rightarrow Z_{\mathcal{G}'_r}(\mathcal{S}')/\mathcal{S}' \rightarrow (Z_{\mathcal{G}'_r}(\mathcal{S}') \rtimes \mathcal{C})/(\mathcal{S}' \times \mathcal{S}) \rightarrow \mathcal{C}/\mathcal{S} \rightarrow 1$$

shows that also $(Z_{\mathcal{G}'_r}(\mathcal{S}') \rtimes \mathcal{C})/(\mathcal{S}' \times \mathcal{S})$ is F -anisotropic. Dividing out the central subgroup $\alpha(\mathcal{T}')$, we find that $Z_{\mathcal{G}_r}(\mathcal{S})/\mathcal{S}$ is also F -anisotropic. Now Proposition 1.1 tells us that $(Z_{\mathcal{G}_r}(\mathcal{S})/\mathcal{S})(F)$ is compact. Since \mathcal{S} is F -split the group $Z_{\mathcal{G}_r}(\mathcal{S})(F)/\mathcal{S}(F)$ is homeomorphic to that, and also compact.

The totally non-reduced factors of \mathcal{G}' (i.e. those of \mathcal{G}_{nr}) can be handled in the same way as the exotic factors, using (14). \square

In general the quotient map $\mathcal{G}' \rtimes \mathcal{C} \rightarrow \mathcal{G}$ need not be surjective on F -rational points, but we can get quite close:

Lemma 2.2. (a) *The natural map $(\mathcal{G}' \rtimes \mathcal{C})(F)/\alpha(\mathcal{T}')(F) \rightarrow \mathcal{G}(F)$ is smooth, injective and open. Its image has finite index in $\mathcal{G}(F)$.*

(b) *Let $\tilde{\mathcal{G}}(F)$ be the image of $\mathcal{G}'(F)$ in $\mathcal{G}(F)$. Then*

$$\tilde{\mathcal{G}}(F)Z_{\mathcal{G}_r}(\mathcal{S})(F) = Z_{\mathcal{G}_r}(\mathcal{S})(F)\tilde{\mathcal{G}}(F) = \mathcal{G}(F).$$

Proof. (a) Since α is an embedding, $\alpha(\mathcal{T}'(F)) = \alpha(\mathcal{T}')(F)$, which implies the injectivity. The remaining properties were shown in Proposition 1.2.

(b) By definition $\mathcal{G}_{nr}(F)$ is a direct factor on both sides, so it suffices to consider the map

$$(22) \quad (\mathcal{G}'_r \rtimes \mathcal{C})(F)/\alpha(\mathcal{T}')(F) \rightarrow \mathcal{G}_r(F).$$

Let $N_{\mathcal{G}_r}(\mathcal{S})$ be the normalizer of the maximal F -split torus \mathcal{S} in \mathcal{G}_r , and let $W = N_{\mathcal{G}_r}(\mathcal{S})/Z_{\mathcal{G}_r}(\mathcal{S})$ be the associated Weyl group. Let \mathcal{U} be the F -split unipotent radical of a minimal pseudo-parabolic subgroup \mathcal{P} of \mathcal{G}_r containing $Z_{\mathcal{G}_r}(\mathcal{S})$. The Bruhat decomposition of $\mathcal{G}_r(F)$ (see [CGP, Theorem C.2.8] or Theorem 5.3) guarantees that any element $g \in \mathcal{G}_r(F)$ can be written as $zu\tilde{w}u'$, where $z \in Z_{\mathcal{G}_r}(\mathcal{S})(F)$, $u, u' \in \mathcal{U}(F)$ and $\tilde{w} \in N_{\mathcal{G}_r}(\mathcal{S})$ is any representative for an element $w \in W$.

Since $\alpha(\mathcal{T}')$ is central in $\mathcal{G}'_r \rtimes \mathcal{C}$, the groups $\mathcal{G}'_r \rtimes \mathcal{C}$ and \mathcal{G}_r have the same F -root system, and the pre-image \mathcal{P}' of \mathcal{P} in $\mathcal{G}'_r \rtimes \mathcal{C}$ is a minimal pseudo-parabolic subgroup over there [CGP, Proposition 2.2.12]. Then (22) induces an isomorphism of algebraic groups $\mathcal{U}' \rightarrow \mathcal{U}$, where \mathcal{U}' denotes the F -split unipotent radical of \mathcal{P}' . By [CGP, Proposition C.2.10] every element of W can be represented by an element of $\mathcal{G}'(F)$, let us choose \tilde{w} in this way. Then u, u' and \tilde{w} lie in $\tilde{\mathcal{G}}(F)$, so $g = zuwu' \in Z_{\mathcal{G}_r}(\mathcal{S})(F)\tilde{\mathcal{G}}(F)$. \square

Let $\lambda : GL_1 \rightarrow \mathcal{S}_\emptyset$ be a cocharacter defined over F . It gives rise to a pseudo-parabolic F -subgroup $\mathcal{P}_\mathcal{G}(\lambda)$ [CGP, §2.1], with F -split unipotent radical $U_\mathcal{G}(\lambda)$ [CGP, Proposition 2.1.10]. The group $\mathcal{Z}_\mathcal{G}(\lambda) = \mathcal{P}_\mathcal{G}(\lambda) \cap \mathcal{P}_\mathcal{G}(\lambda^{-1})$ is again pseudo-reductive over F [CGP, Corollary 2.2.5]. It satisfies $\mathcal{P}_\mathcal{G}(\lambda) = \mathcal{Z}_\mathcal{G}(\lambda) \times U_\mathcal{G}(\lambda)$ and

$$(23) \quad \text{Lie}(\mathcal{G}) = \text{Lie}(U_\mathcal{G}(\lambda)) \oplus \text{Lie}(\mathcal{Z}_\mathcal{G}(\lambda)) \oplus \text{Lie}(U_\mathcal{G}(\lambda^{-1})).$$

Then every pseudo-parabolic F -subgroup containing $Z_\mathcal{G}(\mathcal{S}_\emptyset)$ is of the above form $\mathcal{P}_\mathcal{G}(\lambda)$, see [CGP, Theorem C.2.15]. Since \mathcal{G} is pseudo-reductive over F and \mathcal{S}_\emptyset is maximal F -split torus, the adjoint action of \mathcal{S}_\emptyset on the Lie algebra of \mathcal{G} gives rise to a root system $\Phi(\mathcal{G}, \mathcal{S}_\emptyset)$, whose set of reduced roots we denote by $\Phi(\mathcal{G}, \mathcal{S}_\emptyset)_{\text{red}}$. For every reduced root $\beta \in \Phi(\mathcal{G}, \mathcal{S}_\emptyset)$ there is a root subgroup U_β , which by [CGP, Proposition 2.1.10] is a F -split unipotent group. Choose any ordering of the set of reduced roots $\Phi(U_\mathcal{G}(\lambda), \mathcal{S}_\emptyset)_{\text{red}}$ appearing in the Lie algebra of $U_\mathcal{G}(\lambda)$. Then the multiplication map

$$(24) \quad \prod_{\beta \in \Phi(U_\mathcal{G}(\lambda), \mathcal{S}_\emptyset)_{\text{red}}} U_\beta \rightarrow U_\mathcal{G}(\lambda)$$

is a bijection [CGP, Proposition C.2.26].

Proposition 2.3. *There exists a decreasing sequence $(K_n)_{n=0}^\infty$ of compact open subgroups of $\mathcal{G}(F)$ such that:*

- $\{K_n : n \geq 0\}$ is a neighborhood basis of 1 in $\mathcal{G}(F)$;
- each K_n is a normal subgroup of K_0 ;
- each K_n admits an Iwahori decomposition. That is, for every pseudo-parabolic F -subgroup $\mathcal{P}_\mathcal{G}(\lambda)$ of \mathcal{G} which contains $Z_\mathcal{G}(\mathcal{S}_\emptyset)$, the multiplication map

$$(K_n \cap U_\mathcal{G}(\lambda)(F)) \times (K_n \cap Z_\mathcal{G}(\lambda)(F)) \times (K_n \cap U_\mathcal{G}(\lambda^{-1})(F)) \rightarrow K_n$$

is a homeomorphism.

Proof. We use an extension of the method of [Del]. We fix an ordering on $\Phi(\mathcal{G}, \mathcal{S}_\emptyset)_{\text{red}} \cup \{0\}$, such that the positive roots (with respect to \mathcal{P}_\emptyset) come first, then 0 and finally the negative roots. Writing $U_0 = Z_\mathcal{G}(\mathcal{S}_\emptyset)$, the root space decomposition becomes

$$(25) \quad \text{Lie}(\mathcal{G}) = \bigoplus_{\alpha \in \Phi(\mathcal{G}, \mathcal{S}_\emptyset)_{\text{red}} \cup \{0\}} \text{Lie}(U_\alpha).$$

Moreover the properties of the open Bruhat cell [CGP, Corollary 3.3.6 and p. 624] imply that the multiplication map

$$(26) \quad \prod_{\alpha \in \Phi(\mathcal{G}, \mathcal{S}_\emptyset)_{\text{red}} \cup \{0\}} U_\alpha(F) \rightarrow \mathcal{G}(F)$$

is an open immersion.

Let \mathcal{O}_F be the ring of integers of F and $\omega_F \in \mathcal{O}_F$ a uniformizer. Choose an \mathcal{O}_F -lattice L in $\text{Lie}(\mathcal{G})(F)$ such that in (25) L is the direct sum of the \mathcal{O}_F -modules

$$L_\alpha := L \cap \text{Lie}(U_\alpha)(F) \quad \text{with} \quad \alpha \in \Phi(\mathcal{G}, \mathcal{S})_{\text{red}} \cup \{0\}.$$

Choose \mathcal{O}_F -bases of the L_α , and combine them to an \mathcal{O}_F -basis B of L . For all $b, b' \in B$, the Lie bracket $[b, b']$ is an element of $\text{Lie}(\mathcal{G})(F)$. Pick $i \in 2\mathbb{N}$ such that

$$[\omega_F^i b, \omega_F^i b'] = \omega_F^{2i} [b, b'] \in \omega_F^{i/2} (\omega_F^i L) \quad \forall b, b' \in B.$$

Then the \mathcal{O}_F -lattice $\omega_F^i L$ is a Lie algebra, it even satisfies

$$(27) \quad [x, y] \in \omega_F^{i/2} (\omega_F^i L) \quad \forall x, y \in \omega_F^i L.$$

For each $\alpha \in \Phi(\mathcal{G}, \mathcal{S})_{\text{red}} \cup \{0\}$ we choose an analytic diffeomorphism ϕ_α from a neighborhood of 0 in $\text{Lie}(\mathcal{U}_\alpha)(F)$ to a neighborhood of 1 in $\mathcal{U}_\alpha(F)$, such that $\phi_\alpha(0) = 1$ and the differential of ϕ_α at 0 is the identity map of $\text{Lie}(\mathcal{U}_\alpha)(F)$. For l_α in the domain of ϕ_α we define

$$\phi\left(\sum_\alpha l_\alpha\right) = \prod_\alpha \phi_\alpha(l_\alpha),$$

where we use the ordering of $\Phi(\mathcal{G}, \mathcal{S})_{\text{red}} \cup \{0\}$. By (26) ϕ is an analytic diffeomorphism from a neighborhood of 0 in $\text{Lie}(\mathcal{G})(F)$ to a neighborhood of 1 in $\mathcal{G}(F)$, and $d\phi$ is the identity map. The subsets $\phi(\omega_F^i L)$ of $\mathcal{G}(F)$ (for $i \in \mathbb{N}$ so large that $\omega_F^i L$ is contained in the domain of ϕ) form a neighborhood basis of 1 in $\mathcal{G}(F)$, because the sets $\omega_F^i L$ form a neighborhood basis of 0 in $\text{Lie}(\mathcal{G})(F)$.

Let us endow $\text{Lie}(\mathcal{G})(F)$ with a non-archimedean norm $\|\cdot\|$. Because L is open in $\text{Lie}(\mathcal{G})(F)$, there exists an $\epsilon \in \mathbb{R}_{>0}$ such that L contains the ball of radius $\epsilon \max_{l \in L} \|l\|$ around 0 in $\text{Lie}(\mathcal{G})(F)$. Then, for each $i \in \mathbb{N}$:

$$(28) \quad \omega_F^i L \text{ contains the ball of radius } \epsilon \max_{l \in \omega_F^i L} \|l\|.$$

By the continuity of inversion in $\mathcal{G}(F)$, the domain of ϕ contains $\phi(\omega_F^i L)^{-1}$ for sufficiently large i . For such i we define the map

$$\begin{aligned} \iota : \omega_F^i L &\rightarrow L \\ l &\mapsto \phi^{-1}(\phi(l)^{-1}) + l. \end{aligned}$$

The assumptions on ϕ entail that ι is F -analytic and that its derivative at $0 \in \omega_F^i L$ is zero. In terms of the norm, this says

$$\lim_{l \rightarrow 0} \|\iota(l)\| \|l\|^{-1} = 0.$$

Hence, for $j \in \mathbb{N}$ sufficiently large, $\|\iota(l)\| < \epsilon \|l\|$ on $\omega_F^j L$. With (28) we obtain $\iota(\omega_F^j L) \subset \omega_F^j L$. Furthermore $\omega_F^j L$ is a subgroup of $\text{Lie}(\mathcal{G})(F)$, so

$$\phi^{-1}(\phi(l)^{-1}) = \iota(l) - l \in \omega_F^j L \quad \text{for all } l \in \omega_F^j L.$$

This says that $\phi(\omega_F^j L)$ is closed under taking inverses, for $j \in \mathbb{N}$ sufficiently large.

By the continuity of multiplication in $\mathcal{G}(F)$, the image of ϕ contains $\phi(\omega_F^i L)^2$ for i sufficiently large. For such i we consider the F -analytic map

$$\begin{aligned} \mu : \omega_F^i L \times \omega_F^i L &\rightarrow L \\ (l, m) &\mapsto \phi^{-1}(\phi(l)\phi(m)) - l - m. \end{aligned}$$

Since $\mu(0, m) = 0$ for all m and μ is smooth, there exists $f : \omega_F^i L \rightarrow \mathbb{R}_{\geq 0}$ such that $\|\mu(l, m)\| \leq f(m) \|l\|$ for all $l, m \in \omega_F^i L$. Since $\omega_F^i L$ is compact, we can take

$$(29) \quad f(m) = \sum_{b \in B} \max_{l \in \omega_F^i L} \left\| \frac{\partial \mu}{\partial l_b}(l, m) \right\|,$$

where $\partial \mu / \partial l_b$ denotes the partial derivative with respect to the variable l in the direction of the basis element b of L . Since $l \mapsto \mu(l, 0)$ is identically zero, so is $l \mapsto \frac{\partial \mu}{\partial l_b}(l, 0)$. Therefore the above reasoning can also be applied to $\partial \mu / \partial l_b$. This yields $C_b \in \mathbb{R}_{>0}$ such that $\left\| \frac{\partial \mu}{\partial l_b}(l, m) \right\| \leq C_b \|m\|$ for all $l, m \in \omega_F^i L$, for example

$$(30) \quad C_b = \sum_{a \in B} \max_{l, m \in \omega_F^i L} \left\| \frac{\partial^2 \mu}{\partial m_a \partial l_b}(l, m) \right\|.$$

From (29) and (30) we conclude that

$$(31) \quad \|\mu(l, m)\| \leq f(m) \|l\| \leq \sum_{b \in B} C_b \|m\| \|l\| \quad \text{for all } l, m \in \omega_F^i L.$$

Suppose that $\omega_F^j L$ contains the ball of radius $\sum_{b \in B} C_b \max_{l \in \omega_F^j L} \|l\|^2$ in L . (By (28) this is the case for $j \in \mathbb{N}$ sufficiently large.) Then (31) says that $\mu((\omega_F^j L)^2) \subset \omega_F^j L$. From the definition of μ we see that also $\phi^{-1}(\phi(l)\phi(m)) \in \omega_F^j L$, or equivalently $\phi(l)\phi(m) \in \phi(\omega_F^j L)$ for all $l, m \in \omega_F^j L$.

We showed that, for $j \in \mathbb{N}$ sufficiently large, $\phi(\omega_F^j)$ is closed under multiplication and under inversion. Thus there exists $i_0 \in \mathbb{N}$ such that, for $j \geq i_0$, $\phi(\omega_F^j L)$ is a compact open subgroup of $\mathcal{G}(F)$.

For $k \in \phi(\omega_F^{i_0} L)$ we consider the map

$$(32) \quad \begin{array}{ccc} \gamma_k : \omega_F^{i_0} L & \rightarrow & \omega_F^{i_0} L \\ l & \mapsto & \phi^{-1}(k\phi(l)k^{-1}\phi(l)^{-1}) \end{array} .$$

By the analyticity of ϕ , the γ_k converge to the zero map as k goes to $1 \in \mathcal{G}(F)$. In view of the compactness of $\omega_F^{i_0} L$, this can be formulated more precisely. For every $D \in \mathbb{N}$ there exists an open neighborhood U_D of 1 in $\mathcal{G}(F)$ such that

$$(33) \quad \|\gamma_k(l)\| < D^{-1} \|l\| \quad \forall k \in U_D, l \in \omega_F^{i_0} L.$$

For D such that the ball of radius $D^{-1} \max_{l \in L} \|l\|$ is contained in L , (33) implies that

$$(34) \quad \begin{array}{l} k\phi(l)k^{-1}\phi(l)^{-1} = \phi(\gamma_k(l)) \in \phi(\omega_F^j L) \\ k\phi(\omega_F^j L)k^{-1} \subset \phi(\omega_F^j L) \end{array} \quad \forall k \in U_D, j \geq i_0, l \in \omega_F^j L.$$

Fix such a D and choose $i_D \geq i_0$ such that $\phi(\omega_F^{i_D} L) \subset U_D$ and (27) holds. Then all the $\phi(\omega_F^j L)$ with $j \geq i_D$ are normal subgroups of $\phi(\omega_F^{i_D} L)$. We define

$$K_n := \phi(\omega_F^{n+i_D} L) \text{ for } n \geq 0.$$

Now the first two bullets of the statement are satisfied.

For $\alpha \in \Phi(\mathcal{G}, \mathcal{S}_\emptyset)_{\text{red}} \cup \{0\}$, the above argument also shows that $\phi(\omega_F^j L_\alpha)$ with $j \geq i_D$ is a normal subgroup of $\phi(\omega_F^{i_D} L_\alpha)$. The bijectivity of $\phi : L \rightarrow L$ implies that

$$\phi(\omega_F^j L) \cap \phi(\omega_F^{i_D} L_\alpha) = \phi(\omega_F^j L_\alpha).$$

For any ordering of $\Phi(\mathcal{G}, \mathcal{S}_\emptyset)_{\text{red}} \cup \{0\}$, not necessarily the same as we fixed before, we can consider the multiplication map

$$(35) \quad \prod_{\alpha \in \Phi(\mathcal{G}, \mathcal{S}_\emptyset)_{\text{red}} \cup \{0\}} \phi(\omega_F^j L_\alpha) \longrightarrow \phi(\omega_F^j L).$$

By the above, (35) gives rise to a well-defined map on the lower line of the following diagram:

$$(36) \quad \begin{array}{ccc} \bigoplus_{\alpha} \omega_F^j L_\alpha / \omega_F^{j+1} L_\alpha & \longrightarrow & \omega_F^j L / \omega_F^{j+1} L \\ \downarrow \phi & & \downarrow \phi \\ \prod_{\alpha} \phi(\omega_F^j L_\alpha) / \phi(\omega_F^{j+1} L_\alpha) & \longrightarrow & \phi(\omega_F^j L) / \phi(\omega_F^{j+1} L) \end{array}$$

By (33) the two lower terms in (36) are abelian groups, and the map between them is a group homomorphism. Similarly (27) says that the two upper terms are abelian Lie algebras, and that the map between them is a Lie homomorphism. Consequently the diagram (36) commutes.

Since ϕ was bijective to start with, the downward arrows induced by ϕ are still bijective. The upper horizontal map is a bijection because $\omega_F^j L = \bigoplus_{\alpha} \omega_F^j L_{\alpha}$, so the lower horizontal map is also bijective. Using this for every $j \geq i_D$, in combination with the profiniteness of all the involved groups, we deduce that (35) is bijective. Since it is also an analytic map between compact \mathcal{O}_F -varieties, it is in fact a homeomorphism.

For any pseudo-parabolic F -subgroup $\mathcal{P}_{\mathcal{G}}(\lambda)$ containing $Z_{\mathcal{G}}(\lambda)$, we can focus on the roots α in $\Phi(\mathcal{U}_{\mathcal{G}}(\lambda), \mathcal{S}_{\emptyset})_{\text{red}}$. The above argument shows that multiplication induces a homeomorphism

$$\prod_{\alpha \in \Phi(\mathcal{U}_{\mathcal{G}}(\lambda), \mathcal{S}_{\emptyset})_{\text{red}}} K_n \cap \mathcal{U}_{\alpha}(F) = \prod_{\alpha \in \Phi(\mathcal{U}_{\mathcal{G}}(\lambda), \mathcal{S}_{\emptyset})_{\text{red}}} \phi(\omega_F^{n+i_D} L_{\alpha}) \longrightarrow \phi\left(\prod_{\alpha \in \Phi(\mathcal{U}_{\mathcal{G}}(\lambda), \mathcal{S}_{\emptyset})_{\text{red}}} \omega_F^{n+i_D} L_{\alpha}\right).$$

All this takes place in $\mathcal{U}_{\mathcal{G}}(\lambda)(F)$, and considering that as enveloping group we see that the right hand side equals $K_n \cap \mathcal{U}_{\mathcal{G}}(\lambda)(F)$. The groups $K_n \cap Z_{\mathcal{G}}(\lambda)(F)$ and $K_n \cap \mathcal{U}_{\mathcal{G}}(\lambda^{-1})(F)$ admit analogous expressions.

Finally we use the freedom in the ordering of $\Phi(\mathcal{G}, \mathcal{S}_{\emptyset})_{\text{red}} \cup \{0\}$. We may assume that the roots in $\Phi(\mathcal{U}_{\mathcal{G}}(\lambda), \mathcal{S}_{\emptyset})_{\text{red}}$ come first, then those in $\Phi(Z_{\mathcal{G}}(\lambda), \mathcal{S}_{\emptyset})_{\text{red}}$ and finally the remaining set $\Phi(\mathcal{U}_{\mathcal{G}}(\lambda^{-1}), \mathcal{S}_{\emptyset})_{\text{red}}$. Under those conditions (24) shows that (35) becomes the Iwahori decomposition of K_n with respect to $\mathcal{P}_{\mathcal{G}}(\lambda)$. \square

2.2. Actions on affine buildings.

To understand the geometry of pseudo-reductive groups over non-archimedean local fields better, we first make such groups act on affine buildings.

Lemma 2.4. *Let \mathcal{C} be a commutative pseudo-reductive F -group. Let \mathcal{S} be the maximal F -split torus in \mathcal{C} , and let $X_*(\mathcal{S})$ be its cocharacter lattice. Then $\mathcal{C}(F)$ acts naturally on $X_*(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{R}$, by translations. The action is cocompact and proper (in the sense of topological group actions).*

Proof. By [Loi, Theorem 1.4.3] $\mathcal{C}(F)$ contains a unique maximal compact subgroup, say $K_{\mathcal{C}}$, which is open. Then $\mathcal{C}(F)/K_{\mathcal{C}}$ is a finitely generated free abelian group and $\mathcal{C}(F)$ acts properly and cocompactly on $\mathcal{C}(F)/K_{\mathcal{C}} \otimes_{\mathbb{Z}} \mathbb{R}$, by multiplication. This can be regarded as a translation action of $\mathcal{C}(F)$ on a vector space.

Furthermore $K_{\mathcal{C}} \cap \mathcal{S}(F)$ is the unique maximal compact subgroup of $\mathcal{S}(F)$, and $\mathcal{S}(F)/(K_{\mathcal{C}} \cap \mathcal{S}(F))$ embeds naturally in $\mathcal{C}(F)/K_{\mathcal{C}}$. The image is of finite index in $\mathcal{C}(F)/K_{\mathcal{C}}$ because $\mathcal{C}(F)/\mathcal{S}(F)$ is compact (Lemma 2.1.a) and $K_{\mathcal{C}}$ is open. Hence there is a natural isomorphism

$$(37) \quad \mathcal{S}(F)/(K_{\mathcal{C}} \cap \mathcal{S}(F)) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \mathcal{C}(F)/K_{\mathcal{C}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let ω_F be a uniformizer of F . The map

$$(38) \quad X_*(\mathcal{S}) \rightarrow \mathcal{S}(F) : \nu \mapsto \nu(\omega_F^{-1})$$

and (37) provide canonical isomorphisms $X_*(\mathcal{S}) \rightarrow \mathcal{S}(F)/(K_{\mathcal{C}} \cap \mathcal{S}(F))$ and

$$(39) \quad X_*(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \mathcal{C}(F)/K_{\mathcal{C}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

We transfer the $\mathcal{C}(F)$ -action from the right hand side to the left hand side. Restricted to $\mathcal{S}(F)$, this recovers the action coming from (38) and used in Bruhat–Tits theory, see [ScSt, §1.1]. In this way we get a canonical proper, cocompact $\mathcal{C}(F)$ -action on $X_*(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{R}$. \square

Let $\mathcal{T}_{nr,i}$ be the standard maximal split torus in Sp_{2n_i} , so that $\mathcal{T}_{nr,i}(F)$ is a maximal F -split torus in $R_{F'_i/F}(\mathcal{G}'_{nr,i})(F)$. We write

$$\mathcal{T} = \prod_{i \in I_{nr}} R_{F'_i/F}(\mathcal{T}_{nr,i}) \times \left(\prod_{i \in I_s \cup I_e} R_{F_i/F}(\mathcal{T}_i) \rtimes \mathcal{S} \right) / \alpha(\mathcal{T}').$$

In view (18) we may assume that our maximal F -split torus of \mathcal{G} is given as

$$\mathcal{S}_\emptyset := \prod_{i \in I_{nr}} \mathcal{T}_{nr,i} \times \mathcal{S}.$$

Proposition 2.5. *There exists an affine building $\mathcal{B}(\mathcal{G}, F)$ with an action of the group $\mathcal{G}(F)$ from (15), such that:*

- (a) *The action is isometric, proper and cocompact;*
- (b) *The set of apartments of $\mathcal{B}(\mathcal{G}, F)$ is naturally in bijection with the set of maximal F -split tori in \mathcal{G} .*
- (c) *$\mathcal{C}(F)$ stabilizes an apartment.*

Proof. For notational ease we simplify the description of the totally non-reduced factors and of the exotic pseudo-reductive factors. During this proof we will denote the factors $R_{F'_i/F}(Sp_{2n_i})$ from (12) and (20) and the factors $R_{F_i/F}(\overline{\mathcal{G}}_i)$ from (8) also as $R_{F_i/F}(\mathcal{G}_i)$. That is, we treat them on the same footing as the standard $\mathcal{G}_i(F_i)$.

Let $\mathcal{BT}(\mathcal{G}_i, F_i)$ be the Bruhat–Tits building of $\mathcal{G}_i(F_i)$. We claim that this building admits a natural isometric action of any group H_i that acts on $\mathcal{G}_i(F_i)$ by automorphisms of topological groups. (See Proposition 3.10.c for a generalization and more detailed proof of this claim.) The stabilizer $\mathcal{G}_i(F_i)_{x_i}$ of any point $x_i \in \mathcal{BT}(\mathcal{G}_i, F_i)$ is a compact open subgroup of $\mathcal{G}_i(F_i)$. As \mathcal{G}_i is semisimple and simply connected, [BrTi1, Corollaire 3.3.3] applies, and tells us that $\mathcal{G}_i(F_i)_{x_i}$ is a maximal compact subgroup if and only if x_i is a vertex of $\mathcal{BT}(\mathcal{G}_i, F_i)$. In that case x_i is the only fixed point of $\mathcal{G}_i(F_i)_{x_i}$, for different facets of $\mathcal{BT}(\mathcal{G}_i, F_i)$ have different stabilizers.

For any $h \in H_i$, $h(\mathcal{G}_i(F_i)_{x_i})$ is another maximal compact subgroup of $\mathcal{G}_i(F_i)_{x_i}$, so of the form $\mathcal{G}_i(F_i)_{y_i}$ for a unique vertex y_i of $\mathcal{BT}(\mathcal{G}_i, F_i)$. It is easily seen that the definition $h(x_i) := y_i$ extends uniquely to an isometric action of H_i on $\mathcal{BT}(\mathcal{G}_i, F_i)$. This applies in particular to the conjugation actions of $G' \rtimes C$ and G on $\mathcal{G}_i(F_i)$.

Let $\mathcal{Z} = (\mathcal{S} \cap Z(\mathcal{G}))^\circ$ be the maximal F -split torus in $Z(\mathcal{G})$. Then $\mathcal{S} = \mathcal{Z}(\mathcal{S} \cap \phi_{\mathcal{C}}(\mathcal{T}'))^\circ$ and

$$(40) \quad X_*(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{R} = X_*(\mathcal{S} \cap \phi_{\mathcal{C}}(\mathcal{T}'))^\circ \otimes_{\mathbb{Z}} \mathbb{R} \oplus X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

In the decomposition (40), with the action from Lemma 2.4, $\phi_{\mathcal{C}}(\mathcal{T}')$ acts only the first summand, and the quotient $C/\phi_{\mathcal{C}}(\mathcal{T}')$ acts on the second summand. Thus $G' \rtimes C$ and G act on $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, via the quotient maps

$$(41) \quad G' \rtimes C \rightarrow G \rightarrow C/\phi_{\mathcal{C}}(\mathcal{T}').$$

We define the affine buildings

$$(42) \quad \begin{aligned} \mathcal{B}(G' \rtimes C, F) &= \prod_{i \in I_s \cup I_e \cup I_{nr}} \mathcal{BT}(\mathcal{G}_i, F_i) \times X_*(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{R}, \\ \mathcal{B}(G, F) &= \prod_{i \in I_s \cup I_e \cup I_{nr}} \mathcal{BT}(\mathcal{G}_i, F_i) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}. \end{aligned}$$

- (a) The group $G' \rtimes C$ acts componentwise on these two buildings, and the action on $\mathcal{B}(G, F)$ factors through G . It is known from Bruhat–Tits theory [Tit, §2.2] that the action of $G'(F)$ on $\prod_{i \in I_s \cup I_e \cup I_{nr}} \mathcal{BT}(\mathcal{G}_i, F_i)$ is proper and cocompact. The action of $C/\phi_{\mathcal{C}}(\mathcal{T}')$ (resp. of C) on $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. on $X_*(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{R}$) is proper and

cocompact by Lemma 2.4. Hence the action of $G' \rtimes C$ on $\mathcal{B}(G' \rtimes C, F)$ is proper and cocompact. By Lemma 2.2.a the action of G on $\mathcal{B}(G, F)$ is also proper and cocompact.

Recall [BrTi1, Lemme 2.5.1] that $\mathcal{B}(G_i, F_i)$ is endowed with a $G_i(F_i)$ -invariant metric d_i , which comes from an inner product on an apartment of this building. We endow $\mathcal{B}(G, F)$ with a metric of the form

$$(43) \quad d\left(\left(\prod_i x_i, x_Z\right), \left(\prod_i y_i, y_Z\right)\right) = \sum_i d(x_i, y_i) + \langle x_Z - y_Z, x_Z - y_Z \rangle_Z,$$

where $\langle \cdot, \cdot \rangle_Z$ is an inner product on $X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R}$. Since $G' \rtimes C$ and G act on $X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R}$ by translations, the actions of $G' \rtimes C$ and G on $\mathcal{B}(G', F)$ are isometric. In the same way we can define a $G' \rtimes C$ -invariant metric on $\mathcal{B}(G' \rtimes C, F)$.

(b) By definition an apartment of $\mathcal{B}(G, F)$ is a set of the form $\prod_i \mathbb{A}_i \times X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R}$, where each \mathbb{A}_i is an apartment of $\mathcal{BT}(G_i, F_i)$. According to [Tit, §2.1] there is a canonical bijection between the set of apartments of $\mathcal{BT}(G_i, F_i)$ and the set of maximal F_i -split tori of G_i . By [CGP, Proposition A.5.15.ii] the latter set is in natural bijection with the set of maximal F -split tori in $R_{F_i/F}(G_i)$. Thus the apartments of $\mathcal{BT}(G, F)$ correspond bijectively to the maximal F -split tori in G' .

Clearly the maps $\mathcal{S}' \leftrightarrow \mathcal{S}' \times \mathcal{S}$ provide a bijection between these and the maximal F -split tori of $G' \rtimes C$. As $G = (G' \rtimes C)/\alpha(\mathcal{T}')$, the natural map $G' \rtimes C \rightarrow G$ sends maximal F -split tori to maximal F -split tori. As $\alpha(\mathcal{T}')$ central, [CGP, Proposition 2.2.12.i] shows that this induces a bijection between the set of maximal F -split tori of $G' \rtimes C$ and the analogous set for G .

(c) Since we assumed that every \mathcal{T}_i contains a maximal F_i -split torus, we can define $\mathbb{A}_{\mathcal{T}_i}$ to be the apartment of $\mathcal{BT}(G_i, F_i)$ associated to that maximal F_i -split torus. The sets

$$\mathbb{A}_{\mathcal{T}} = \prod_i \mathbb{A}_{\mathcal{T}_i} \times X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad \mathbb{A}_{\emptyset} = \prod_i \mathbb{A}_{\mathcal{T}_i} \times X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R}$$

are apartments in, respectively, $\mathcal{B}(G' \rtimes C, F)$ and $\mathcal{B}(G, F)$. The latter is associated to the maximal F -split torus \mathcal{S}_{\emptyset} . The action of C on $\mathcal{B}(G', F)$ factors via $\mathcal{T}'(F)/Z(G'(F))$, so it stabilizes $\prod_i \mathbb{A}_{\mathcal{T}_i}$. It follows that $C \subset G' \rtimes C$ stabilizes $\mathbb{A}_{\mathcal{T}}$ and that $C \subset G$ stabilizes \mathbb{A}_{\emptyset} . \square

The building $\prod_i \mathcal{BT}(G_i, F_i)$ is endowed with a canonical polysimplicial structure, but $X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R}$ is not. A facet of $\mathcal{B}(G, F)$ is by definition a subset of the form $\mathfrak{f} = \prod_i \mathfrak{f}_i \times X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R}$, where each \mathfrak{f}_i is a facet of $\mathcal{BT}(G_i, F_i)$. By [Tit, §3.1] and Lemma 2.4

(44) the pointwise stabilizer of any facet of $\mathcal{B}(G, F)$ is an open subgroup of $G(F)$.

We will call a point $x = ((x_i)_i, x_Z) \in \mathcal{B}(G, F)$ a vertex if each x_i is a vertex of $\mathcal{BT}(G_i, F_i)$ and $x_Z \in X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R}$. Recall that x_i is a special vertex of $\mathcal{BT}(G_i, F_i)$ if there is an apartment \mathbb{A}_i such that for every wall in \mathbb{A}_i there is a parallel wall in \mathbb{A}_i which contains x_i . We say that x is a special vertex if each x_i is so. By [BrTi1, §1.3.7] every apartment of $\mathcal{BT}(G_i, F_i)$, and hence every apartment of $\mathcal{B}(G, F)$, contains special vertices.

As a first consequence of the existence of appropriate affine buildings for $G(F)$, we analyse the structure of the centralizer of a maximal F -split torus.

Lemma 2.6. $Z_G(S_{\emptyset}) = Z_G(S_{\emptyset})(F)$ has a unique maximal compact subgroup, say $Z_G(S_{\emptyset})_{\text{cpt}}$.

Proof. The group $Z_G(S_\emptyset)$ stabilizes the apartment \mathbb{A}_\emptyset of $\mathcal{B}(\mathcal{G}, F)$ and acts by translations on it (see the proof of Lemma 2.4 for C and [Tit, §1.2] for $G_{nr} \times G'_r$). By the Bruhat–Tits fixed point theorem every compact subgroup of $Z_G(S_\emptyset)$ fixes a point of \mathbb{A}_\emptyset . By Proposition 2.5 the action is proper, so the stabilizer of a point of \mathbb{A}_\emptyset is compact, and every maximal compact subgroup of $Z_G(S_\emptyset)$ is of this form. But the action is by translations, so every point of \mathbb{A}_\emptyset has the same isotropy group. That is the unique maximal compact subgroup of $Z_G(S_\emptyset)$. \square

2.3. The Iwasawa and Cartan decompositions.

The above actions on buildings can be used to construct maximal compact subgroups of pseudo-reductive F -groups. Recall that K_C is the unique maximal compact subgroup of C , and denote the unique maximal compact subgroup of T' by T'_{cpt} .

Lemma 2.7. *Let $x = ((x_i)_i, x_Z)$ be a vertex in the apartment \mathbb{A}_\emptyset of $\mathcal{B}(\mathcal{G}, F)$.*

- (a) $G'_{(x_i)_i}$ is a maximal compact subgroup of G' and $G'_{(x_i)_i} \rtimes K_C$ is a maximal compact subgroup of $G' \rtimes C$.
- (b) G_x is a maximal compact subgroup of G . It contains $(G'_{(x_i)_i} \rtimes K_C) / \alpha(T'_{\text{cpt}})$ as an open subgroup of finite index.

Proof. (a) As a consequence of the Bruhat–Tits fixed point theorem [BrTi1, Corollaire 3.3.2], the group $G'_{(x_i)_i} = \prod_i \mathcal{G}_i(F_i)_{x_i}$ is a maximal compact subgroup of $G' = \prod_i \mathcal{G}_i(F_i)$. The action of C on $\prod_i \mathbb{A}_{\mathcal{T}_i}$ factors via $\mathcal{T}'(F)/Z(\mathcal{G}'(F))$, so it acts by translations and every compact subgroup of C acts trivially. In particular the maximal compact subgroup K_C of C fixes $\prod_i \mathbb{A}_{\mathcal{T}_i}$ pointwise. Hence K_C normalizes $G'_{(x_i)_i}$, and $G'_{(x_i)_i} \rtimes K_C$ is a compact open subgroup of $G' \rtimes C$.

Since the image of $G'_{(x_i)_i} \rtimes K_C$ in $C \cong G' \rtimes C / G'$ is the unique maximal compact subgroup of C and since $G'_{(x_i)_i}$ is maximal compact in G' , $G'_{(x_i)_i} \rtimes K_C$ is maximal compact in $G' \rtimes C$.

(b) G_x is maximal compact in G for the same reason as $G'_{(x_i)_i}$. By construction the image of $G'_{(x_i)_i} \rtimes K_C$ in G is contained in G_x . By Proposition 1.2 it is open and of finite index in G_x . By Lemma 2.2.a this image is isomorphic to

$$(G'_{(x_i)_i} \rtimes K_C) / (\alpha(T') \cap G'_{(x_i)_i} \rtimes K_C).$$

Since $G'_{(x_i)_i}$ is associated to a point in the apartment of $\mathcal{B}(\mathcal{G}', F)$ stabilized by T' , $\alpha(T') \cap (G'_{(x_i)_i} \rtimes K_C)$ is the maximal compact subgroup $\alpha(T'_{\text{cpt}})$ of $\alpha(T')$. \square

The Iwasawa decomposition and the Cartan decomposition are two of the most useful results abouts the structure of reductive groups over local fields. We thank Gopal Prasad for explaining to us how the next result can be proven using the universal smooth F -tame central extension of \mathcal{G} from [CoPr1]. Nevertheless, with an eye on later results we find it useful to provide a different argument.

Let x' be a special vertex of the apartment \mathbb{A}_\emptyset and let $K = G_{x'}$ be its stabilizer in G . Let K' be the stabilizer in $G' = G_{nr} \times G'_r$ of the image of x' in $\mathcal{B}(\mathcal{G}' \rtimes C, F)$. The maximal compact subgroups $K \subset G$ and $K' \subset G'$ contain representatives for all elements of $W(\mathcal{G}, \mathcal{S}_\emptyset)$, because x' and its image in $\mathcal{B}(\mathcal{G}' \rtimes C, F)$ are special vertices [BrTi1, Proposition 4.4.6].

Theorem 2.8. *The group G admits an Iwasawa decomposition. More precisely, for every pseudo-parabolic F -subgroup \mathcal{P} of \mathcal{G} we have*

$$\mathcal{G}(F) = \mathcal{P}(F)K = K\mathcal{P}(F).$$

Proof. First we prove the theorem for $(\mathcal{G}_{nr} \times \mathcal{G}'_r)(F)$ with the compact subgroup K' . By construction

$$\begin{aligned} \mathcal{B}(\mathcal{G}_{nr} \times \mathcal{G}'_r, F) &= \mathcal{B}(\mathcal{G}_{nr}, F) \times \mathcal{B}(\mathcal{G}'_r, F), \\ K' &= (K' \cap \mathcal{G}_{nr}(F)) \times (K' \cap \mathcal{G}'_r(F)). \end{aligned}$$

Moreover both factors of K' are associated to special vertices of the appropriate affine buildings. Thus we may treat the two groups \mathcal{G}_{nr} and \mathcal{G}'_r separately.

We note that $\mathcal{G}_{nr}(F) = \prod_{i \in I_{nr}} Sp_{2n_i}(F_i'')$ and that each $Sp_{2n_i}(F_i'')$ satisfies an Iwasawa decomposition. As compact subgroup of $Sp_{2n_i}(F_i'')$ one can use the isotropy group of a special vertex in the Bruhat–Tits building $\mathcal{B}(Sp_{2n_i}, F_i'')$. Now $K' \cap \mathcal{G}_{nr}(F)$ is a direct product of such isotropy groups, so $(\mathcal{G}_{nr}(F), K' \cap \mathcal{G}_{nr}(F))$ satisfies the Iwasawa decomposition, for every pseudo-parabolic F -subgroup.

Similarly, for $\mathcal{G}'_r(F)$ the problem reduces to the direct factors $\mathcal{G}_i(F_i)$, which by (8) are isomorphic to reductive groups over local fields. The same argument as for $\mathcal{G}_{nr}(F)$ applies here. This concludes the case $(\mathcal{G}_{nr} \times \mathcal{G}'_r)(F)$, now we consider $\mathcal{G}(F)$.

Every pseudo-parabolic F -subgroup arises from a F -rational cocharacter $\lambda : GL_1 \rightarrow \mathcal{G}$. Since all maximal F -split tori in $\mathcal{G}(F)$ are conjugate [CGP, Theorem C.2.3], \mathcal{P} is conjugate to $\mathcal{P}_{\mathcal{G}}(\lambda)$ for some $\lambda : GL_1 \rightarrow \mathcal{T}$.

In particular \mathcal{P} is $\mathcal{G}(F)$ -conjugate to a pseudo-parabolic F -subgroup of \mathcal{G} containing \mathcal{C} . By [CGP, Proposition 2.2.12] the pre-image \mathcal{P}_1 of \mathcal{P} in $\mathcal{G}_{nr} \times \mathcal{G}'_r \rtimes \mathcal{C}$ is again pseudo-parabolic. It follows that there exists a pseudo-parabolic F -subgroup \mathcal{P}_2 of $\mathcal{G}_{nr} \times \mathcal{G}'_r$ such that \mathcal{P}_1 is $(\mathcal{G}_{nr} \times \mathcal{G}'_r \rtimes \mathcal{C})(F)$ -conjugate to $\mathcal{P}_2 \rtimes \mathcal{C}$, say by an element g_2 . Then \mathcal{P}_1 contains $g_2 \mathcal{P}_2 g_2^{-1}$, which is a pseudo-parabolic F -subgroup of $\mathcal{G}_{nr} \times \mathcal{G}'_r$.

From the case $(\mathcal{G}_{nr} \rtimes \mathcal{G}'_r)(F)$ we know that $K' \mathcal{P}_1(F)$ contains

$$K' g_2 \mathcal{P}_2(F) g_2^{-1} = (\mathcal{G}_{nr} \times \mathcal{G}'_r)(F).$$

The image of $K' \mathcal{P}_1(F)$ in

$$\mathcal{C}(F) = (\mathcal{G}_{nr} \times \mathcal{G}'_r \rtimes \mathcal{C})(F) / (\mathcal{G}_{nr} \times \mathcal{G}'_r)(F)$$

contains

$$(\mathcal{G}_{nr} \times \mathcal{G}'_r)(F) g_2 (\mathcal{P}_2 \times \mathcal{C})(F) g_2^{-1} / (\mathcal{G}_{nr} \times \mathcal{G}'_r)(F) = \bar{g}_2 \mathcal{C}(F) \bar{g}_2^{-1} = \mathcal{C}(F),$$

where \bar{g}_2 the image of g_2 in the quotient. Hence

$$(K' \rtimes K_{\mathcal{C}}) \mathcal{P}_1(F) = (\mathcal{G}_{nr} \times \mathcal{G}'_r \rtimes \mathcal{C})(F).$$

Since \mathcal{P}_1 contains the central subgroup $\alpha(\mathcal{T}')$ we conclude with Lemma 2.7 that

$$(45) \quad K\mathcal{P}(F) \supset ((K' \rtimes K_{\mathcal{C}}) \mathcal{P}_1(F) / \alpha(\mathcal{T}')) \mathcal{P}(F) = ((\mathcal{G}_{nr} \times \mathcal{G}'_r \rtimes \mathcal{C})(F) / \alpha(\mathcal{T}')) \mathcal{P}(F).$$

Since $Z_{\mathcal{G}_r}(\mathcal{S}) \subset Z_{\mathcal{G}}(\mathcal{S}_\emptyset) \subset \mathcal{P}$, we can use Lemma 2.2.b to conclude that the right hand side is none other than $\mathcal{G}(F)$. \square

Related to the Iwasawa decomposition is the question whether $(\mathcal{G}/\mathcal{P})(F)$ is compact, for any pseudo-parabolic F -subgroup \mathcal{P} . We note that by [CGP, Proposition 3.5.7], $\mathcal{G}(F)/\mathcal{P}(F)$ can be identified with the set of pseudo-parabolic F -subgroups of \mathcal{G} that are $\mathcal{G}(F)$ -conjugate to \mathcal{P} . It is known from [CGP, Proposition C.1.6] that

the variety \mathcal{G}/\mathcal{P} is pseudo-complete, but it is unclear (to the author) whether or not this implies compactness of $(\mathcal{G}/\mathcal{P})(F)$. Fortunately Theorem 2.8 solves this.

Corollary 2.9. *Let \mathcal{P} be any pseudo-parabolic F -subgroup of the pseudo-reductive group \mathcal{G} . The natural map $\mathcal{G}(F)/\mathcal{P}(F) \rightarrow (\mathcal{G}/\mathcal{P})(F)$ is a homeomorphism and this space is compact.*

Proof. By [Spr, Corollary 15.1.4] the map $\mathcal{G}(F) \rightarrow (\mathcal{G}/\mathcal{P})(F)$ is surjective. The induced map $\mathcal{G}(F)/\mathcal{P}(F) \rightarrow (\mathcal{G}/\mathcal{P})(F)$ is a bijective submersion of analytic F -varieties, in particular a homeomorphism. By Theorem 2.8 already $K \rightarrow \mathcal{G}(F)/\mathcal{P}(F)$ is surjective. Since K is compact, so is its image $\mathcal{G}(F)/\mathcal{P}(F)$. \square

Recall that \mathcal{S}_\emptyset is a maximal F -split torus in \mathcal{G} and that \mathcal{P}_\emptyset is a minimal pseudo-parabolic F -subgroup of \mathcal{G} containing \mathcal{C} and \mathcal{S}_\emptyset . We also recall $Z_G(\mathcal{S}_\emptyset)_{\text{cpt}}$ from Lemma 2.6.

Theorem 2.10. *The group G admits a Cartan decomposition. More explicitly, there exists a finitely generated semigroup $A \subset Z_G(\mathcal{S}_\emptyset)$ such that*

- $G = KAK$ and the natural map $A \rightarrow K \backslash G / K$ is bijective;
- A represents the orbits of the Weyl group $W(\mathcal{G}, \mathcal{S}_\emptyset)$ on $Z_G(\mathcal{S}_\emptyset) / Z_G(\mathcal{S}_\emptyset)_{\text{cpt}}$.

Proof. First we consider $G_{nr} \times G'_r \rtimes C$, with the maximal compact subgroup $K' \rtimes K_C$.

By (20) and (8) $G_{nr} \times G'_r$ is a direct product of semisimple groups (we call them just $\mathcal{G}_i(F_i)$ for convenience) over non-archimedean local fields. By [BrTi1, Proposition 4.4.3] each of these $\mathcal{G}_i(F_i)$ satisfies a Cartan decomposition, with respect to the good compact subgroup $\mathcal{G}_i(F_i) \cap K'$. It also says that for $A \cap \mathcal{G}_i(F_i)$ we should take a set of representatives for the orbits of the Weyl group $W(\mathcal{G}_i, \mathcal{S}_i)$ on

$$Z_{\mathcal{G}_i}(\mathcal{S}_i)(F_i) / Z_{\mathcal{G}_i}(\mathcal{S}_i)(F_i)_{\text{cpt}},$$

namely a cone (essentially the intersection with a positive Weyl chamber) in this lattice. For the moment, it is more convenient to let A_i be a group in $Z_{\mathcal{G}_i}(\mathcal{S}_i)(F_i)$ representating the cosets with respect to $Z_{\mathcal{G}_i}(\mathcal{S}_i)(F_i)_{\text{cpt}}$.

We put $A' = \prod_i A_i$, a finitely generated abelian subgroup of $Z_{G'}(\mathcal{S}_\emptyset)$ which represents $Z_{G'}(\mathcal{S}_\emptyset) / Z_{G'}(\mathcal{S}_\emptyset)_{\text{cpt}}$. It intersects $\phi_C^{-1}(Z_G(\mathcal{S}_\emptyset)_{\text{cpt}}) \subset Z_{G'}(\mathcal{S}_\emptyset)_{\text{cpt}}$ trivially, so dividing out $\alpha(T')$ and then $Z_G(\mathcal{S}_\emptyset)_{\text{cpt}}$ sends A' injectively to $Z_G(\mathcal{S}_\emptyset) / Z_G(\mathcal{S}_\emptyset)_{\text{cpt}}$. We extend the image of A' in $Z_G(\mathcal{S}_\emptyset)$ to a set of representatives A_C for $Z_G(\mathcal{S}_\emptyset) / Z_G(\mathcal{S}_\emptyset)_{\text{cpt}}$. Since K' contains $Z_{G'}(\mathcal{S}_\emptyset)_{\text{cpt}}$, we obtain $K'A' = K'Z_{G'}(\mathcal{S}_\emptyset)$. We calculate inside G , temporarily identifying subsets of G' with their image in G :

$$(46) \quad KA_CK = KA'A_CK'K = \bigcup_{a \in A_C} KK'Z_{G'}(\mathcal{S}_\emptyset)(aK'a^{-1})aK.$$

Here $aK'a^{-1}$ is another good maximal compact subgroup of G' , namely the stabilizer of the special vertex $ax' \in \mathbb{A}_\emptyset$. By a more general version of the Cartan decomposition for the $\mathcal{G}_i(F_i)$ [BrTi1, Proposition 7.4.15], $K'Z_{G'}(\mathcal{S}_\emptyset)(aK'a^{-1}) = G'$. As K contains $Z_G(\mathcal{S}_\emptyset)_{\text{cpt}}$, we have $A_CK = Z_G(\mathcal{S}_\emptyset)K$. Hence the right hand side of (46) equals

$$\bigcup_{a \in A_C} KG'aK = KG'Z_G(\mathcal{S}_\emptyset)K.$$

By Lemma 2.2.b the last expression equals G , which together with the above implies

$$(47) \quad KA_CK = G.$$

Now that we know this, we may replace $A_C \subset Z_G(S_\emptyset)$ by any set of representatives for $Z_G(S_\emptyset)/Z_G(S_\emptyset)_{\text{cpt}}$, and then (47) remains true. In the proof of Lemma 2.6 we already observed that $Z_G(S_\emptyset)/Z_G(S_\emptyset)_{\text{cpt}}$ is a lattice in \mathbb{A}_\emptyset . Hence we can represent it by a finitely generated subgroup of $Z_G(S_\emptyset)$.

Of course, some redundancy remains in (47). The Weyl group $W(\mathcal{G}, \mathcal{S}_\emptyset)$ can be identified with $\prod_i W(\mathcal{G}_i, \mathcal{S}_i)$, and since x' is special we can represent $W(\mathcal{G}, \mathcal{S}_\emptyset)$ by elements of K . For such a $w \in W(\mathcal{G}, \mathcal{S}_\emptyset)$ represented in K ,

$$KaK = Kawaw^{-1}K.$$

Therefore we may replace A_C by a set A of representatives for the orbits of $W(\mathcal{G}, \mathcal{S}_\emptyset)$ on $Z_G(S_\emptyset)/Z_G(S_\emptyset)_{\text{cpt}}$, and then (47) implies $G = KAK$. We take A so that Ax' is the intersection of $Z_G(S_\emptyset)x' = A_Cx'$ with a positive Weyl chamber for $W(\mathcal{G}, \mathcal{S}_\emptyset)$ in \mathbb{A}_\emptyset (with x' as origin). Since $W(\mathcal{G}, \mathcal{S}_\emptyset)$ is a subgroup of the (extended) affine Weyl group

$$N_G(S_\emptyset)/Z_G(S_\emptyset)_{\text{cpt}} \cong W(\mathcal{G}, \mathcal{S}_\emptyset) \rtimes Z_G(S_\emptyset)/Z_G(S_\emptyset)_{\text{cpt}},$$

A is still finitely generated (but now as semigroup).

We claim that this positive Weyl chamber in $Z_G(S_\emptyset)x'$ is a fundamental domain for the action of K on $Gx' \subset \mathcal{B}(\mathcal{G}, F)$. Notice that $W(\mathcal{G}, \mathcal{S}_\emptyset)$ and K fix the factor $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ of $\mathcal{B}(\mathcal{G}, F)$ pointwise. Hence the projection of Gx' on $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ equals the projection of $Z_G(S_\emptyset)x'$, which is also the same as the projections of A_Cx' and of Ax' on $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$.

Therefore it suffices to check our claim for the other factor $\prod_i \mathcal{BT}(\mathcal{G}_i, F_i)$ of $\mathcal{B}(\mathcal{G}, F)$. There it reduces to a claim about the action of G on $\mathcal{BT}(\mathcal{G}_i, F_i)$. Then it follows from (in fact is equivalent to) the Cartan decomposition with respect to the stabilizer of the special vertex $x_i \in \mathbb{A}_{\mathcal{T}_i} \subset \mathcal{BT}(\mathcal{G}_i, F_i)$.

Finally, suppose that $a, a' \in A$ and $KaK = Ka'K$. Then

$$Kax' = KaKx' = Ka'Kx' = Ka'x',$$

where both ax' and $a'x'$ lie in the positive Weyl chamber in A_\emptyset . Our claim entails that $a = a'$, which establishes the bijectivity of $A \rightarrow K \backslash G / K$. \square

3. QUASI-REDUCTIVE GROUPS

Throughout this section we suppose that F is a non-archimedean local field and that \mathcal{Q} is a quasi-reductive F -group, as defined in [CGP, C.2.11]. As an F -group, $\mathcal{Q}(F)$ is certainly locally compact and totally disconnected. By Lemma 1.3 and (16), it is unimodular.

Let us recall some important results about the F -unipotent radical of \mathcal{Q} , and prove two new ones.

- Theorem 3.1.** (a) $\mathcal{R}_{u,F}(\mathcal{Q})$ is a F -wound unipotent group [CGP, Corollary B.3.5].
 (b) $\mathcal{R}_{u,F}(\mathcal{Q})(F)$ is compact [Oes, Théorème VI.1].
 (c) Every F -torus in \mathcal{Q} centralizes $\mathcal{R}_{u,F}(\mathcal{Q})$ [CGP, Proposition B.4.4].
 (d) Let $\mathcal{U} \subset \mathcal{Q}$ be a root subgroup with respect to a nontrivial root of a F -torus in \mathcal{Q} . Then \mathcal{U} commutes with $\mathcal{R}_{u,F}(\mathcal{Q})$.
 (e) $\mathcal{R}_{u,F}(\mathcal{Q})(F)$ has arbitrarily small compact open subgroups which are normalized by $\mathcal{Q}(F)$.

Proof. (d) Let \mathcal{T} be the relevant F -torus in \mathcal{Q} . By part (c) $\mathcal{R}_{u,F}(\mathcal{Q}) \subset Z_{\mathcal{Q}}(\mathcal{T})$, so $\mathcal{R}_{u,F}(\mathcal{Q})$ normalizes \mathcal{U} . But at the same time $\mathcal{R}_{u,F}(\mathcal{Q})$ is normal in \mathcal{Q} and

$\mathcal{R}_{u,F}(\mathcal{Q}) \cap \mathcal{U} = \{1\}$, so $\mathcal{R}_{u,F}(\mathcal{Q})$ commutes with \mathcal{U} .

(e) Since $\mathcal{R}_{u,F}(\mathcal{Q})(F)$ is totally disconnected, compact and Hausdorff, it is a profinite group and has small compact open subgroups, say K .

Let \mathcal{P}_\emptyset be a minimal pseudo-parabolic subgroup of \mathcal{Q} containing $Z_{\mathcal{Q}}(\mathcal{S}_\emptyset)$. By [CGP, Propositions C.2.4 and C.2.26]

$$(48) \quad \mathcal{P}_\emptyset = Z_{\mathcal{Q}}(\mathcal{S}_\emptyset) \times \mathcal{R}_{us,F}(\mathcal{P}_\emptyset) = Z_{\mathcal{Q}}(\mathcal{S}_\emptyset) \times \prod_{\alpha \in \Phi(\mathcal{Q}, \mathcal{S}_\emptyset)_{\text{red}}^+} \mathcal{U}_\alpha,$$

where $\Phi(\mathcal{Q}, \mathcal{S}_\emptyset)^+$ is the positive subsystem of $\Phi(\mathcal{Q}, \mathcal{S}_\emptyset)$ consisting of all roots that appear in $\text{Lie}(\mathcal{R}_{us,F}(\mathcal{P}_\emptyset))$. By part (d) $\mathcal{R}_{u,F}(\mathcal{Q})$ commutes with $\mathcal{R}_{us,F}(\mathcal{P}_\emptyset)$. Now the Bruhat decomposition (Theorem 5.3 for $\mathcal{Q}, \mathcal{P}_\emptyset$) shows that

$$(49) \quad \{qKq^{-1} : q \in \mathcal{Q}\} = \{nKn^{-1} : n \in N_{\mathcal{Q}}(\mathcal{S}_\emptyset)\}.$$

By part (c) \mathcal{S}_\emptyset centralizes $\mathcal{R}_{u,F}(\mathcal{Q})$. The argument with (32), (33) and (34) shows that there exists an open neighborhood U_D of 1 in \mathcal{Q} , all whose elements normalize K . By Proposition 2.3 for $Z_{\mathcal{Q}}(\mathcal{S}_\emptyset)$ we may assume that U_D is a compact open subgroup of $Z_{\mathcal{Q}}(\mathcal{S}_\emptyset)$. Then $N_{N_{\mathcal{Q}}(\mathcal{S}_\emptyset)}(K)$ contains the open subgroup $\mathcal{S}_\emptyset U_D$. By (57) (whose proof does not rely on the current part e) $\mathcal{S}_\emptyset U_D$ has finite index in $N_{\mathcal{Q}}(\mathcal{S}_\emptyset)$. Hence $[N_{N_{\mathcal{Q}}(\mathcal{S}_\emptyset)}(K) : \mathcal{S}_\emptyset U_D]$ and the set (49) are finite. Then $K' := \bigcap_{q \in \mathcal{Q}} qKq^{-1}$ is a finite intersection, so in particular another compact open subgroup of $\mathcal{R}_{u,F}(\mathcal{Q})(F)$. Clearly it is normalized by \mathcal{Q} . Starting with a very small $K \subset \mathcal{R}_{u,F}(\mathcal{Q})(F)$, we can make $K' \subset K$ arbitrarily small. \square

The definition of the unipotent radical implies that $\mathcal{G} := \mathcal{Q}/\mathcal{R}_{u,F}(\mathcal{Q})$ is a pseudo-reductive F -group. One could be inclined to deduce from this that the quasi-reductive group $\mathcal{Q}(F)$ is an extension of the pseudo-reductive group $\mathcal{G}(F)$ by the compact group $\mathcal{R}_{u,F}(\mathcal{Q})(F)$. However, it seems that this need not be true in general, the correct analogue is more complicated. Consider the short exact sequence of linear algebraic groups

$$(50) \quad 1 \rightarrow \mathcal{R}_{u,F}(\mathcal{Q}) \rightarrow \mathcal{Q} \xrightarrow{\pi} \mathcal{G} = \mathcal{Q}/\mathcal{R}_{u,F}(\mathcal{Q}) \rightarrow 1.$$

It induces an exact sequence of F -rational points

$$(51) \quad 1 \rightarrow \mathcal{R}_{u,F}(\mathcal{Q})(F) \rightarrow \mathcal{Q}(F) \xrightarrow{\pi} \mathcal{G}(F),$$

where the last map need not be surjective. We denote the image of $\pi : \mathcal{Q}(F) \rightarrow \mathcal{G}(F)$ by $\tilde{\mathcal{Q}}$.

Theorem 3.2. (a) *The map $\pi : \mathcal{Q}(F) \rightarrow \mathcal{G}(F)$ is smooth and open with respect to the locally compact topology. The group $\tilde{\mathcal{Q}}$ has finite index in $\mathcal{G}(F)$.*

(b) *The group $\tilde{\mathcal{Q}}$ contains all maximal F -tori of $\mathcal{G}(F)$ and all groups $\mathcal{R}_{us,F}(\mathcal{P})(F)$, where \mathcal{P} is a pseudo-parabolic F -subgroup of \mathcal{G} .*

(c) *Let \mathcal{S}_\emptyset be a maximal F -split torus in \mathcal{G} and let \mathcal{P} be a pseudo-parabolic F -subgroup containing $Z_{\mathcal{G}}(\mathcal{S}_\emptyset)$. Then $\tilde{\mathcal{Q}}$ satisfies the Bruhat decomposition*

$$\tilde{\mathcal{Q}} = (\mathcal{P}(F) \cap \tilde{\mathcal{Q}}) N_{\tilde{\mathcal{Q}}}(\mathcal{S}_\emptyset) (\mathcal{P}(F) \cap \tilde{\mathcal{Q}}).$$

Proof. (a) This is an instance of Proposition 1.2.

(b) Let \mathcal{S} be a maximal F -torus of \mathcal{G} . By [CGP, Proposition 2.2.10] $\pi^{-1}(\mathcal{S})$ contains a maximal F -torus \mathcal{S}' of \mathcal{Q} . By Theorem 3.1.c $\mathcal{R}_{u,F}(\mathcal{Q})$ commutes with \mathcal{S}' , and $\mathcal{R}_{u,F}(\mathcal{Q}) \cap \mathcal{S}'$ equals $\{1\}$ because one group is unipotent and the other is semisimple.

Then $\mathcal{S}' \times \mathcal{R}_{u,F}(\mathcal{Q})$ is a subgroup of $\pi^{-1}(\mathcal{S})$, which by Proposition 1.2.b and (50) in fact exhausts $\pi^{-1}(\mathcal{S})$. Hence

$$(52) \quad \mathcal{S} \cong (\mathcal{S}' \times \mathcal{R}_{u,F}(\mathcal{Q})) / \mathcal{R}_{u,F}(\mathcal{Q}) \cong \mathcal{S}'$$

and $\pi : \mathcal{S}'(F) \rightarrow \mathcal{S}(F)$ is an isomorphism.

By [CGP, Proposition 2.2.10] $\mathcal{P}' := \pi^{-1}(\mathcal{P})$ is a pseudo-parabolic F -subgroup of \mathcal{Q} . In view of [CGP, Corollary 2.2.5],

$$\mathcal{R}_{u,F}(\mathcal{P}') = \pi^{-1}(\mathcal{R}_{u,F}(\mathcal{P})) = \pi^{-1}(\mathcal{R}_{us,F}(\mathcal{P}))$$

is generated by two normal unipotent subgroups: $\ker \pi = \mathcal{R}_{u,F}(\mathcal{Q})$ and $\mathcal{R}_{us,F}(\mathcal{P}')$. These two subgroups commute by Theorem 3.1.d. As in [CGP, §2.2], let $\lambda : GL_1 \rightarrow \mathcal{Q}$ be a cocharacter such that $\mathcal{P}' = \mathcal{P}_{\mathcal{Q}}(\lambda)\mathcal{R}_{u,F}(\mathcal{Q})$. Then $\mathcal{R}_{u,F}(\mathcal{Q}) \subset \mathcal{Z}_{\mathcal{Q}}(\lambda)$ and $\mathcal{R}_{us,F}(\mathcal{P}') \subset \mathcal{U}_{\mathcal{Q}}(\lambda)$, so by [CGP, Proposition 2.1.8.(2)] their intersection is trivial. We deduce that

$$\mathcal{R}_{u,F}(\mathcal{P}') = \mathcal{R}_{u,F}(\mathcal{Q}) \times \mathcal{R}_{us,F}(\mathcal{P}')$$

and $\mathcal{R}_{us,F}(\mathcal{P}') \cong \mathcal{R}_{us,F}(\mathcal{P})$. In particular

$$(53) \quad \pi : \mathcal{R}_{us,F}(\mathcal{P}')(F) \rightarrow \mathcal{R}_{us,F}(\mathcal{P})(F) \text{ is an isomorphism.}$$

This proves that $\pi(\mathcal{Q}(F))$ contains the required subgroups.

(c) It suffices to consider the case of a minimal pseudo-parabolic F -subgroup \mathcal{P}_{\emptyset} of \mathcal{G} containing $Z_{\mathcal{G}}(\mathcal{S}_{\emptyset})$. Let N be the smallest normal subgroup of $\mathcal{G}(F)$ which contains all the subgroups $\mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)$. By [CGP, Proposition C.2.2.4] N contains representatives for the Weyl group of $(\mathcal{G}, \mathcal{S}_{\emptyset})$. So does $\tilde{Q}(F)$, because $N \subset \tilde{Q}$ by part (b). Recall the Bruhat decomposition – Theorem 5.3 with representatives for $W(\mathcal{G}, \mathcal{S}_{\emptyset})$ in \tilde{Q} :

$$\mathcal{G}(F) = \mathcal{P}_{\emptyset}(F)N_{\mathcal{G}}(\mathcal{S}_{\emptyset})(F)\mathcal{P}_{\emptyset}(F) = \mathcal{P}_{\emptyset}(F)Z_{\mathcal{G}}(\mathcal{S}_{\emptyset})(F)W(\mathcal{G}, \mathcal{S}_{\emptyset})\mathcal{P}_{\emptyset}(F).$$

With (48) we can rewrite this as

$$(54) \quad \mathcal{G}(F) = Z_{\mathcal{G}}(\mathcal{S}_{\emptyset})(F)\mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)W(\mathcal{G}, \mathcal{S}_{\emptyset})\mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F).$$

By part (b) the three rightmost terms are contained in \tilde{Q} , so we obtain

$$\begin{aligned} \tilde{Q} &= Z_{\tilde{Q}}(\mathcal{S}_{\emptyset})\mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)W(\mathcal{G}, \mathcal{S}_{\emptyset})\mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F) \\ &= \mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)Z_{\tilde{Q}}(\mathcal{S}_{\emptyset})W(\mathcal{G}, \mathcal{S}_{\emptyset})Z_{\tilde{Q}}(\mathcal{S}_{\emptyset})\mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F) \\ &= (\mathcal{P}_{\emptyset}(F) \cap \tilde{Q})N_{\tilde{Q}}(\mathcal{S}_{\emptyset})(\mathcal{P}_{\emptyset}(F) \cap \tilde{Q}). \end{aligned} \quad \square$$

3.1. Valuated root data.

We would like to generalize our results for pseudo-reductive groups to quasi-reductive groups. In spite of Theorem 3.2 this is not so trivial, and we find it convenient to use more Bruhat–Tits theory. Let \mathcal{S}_{\emptyset} be a maximal F -split torus in \mathcal{Q} . By (52) we may identify it with its image in \mathcal{G} , which we also denote by \mathcal{S}_{\emptyset} .

Let $\Phi(\mathcal{Q}, \mathcal{S}_{\emptyset})$ be the root system of \mathcal{Q} with respect to the maximal F -split torus \mathcal{S}_{\emptyset} . Since $\mathcal{R}_{u,F}(\mathcal{Q})$ commutes with \mathcal{S}_{\emptyset} , $\Phi(\mathcal{Q}, \mathcal{S}_{\emptyset})$ can be identified with $\Phi(\mathcal{G}, \mathcal{S}_{\emptyset})$ [CGP, Theorem C.2.15]. In particular they have the same Weyl group

$$(55) \quad N_{\mathcal{Q}}(\mathcal{S}_{\emptyset})/Z_{\mathcal{Q}}(\mathcal{S}_{\emptyset}) = W(\mathcal{Q}, \mathcal{S}_{\emptyset}) \cong W(\mathcal{G}, \mathcal{S}_{\emptyset}) = N_{\mathcal{G}}(\mathcal{S}_{\emptyset})/Z_{\mathcal{G}}(\mathcal{S}_{\emptyset}).$$

Lemma 3.3. (a) $Z_{\mathcal{Q}}(\mathcal{S}_{\emptyset})/\mathcal{S}_{\emptyset}$ is F -anisotropic and $Z_{\mathcal{Q}}(\mathcal{S}_{\emptyset})/\mathcal{S}_{\emptyset}$ is compact.

(b) $Z_{\mathcal{Q}}(\mathcal{S}_{\emptyset})$ has a unique maximal compact subgroup $Z_{\mathcal{Q}}(\mathcal{S}_{\emptyset})_{\text{cpt}}$, and the quotient $Z_{\mathcal{Q}}(\mathcal{S}_{\emptyset})/Z_{\mathcal{Q}}(\mathcal{S}_{\emptyset})_{\text{cpt}}$ is a finitely generated free abelian group.

(c) $Z_Q(\mathcal{S}_\theta)$ has arbitrarily small compact open subgroups which are normal in $N_Q(\mathcal{S}_\theta)$.

Proof. (a) Since $\mathcal{R}_{u,F}(\mathcal{Q}) \cap \mathcal{S}_\theta = \{1\}$, there is a short exact sequence

$$1 \rightarrow \mathcal{R}_{u,F}(\mathcal{Q}) \rightarrow Z_{\mathcal{Q}}(\mathcal{S}_\theta)/\mathcal{S}_\theta \rightarrow Z_G(\mathcal{S}_\theta)/\mathcal{S}_\theta \rightarrow 1.$$

By Lemma 2.1.b the outer terms are F -anisotropic, so the middle term is also F -anisotropic. By Proposition 1.1 $(Z_{\mathcal{Q}}(\mathcal{S}_\theta)/\mathcal{S}_\theta)(F)$ and its closed subgroup $Z_Q(\mathcal{S}_\theta)/\mathcal{S}_\theta$ are compact.

(b) Clearly $\pi(Z_{\mathcal{Q}}(\mathcal{S}_\theta)) \subset Z_G(\mathcal{S}_\theta)$. By Theorem 3.2, the Bruhat decompositions for \mathcal{Q} and \mathcal{G} with respect to \mathcal{S}_θ and (55), $Z_{\mathcal{Q}}(\mathcal{S}_\theta)$ must be as large as possible with this constraint. That is,

$$(56) \quad Z_{\mathcal{Q}}(\mathcal{S}_\theta) = \pi^{-1}(Z_G(\mathcal{S}_\theta)).$$

Now first the statement follows immediately from Lemma 2.6 and Theorem 3.1.b. Let $Z_Q(\mathcal{S}_\theta)$ act on \mathbb{A}_θ via $\pi : Z_Q(\mathcal{S}_\theta) \rightarrow Z_G(\mathcal{S}_\theta)$. In the proof of Lemma 2.6 we noted that $Z_G(\mathcal{S}_\theta)_{\text{cpt}}$ equals the $Z_G(\mathcal{S}_\theta)$ -stabilizer of any point of \mathbb{A}_θ , so the same goes for its preimage $Z_Q(\mathcal{S}_\theta)_{\text{cpt}}$ in $Z_Q(\mathcal{S}_\theta)$. Consequently $Z_Q(\mathcal{S}_\theta)/Z_Q(\mathcal{S}_\theta)_{\text{cpt}}$ acts freely on \mathbb{A}_θ by translations. From Lemma 2.4 we see that any $Z_G(\mathcal{S}_\theta)$ -orbit in \mathbb{A}_θ is discrete and cocompact, so $Z_Q(\mathcal{S}_\theta)/Z_Q(\mathcal{S}_\theta)_{\text{cpt}} \cong \mathbb{Z}^{\dim \mathbb{A}_\theta}$.

(c) Proposition 2.3 for $Z_{\mathcal{Q}}(\mathcal{S}_\theta)$ (with the empty root system) yields a decreasing family of compact open subgroups $K_n \subset Z_Q(\mathcal{S}_\theta)$ with $\bigcap_{n=1}^{\infty} K_n = \{1\}$. By part (a) and (95)

$$(57) \quad N_Q(\mathcal{S}_\theta)/\mathcal{S}_\theta \text{ is compact.}$$

The normalizer of K_n in $N_Q(\mathcal{S}_\theta)$ contains the cocompact open subgroup $\mathcal{S}_\theta K_n$, so it has finite index in $N_Q(\mathcal{S}_\theta)$. Therefore

$$\bigcap_{q \in N_Q(\mathcal{S}_\theta)} q K_n q^{-1}$$

is an intersection of only finitely many terms, and is open in $Z_Q(\mathcal{S}_\theta)$. By construction this intersection is also a compact normal subgroup of $N_Q(\mathcal{S}_\theta)$. \square

Each $\alpha \in \Phi(\mathcal{Q}, \mathcal{S}_\theta)$ gives rise to a root subgroup \mathcal{U}_α in \mathcal{Q} , see [CGP, C.2.21]. Let $m_\alpha = m_\alpha(u_\alpha) \in N_Q(\mathcal{S}_\theta)$ be as in [CGP, Proposition C.2.24] and define $M_\alpha = m_\alpha Z_Q(\mathcal{S}_\theta)$.

Theorem 3.4. *The group $Q = \mathcal{Q}(F)$ has a generating root datum*

$$(Z_Q(\mathcal{S}_\theta), (U_\alpha, M_\alpha)_{\alpha \in \Phi(\mathcal{Q}, \mathcal{S}_\theta)})$$

in the sense of [BrTi1, §6.1]. It admits a valuation [BrTi1, §6.2].

Proof. It was noted in [CGP, Remark C.2.28] that these F -subvarieties satisfy the axioms of a generating root datum.

By (53) every U_α maps isomorphically to its image in $G = \mathcal{G}(F)$ and by Theorem 3.2.c the U_α and $Z_{\tilde{Q}}(\mathcal{S}_\theta)$ generate $\tilde{Q} = \pi(\mathcal{Q}(F))$. Hence we can transform the above into a generating root datum for the group \tilde{Q} :

$$(58) \quad (Z_{\tilde{Q}}(\mathcal{S}_\theta), (U_\alpha, \pi(m_\alpha)Z_{\tilde{Q}}(\mathcal{S}_\theta))_{\alpha \in \Phi(\mathcal{Q}, \mathcal{S}_\theta)}).$$

A valuation of either of these two generating root data consists of a family of maps $\phi_\alpha : U_\alpha \rightarrow \mathbb{R} \cup \{\infty\}$ satisfying several conditions [BrTi1, Définition 6.2.1]. These conditions involve only the groups U_α and the sets M_α . Since $\pi : \mathcal{Q} \rightarrow G$ is bijective

on the U_α and $\pi(M_\alpha) = \pi(m_\alpha)Z_{\tilde{Q}}(S_\emptyset)$, it suffices to show that the root datum for \tilde{Q} admits a valuation. This in turn follows if we show that the generating root datum

$$(59) \quad (Z_G(S_\emptyset), (U_\alpha, \pi(m_\alpha)Z_G(S_\emptyset))_{\alpha \in \Phi(\mathcal{G}, S_\emptyset)})$$

for G admits a valuation.

Recall the shape of the pseudo-reductive group \mathcal{G} , which we described in (15) and the lines before that. Equations (13), (8) and (17) show how $\mathcal{G}(F)$ is constructed from some reductive F_i -groups $\mathcal{G}_i(F_i)$. As before, let \mathcal{T}_i be a maximal F_i -torus in \mathcal{G}_i , such that $\mathcal{S}_\emptyset \cap R_{F_i/F}(\mathcal{G}_i)$ is contained in $R_{F_i/F}(\mathcal{T}_i)$. By [CGP, Proposition A.5.15] \mathcal{T}_i contains a maximal F_i -split torus \mathcal{S}_i of \mathcal{G}_i . Then

$$\Phi(\mathcal{G}, \mathcal{S}_\emptyset) = \bigsqcup_i \Phi(R_{F_i/F}(\mathcal{G}_i)\mathcal{S}_\emptyset, \mathcal{S}_\emptyset) \quad \text{can be identified with} \quad \bigsqcup_i \Phi(\mathcal{G}_i, \mathcal{S}_i).$$

Suppose that $\alpha_i \in \Phi(\mathcal{G}_i, \mathcal{S}_i)$ corresponds to $\alpha \in \Phi(\mathcal{G}, \mathcal{S}_\emptyset)$. From (15) we see that the root subgroup $\mathcal{U}_{\alpha_i}(F_i) \subset \mathcal{G}_i(F_i)$ is mapped isomorphically to $\mathcal{U}_\alpha(F) \subset \mathcal{G}(F)$. The element $m_\alpha(u)$ can also be constructed entirely in $\mathcal{G}_i(F_i)$, and then be mapped to $\mathcal{G}(F)$ via (15).

The main result of [BrTi2] says that the generating root datum

$$(60) \quad (Z_{\mathcal{G}_i}(\mathcal{S}_i)(F_i), (\mathcal{U}_{\alpha_i}(F_i), M_{\alpha_i})_{\alpha_i \in \Phi(\mathcal{G}_i, \mathcal{S}_i)})$$

for $\mathcal{G}_i(F_i)$ admits a valuation. We use the maps $\phi_{\alpha_i} : \mathcal{U}_{\alpha_i}(F_i) \rightarrow \mathbb{R} \cup \{\infty\}$ from such a valuation, and we retract them to $\phi_\alpha : \mathcal{U}_\alpha(F) \rightarrow \mathbb{R} \cup \{\infty\}$. Then the shape of \mathcal{G} , as in Lemma 2.2, immediately implies that the all conditions of [BrTi1, Définition 6.2.1], except possibly (V2), remain valid for the ϕ_α . The condition (V2) states that, for all $\alpha \in \Phi(\mathcal{G}, \mathcal{S}_\emptyset)$ and all $m \in M_\alpha$, the function

$$\mathcal{U}_{-\alpha}(F) \setminus \{1\} \rightarrow \mathbb{R} : u \mapsto \phi_{-\alpha}(u) - \phi_\alpha(mum^{-1})$$

is constant. Recall that $M_\alpha = \pi(m_\alpha)Z_G(S_\emptyset)$. Condition (V2) holds for $\pi(m_\alpha)$ because it boils down to a statement entirely in $\mathcal{G}_i(F_i)$, where we know that we have a valuation. Therefore it suffices to show that, for every $z \in Z_G(S_\emptyset)$,

$$(61) \quad \mathcal{U}_\alpha(F) \setminus \{1\} \rightarrow \mathbb{R} : u \mapsto \phi_\alpha(u) - \phi_\alpha(zuz^{-1}) \quad \text{is constant.}$$

The construction of \mathcal{G} entails that the conjugation action $c(z)$ of z on \mathcal{U}_α is computed via elements of $Z_{\mathcal{G}'}(S_\emptyset) = \prod_i R_{F_i/F}Z_{\mathcal{G}_i}(S_\emptyset)$ and of $\prod_i R_{F_i/F}(\mathcal{T}_i/Z(\mathcal{G}_i))$. By the surjectivity of $\mathcal{G}' \times \mathcal{C} \rightarrow \mathcal{G}$ and $\mathcal{T}_i \rightarrow \mathcal{T}_i/Z(\mathcal{G}_i)$ on rational points over a separably closed field, there exists a finite Galois extension F'/F and elements $z_1 \in Z_{\mathcal{G}'}(S_\emptyset)(F')$, $z_2 \in \prod_i R_{F_i/F}(\mathcal{T}_i)(F')$ such that $c(z)$ is conjugation by z_1 composed with conjugation by z_2 . The valuations of (60) constructed by Bruhat and Tits are schematic, in particular they are compatible with finite Galois extensions, in the sense that the F -rational points can be obtained from the F' -rational points by intersecting with $\mathcal{U}_\alpha(F)$. Hence conjugation with z_1 satisfies (61), and the same for z_2 . Thus (V2) holds, which provides the desired valuations on the root data (59) and (58). \square

Bruhat and Tits also devised the notion of a prolonged valuation [BrTi1, Définition 6.4.38]. It presumes the existence of a valued root datum for a group G , with in particular a subgroup U_0 (called T in [BrTi1]), a root system Φ and root subgroups U_α for $\alpha \in \Phi$. The valuation gives rise to subgroups

$$(62) \quad U_{\alpha,r} := \phi_\alpha^{-1}([r, \infty]) \quad \alpha \in \Phi, r \in \mathbb{R}.$$

As observed in [ScSt, p. 111] any “good filtration” of U_0 gives rise to a prolongation of the valuation of a generating root datum for a locally compact group G .

Definition 3.5. A good filtration of U_0 is a family of subgroups $U_{0,k}$ ($k \in \mathbb{R}$) of U_0 , such that:

- (i) for $k \leq 0$, $U_{0,k}$ is a maximal compact subgroup of U_0 ;
- (ii) $U_{0,k} \supset U_{0,\ell}$ for $k \leq \ell$;
- (iii) $U_{0,k}$ lies between certain subgroups $H_{[k]}$ and $H_{(k)}$ of U_0 .
- (iv) $[U_{0,k}, U_{0,\ell}] \subset U_{0,k+\ell}$ for all $k, \ell \in \mathbb{R}_{\geq 0}$.

We recall from [BrTi1, 6.4.13] that $H_{(k)}$ consists of those $h \in U_0$ such that

$$(63) \quad [h, U_{\alpha,r}] \subset U_{\alpha,r+k} U_{2\alpha,2r+k} \quad \forall \alpha \in \Phi, \forall r \in \mathbb{R}.$$

By [BrTi1, 6.4.14] the group $H_{[k]}$ for G is the intersection of U_0 with the group generated by certain compact subgroups of the U_α for $\alpha \in \Phi$.

Proposition 3.6. *There exist compact open normal subgroups of $N_Q(S_\emptyset)$ which form a good filtration of $Z_Q(S_\emptyset)$ and provide a prolongation of the valuation of the generating root datum from Theorem 3.4.*

Proof. First we will settle this for the pseudo-reductive group \mathcal{G} , with $U_0 = Z_G(S_\emptyset)$. It was shown in [ScSt, Proposition I.2.6] that the valuations of the root data from (60) used above can be prolonged. In fact there exists a natural prolongations [Yu, §5], which give compact open subgroups of $Z_{\mathcal{G}_i}(\mathcal{S}_i)(F_i)$ that are stable under all automorphisms of $Z_{\mathcal{G}_i}(\mathcal{S}_i)$ as an \mathcal{O}_{F_i} -scheme.

Via the same procedure as in Theorem 3.4, these give rise to a prolongation of the valuation of the generating root datum for $\mathcal{G}'(F)$ (which is just like (59), only with G replaced by G').

This yields such subgroups $U'_{0,k}$ for $\mathcal{G}'(F)$, that is, in $Z_{G'}(S_\emptyset)$. In fact these come from \mathcal{O}_F -schemes [Yu, Corollary 8.8], and they are characteristic subgroups of $Z_{\mathcal{G}'}(\mathcal{S}_\emptyset)$ in the category of \mathcal{O}_F -schemes. The actions of $(N_{\mathcal{G}'}(\mathcal{S}_\emptyset) \rtimes \mathcal{C})(F)$ and of $N_G(\mathcal{S}_\emptyset)(F)$ on $Z_{G'}(S_\emptyset)$ are via automorphisms of \mathcal{O}_F -schemes, so $N_{G'}(S_\emptyset) \rtimes \mathcal{C}$ and $N_G(S_\emptyset)$ normalize every $U'_{0,k}$.

Recall from (10) and (8) that the data for \mathcal{C} involve a homomorphism

$$\psi_{\mathcal{C}} : \mathcal{C}(F) \rightarrow \prod_{i \in I_s \cup I_e} R_{F_i/F}(\overline{\mathcal{T}}_i/Z(\overline{\mathcal{G}}_i))(F) = \prod_{i \in I_s \cup I_e} (\overline{\mathcal{T}}_i/Z(\overline{\mathcal{G}}_i))(F_i).$$

As above, from [Yu, §5] we get analogous subgroups of $(\overline{\mathcal{T}}_i/Z(\overline{\mathcal{G}}_i))(F_i)$ for every finite extension F'_i/F_i . This yields a consistent definition of a subgroup $\psi_{\mathcal{C}}^{-1}(U'_{0,k}) \subset \mathcal{C}$, for every $k \in \mathbb{R}_{\geq 0}$. The same argument as in the proof of Theorem 3.4 shows that

$$(64) \quad [\psi_{\mathcal{C}}^{-1}(U'_{0,k}), U'_{0,\ell}] \subset U'_{0,k+\ell} \text{ for all } k, \ell \in \mathbb{R}_{\geq 0}.$$

Hence the $U'_{0,k} \rtimes \psi_{\mathcal{C}}^{-1}(U'_{0,k})$ with $k \in \mathbb{R}_{\geq 0}$ form a family of subgroups of $G' \rtimes \mathcal{C}$ which satisfies (ii) and (iv). Notice that for $k = 0$ we obtain

$$(65) \quad Z_{G'}(S_\emptyset)_{\text{cpt}} \rtimes \psi_{\mathcal{C}}^{-1}(U'_{0,0}) = (Z_{G'}(S_\emptyset) \rtimes \mathcal{C})_x,$$

for any point x in the standard apartment of $\mathcal{B}(\mathcal{G}', F)$.

However, $\psi_{\mathcal{C}}^{-1}(U'_{0,k})$ is compact if and only if $Z(\mathcal{G})(F)$ is compact. When $Z(\mathcal{G})(F)$ is not compact, we need to do more work. Applying Proposition 2.3 to \mathcal{C} with the trivial root system gives us a decreasing sequence of compact open subgroups C_n ($n \in \mathbb{N}$) of \mathcal{C} , with $\bigcap_{n=1}^{\infty} C_n = \{1\}$. We consider

$$(66) \quad U'_{0,k} \rtimes (\psi_{\mathcal{C}}^{-1}(U'_{0,k}) \cap C_{[k]}) \quad k \in \mathbb{R},$$

where $\lceil k \rceil \in \mathbb{Z}$ denotes the ceiling of k .

The group $(\mathcal{S}' \rtimes \mathcal{S})(F)$, with \mathcal{S}' as in (19), is contained in $Z_{G'}(S_\emptyset) \rtimes C$ and centralizes \mathcal{C} . Thus it normalizes (66). By Lemma 2.1.b $(\mathcal{S}' \rtimes \mathcal{S})(F)$ is cocompact in $N_{G'}(S_\emptyset) \rtimes C$, and (66) is compact and open in $N_{G'}(S_\emptyset) \rtimes C$. Therefore the normalizer of (66) has finite index in $N_{G'}(S_\emptyset) \rtimes C$. In view of Proposition 1.2.c, dividing out the central subgroup $\alpha(T')$ maps $N_{G'}(S_\emptyset) \rtimes C$ to a finite index subgroup of $N_G(S_\emptyset)$. Hence the normalizer of (66) in $N_G(S_\emptyset)$ has finite index in there. Then the intersection in

$$(67) \quad \bigcap_{g \in N_G(S_\emptyset)} g(U'_{0,k} \rtimes (\psi_{\mathcal{C}}^{-1}(U'_{0,k}) \cap C_{\lceil k \rceil}))g^{-1}$$

is essentially over finitely many terms, and the result is again open in $Z_{G'}(S_\emptyset) \rtimes C$. Moreover (67) is a compact subgroup of $N_{G'}(S_\emptyset) \rtimes C$, normalized by $N_G(S_\emptyset)$. Since $N_G(S_\emptyset) \rtimes C$ normalizes $U'_{0,k}$, the coordinates of the elements of (67) in $Z_{G'}(S_\emptyset)$ are precisely $U'_{0,k}$. Therefore (67) has the form $U'_{0,k}C'_k$ for a compact open subgroup

$$C'_k \subset \psi_{\mathcal{C}}^{-1}(U'_{0,k}) \cap C_{\lceil k \rceil}.$$

For $k \in \mathbb{R}_{\leq 0}$, we put $U_{0,k} = Z_G(S_\emptyset)_{\text{cpt}}$, which is compact, open and normal in $N_G(S_\emptyset)$ by Lemma 2.6. For $k \in \mathbb{R}_{> 0}$ we define $U_{0,k}$ to be the image of $U'_{0,k}C'_k$ in G . By the above construction and Proposition 1.2, every $U_{0,k}$ is a compact open normal subgroup of $N_G(S_\emptyset)$.

Now property (i) holds by definition, while the property (ii) for $\tilde{U}_{0,k} \rtimes \psi_{\mathcal{C}}^{-1}(U'_{0,k})$ immediately implies it also for $U_{0,k}$. Since C is commutative, (64) entails that $U'_{0,k}C'_k$ and $U_{0,k}$ satisfy (iv). The description of G as in (15) implies that $H_{\lceil k \rceil}$ for G is just the image of $H_{\lceil k \rceil}$ for G' , which is contained in $U'_{0,k}$ by (iii). So $H_{\lceil k \rceil} \subset U_{0,k}$, as required. All elements of $U'_{0,k}$ have the property (63) and

$$U'_{0,k}C'_k \subset U'_{0,k}\psi_{\mathcal{C}}^{-1}(U'_{0,k}).$$

Hence taking commutators with any element of $U_{0,k}$ also behaves as in (63) and $U_{0,k} \subset H_{(k)}$. We conclude that (iii) holds for the groups $U_{0,k}$, and that they define a prolongation of the valuated root datum for G .

To prolong the valuation of the generating root datum for Q , we can take the groups $\pi^{-1}(U_{0,k})$ with $k \in \mathbb{R}$. By (56) these are subgroups of the group $Z_Q(S_\emptyset)$, which plays the role of U_0 for Q . The properties (ii)–(iv) of the groups $U_{0,k}$ immediately imply them also for the groups $\pi^{-1}(U_{0,k})$. By Theorem 3.1.c $\pi^{-1}(N_G(S_\emptyset)) = N_Q(S_\emptyset)$, so the $\pi^{-1}(U_{0,k})$ are open normal subgroups of $N_Q(S_\emptyset)$. Theorem 3.1.b shows that they are compact, and that

$$\pi^{-1}(U_{0,k}) = \pi^{-1}(Z_G(S_\emptyset)_{\text{cpt}}) = Z_Q(S_\emptyset)_{\text{cpt}} \quad \text{for } k \leq 0. \quad \square$$

Notice that, when \mathcal{Q} is not pseudo-reductive, the intersection $\bigcap_{k \in \mathbb{R}} U_{0,k} = \mathcal{R}_{u,F}(\mathcal{Q})(F)$ is more than just the identity element. It would be useful to find a prolonged valuation with $\bigcap_{k \in \mathbb{R}} U_{0,k} = \{1\}$, but we did not achieve this in general.

3.2. Affine buildings.

Let \mathcal{Z} be a maximal F -split torus in $Z(\mathcal{Q})$. Its image in the commutative quasi-reductive F -group $\mathcal{Q}/\mathcal{D}(\mathcal{Q})\mathcal{R}_{u,F}(\mathcal{Q})$ is a maximal F -split torus \mathcal{Z}' of the same rank. In particular we can identify $X_*(\mathcal{Z}') \otimes_{\mathbb{Z}} \mathbb{R}$ with $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. We let $\mathcal{Q}(F)$ act on this space via $\mathcal{Q} \rightarrow \mathcal{Q}/\mathcal{D}(\mathcal{Q})\mathcal{R}_{u,F}(\mathcal{Q})$ and then as in Lemma 2.4. Notice that this action is by translations, and in particular by isometries.

For use in Paragraph 4.2 we introduce and investigate a subgroup of Q which plays a role analogous to the kernel of all unramified characters of Q (if Q were reductive). The action of Q on $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ yields a group homomorphism

$$(68) \quad Q \rightarrow X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R} : q \mapsto q \cdot 0.$$

We define 0Q as the kernel of this homomorphism, or equivalently as the isotropy group of $0 \in X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ in Q .

- Lemma 3.7.** (a) *The group 0Q is an open normal, unimodular subgroup of Q .*
 (b) *0Q contains every compact subgroup of Q .*
 (c) *The intersection ${}^0Q \cap Z(Q)$ is compact and $Q/{}^0QZ(Q)$ is finite.*

Proof. (a) From the proof of Lemma 2.4 we see that the stabilizer in $(\mathcal{Q}/\mathcal{D}(\mathcal{Q})\mathcal{R}_{u,F}(\mathcal{Q}))(F)$ of any point of $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ is open and normal in that group. Hence 0Q is an open normal subgroup of Q . Since Q is unimodular (Lemma 1.3), so is 0Q .

(b) The vector space $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ has no compact subgroups besides $\{0\}$. Hence the image of any compact subgroup K of Q under (68) is $\{0\}$, which means that $K \subset {}^0Q$.

(c) The quotient $Q/{}^0Q$ injects into $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. We note that by (39) and (40) the image is a lattice of full rank in $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. The image of $\mathcal{Z}(F) \subset Z(Q)$ is the sublattice $X_*(\mathcal{Z})$, which has finite index in any larger lattice in $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Hence $Q/{}^0Q\mathcal{Z}(F)$ and $Q/{}^0QZ(Q)$ are finite.

The group $Z(Q)$ acts on $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ via $\pi : Z(Q) \rightarrow \mathcal{C}(F)$. The $\mathcal{C}(F)$ -action is proper by Lemma 2.4, so the isotropy group $\mathcal{C}(F)_0$ is compact. By Theorem 3.1.b so are $\pi^{-1}(\mathcal{C}(F)_0)$ and its closed subgroup ${}^0Q \cap Z(Q)$. \square

The big advantage of Theorem 3.4 is that now many results of Bruhat and Tits apply to $Q = \mathcal{Q}(F)$, in particular it gives rise an affine building $\mathcal{BT}(\mathcal{Q}, F)$. Then $\mathcal{BT}(\mathcal{Q}, F) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ is an affine building – maybe not literally according to [BrTi1], but in the sense of [Tit, §2.1]. The remarks in [Tit] entail that the results from [BrTi1] are valid just as well for such an extended building.

- Theorem 3.8.** (a) *Q contains a double Tits system.*
 (b) *This double Tits system gives rise to a thick affine building $\mathcal{BT}(\mathcal{Q}, F)$ with an isometric, cocompact Q -action.*
 (c) *$\mathcal{BT}(\mathcal{Q}, F) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ is canonically homeomorphic to $\mathcal{B}(\mathcal{G}, F)$.*
 (d) *We endow $\mathcal{BT}(\mathcal{Q}, F) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ with the diagonal Q -action and we let Q act on $\mathcal{B}(\mathcal{G}, F)$ via $\pi : \mathcal{Q}(F) \rightarrow \mathcal{G}(F)$. Then part (c) becomes an isomorphism of Q -spaces.*
 (e) *The Q -action on $\mathcal{BT}(\mathcal{Q}, F) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ is isometric, proper and cocompact. Moreover $\mathcal{BT}(\mathcal{Q}, F) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ is a universal space for proper Q -actions in the sense of [BCH, §1].*

Proof. (a) With Theorem 3.4 at hand, this is [BrTi1, Théorème 6.5.i].

(b) This is worked out in [BrTi1, §2 and §6]. For later use we recall some parts of the construction. Firstly, a standard apartment \mathbb{A}_0 with an action of $N_Q(S_\emptyset)$ is exhibited in [BrTi1, Proposition 6.2.5]. Next, a standard Iwahori subgroup B of Q is built, and a set of chambers is defined as the collection of subgroups of Q that are conjugate to B (in other words, the set of Iwahori subgroups of Q). This yields an affine building $\mathcal{BT}(\mathcal{Q}, F)$, on which Q acts by [BrTi2, Théorème 6.5.ii].

The space $\mathcal{BT}(\mathcal{Q}, F)$ is naturally a direct product of affine buildings whose facets are simplices and whose underlying affine Weyl group is irreducible [BrTi1, Proposition 2.6.4]. Every such affine building is thick, in the sense that every simplex of codimension one is contained in the closure of at least three chambers.

(c) The group $Z_Q(S_\emptyset)$ acts by translations on \mathbb{A}_0 , and Bruhat and Tits call the isotropy subgroup (of any point of \mathbb{A}_0) H . Comparing with the proof of Lemma 2.6, we see that H contains the unique maximal compact subgroup of $Z_Q(S_\emptyset)$. In particular H contains the compact normal subgroup $\mathcal{R}_{u,F}(\mathcal{Q})(F)$ of Q .

The standard Iwahori subgroup B contains H , so also $\mathcal{R}_{u,F}(\mathcal{Q})(F) \subset B$. Since all Iwahori subgroups of Q are conjugate, $\mathcal{R}_{u,F}(\mathcal{Q})(F)$ is contained in every one of them. But every Iwahori subgroup fixes the associated chamber of $\mathcal{BT}(\mathcal{Q}, F)$ pointwise, so $\mathcal{R}_{u,F}(\mathcal{Q})(F)$ fixes this entire building pointwise. In view of the construction of the valuated root data in the proof of Theorem 3.4 (first for \tilde{Q} , from there for Q), this entails that $\mathcal{BT}(\mathcal{Q}, F)$ is the same as the affine building $\mathcal{BT}(\tilde{Q})$ associated to the analogous double Tits system for \tilde{Q} .

The latter double Tits system is contained in the subgroup \tilde{Q}' of \tilde{Q} generated by the root subgroups $U_\alpha(F)$ and \tilde{H} (the image of H in \tilde{Q}). In effect, restriction from \tilde{Q} to \tilde{Q}' means that $Z_{\tilde{Q}}(S_\emptyset)$ is replaced by $\tilde{H}X$, where X is a finitely generated free abelian subgroup of $Z_{\tilde{Q}}(S_\emptyset)$ which acts on \mathbb{A}_0 as the translation part of the affine Weyl group \mathbb{W} from [BrTi1, 6.2.11]. Here the \mathbb{W} -action is generated by the affine reflections of \mathbb{A}_0 coming from elements $m_\alpha(u) \in N_{\tilde{Q}}(S_\emptyset)$ as in the definition of M_α . Thus the entire building $\mathcal{BT}(\mathcal{Q}, F)$ depends only on \tilde{Q}' .

Similarly the affine building $\mathcal{BT}(\mathcal{G}, F)$ depends only on the subgroup G'' of G generated by the U_α and H'' (the H for G). From Theorem 3.2 we know that the difference between \tilde{Q}' and G'' is small, in a precise sense. Together with [BrTi1, 6.2.11] we find first that \tilde{Q}' and G'' have the same image in the group of affine transformations of \mathbb{A}_0 , and then that their difference comes entirely from the finite index inclusion $\tilde{H} \subset H''$. As in the proof of Theorem 3.2.c, one can find Bruhat decompositions for \tilde{Q}' and G'' :

$$(69) \quad \begin{aligned} G'' &= H'' X \mathcal{R}_{u,F}(\mathcal{P}_\emptyset)(F) W(\mathcal{G}, \mathcal{S}_\emptyset) \mathcal{R}_{u,F}(\mathcal{P}_\emptyset)(F), \\ \tilde{Q}' &= \tilde{H} X \mathcal{R}_{u,F}(\mathcal{P}_\emptyset)(F) W(\mathcal{G}, \mathcal{S}_\emptyset) \mathcal{R}_{u,F}(\mathcal{P}_\emptyset)(F). \end{aligned}$$

Since H'' and \tilde{H} are contained in the respective standard Iwahori subgroups, we deduce from (69) that there is a natural bijection between the chambers of $\mathcal{BT}(\mathcal{G}, F)$ and those of $\mathcal{BT}(\tilde{Q})$. The standard apartments in these buildings are the same, because they depend only on the root subgroups U_α and on the valuation. We conclude that $\mathcal{BT}(\tilde{Q})$ can be identified canonically with $\mathcal{BT}(\mathcal{G}, F)$.

From the above argument we also see that we obtain the same building $\mathcal{BT}(\mathcal{G}, F)$ if we replace $Z_G(S_\emptyset)$ by any subgroup containing the intersection of $Z_G(S_\emptyset)$ with the subgroup of G generated by the root subgroups. In particular $\mathcal{BT}(\mathcal{G}', F)$ is just the same as $\mathcal{BT}(\mathcal{G}, F)$.

Recall that the building $\mathcal{B}(\mathcal{G}', F)$ is constructed in (42) and Proposition 2.5, as the direct product of Bruhat–Tits buildings for the semisimple groups $\mathcal{G}_i(F_i)$. Hence there are canonical homeomorphisms

$$(70) \quad \prod_i \mathcal{BT}(\mathcal{G}_i, F_i) = \mathcal{B}(\mathcal{G}', F) = \mathcal{BT}(\mathcal{G}', F) = \mathcal{BT}(\mathcal{G}, F) = \mathcal{BT}(\tilde{Q}) = \mathcal{BT}(\mathcal{Q}, F).$$

Recall from (42) that $\mathcal{B}(\mathcal{G}, F) = \mathcal{B}(\mathcal{G}', F) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, where \mathcal{Z} denotes a maximal F -split torus in $Z(\mathcal{G})$. From the proof of Theorem 3.2.b, in particular (52), we see that the maximal F -split torus in $Z(\mathcal{Q})$ is mapped isomorphically to \mathcal{Z} by $\pi : \mathcal{Q} \rightarrow \mathcal{G}$. Therefore we may identify this \mathcal{Z} with the \mathcal{Z} in the statement of the proposition. Together with (70) this gives the desired homeomorphism.

(d) The Q -action on $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ as subset of $\mathcal{B}(\mathcal{G}, F)$ goes via $\mathcal{Q} \rightarrow \mathcal{G} \rightarrow \mathcal{C}/\phi(\mathcal{T}')$ and then as in (40). As noted after (15), the image of \mathcal{G}' in \mathcal{G} is $\mathcal{D}(\mathcal{G})$. Hence

$$\mathcal{C}/\phi(\mathcal{T}') \cong \mathcal{G}/\mathcal{D}(\mathcal{G}) \cong \mathcal{Q}/\mathcal{D}(\mathcal{Q})\mathcal{R}_{u,F}(\mathcal{Q}),$$

showing that the action of Q on $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ as defined just before the current proposition agrees with the action via G .

We already observed that the action of Q on $\mathcal{BT}(\mathcal{Q}, F)$ can be identified with the action of $\tilde{Q} = Q/\mathcal{R}_{u,F}(\mathcal{Q})(F)$ on $\mathcal{BT}(\mathcal{G}, F)$. In (70) the action of \tilde{Q} on $\mathcal{BT}(\mathcal{G}', F)$ is defined as the canonical extension of the \tilde{Q}' -action (coming from the valuated root datum for that group) to \tilde{Q} . By [BrTi2, Théorème 6.5.ii] the embedding $\tilde{Q}' \rightarrow \tilde{Q}$ has the correct type for such an extension.

Similarly the G -action on $\mathcal{BT}(\mathcal{G}', F)$ is defined by first considering the action of G'' derived from the valuated root datum, and then extending to G via the embedding $G'' \rightarrow G$. The two actions of \tilde{Q}' on these buildings can be identified via (70), and they are extended in the same way to \tilde{Q} . Therefore the actions of \tilde{Q} on $\mathcal{BT}(\mathcal{Q}, F) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ and on $\mathcal{B}(\mathcal{G}, F)$ can be identified as well.

(e) The claim about the properties of the Q -action follows immediately from Theorem 3.2 and the compactness of $\mathcal{R}_{u,F}(\mathcal{Q})(F)$. Bruhat and Tits showed that $\mathcal{BT}(\mathcal{Q}, F) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ is a CAT(0)-space [BrTi1, 3.2.1], that it has unique geodesics [BrTi1, 2.5.13] and that every compact subgroup of Q fixes a point of this affine building [BrTi1, Proposition 3.2.4]. By [BCH, Proposition 1.8] these properties guarantee that $\mathcal{BT}(\mathcal{Q}, F) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ is a universal space for proper Q -actions. \square

Theorem 3.8.c justifies the definition

$$(71) \quad \mathcal{B}(\mathcal{Q}, F) := \mathcal{BT}(\mathcal{Q}, F) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R},$$

with the Q -action from Theorem 3.8.d. Notice that with these conventions

$$X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = \mathcal{B}(\mathcal{Q}/\mathcal{D}(\mathcal{Q})\mathcal{R}_{u,F}(\mathcal{Q}), F) = \mathcal{B}(\mathcal{Q}/\mathcal{D}(\mathcal{Q}), F).$$

An apartment in the affine building $\mathcal{B}(\mathcal{Q}, F)$ is a subset of the form $\mathbb{A} \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, where \mathbb{A} is an apartment in $\mathcal{BT}(\mathcal{Q}, F)$. By Proposition 2.5.c, Theorem 3.8.d and (52) every apartment of $\mathcal{B}(\mathcal{Q}, F)$ is naturally associated to a maximal F -split torus in \mathcal{Q} , and conversely. As all maximal F -split tori in \mathcal{Q} are Q -conjugate, all apartments of $\mathcal{B}(\mathcal{G}, F)$ are Q -associate. We note also that $N_Q(S_\emptyset)$ stabilizes the apartment associated to S_\emptyset .

3.3. Compact open subgroups.

It was shown in [Loi, Theorem 1.1.2] that a connected linear algebraic F -group has a maximal compact subgroup if and only if it is quasi-reductive. We can use the affine building of $\mathcal{Q}(F)$ to construct many compact open subgroups with very specific nice properties.

Proposition 3.9. *Let $x = (x_s, x_Z) \in \mathcal{BT}(\mathcal{Q}, F) \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$.*

(a) Q_x is a compact open subgroup of Q .

- (b) The stabilizer Q_{x_s} of $x_s \in \mathcal{BT}(\mathcal{Q}, F)$ equals $Z_Q(S_\emptyset)_{x_s}Q_x$. It contains $Z(Q)Q_x$ as a cocompact subgroup.
- (c) Suppose that x_s is a vertex of $\mathcal{BT}(\mathcal{Q}, F)$, so that x is a vertex of $\mathcal{B}(\mathcal{Q}, F)$. Then Q_x is a maximal compact subgroup of Q and Q_{x_s} is a maximal compact modulo center subgroup of Q .

Proof. (a) The group Q_x is compact because the action of Q on $\mathcal{B}(\mathcal{Q}, F)$ is proper (Theorem 3.8.e). It is open in Q because this action factors via $\pi : Q \rightarrow G$ and G_x is open in G – see (44).

(b) By definition (see Theorem 3.8.d)

$$Q_x = Q_{x_s} \cap Q_{x_z}, \text{ so } Z_Q(S_\emptyset)_{x_s}Q_x \subset Q_{x_s}.$$

The Q -action on $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ factors via $(\mathcal{Q}/\mathcal{D}(\mathcal{Q}))(F)$, so for any $q \in Q_{x_s}$ we can find $q' \in Z_Q(S_\emptyset)$ with $q'qx_z = x_z$. In view of Theorem 3.8.d and the decomposition (40), we can even achieve this with $q' \in Z_Q(S_\emptyset)_{x_s}$. Then $q = (q')^{-1}(q'q)$ with $q'q \in Q_{x_s} \cap Q_{x_z} = Q_x$.

The center $Z(Q)$ acts trivially on $\mathcal{BT}(\mathcal{Q}, F)$ because the Q -action is defined in terms of conjugating certain subgroups of Q – see the proof of Theorem 3.8.b. Thus

$$Z(Q)Q_x \subset Q_{x_s} = Z_Q(S_\emptyset)_{x_s}Q_x.$$

By Lemma 3.3.a $(S_\emptyset)_{x_s}$ is cocompact in $Z_Q(S_\emptyset)_{x_s}$ and from (40) we see that $Z(Q) \cap S_\emptyset$ is cocompact in $(S_\emptyset)_{x_s}$. Hence $Q_{x_s}/Z(Q)Q_x \cong Z_Q(S_\emptyset)_{x_s}/Z(Q)$ is compact.

(c) By the Bruhat–Tits fixed point theorem [BrTi1, Proposition 3.2.4] every compact subgroup of Q fixes a point of $\mathcal{B}(\mathcal{Q}, F)$. From part (a) we know that the Q -stabilizer of any point is compact. The construction of isotropy groups for affine buildings associated to double Tits systems [BrTi1, §2.1.2] entails that the stabilizers of vertices are maximal among the stabilizers of points. Therefore Q_x is maximal compact.

As observed above, $Z(Q)$ acts trivially on $\mathcal{BT}(\mathcal{Q}, F)$. Hence we may replace Q by $Q/Z(Q)$ when it comes to the isotropy group of x_s . Using part (b) as input, the same argument as for Q_x applies. \square

We can describe maximal compact (modulo center) subgroups of Q more precisely, in terms of its building. Their characterization enables one to lift automorphisms of Q to isometries of $\mathcal{BT}(\mathcal{Q}, F)$.

- Proposition 3.10.** (a) Every maximal compact modulo center subgroup of Q is of the form Q_{y_s} for a unique $y_s \in \mathcal{BT}(\mathcal{Q}, F)$, which is the barycenter of a polysimplex.
- (b) Every maximal compact subgroup of Q is of the form $Q_{(y_s, 0)}$ for a unique $y_s \in \mathcal{BT}(\mathcal{Q}, F)$, which is the barycenter of a polysimplex.
- (c) Every group which acts on Q by automorphisms of topological groups has a natural isometric action on $\mathcal{BT}(\mathcal{Q}, F)$.

Proof. (a) Consider a maximal compact modulo center subgroup of Q . By the Bruhat–Tits fixed point theorem and Proposition 3.9.b, it equals the isotropy group Q_{y_s} , for a point $y_s \in \mathcal{BT}(\mathcal{Q}, F)$. Let \mathfrak{f} be the polysimplex of $\mathcal{BT}(\mathcal{Q}, F)$ containing y_s . By [BrTi1, §2.1], Q_{y_s} contains the parahoric subgroup $P_{\mathfrak{f}}$ associated to \mathfrak{f} .

Suppose that Q_{y_s} also fixes another point y'_s . Since $P_{\mathfrak{f}}$ only fixes $\bar{\mathfrak{f}}$, y_s lies in the closure of \mathfrak{f} . By the uniqueness of geodesics [BrTi1, Remarque 3.2.5] Q_{y_s} fixes the line segment $[y_s, y'_s]$ pointwise. Then it would also fix the intersection of $\bar{\mathfrak{f}}$ with a geodesic containing $[y_s, y'_s]$. In particular Q_{y_s} would fix a point $x_s \in \bar{\mathfrak{f}} \setminus \mathfrak{f}$ of a facet

$\mathfrak{f}' \subset \bar{\mathfrak{f}}$, and $Q_{x_s} \supset P_{\mathfrak{f}'}$. Since \mathfrak{f}' is a boundary facet of \mathfrak{f} , $P_{\mathfrak{f}'}$ is strictly larger than $P_{\mathfrak{f}}$. The construction of the Q -action on its building [BrTi1, Théorème 6.5] entails that there exists a subgroup $Q' \subset Q$ such that $Q' \cap Q_{y_s} = P_{\mathfrak{f}}$ and $Q' \cap Q_{x_s} = P_{\mathfrak{f}'}$. Thus $Q_{x_s} \supsetneq Q_{y_s}$, contradicting the maximality (modulo center) of Q_{y_s} .

Therefore y_s is the unique fixed point of Q_{y_s} . Since Q_{y_s} stabilizes \mathfrak{f} and acts on it by polysimplicial automorphisms, this is only possible if y_s is the barycenter of \mathfrak{f} . (b) Consider any maximal compact subgroup of G . By the Bruhat–Tits fixed point theorem it is of the form Q_y for some $y \in \mathcal{B}(\mathcal{Q}, F)$. Since Q acts by translations on $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, Q_y fixes $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ pointwise. Hence $Q_y = Q_{(y_s, 0)}$ for some $y_s \in \mathcal{BT}(\mathcal{Q}, F)$.

If Q_y would fix another $y'_s \in \mathcal{BT}(\mathcal{Q}, F)$, then we proceed as in the proof of part (a) to find an x_s fixed by Q_y . We obtain $Q' \cap Q_y = P_{\mathfrak{f}} \cap {}^0Q$ and $Q' \cap Q_{(x_s, 0)} = P_{\mathfrak{f}'} \cap {}^0Q$. The difference between $P_{\mathfrak{f}}$ and $P_{\mathfrak{f}'}$ can already be detected in some root subgroup (see [Tit, §3.1] and [BrTi1, Proposition 7.4.4]) and those are contained in 0Q (Lemma 3.7). Therefore $Q' \cap Q_{(x_s, 0)} \supsetneq Q' \cap Q_y$, which would contradict the maximality of G_y . This remainder of the argument is exactly as in part (a).

(c) Let $X_M \subset \mathcal{BT}(\mathcal{Q}, F)$ be the set of points whose isotropy group is a maximal compact modulo center subgroup of Q . Let Γ be a group acting on Q by automorphisms of topological groups. For $\gamma \in \Gamma$ and $y_s \in X_M$, $\gamma(Q_{y_s})$ is another maximal compact modulo center subgroup of Q . By part (a) it is of the form Q_{z_s} , for a unique point $z_s \in \mathcal{BT}(\mathcal{Q}, F)$. The definition $\tilde{\gamma}(y_s) := z_s$ provides a permutation of X_M .

Every $y_s \in X_M$ determines a set of “neighbors”

$$Nb(y_s) := \{x_s \in X_M \setminus \{y_s\} : [x_s, y_s] \cap X_M = \{x_s, y_s\}\}$$

In view of part (a), $Nb(y_s)$ is finite and equals the set of $x_s \in X_M \setminus \{y_s\}$ such that Q_{x_s} and Q_{y_s} are the only maximal compact modulo center subgroups of Q containing $Q_{x_s} \cap Q_{y_s}$. As $Q_{\tilde{\gamma}(y_s)} = \gamma(Q_{y_s})$, we have $\tilde{\gamma}(Nb(y_s)) = Nb(\tilde{\gamma}(y_s))$. Using part (a), this implies that $\tilde{\gamma}$ is an isometry of X_M , and that it maps the collection of vertices of any single polysimplex to the collection of vertices of a single (other) polysimplex. Thus $\tilde{\gamma}$ extends uniquely to an isometry of $\mathcal{BT}(\mathcal{Q}, F)$ which is affine on every polysimplex.

For $\gamma' \in \Gamma$, the uniqueness of $\tilde{\gamma}$ implies that $\tilde{\gamma\gamma'} = \tilde{\gamma}\tilde{\gamma}'$. Hence $\gamma \mapsto \tilde{\gamma}$ defines a group homomorphism $\Gamma \rightarrow \text{Isom}(\mathcal{BT}(\mathcal{Q}, F))$. \square

Now we can formulate and prove a more precise version of Proposition 2.3. The main advantage over Proposition 2.3 is that now we can find arbitrarily small compact open subgroups that are normal in the isotropy group of a point of $\mathcal{B}(\mathcal{Q}, F)$. This property will be used in Theorem 4.9, via [Ber].

Theorem 3.11. *Let \mathcal{S} be a maximal F -split torus of \mathcal{Q} and let $\mathbb{A}_{\mathcal{S}} = X_*(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ be the associated apartment of $\mathcal{B}(\mathcal{Q}, F)$. Let $x \in \mathbb{A}_{\mathcal{S}}$ and let x_s be its image in $\mathcal{BT}(\mathcal{Q}, F)$ via (71). There exists a sequence of compact open subgroups $(K_n)_{n=1}^{\infty}$ of Q such that:*

- (a) $K_n \supset K_{n+1}$ and the K_n form a neighborhood basis of 1 in Q .
- (b) $K_n \subset Q_x \subset Q_{x_s}$ and K_n is normal in Q_{x_s} .
- (c) K_n satisfies the Iwahori decomposition: Suppose that $\mathcal{P}_{\mathcal{Q}}(\lambda)$ is a pseudo-parabolic F -subgroup of \mathcal{Q} containing $Z_{\mathcal{Q}}(\mathcal{S})$. Then the multiplication map

$$(K_n \cap U_{\mathcal{Q}}(\lambda)(F)) \times (K_n \cap Z_{\mathcal{Q}}(\lambda)(F)) \times (K_n \cap U_{\mathcal{Q}}(\lambda^{-1})(F)) \rightarrow K_n$$

is a homeomorphism.

(d) Suppose that $a \in Z_Q(\mathcal{S})$ is positive with respect to $\mathcal{P}_Q(\lambda)$:

$$\begin{aligned} \langle \alpha, a \cdot x - x \rangle &\geq 0 \quad \forall \alpha \in \Phi(\mathcal{P}_Q(\lambda), \mathcal{S}), \\ \langle \alpha, a \cdot x - x \rangle &= 0 \quad \forall \alpha \in \Phi(\mathcal{Z}_Q(\lambda), \mathcal{S}). \end{aligned}$$

Then a normalizes $K_n \cap Z_Q(\lambda)(F)$ and

$$\begin{aligned} a(K_n \cap U_Q(\lambda)(F))a^{-1} &\supset (K_n \cap U_Q(\lambda)(F)), \\ a(K_n \cap U_Q(\lambda^{-1})(F))a^{-1} &\subset (K_n \cap U_Q(\lambda^{-1})(F)). \end{aligned}$$

(e) Suppose that $a \in Z_Q(\mathcal{S})$ is positive with respect to $\mathcal{P}_Q(\lambda)$ and

$$\langle \alpha, a \cdot x - x \rangle > 0 \quad \forall \alpha \in \Phi(\mathcal{P}_Q(\lambda), \mathcal{S}).$$

Then $\{a^m(K_n \cap U_Q(\lambda)(F))a^{-m} : m \in \mathbb{N}\}$ forms a neighborhood basis of 1 in $U_Q(\lambda)(F)$ and

$$\bigcup_{m \in \mathbb{N}} a^{-m}(K_n \cap U_Q(\lambda)(F))a^m = U_Q(\lambda)(F).$$

Similarly $\{a^{-m}(K_n \cap U_Q(\lambda^{-1})(F))a^m : m \in \mathbb{N}\}$ forms a neighborhood basis of 1 in $U_Q(\lambda^{-1})(F)$ and

$$\bigcup_{m \in \mathbb{N}} a^m(K_n \cap U_Q(\lambda^{-1})(F))a^{-m} = U_Q(\lambda^{-1})(F).$$

Proof. Recall that in Proposition 3.6 we exhibited normal subgroups of $N_G(S_\emptyset)$ to prolong the valuation for the generating root datum of G from Theorem 3.4. Given this, $n \in \mathbb{Z}_{\geq 0}$ and a facet $\mathfrak{f} \subset \mathcal{BT}(\mathcal{G}, F) = \mathcal{BT}(\mathcal{Q}, F)$ containing \tilde{x} , Schneider and Stuhler [ScSt, p.114] construct a subgroup $U_{\mathfrak{f}}^{(n)} \subset G$. It is checked in [ScSt, §I.2] that it has the properties (a), (b) and (c) required in the theorem, for \mathcal{G}, F and \tilde{x} . Let us note here that

$$(72) \quad Z_G(S_\emptyset) \cap U_{\mathfrak{f}}^{(n)} = U_{0, n+} := \bigcap_{k \in \mathbb{R}_{> n}} U_{0, k}.$$

For an arbitrary quasi-reductive group Q we could use the same construction, that would produce a family of subgroups $\pi^{-1}(U_{\mathfrak{f}}^{(n)}) \subset Q$ which almost satisfies the theorem. The only problem would be part (a): the intersection of those groups would be $\mathcal{R}_{u, F}(Q)(F)$, which is nontrivial if Q is not pseudo-reductive. Below we will use the properties (b) and (c) of the groups $\pi^{-1}(U_{\mathfrak{f}}^{(n)})$. By adjusting these groups, we construct the K_n .

In view of the conjugacy of maximal F -split tori in Q [CGP, Theorem C.2.3], we may assume that $\mathcal{S} = \mathcal{S}_\emptyset$, $\mathbb{A}_{\mathcal{S}} = \mathbb{A}_\emptyset$ and $\mathfrak{f} \subset \mathbb{A}_\emptyset$. With Lemma 3.3.c we can find a decreasing sequence of compact open subgroups Z_r ($r \in \mathbb{R}$) of $Z_Q(S_\emptyset)$, such that every Z_r is normal in $N_Q(S_\emptyset)$ and $\bigcap_{r \in \mathbb{R}} Z_r = \{1\}$.

For every $\alpha \in \Phi(Q, \mathcal{S}_\emptyset)$ and every $n \in \mathbb{N}$ we have a set of commutators

$$(73) \quad [U_\alpha \cap Q_x, U_{-\alpha} \cap \pi^{-1}(U_{\mathfrak{f}}^{(n)})] \subset \pi^{-1}(U_{\mathfrak{f}}^{(n)}) \subset Q.$$

Define the positive roots $\Phi(Q, \mathcal{S}_\emptyset)^+$ as in (79). In view of the Iwahori decomposition of $\pi^{-1}(U_{\mathfrak{f}}^{(n)})$, for sufficiently large n we can find $k(n+) \in \mathbb{R}$ such that

$$\left(\prod_{\alpha \in -\Phi(Q, \mathcal{S}_\emptyset)_{\text{red}}^+} U_\alpha \cap \pi^{-1}(U_{\mathfrak{f}}^{(n)}) \right) Z_{k(n+)} \left(\prod_{\alpha \in \Phi(Q, \mathcal{S}_\emptyset)_{\text{red}}^+} U_\alpha \cap \pi^{-1}(U_{\mathfrak{f}}^{(n)}) \right)$$

contains the union, over $\alpha \in \Phi(\mathcal{Q}, \mathcal{S}_\emptyset)$, of the sets (73). In the notation of (72), this says precisely that $Z_{k(n+)}$ contains the group $H_{[n+]}$ from Definition 3.5. Since $\pi^{-1}(U_{0,n+})$ comes from a prolonged valuation, it also contains $H_{[n+]}$.

For each α the set (73) decreases to $\{1\}$ when $n \rightarrow \infty$, so we can achieve that $k((n+1)+) > k(n+)$ and $\lim_{n \rightarrow \infty} k(n+) = \infty$. For sufficiently large n we consider the group

$$(74) \quad \tilde{U}_{0,n+} := \pi^{-1}(U_{0,n+}) \cap Z_{k(n+)}.$$

Both terms on the right hand side are open and normal in $N_Q(S_\emptyset)$ (the former by Proposition 3.6) and $Z_{k(n+)}$ is compact, so $\tilde{U}_{0,n+}$ is compact open and normal in $N_Q(S_\emptyset)$. The $\tilde{U}_{0,n+}$ have intersection $\{1\}$ and by Proposition 3.6 they fulfill (ii) and (iii) of Definition 3.5. (However, it is not clear whether they satisfy (iv), we have too little control over $[\tilde{U}_{0,n+}, \tilde{U}_{0,m+}]$.) Slightly varying on the construction of $U_{\mathfrak{f}}^{(n)}$ in [ScSt], we define

$$K_n := \left(\prod_{\alpha \in -\Phi(\mathcal{Q}, \mathcal{S}_\emptyset)_{\text{red}}^+} U_\alpha \cap \pi^{-1}(U_{\mathfrak{f}}^{(n)}) \right) \tilde{U}_{0,n+} \left(\prod_{\alpha \in \Phi(\mathcal{Q}, \mathcal{S}_\emptyset)_{\text{red}}^+} U_\alpha \cap \pi^{-1}(U_{\mathfrak{f}}^{(n)}) \right).$$

The arguments in [ScSt, §I.2] show that K_n is a normal subgroup of Q_x and that it admits an Iwahori decomposition. Recall from Proposition 3.6.b that $Q_{x_s} = Z_Q(S_\emptyset)_{x_s} Q_x$. Since $\pi : Q \rightarrow G$ becomes an isomorphism on root subgroups (53), the properties of $U_{\mathfrak{f}}^{(n)}$ entail that $Z_Q(S_\emptyset)_{\bar{x}}$ normalizes

$$\prod_{\alpha \in \Phi(\mathcal{Q}, \mathcal{S}_\emptyset)_{\text{red}}^+} U_\alpha \cap \pi^{-1}(U_{\mathfrak{f}}^{(n)}),$$

and similarly with negative roots. As $\tilde{U}_{0,n+}$ is a normal subgroup of $N_Q(S_\emptyset)$, $Z_Q(S_\emptyset)_{\bar{x}}$ also normalizes $\tilde{U}_{0,n+}$. Now we see from the Iwahori decomposition of K_n that $Z_Q(S_\emptyset)_{x_s}$ and Q_{x_s} normalize it.

(d) In view of the Iwahori decomposition of K_n , we only have to check the corresponding statements for each $K_n \cap U_\alpha$ with $\alpha \in \Phi(\mathcal{Q}, \mathcal{S}_\emptyset)_{\text{red}} \cup \{0\}$. As explained after (74), $a \in Z_Q(S_\emptyset)$ normalizes $K_n \cap Z_Q(S_\emptyset) = \tilde{U}_{0,n+}$.

For $\alpha \in \Phi(\mathcal{G}, \mathcal{S}_\emptyset)$, $\pi : \mathcal{Q} \rightarrow \mathcal{G}$ restricts to an isomorphism on \mathcal{U}_α and a normalizes \mathcal{U}_α . Therefore it suffices to consider the effect of conjugation by $\pi(a) \in Z_G(S_\emptyset)$ on $U_\alpha \cap \pi(K_n) = U_\alpha \cap U_{\mathfrak{f}}^{(n)}$. By [ScSt, p. 114]

$$\pi(a) U_{\mathfrak{f}}^{(n)} \pi(a)^{-1} = U_{\pi(a)\mathfrak{f}}^{(n)}.$$

Furthermore [ScSt, Corollary I.2.8] implies that

$$(75) \quad U_\alpha \cap U_{\mathfrak{f}}^{(n)} \subset U_\alpha \cap U_{\pi(a)\mathfrak{f}}^{(n)} \iff \langle \alpha, x \rangle \leq \langle \alpha, \pi(a)x \rangle.$$

The action of $Z_Q(S_\emptyset)$ on \mathbb{A}_\emptyset factors through π , so the right hand side is equivalent to $\langle \alpha, a \cdot x - x \rangle \geq 0$.

(e) The equivalence (75) can be expressed in terms of the compact open subgroups $U_{\alpha,r} \subset U_\alpha$ defined in (62). They satisfy

$$U_{\alpha,r} \subset U_{\alpha,s} \text{ if } r \geq s, \quad \bigcup_{r \in \mathbb{R}} U_{\alpha,r} = U_\alpha \text{ and } \bigcap_{r \in \mathbb{R}} U_{\alpha,r} = \{1\}.$$

As explained above, we can interpret (75) also in Q . Then it implies

$$(76) \quad \lim_{\langle \alpha, ax \rangle \rightarrow \infty} U_\alpha \cap U_{a\mathfrak{f}}^{(n)} = U_\alpha \quad \text{and} \quad \lim_{\langle \alpha, ax \rangle \rightarrow -\infty} U_\alpha \cap U_{a\mathfrak{f}}^{(n)} = \{1\}.$$

Now apply the Iwahori decomposition of K_n to decompose $K_n \cap \mathcal{U}_{\mathcal{Q}}(\lambda^{\pm 1})(F)$ as products over root subgroups, then (76) becomes the required statement. \square

3.4. Decompositions.

We recall some results of Bruhat and Tits about $\mathcal{BT}(\mathcal{Q}, F)$ and lift them to $\mathcal{B}(\mathcal{Q}, F)$. We use this to decompose Q in various ways, generalizing the decompositions named after Iwasawa, Cartan and Iwahori–Bruhat.

Theorem 3.12. *Let x be any point of the standard apartment \mathbb{A}_0 of $\mathcal{BT}(\mathcal{Q}, F)$.*

- (a) $Q = \mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)N_Q(S_{\emptyset})Q_x = Q_xN_Q(S_{\emptyset})\mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)$.
- (b) $Q \cdot x \cap \mathbb{A}_0 = N_Q(S_{\emptyset}) \cdot x$.
- (c) \mathbb{A}_0 is a fundamental domain for the action of $\mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)$ on $\mathcal{BT}(\mathcal{Q}, F)$.
- (d) Q_x acts transitively on the collection of apartments of $\mathcal{BT}(\mathcal{Q}, F)$ containing x .
- (e) For every $y \in \mathcal{BT}(\mathcal{Q}, F)$ there exists an apartment containing $\{x, y\}$.

Proof. (a) The first equality is the Iwasawa decomposition for $\mathcal{BT}(\mathcal{Q}, F)$ [BrTi1, Proposition 7.3.1.i]. The second equality follows by considering the inverses of elements of Q .

(b) This is a direct consequence of the definition of $\mathcal{BT}(\mathcal{Q}, F)$ in [BrTi1, 7.4.2] (which is equivalent to the earlier definition).

(c) By [BrTi1, Proposition 7.3.1.ii] the map

$$\begin{aligned} \mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F) \backslash Q / Q_x &\rightarrow \mathbb{A}_0 \\ \mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)qQ_x &\mapsto qx \quad \text{for } q \in N_Q(S_{\emptyset}) \end{aligned}$$

is well-defined and injective. Hence $\mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F) \cdot x$ contains exactly one element of \mathbb{A}_0 , namely x itself.

Let $y \in \mathcal{BT}(\mathcal{Q}, F)$. By definition [BrTi1, 7.4.2] $Q \cdot \mathbb{A}_0 = \mathcal{BT}(\mathcal{Q}, F)$, so $y \in Q \cdot x$ for some $x \in \mathbb{A}_0$. With part (a) we can find $u \in \mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)$, $n \in N_Q(S_{\emptyset})$ and $k \in Q_x$ such that

$$y = unx = unx \in uN_Q(S_{\emptyset}) \cdot x \subset u\mathbb{A}_0 \subset \mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)\mathbb{A}_0.$$

Thus $\mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)\mathbb{A}_0$ contains every element of $\mathcal{BT}(\mathcal{Q}, F)$.

(d) Let \mathbb{A}_x be an apartment of $\mathcal{BT}(\mathcal{Q}, F)$ containing x . By the definition of apartments [BrTi1, 2.2.2], there exists a $q \in Q$ with $\mathbb{A}_x = q\mathbb{A}_0$. Using part (a) we write $q = knu$ with $k \in Q_x$, $n \in N_Q(S_{\emptyset})$ and $u \in \mathcal{R}_{us,F}(\mathcal{P}_{\emptyset})(F)$. Then $x = k^{-1}x \in k^{-1}\mathbb{A}_x$ and we can rewrite this apartment as

$$k^{-1}\mathbb{A}_x = k^{-1}q\mathbb{A}_0 = nu\mathbb{A}_0 = (nun^{-1})n\mathbb{A}_0 = (nun^{-1})\mathbb{A}_0.$$

Notice that $n\mathcal{P}_{\emptyset}n^{-1}$ is another minimal pseudo-parabolic F -subgroup of Q containing $Z_Q(S_{\emptyset})$. As $x \in \mathbb{A}_0 \cap (nun^{-1})\mathbb{A}_0$ and $nun^{-1} \in \mathcal{R}_{us,F}(n\mathcal{P}_{\emptyset}n^{-1})(F)$, part (c) implies that $(nun^{-1})x = x$. Thus $\mathbb{A}_x = k(nun^{-1})\mathbb{A}_0$ where $k(nun^{-1}) \in Q_x$.

(e) See [BrTi1, Proposition 2.3.1]. \square

For notational convenience we formulated Theorem 3.12 only in the “standard” situation. After (71) we remarked that all apartments of $\mathcal{BT}(\mathcal{Q}, F)$ are Q -associate. Hence the above also holds for other apartments. Since the collection of apartments of $\mathcal{B}(\mathcal{Q}, F)$ is canonically in bijection with that for $\mathcal{BT}(\mathcal{Q}, F)$ via (71), the transfer of Theorem 3.12 to $\mathcal{B}(\mathcal{Q}, F)$ is straightforward. We refrain from writing it down here, since the statements are exactly the same as for $\mathcal{BT}(\mathcal{Q}, F)$.

The standard apartment of $\mathcal{B}(\mathcal{Q}, F)$ is

$$\mathbb{A}_\emptyset := \mathbb{A}_0 \times X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The unique maximal compact subgroup $Z_Q(S_\emptyset)_{\text{cpt}}$ of $Z_Q(S_\emptyset)$ (from Lemma 3.3.b) is the isotropy group of any point of \mathbb{A}_\emptyset . The quotient $Z_Q(S_\emptyset)/Z_Q(S_\emptyset)_{\text{cpt}}$ acts co-compactly on \mathbb{A}_\emptyset by translations, so it is a finitely generated free abelian group. The group $N_Q(S_\emptyset)/Z_Q(S_\emptyset)_{\text{cpt}}$ is an extension of the Weyl group $W(\mathcal{Q}, S_\emptyset)$ by the lattice $Z_Q(S_\emptyset)/Z_Q(S_\emptyset)_{\text{cpt}}$, it can be regarded as an (extended) affine Weyl group for $(\mathcal{Q}, S_\emptyset)$.

Recall that a maximal compact subgroup K of a reductive group H over a local field is called good if the Iwasawa decomposition $H = PK$ holds for any parabolic subgroup P of H . Equivalently [BrTi1, Proposition 4.4.2], it contains representatives for all elements of the Weyl group of H (with respect to a maximal split torus).

By the Bruhat–Tits fixed point theorem [BrTi1, Corollaire 3.3.2], every maximal compact subgroup of Q is the full stabilizer of a point of $\mathcal{B}(\mathcal{Q}, F)$. Recall also that every Q -orbit in $\mathcal{B}(\mathcal{Q}, F)$ intersects every apartment. With this in mind, we define a maximal compact subgroup of Q to be *good* if it is conjugate to a group $Q_{x'}$ with

$$(77) \quad x' \in \mathbb{A}_\emptyset \quad \text{and} \quad N_Q(S_\emptyset)_{x'} Z_Q(S_\emptyset) = N_Q(S_\emptyset).$$

Every quasi-reductive F -group Q has good maximal compact subgroups, for instance the isotropy groups of special vertices [BrTi1, §4.4.6].

Theorem 3.13. *Let K be any good maximal compact subgroup of Q .*

- (a) *Let \mathcal{P} be any pseudo-parabolic F -subgroup of \mathcal{Q} . Then Q admits the Iwasawa decomposition $Q = K\mathcal{P}(F) = \mathcal{P}(F)K$.*
- (b) *Suppose that K fixes a point of \mathbb{A}_\emptyset . Then the map*

$$\begin{aligned} Z_Q(S_\emptyset)/Z_Q(S_\emptyset)_{\text{cpt}} &\rightarrow \mathcal{R}_{us,F}(\mathcal{P}_\emptyset)(F) \backslash Q/K \\ z Z_Q(S_\emptyset)_{\text{cpt}} &\mapsto \mathcal{R}_{us,F}(\mathcal{P}_\emptyset)(F) z K \end{aligned}$$

is bijective.

Remark. The proof of part (a) will show that, if \mathcal{P} is minimal, one can always come to the setting of part (b) by conjugation in Q . Since then $Z_Q(S_\emptyset) \cap K = Z_Q(S_\emptyset)_{\text{cpt}}$ and $\mathcal{P}_\emptyset = \mathcal{R}_{us,F}(\mathcal{P}_\emptyset) \rtimes Z_Q(S_\emptyset)$, part (b) describes the uniqueness in the Iwasawa decomposition.

Proof. (a) Conjugating K and \mathcal{P} by an element of Q , we may assume that $K = Q_{x'}$ for some $x' \in \mathbb{A}_\emptyset$. We may and will also assume that \mathcal{P} is a minimal pseudo-parabolic F -subgroup. Let \mathcal{S}_1 be a maximal F -split torus in \mathcal{P} and let \mathbb{A}_1 be the associated apartment of $\mathcal{B}(\mathcal{Q}, F)$, that is, the apartment of $\mathcal{BT}(\mathcal{Q}, F)$ associated to $Z_Q(\mathcal{S}_1)(F)$ times $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$.

We abbreviate $U := \mathcal{R}_{us,F}(\mathcal{P})(F)$. With Theorem 3.12.c (for \mathbb{A}_1) we can find a $u_1 \in U$ such that $x' \in u_1 \mathbb{A}_1$. Then $\mathcal{S}_2 := u_1 \mathcal{S}_1 u_1^{-1}$ is another maximal F -split torus in \mathcal{P} , and $\mathbb{A}_2 = u_1 \mathbb{A}_1$ contains x' . By (48) $\mathcal{P}(F) = Z_Q(\mathcal{S}_2) \rtimes U$.

Now we consider any $q \in Q$. By Theorem 3.12.c (for \mathbb{A}_2) we can find $u_2 \in U$ such that $u_2 q x' \in \mathbb{A}_2$. Since $K = Q_{x'}$ is good, $N_Q(\mathcal{S}_2) \subset Z_Q(\mathcal{S}_2)K$. With Theorem 3.12.b for $x' \in \mathbb{A}_2$ we get

$$Q \cdot x' \cap \mathbb{A}_2 = N_Q(\mathcal{S}_2) \cdot x' \cap \mathbb{A}_2 = Z_Q(\mathcal{S}_2)K \cdot x' \cap \mathbb{A}_2 = Z_Q(\mathcal{S}_2) \cdot x'.$$

Pick $z \in Z_Q(S_2)$ such that $u_2qx' = zx'$. Then $k := z^{-1}u_2q$ lies in K . We conclude that

$$q = u_2^{-1}zk \in UZ_Q(S_2)K = \mathcal{P}(F)K.$$

(b) By the above remark and by part (a), the map in the statement is well-defined and surjective. Suppose that $z, z' \in Z_Q(S_\emptyset)$ have the same image in $\mathcal{R}_{us,F}(\mathcal{P}_\emptyset)(F) \backslash Q/K$. Then there are $u \in \mathcal{R}_{us,F}(\mathcal{P}_\emptyset)(F)$ and $k' \in K$ such that $z' = uz k'$. In particular $uzx' = z'x' \in \mathbb{A}_\emptyset$. By Theorem 3.12.c for $\mathcal{B}(\mathcal{Q}, F)$, we must have $uzx' = zx'$. Hence $zx' = z'x'$, which is equivalent to $zZ_Q(S_\emptyset)_{\text{cpt}} = z'Z_Q(S_\emptyset)_{\text{cpt}}$. Thus the map of the theorem is also injective. \square

Notice that the algebraic variety \mathcal{Q}/\mathcal{P} is naturally isomorphic to $\mathcal{G}/\pi(\mathcal{P})$, because $\mathcal{R}_{u,F}(\mathcal{Q}) \subset \mathcal{P}$. With the surjectivity of $\mathcal{Q}(F) \rightarrow (\mathcal{Q}/\mathcal{P})(F)$ [CGP, Lemma C.2.1] and Theorem 3.13, the proof of Corollary 2.9 can be applied. It shows that

$$(78) \quad (\mathcal{Q}/\mathcal{P})(F) \cong \mathcal{Q}(F)/\mathcal{P}(F) \text{ is compact.}$$

From Theorem 5.1 one sees that \mathcal{Q}/\mathcal{P} is the variety of pseudo-parabolic subgroups of \mathcal{Q} conjugate to \mathcal{P} .

Theorem 3.14. *Let x and y be any points of \mathbb{A}_\emptyset .*

- (a) $Q = Q_x N_Q(S_\emptyset) Q_y$.
- (b) *The natural map $N_Q(S_\emptyset)_x \backslash N_Q(S_\emptyset) / N_Q(S_\emptyset)_y \rightarrow Q_x \backslash Q / Q_y$ is bijective.*
- (c) *Suppose that $x \in \mathbb{A}_\emptyset$ is generic, in the sense that $N_Q(S_\emptyset)_x = Z_Q(S_\emptyset)_{\text{cpt}}$. (Thus Q_x can be regarded as an Iwahori subgroup of Q , namely the pointwise stabilizer of the unique chamber of \mathbb{A}_\emptyset containing x .) Choosing representatives w for $N_Q(S_\emptyset) / Z_Q(S_\emptyset)_{\text{cpt}}$ in $N_Q(S_\emptyset)$, we obtain the Iwahori–Bruhat decomposition*

$$Q = \bigsqcup_{w \in N_Q(S_\emptyset) / Z_Q(S_\emptyset)_{\text{cpt}}} Q_x w Q_x.$$

- (d) *Let $x', y' \in \mathbb{A}_\emptyset$ be such that $Q_{x'}$ and $Q_{y'}$ are good maximal compact subgroups of Q , for example special vertices. Then $N_Q(S_\emptyset)_{x'} \backslash N_Q(S_\emptyset) / N_Q(S_\emptyset)_{y'}$ is canonically in bijection with the set of $W(\mathcal{Q}, S_\emptyset)$ -orbits in $Z_Q(S_\emptyset) / Z_Q(S_\emptyset)_{\text{cpt}}$.*

In case $x' = y'$, this orbit space can be represented by a finitely generated semigroup of $Z_Q(S_\emptyset) / Z_Q(S_\emptyset)_{\text{cpt}}$ and by a subset $A \subset Z_Q(S_\emptyset)$. We obtain the Cartan decomposition

$$Q = \bigsqcup_{a \in A} Q_{x'} a Q_{x'}.$$

Proof. (a) Consider any $q \in Q$, and let x_s and $qy_s = (qy)_s$ be the components of, respectively, x and qy in $\mathcal{BT}(\mathcal{Q}, F)$. By Theorem 3.12.e there exists an apartment of $\mathcal{BT}(\mathcal{Q}, F)$ containing $\{x_s, qy_s\}$. Taking the Cartesian product with $X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, we obtain an apartment \mathbb{A}_q of $\mathcal{B}(\mathcal{Q}, F)$ containing $\{x, qy\}$. By Theorem 3.12.d there is a $k \in Q_x$ such that $\mathbb{A}_q = \mathbb{A}_\emptyset$. Then

$$k^{-1}qy \in \mathbb{A}_\emptyset \cap Q \cdot y,$$

so by Theorem 3.12.b there exists a $n \in N_Q(S_\emptyset)$ with $k^{-1}qy = ny$. Then $k' := n^{-1}k^{-1}q$ lies in Q_y and

$$q = knk' \in Q_x N_Q(S_\emptyset) Q_y.$$

(b) Part (a) says that this natural map is surjective. Suppose that $n, n' \in N_Q(S_\emptyset)$ determine the same double coset $Q_x n Q_y = Q_x n' Q_y$. As $Q_x \subset Q_{x_s}$ and $Q_y \subset Q_{y_s}$ also, $Q_{x_s} n Q_{y_s} = Q_{x_s} n' Q_{y_s}$. Then [BrTi1, Propostion 4.2.1] implies that

$$N_Q(S_\emptyset)_x n N_Q(S_\emptyset)_y = N_Q(S_\emptyset)_x n' N_Q(S_\emptyset)_y.$$

(c) This is a direct consequence of part (b) and the normality of $Z_Q(S_\emptyset)_{\text{cpt}}$ in $N_Q(S_\emptyset)$.

(d) Since $Q_{y'}$ is good, the canonical map

$$N_Q(S_\emptyset)_{y'} \rightarrow W(\mathcal{Q}, \mathcal{S}_\emptyset) = N_Q(S_\emptyset)/Z_Q(S_\emptyset)$$

is surjective, with kernel $Z_Q(S_\emptyset)_{\text{cpt}}$. Choose a set of representatives $W_{y'} \subset N_Q(S_\emptyset)_{y'}$ for $W(\mathcal{Q}, \mathcal{S}_\emptyset)$, then

$$(79) \quad Z_Q(S_\emptyset)W_{y'} = W_{y'}Z_Q(S_\emptyset) = N_Q(S_\emptyset),$$

and similarly for x' . Consider the injection

$$N_Q(S_\emptyset)/N_Q(S_\emptyset)_{y'} \rightarrow \mathbb{A}_\emptyset : nN_Q(S_\emptyset)_{y'} \mapsto ny'.$$

It is left $N_Q(S_\emptyset)_{x'}$ -equivariant and by (79) its image is $Z_Q(S_\emptyset)W_{y'} \cdot y' = Z_Q(S_\emptyset) \cdot y'$. Dividing out by that left action and using $N_Q(S_\emptyset)_{x'}/Z_Q(S_\emptyset)_{x'} \cong W_{x'}$, we obtain a bijection

$$(80) \quad N_Q(S_\emptyset)_{x'} \backslash N_Q(S_\emptyset)/N_Q(S_\emptyset)_{y'} \rightarrow W_{x'} \backslash Z_Q(S_\emptyset) \cdot y'.$$

Similarly the injection

$$Z_Q(S_\emptyset)/Z_Q(S_\emptyset)_{\text{cpt}} \rightarrow \mathbb{A}_\emptyset : zZ_Q(S_\emptyset)_{\text{cpt}} \mapsto z \cdot y'$$

induces a bijection

$$(81) \quad W(\mathcal{Q}, \mathcal{S}_\emptyset) \backslash Z_Q(S_\emptyset)/Z_Q(S_\emptyset)_{\text{cpt}} \rightarrow W_{x'} \backslash Z_Q(S_\emptyset) \cdot y'.$$

We regard y' as the origin of the affine space \mathbb{A}_\emptyset . Then

$$(82) \quad Z_Q(S_\emptyset)/Z_Q(S_\emptyset)_{\text{cpt}} \cong Z_Q(S_\emptyset) \cdot y'$$

becomes a lattice in \mathbb{A}_\emptyset , and we represent it by a subset $A' \subset Z_Q(S_\emptyset)$. Since $W(\mathcal{Q}, \mathcal{S})$ is the Weyl group of a root system in \mathbb{A}_\emptyset , a fundamental domain for the $W_{x'}$ -action is formed by a Weyl chamber with respect to x' as base point. Choose a Weyl chamber \mathbb{A}_\emptyset^+ which contains y' . Then

$$(83) \quad Z_Q(S_\emptyset) \cdot y' \cap \mathbb{A}_\emptyset^+ \text{ represents } W_{x'} \backslash Z_Q(S_\emptyset) \cdot y'.$$

Let A be the subset of A' corresponding to $\mathbb{A}_\emptyset^+ \cap Z_Q(S_\emptyset) \cdot y'$. With (80), (81) and part (a) we obtain bijections

$$A \rightarrow W(\mathcal{Q}, \mathcal{S}_\emptyset) \backslash Z_Q(S_\emptyset)/Z_Q(S_\emptyset)_{\text{cpt}} \rightarrow N_Q(S_\emptyset)_{x'} \backslash N_Q(S_\emptyset)/N_Q(S_\emptyset)_{y'} \rightarrow Q_{x'} \backslash Q/Q_{y'}.$$

The second bijection is canonical if we regard $W(\mathcal{Q}, \mathcal{S}_\emptyset)$ as $N_Q(S_\emptyset)_{x'}/Z_Q(S_\emptyset)_{\text{cpt}}$.

By [BrTi1, 1.3.8 and Proposition 6.2.10] the root system for $W_{x'}$ is contained in $Z_Q(S_\emptyset) \cdot x'$. If $x' = y'$, then this implies that the image of $Z_Q(S_\emptyset) \cdot y' \cap \mathbb{A}_\emptyset^+$ under (82) is a finitely generated semigroup in $Z_Q(S_\emptyset)/Z_Q(S_\emptyset)_{\text{cpt}}$. \square

4. REPRESENTATION THEORY

Let \mathcal{G} be a quasi-reductive group defined over a non-archimedean local field F . We will investigate the representations of locally compact groups like $G = \mathcal{G}(F)$. We work in the category $\text{Rep}(G)$ of smooth G -representations on complex vector spaces. Thanks to Proposition 2.3, this category is quite large.

4.1. Parabolic induction and restriction.

We will show that certain results involving parabolic induction and Jacquet restriction hold for all quasi-reductive F -groups. We note that for pseudo-reductive groups these results (except Theorem 4.5) only use Section 2, and in particular do not rely on the Bruhat–Tits theory developed in Paragraph 3.1.

Suppose that $H \subset G$ is a closed subgroup and that $(\pi, V) \in \text{Rep}(H)$. The vector space $\text{ind}_H^G(V)$ consists of all locally constant functions $G \rightarrow V$ such that

$$f(gh^{-1}) = \pi(h)f(g) \quad \forall h \in H, g \in G \quad \text{and } \text{supp}(f)/H \text{ is compact.}$$

This is a G -representation for the action by left translations. Obviously the condition on the support is empty if G/H is compact, which by Corollary 2.9 and (78) happens if H is a pseudo-parabolic F -subgroup of G .

Let $\mathcal{P}_{\mathcal{G}}(\lambda)$ be a pseudo-parabolic F -subgroup of \mathcal{G} , and recall that

$$\mathcal{P}_{\mathcal{G}}(\lambda) = \mathcal{U}_{\mathcal{G}}(\lambda) \rtimes \mathcal{Z}_{\mathcal{G}}(\lambda) \quad \text{with } \mathcal{U}_{\mathcal{G}}(\lambda) = \mathcal{R}_{us,F}(\mathcal{P}_{\mathcal{G}}(\lambda)).$$

In this paragraph we will often abbreviate

$$\mathcal{P} = \mathcal{P}_{\mathcal{G}}(\lambda), \quad \overline{\mathcal{P}} = \mathcal{P}_{\mathcal{G}}(\lambda^{-1}), \quad M = \mathcal{Z}_{\mathcal{G}}(\lambda)(F), \quad U = \mathcal{U}_{\mathcal{G}}(\lambda)(F) \quad \text{and} \quad \overline{U} = \mathcal{U}_{\mathcal{G}}(\lambda^{-1})(F).$$

Notice that M and \overline{U} are not canonically determined by \mathcal{P} , this really involves $\lambda : GL_1 \rightarrow \mathcal{G}$. By [CGP, Corollary 2.2.5] the group

$$\mathcal{Z}_{\mathcal{G}}(\lambda)/\mathcal{R}_{u,F}(\mathcal{G}) = (\mathcal{P}_{\mathcal{G}}(\lambda)/\mathcal{R}_{u,F}(\mathcal{G})) \cap (\overline{\mathcal{P}}_{\mathcal{G}}(\lambda)/\mathcal{R}_{u,F}(\mathcal{G}))$$

inherits the pseudo-reductivity of $\mathcal{G}/\mathcal{R}_{u,F}(\mathcal{G})$. As $\mathcal{R}_{u,F}(\mathcal{G})$ is F -wound, this implies that $\mathcal{Z}_{\mathcal{G}}(\lambda)$ is quasi-reductive. Then Lemma 1.3 and (16) say that M is unimodular. Let $\delta_P : P \rightarrow \mathbb{R}_{>0}$ be the modular function. Since the unipotent group U is a union of compact subgroups, δ_P is trivial on U and factors through $P/U \cong M$.

We define the normalized parabolic induction of $(\pi, V) \in \text{Rep}(P/U)$ as

$$i_P^G(\pi) = \text{ind}_P^G(\delta_P^{-1/2} \otimes \pi) \in \text{Rep}(G).$$

The space of U -coinvariants of $(\rho, W) \in \text{Rep}(G)$ is defined as

$$W_U = W / \text{span}\{\rho(u)w - w : w \in W, u \in U\}.$$

This space carries a natural representation ρ_U of P/U . We define the normalized Jacquet restriction of (ρ, W) as $r_P^G(\rho) = \delta_P^{1/2} \otimes \rho_U$. The functors

$$i_P^G : \text{Rep}(P/U) \rightarrow \text{Rep}(G) \quad \text{and} \quad r_P^G : \text{Rep}(G) \rightarrow \text{Rep}(P/U)$$

for quasi-reductive F -groups have many of the properties that are known in the setting of reductive groups. Indeed, thanks to Proposition 2.3, Lemma 2.2.b and Corollary 2.9 (as well as their analogues for quasi-reductive F -groups), everything in [Ren, §VI.1] and [Cas, Chapters 1–4] remains valid in our generality. Let us list some of these results:

$$(84) \quad i_P^G \text{ and } r_P^G \text{ are exact functors,}$$

$$(85) \quad i_P^G \text{ is left adjoint to } r_P^G \text{ (for both see [Ren, Proposition VI.1.2]),}$$

$$(86) \quad i_P^G \text{ preserves unitarity (see [Ren, IV.2.3] or [Cas, Proposition 3.1.4]),}$$

$$(87) \quad r_P^G \text{ preserves finite type [Ren, Proposition VI.1.3].}$$

(Recall that a G -representation has finite type if it is generated by a finite subset.)

It is easy to see in examples (e.g. the Steinberg representation of $SL_2(F)$) that Jacquet restriction does not preserve unitarity. Nevertheless, there is an analogue. Let $(\pi, V) \in \text{Rep}(V)$ and let $(\check{\pi}, \check{V})$ be its smooth contragredient representation. Let $j_U : V \rightarrow V_U$ and $j_{\bar{U}} : \check{V} \rightarrow \check{V}_{\bar{U}}$ be the quotient maps.

Let S_M be the maximal F -split torus in $Z(M)$. The negative chamber of S_M consists of all elements m such that $\|\alpha(m)\|_F < 1$ for all roots α of U with respect to S_M . We say that $m \in S_M$ lies deep in the negative chamber if there exists an $\epsilon < 1$ such that $\|\alpha(m)\|_F < \epsilon$ for all such α . The smaller ϵ , the deeper.

Theorem 4.1. (Casselman)

There exists a canonical M -invariant, non-degenerate pairing

$$\langle \cdot, \cdot \rangle_P : r_P^G(V) \times r_{\bar{P}}^G(\check{V}) \rightarrow \mathbb{C}.$$

It is characterized by

$$\langle \pi(m)v, \check{v} \rangle = \langle r_{\bar{P}}^G(\pi)(m)j_U(v), j_{\bar{U}}(\check{v}) \rangle_P$$

for all $v \in V, \check{v} \in \check{V}$ and all m sufficiently deep in the negative chamber of S_M .

Proof. The statement is a combination of Proposition 4.2.3, Theorem 4.2.4 and Theorem 4.3.3 of [Cas]. The proof goes exactly the same, using Proposition 2.3. \square

Lemma 4.2. *Let $P' = M'U' \supset P$ be another pseudo-parabolic F -subgroup of G , with opposite $\bar{P}' = M'\bar{U}' \supset \bar{P}$. Then $P \cap M' = P \cap \bar{P}'$ is a pseudo-parabolic subgroup of $M' = P' \cap \bar{P}'$ and the above functors are transitive in the following sense:*

$$i_P^G = i_{P'}^G \circ i_{P \cap M'}^{M'}, \quad r_P^G = r_{P \cap M'}^{M'} \circ r_{P'}^G.$$

Proof. For reductive groups this is shown in [Ren, Lemme VI.I.4]. The proof relies on the structure of parabolic subgroups and of root subgroups. By (53) the latter are the same for quasi-reductive groups as for pseudo-reductive groups. Using the structure of pseudo-parabolic subgroups, as described in [CGP, §2.2], the argument from [Ren, Lemme VI.I.4] goes through. \square

Another result that remains valid for quasi-reductive F -groups is Bernstein's geometric lemma, which we now formulate. Let \mathcal{S}_θ and \mathcal{P}_θ be as in (48). First we assume that \mathcal{P} and \mathcal{P}' are pseudo-parabolic F -subgroups of \mathcal{G} containing \mathcal{P}_θ . Let $W(\mathcal{G}, \mathcal{S}_\theta)^{\mathcal{P}', \mathcal{P}} \subset W(\mathcal{G}, \mathcal{S}_\theta)$ be a set of representatives for

$$W(\mathcal{P}', \mathcal{S}_\theta) \backslash W(\mathcal{G}, \mathcal{S}_\theta) / W(\mathcal{P}, \mathcal{S}_\theta).$$

For each $w \in W(\mathcal{G}, \mathcal{S}_\theta)^{\mathcal{P}', \mathcal{P}}$ we choose a lift $\bar{w} \in N_G(\mathcal{S}_\theta)$, and we let $\overline{W(\mathcal{G}, \mathcal{S}_\theta)^{\mathcal{P}', \mathcal{P}}}$ be the collection of all these \bar{w} . The Bruhat decomposition (see Theorem 5.3) says that

$$(88) \quad G = \bigsqcup_{\bar{w} \in \overline{W(\mathcal{G}, \mathcal{S}_\theta)^{\mathcal{P}', \mathcal{P}}}} \bar{w} P' \bar{w} P.$$

When \mathcal{P} and \mathcal{P}' are general pseudo-parabolic F -subgroups of \mathcal{G} , we can adjust the construction of the \bar{w} as in [Ren, §V.4.7], so that (88) remains true with the new interpretation of $\overline{W(\mathcal{G}, \mathcal{S}_\theta)^{\mathcal{P}', \mathcal{P}}}$.

Theorem 4.3. *Let P, P' be pseudo-parabolic F -subgroups of G and let $(\pi, V) \in \text{Rep}(P/U)$. Then $r_P^G, i_P^G(\pi, V)$ admits a filtration with terms $F(V)_i$, such that the successive subquotients $F(V)_i / F(V)_{i-1}$ are*

$$i_{M' \cap \bar{w} P \bar{w}^{-1}}^{M'} \circ \bar{w} \circ r_{M' \cap \bar{w} P' \bar{w}^{-1}}^M(\pi, V).$$

Here \bar{w} runs through $\overline{W(\mathcal{G}, \mathcal{S}_\emptyset)^{\mathcal{P}', \mathcal{P}}}$ and the functor \bar{w} stands for $(\bar{w} \cdot \rho)(g) = \rho(\bar{w}^{-1} g \bar{w})$.

Proof. For reductive groups see [BeRu, §III.1.2] or [Ren, §VI.5.1]. With the small compact open subgroups from Proposition 2.3, Theorem 3.2 and the structure of pseudo-parabolic subgroups from [CGP, §2.2], these two proofs remain valid for all quasi-reductive F -groups. \square

We note one special case of Theorem 4.3, where P and P' both equal the minimal pseudo-parabolic F -subgroup P_\emptyset and $P_\emptyset \cap \overline{P_\emptyset} = Z_G(\mathcal{S}_\emptyset)$. It says that $r_{P_\emptyset}^G i_{P_\emptyset}^G(\pi, V)$ has a filtration with successive subquotients

$$(89) \quad (\bar{w} \cdot \pi, V) \quad w \in W(\mathcal{G}, \mathcal{S}_\emptyset).$$

In particular, when π is irreducible, the $Z_G(\mathcal{S}_\emptyset)$ -representation $r_{P_\emptyset}^G i_{P_\emptyset}^G(\pi)$ has finite length and the $\bar{w} \cdot \pi$ with $w \in W(\mathcal{G}, \mathcal{S}_\emptyset)$ form precisely its Jordan–Hölder content.

Following the usual terminology for reductive groups, we say that $(\pi, V) \in \text{Rep}(G)$ is supercuspidal if $r_P^G(\pi, V) = 0$ for all proper pseudo-parabolic F -subgroups \mathcal{P} of \mathcal{G} . The following result and its proof are analogues of [Ren, Corollaire VI.2.1].

Lemma 4.4. *Every irreducible smooth G -representation embeds in the parabolic induction of an irreducible supercuspidal representation.*

Proof. For any irreducible $(\rho, W) \in \text{Rep}(G)$, Lemma 4.2 implies that there exists a pseudo-parabolic F -subgroup \mathcal{P} of \mathcal{G} such that $r_{\mathcal{P}}^G(\rho, W)$ is a nonzero supercuspidal representation of P/U . By (87) $r_{\mathcal{P}}^G(\rho, W)$ has finite type, and hence admits an irreducible quotient, say $(\pi, V) \in \text{Rep}(P/U)$. By the exactness of $r_{\mathcal{P}}^G$ that is again a supercuspidal P/U -representation. By Frobenius reciprocity (85)

$$0 \neq \text{Hom}_{P/U}(r_{\mathcal{P}}^G(\rho), \pi) \cong \text{Hom}_G(\rho, i_{\mathcal{P}}^G(\pi)).$$

In view of the irreducibility of ρ , any nonzero G -homomorphism $(\rho, W) \rightarrow i_{\mathcal{P}}^G(\pi, V)$ gives the required embedding. \square

Using the group 0G from (68) we can formulate different, equivalent characterizations of supercuspidality.

Theorem 4.5. (Bernstein)

For $(\pi, V) \in \text{Rep}(G)$ the following are equivalent:

- (π, V) is supercuspidal.
- The restriction of (π, V) to 0G is a compact representation.
- Every matrix coefficient of (π, V) has compact support modulo $Z(G)$.

Proof. The proof in the case of reductive groups is given in [Ren, Théorème VI.2.1]. It uses:

- the properties of 0G from Lemma 3.7,
- the small subgroups with an Iwahori decomposition from Theorem 3.11,
- the Cartan decomposition from Theorem 3.14.d.

With this input, the argument from [Ren] is also valid for quasi-reductive groups. \square

4.2. Uniform admissibility.

Recall that a G -representation (π, V) is admissible if V^K has finite dimension for every compact open subgroup $K \subset G$. This property is important in harmonic analysis. To establish the upcoming results, we make essential use of valuated root data and Bruhat–Tits theory, even in the case of pseudo-reductive groups.

Theorem 4.6. *Every irreducible smooth G -representation is admissible.*

Proof. By Lemma 4.4 such a representation embeds in the parabolic induction of an irreducible supercuspidal representation, say in $i_P^G(\pi, V)$. The compactness of G/P (78) implies that the functor i_P^G preserves admissibility [Ren, Lemme III.2.3]. Therefore it suffices to prove that every irreducible supercuspidal representation is admissible. The argument for that given in [Ren, Théorème VI.2.2] uses only Theorem 4.5, Lemma 3.7 and general properties of totally disconnected groups, so it also works for the quasi-reductive group G . \square

We fix a Haar measure on G , so that a convolution product is defined on $C_c(G)$. For any compact open subgroup K , let $\mathcal{H}(G, K)$ be the convolution algebra of compactly supported K -biinvariant functions $G \rightarrow \mathbb{C}$. It is an involutive algebra with $f^*(g) = f(g^{-1})$.

The Hecke algebra $\mathcal{H}(G)$ is defined as the union of the algebras $\mathcal{H}(G, K)$, over all compact open subgroups K of G . It is well-known that $\text{Rep}(G)$ is equivalent with the category of nondegenerate $\mathcal{H}(G)$ -modules, see [Ren, Théorème III.1.4].

Lemma 4.7. *The algebra $\mathcal{H}(G)$ is dense in $C_c(G)$ and in $L^1(G)$.*

Proof. By definition $\mathcal{H}(G)$ is a subalgebra of $C_c(G) \subset L^1(G)$. By Proposition 2.3 the compact open subgroups K form a neighborhood basis of 1 in G , so $\mathcal{H}(G)$ separates the points of G . The Stone–Weierstrass Theorem says that $\mathcal{H}(G)$ is dense in $C_c(G)$. Since $L^1(G)$ is a completion of $C_c(G)$, $\mathcal{H}(G)$ also lies dense in there. \square

Let K be any compact open subgroup of G and write

$$(90) \quad \text{Irr}(G, K) = \{V \in \text{Rep}(G) : V \text{ is irreducible and } V^K \neq 0\}.$$

Proposition 4.8. (a) *There are functors*

$$\begin{array}{ccc} \text{Rep}(G) & \longleftrightarrow & \text{Mod}(\mathcal{H}(G, K)) \\ V & \mapsto & V^K \\ \mathcal{H}(G) \otimes_{\mathcal{H}(G, K)} W & \leftarrow & W \end{array}$$

- (b) *These functors provide a bijection between (isomorphism classes in) $\text{Irr}(G, K)$ and (isomorphism classes of) irreducible representations of $\mathcal{H}(G, K)$.*
- (c) *Suppose that π is a topologically irreducible unitary representation of G on a Hilbert space V . Then V^K is a topologically irreducible $*$ -representation of $\mathcal{H}(G, K)$, or $V^K = 0$.*

Recall that a representation of a group or an algebra on a topological vector space is called topologically irreducible if every nonzero invariant subspace is dense. We also note that such a representation is usually not smooth.

Proof. For parts (a) and (b) see [Ren, §I.3.2 and Théorème III.1.5].

(c) Since (π, V) is unitary, V^K is a $*$ -representation of $\mathcal{H}(G, K)$. Suppose that

$V^K \neq 0$ and take $v \in V^K \setminus \{0\}$. Let $\langle K \rangle \in \mathcal{H}(G)$ be the idempotent corresponding to K , it is the unit element of $\mathcal{H}(G, K) = \langle K \rangle \mathcal{H}(G) \langle K \rangle$. Then inside V^K :

$$\overline{\mathcal{H}(G, K)v} = \overline{\langle K \rangle \mathcal{H}(G) \langle K \rangle v} = \langle K \rangle \overline{\mathcal{H}(G)v}.$$

By Lemma 4.7 and the topological irreducibility of V , the right hand side equals $\langle K \rangle V = V^K$. Since $v \in V^K \setminus \{0\}$ was arbitrary, this shows that V^K is topologically irreducible. \square

The quasi-reductive F -group G satisfies the following uniform admissibility property:

Theorem 4.9. *For every compact open subgroup K of G there exists a bound $N(G, K) \in \mathbb{N}$ such that:*

- (a) *every irreducible $\mathcal{H}(G, K)$ -module has dimension at most $N(G, K)$,*
- (b) *$\dim_{\mathbb{C}}(V^K) \leq N(G, K)$ for every irreducible smooth G -representation V ,*
- (c) *$\dim_{\mathbb{C}}(V^K) \leq N(G, K)$ for every topologically irreducible unitary G -representation on a Hilbert space V .*

Proof. Let $W \in \text{Irr}(\mathcal{H}(G, K))$ be any irreducible module. By Proposition 4.8.b $\mathcal{H}(G) \otimes_{\mathcal{H}(G, K)} W$ is an irreducible smooth G -representation. By Theorem 4.6 it is admissible, so again using Proposition 4.8.b we find that

$$(91) \quad W = (\mathcal{H}(G) \otimes_{\mathcal{H}(G, K)} W)^K \text{ has finite dimension.}$$

Let $K_n \subset G$ be a compact open subgroup as in Theorem 3.11, such that $K_n \subset K$. Then $\text{Irr}(G, K) \subset \text{Irr}(G, K_n)$ and $\mathcal{H}(G, K) \subset \mathcal{H}(G, K_n)$. From Proposition 4.8.b we get an injection

$$\text{Irr}(\mathcal{H}(G, K)) \rightarrow \text{Irr}(\mathcal{H}(G, K_n)) : W \mapsto \mathcal{H}(G, K_n) \otimes_{\mathcal{H}(G, K)} W,$$

which increases the dimensions. Hence we may replace K by K_n .

Now we check that the assumptions needed for the arguments in [Ber] are fulfilled. Let $x \in \mathbb{A}_\theta$ be such that G_x is a good maximal compact subgroup of G . Then $G = G_x A G_x$ by Theorem 3.14.d. By Lemma 3.3.a we can write $A = A_{S_\theta} \Omega_1$, where $\Omega_1 \subset Z_G(S_\theta)$ is finite and $A_{S_\theta} \subset S_\theta$ is a finitely generated semigroup representing $W(\mathcal{G}, \mathcal{S}_\theta) \backslash S_\theta / (S_\theta)_{\text{cpt}}$.

Write $S_\theta = \mathcal{S} \mathcal{Z}$ with \mathcal{S} a maximal F -split torus in the subgroup of \mathcal{G} generated by the root subgroups (with respect to S_θ). Accordingly

$$A_{S_\theta} = A^+(A_{S_\theta} \cap \mathcal{Z}(F)) \Omega_2,$$

where Ω_2 is finite, $A_{S_\theta} \cap \mathcal{Z}(F)$ is a subgroup of $\mathcal{Z}(F)$ representing $\mathcal{Z}(F) / \mathcal{Z}(F)_{\text{cpt}}$ and A^+ represents $W(\mathcal{G}, \mathcal{S}_\theta) \backslash S / S_{\text{cpt}}$. We obtain

$$(92) \quad G = G_x A^+(A_{S_\theta} \cap \mathcal{Z}(F)) (\Omega_2 \Omega_1) G_x$$

where $A^+ \subset S_\theta$ is a finitely generated semigroup, $A_{S_\theta} \cap \mathcal{Z}(F)$ central and $\Omega_2 \Omega_1$ finite. This and Theorem 3.11 (parts a–d) take care of all the assumptions on G and the K_n made in [Ber, p.91].

Then we may use [Ber, Assertion A], which together with (91) gives us precisely part (a). (See also [Rüd, §5] for more details about Bernstein's arguments.) Applying Proposition 4.8.b, we get part (b). By part (a) and [FiRü, Theorem 2.4], every topologically irreducible $*$ -representation of $\mathcal{H}(G, K)$ has dimension at most $N(G, K)$. We combine this with Proposition 4.8.c to obtain part (c). \square

We remark that in Theorem 4.9.c the finite dimensional module V^K is also irreducible as a representation of $\mathcal{H}(G, K)$ in the purely algebraic sense – in finite dimensional vector spaces the topology barely makes any difference. From Proposition 4.8.b we then see that $\mathcal{H}(G)V^K$ is an irreducible smooth G -subrepresentation of V . It is the same for every compact open subgroup K , namely the space of smooth vectors

$$V_{sm} := \{v \in V : \{g \in G : \pi(g)v = v\} \text{ is open in } G\}.$$

With Lemma 4.7 we find that, for every topologically irreducible unitary G -representation π on a Hilbert space V ,

$$(93) \quad V_{sm} \text{ is a dense irreducible smooth } G\text{-subrepresentation of } V.$$

Finally, we turn to analytic properties of group C^* -algebras for quasi-reductive groups. We refer to [Dix, §13.9] for the terminology.

Corollary 4.10. *Let $C^*(G)$ be the maximal C^* -algebra of G .*

- (a) *Let π be a topologically irreducible unitary G -representation on a Hilbert space V . For every $f \in C^*(G)$, $\pi(f)$ is a compact operator on V . In other words, $C^*(G)$ is a liminal C^* -algebra.*
- (b) *The group G has type I.*

Proof. (a) Consider any $f \in \mathcal{H}(G, K)$. Then $\pi(f)V \subset V^K$, so by Theorem 4.9.c $\pi(f)$ has finite rank. Thus $\pi(\mathcal{H}(G))$ consists of finite rank operators on V . By Lemma 4.7 $\mathcal{H}(G)$ is dense in $C^*(G)$, so $\pi(C^*(G))$ is contained in the closure of the finite rank operators in the C^* -algebra of bounded linear operators on V . By definition that is the algebra of compact operators on V .

(b) By part (a) and [Dix, Théorème 5.5.2], $C^*(G)$ is of type I, which by definition means that G has type I. \square

5. DISCONNECTED GROUPS

For the applications to the Baum–Connes conjecture that we have in mind, we need to consider disconnected algebraic groups. Let F be an arbitrary field. The Bruhat decomposition was announced for any connected linear algebraic F -group \mathcal{G} by Borel and Tits [BoTi]. Based on [CGP] and for use in Paragraph 5.2, we will show that the connectedness assumption is superfluous.

Let \mathcal{G}^+ be a linear algebraic F -group (not necessarily smooth), with connected component \mathcal{G} (which is smooth, see [Spr, Theorem 4.3.7.i]). Let \mathcal{S} be a maximal F -split torus of \mathcal{G} , let $Z_{\mathcal{G}^+(F)}(\mathcal{S})$ (resp. $N_{\mathcal{G}^+(F)}(\mathcal{S})$) be its centralizer (resp. normalizer) in $\mathcal{G}^+(F)$. The group $\mathcal{G}^+(F)$ acts by conjugation on the set of maximal F -split tori of \mathcal{G} . Since all maximal F -split tori of \mathcal{G} are already conjugate by elements of $\mathcal{G}(F)$ [CGP, Theorem C.2.3],

$$(94) \quad \mathcal{G}^+(F) = \mathcal{G}(F)N_{\mathcal{G}^+(F)}(\mathcal{S}) = N_{\mathcal{G}^+(F)}(\mathcal{S})\mathcal{G}(F).$$

The Weyl group of $(\mathcal{G}, \mathcal{S})$ is [CGP, Proposition C.2.10]

$$(95) \quad W(\mathcal{G}, \mathcal{S}) := N_{\mathcal{G}}(\mathcal{S})/Z_{\mathcal{G}}(\mathcal{S}) \cong N_{\mathcal{G}(F)}(\mathcal{S})/Z_{\mathcal{G}(F)}(\mathcal{S}).$$

The finite group

$$W(\mathcal{G}^+(F), \mathcal{S}) := N_{\mathcal{G}^+(F)}(\mathcal{S})/Z_{\mathcal{G}^+(F)}(\mathcal{S})$$

contains $W(\mathcal{G}, \mathcal{S})$ as a normal subgroup.

5.1. Pseudo-parabolic subgroups.

For possibly disconnected groups, [CGP, Definition 2.2.1] says that every pseudo-parabolic F -subgroup of \mathcal{G}^+ is of the form $\mathcal{P}^+ = \mathcal{P}_{\mathcal{G}^+}(\lambda)\mathcal{R}_{u,F}(\mathcal{G})$. Here $\lambda : GL_1 \rightarrow \mathcal{G}$ is a F -rational cocharacter and $\mathcal{P}_{\mathcal{G}^+}(\lambda)$ is defined by

$$(96) \quad \mathcal{P}_{\mathcal{G}^+}(\lambda)(F) = \{g \in \mathcal{G}^+(F) : \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists in } \mathcal{G}^+(F)\}.$$

We note that \mathcal{P}^+ need not be connected if \mathcal{G}^+ is not, for $Z_{\mathcal{G}^+}(\lambda) := \mathcal{P}_{\mathcal{G}^+}(\lambda) \cap \mathcal{P}_{\mathcal{G}^+}(-\lambda)$ can contain elements of $\mathcal{G}^+(F) \setminus \mathcal{G}(F)$. In particular $\mathcal{P}^+(F)$ always contains the full $\mathcal{G}^+(F)$ -centralizer of the smallest F -torus of \mathcal{G} through which λ factors.

Let \mathcal{P}_\emptyset be a minimal pseudo-parabolic F -subgroup of \mathcal{G} . We may assume that it contains \mathcal{S} . From (96) we see that $\mathcal{P}_\emptyset \cap N_{\mathcal{G}}(\mathcal{S}) = Z_{\mathcal{G}}(\mathcal{S})$. By [CGP, Proposition C.2.4]

$$(97) \quad \mathcal{P}_\emptyset = Z_{\mathcal{G}}(\mathcal{S}) \rtimes \mathcal{R}_{u,F}(\mathcal{P}_\emptyset).$$

The following results were announced in [BoTi], see [CGP, Theorem C.2.5] and [CGP, Proposition 3.5.7] for a proof.

Theorem 5.1. *Let \mathcal{G} be any connected linear algebraic group defined over F .*

- (a) *All minimal pseudo-parabolic F -subgroups of \mathcal{G} are conjugate under $\mathcal{G}(F)$.*
- (b) *Every pseudo-parabolic F -subgroup of \mathcal{G} equals its own normalizer.*

Let $N_{\mathcal{G}^+(F)}(\mathcal{P}_\emptyset, \mathcal{S})$ be the simultaneous normalizer of \mathcal{P}_\emptyset and \mathcal{S} in $\mathcal{G}^+(F)$, and put

$$W(\mathcal{G}^+(F), \mathcal{P}_\emptyset, \mathcal{S}) = N_{\mathcal{G}^+(F)}(\mathcal{P}_\emptyset, \mathcal{S}) / Z_{\mathcal{G}^+(F)}(\mathcal{S}).$$

The group $Z_{\mathcal{G}^+(F)}(\mathcal{S})$ normalizes every pseudo-parabolic F -subgroup \mathcal{P} of \mathcal{G} containing $Z_{\mathcal{G}}(\mathcal{S})$, for it fixes any cocharacter $GL_1 \rightarrow \mathcal{S}$ determining \mathcal{P} . Hence the subgroup $W(\mathcal{G}^+(F), \mathcal{P}_\emptyset, \mathcal{S})$ of $W(\mathcal{G}^+(F), \mathcal{S})$ is precisely the stabilizer of \mathcal{P}_\emptyset , and it acts naturally on the set of pseudo-parabolic subgroups of \mathcal{G} containing $Z_{\mathcal{G}}(\mathcal{S})$.

Lemma 5.2. $W(\mathcal{G}^+(F), \mathcal{S}) = W(\mathcal{G}^+(F), \mathcal{P}_\emptyset, \mathcal{S}) \rtimes W(\mathcal{G}, \mathcal{S})$.

Proof. The F -group $\bar{\mathcal{G}} = \mathcal{G}/\mathcal{R}_{u,F}(\mathcal{G})$ is pseudo-reductive and the image of \mathcal{S} is a maximal F -split torus in there, say $\bar{\mathcal{S}}$. These groups give rise to a root system $\Phi(\bar{\mathcal{G}}, \bar{\mathcal{S}})$, whose Weyl group can be identified with $W(\mathcal{G}, \mathcal{S})$ [CGP, §C.2.13]. By [CGP, Theorem C.2.15] the set of minimal pseudo-parabolic F -subgroups of \mathcal{G} containing $Z_{\mathcal{G}}(\mathcal{S})$ is naturally in bijection with the collection of positive subsystems of $\Phi(\bar{\mathcal{G}}, \bar{\mathcal{S}})$.

Hence the Weyl group $W(\mathcal{G}, \mathcal{S})$ acts simply transitively on both these sets. This implies that $W(\mathcal{G}^+(F), \mathcal{S})$ is the semidirect product of its normal subgroup $W(\mathcal{G}, \mathcal{S})$ and the stabilizer of any minimal pseudo-parabolic F -subgroup. \square

In contrast to Theorem 5.1, a (minimal) pseudo-parabolic F -subgroup of \mathcal{G}^+ can be properly contained in its own normalizer. We say that a pseudo-parabolic F -subgroup \mathcal{P}^+ of \mathcal{G}^+ is good, or more precisely F -good, if

$$\mathcal{P}^+(F)\mathcal{G}(F) = \mathcal{G}^+(F).$$

It follows from Theorem 5.1 that every good pseudo-parabolic F -subgroup of \mathcal{G}^+ does equal its own normalizer.

By definition every pseudo-parabolic F -subgroup of the connected algebraic group \mathcal{G} is good. It is easy to see that good pseudo-parabolic F -subgroups always exist in \mathcal{G}^+ . Namely, let $\lambda : GL_1 \rightarrow \mathcal{S}$ be a cocharacter with $\mathcal{P}_\emptyset = \mathcal{P}_{\mathcal{G}}(\lambda)\mathcal{R}_{u,F}(\mathcal{G})$.

For any $w \in W(\mathcal{G}^+(F), \mathcal{P}_\emptyset, \mathcal{S})$, $w(\lambda)$ is another such cocharacter. Consider the $W(\mathcal{G}^+(F), \mathcal{P}_\emptyset, \mathcal{S})$ -invariant cocharacter

$$\lambda^+ = \prod_{w \in W(\mathcal{G}^+(F), \mathcal{P}_\emptyset, \mathcal{S})} w(\lambda).$$

With [CGP, Proposition 2.2.4], one sees that $\mathcal{P}_{\mathcal{G}}(\lambda^+) = \mathcal{P}_{\mathcal{G}}(\lambda)$. By the $W(\mathcal{G}^+(F), \mathcal{P}_\emptyset, \mathcal{S})$ -invariance, $\mathcal{P}_{\mathcal{G}^+}(\lambda^+)$ contains $N_{\mathcal{G}^+(F)}(\mathcal{P}_\emptyset, \mathcal{S})$. By (94) and Lemma 5.2 this means that

$$(98) \quad \mathcal{P}_\emptyset^+ := \mathcal{P}_{\mathcal{G}^+}(\lambda^+) \mathcal{R}_{us, F}(\mathcal{G})$$

is a good pseudo-parabolic F -subgroup of \mathcal{G}^+ .

Now we can prove a Bruhat decomposition for possibly disconnected linear algebraic groups over arbitrary fields.

Theorem 5.3. *Let \mathcal{P}^+ and \mathcal{P}' be pseudo-parabolic F -subgroups of \mathcal{G}^+ containing \mathcal{P}_\emptyset .*

(a) $\mathcal{G}^+(F) = \mathcal{P}'(F)N_{\mathcal{G}^+(F)}(\mathcal{S})\mathcal{P}^+(F)$.

(b) Write $W(\mathcal{P}, \mathcal{S}) = N_{\mathcal{P}}(\mathcal{S})/Z_{\mathcal{P}}(\mathcal{S})$. Suppose that \mathcal{P}^+ is F -good. The map

$$W(\mathcal{P}' \cap \mathcal{G}, \mathcal{S}) \backslash W(\mathcal{G}, \mathcal{S}) / W(\mathcal{P}^+ \cap \mathcal{G}, \mathcal{S}) \rightarrow (\mathcal{P}' \cap \mathcal{G})(F) \backslash \mathcal{G}^+(F) / \mathcal{P}^+(F)$$

is bijective.

Proof. By [CGP, Theorem C.2.20] there is a Tits system $(\mathcal{G}(F), \mathcal{P}_\emptyset(F), \mathcal{S}(F), R)$ with Weyl group $W(\mathcal{G}, \mathcal{S})$. Here R denotes the set of simple reflections in $W(\mathcal{G}, \mathcal{S})$ determined by \mathcal{P}_\emptyset . The theorem for $\mathcal{G}(F)$ then follows from the properties of Tits systems [Bou, §IV.2.5].

Choose a set of representatives $W^+ \subset N_{\mathcal{G}^+(F)}(\mathcal{P}_\emptyset, \mathcal{S})$ for $W(\mathcal{G}^+(F), \mathcal{P}_\emptyset, \mathcal{S})$. By Lemma 5.2 and (94)

$$(99) \quad \mathcal{G}^+(F) = \sqcup_{w \in W^+} \mathcal{G}(F)Z_{\mathcal{G}^+(F)}(\mathcal{S})w.$$

With the results for $\mathcal{G}(F)$ and $Z_{\mathcal{G}^+(F)}(\mathcal{S}) \subset N_{\mathcal{G}^+(F)}(\mathcal{P}_\emptyset)$ we obtain

$$\begin{aligned} \mathcal{G}^+(F) &= \mathcal{P}_\emptyset(F)N_{\mathcal{G}(F)}(\mathcal{S})\mathcal{P}_\emptyset(F)Z_{\mathcal{G}^+(F)}(\mathcal{S})W^+ \\ &\subset \bigcup_{w \in W^+} \mathcal{P}_\emptyset(F)N_{\mathcal{G}(F)}(\mathcal{S})Z_{\mathcal{G}^+(F)}(\mathcal{S})w(w^{-1}\mathcal{P}_\emptyset(F)w) \\ &= \bigcup_{w \in W^+} \mathcal{P}_\emptyset(F)Z_{\mathcal{G}^+(F)}(\mathcal{S})N_{\mathcal{G}(F)}(\mathcal{S})w\mathcal{P}_\emptyset(F) \\ &= \mathcal{P}_\emptyset(F)N_{\mathcal{G}^+(F)}(\mathcal{S})\mathcal{P}_\emptyset(F). \end{aligned}$$

Clearly this implies part (a). Moreover $Z_{\mathcal{G}^+(F)}(\mathcal{S}) \subset \mathcal{P}^+$ and \mathcal{P}^+ is F -good, so it contains $\mathcal{P}_\emptyset^+ \supset N_{\mathcal{G}^+(F)}(\mathcal{P}_\emptyset, \mathcal{S})$. Therefore the map in part (b) is surjective.

Furthermore the goodness of \mathcal{P}^+ implies that $\mathcal{G}^+(F)/\mathcal{P}^+(F)$ can be identified with $\mathcal{G}(F)/(\mathcal{P}^+ \cap \mathcal{G})(F)$. Now the injectivity in (b) follows from part (b) for \mathcal{G} . \square

5.2. Good maximal compact subgroups.

In this paragraph we assume that F is a non-archimedean local field and that \mathcal{G}^+ is a possibly disconnected linear algebraic F -group (not necessarily smooth), whose identity component \mathcal{G} is quasi-reductive over F . We continue the use of the notation of Section 4 for \mathcal{G} and G . The locally compact group $G^+ = \mathcal{G}^+(F)$ contains G as an open normal subgroup of finite index. By Lemma 1.3 and (16)

$$(100) \quad G \text{ is unimodular, hence so is } G^+.$$

It would be desirable that G^+ acts naturally on the affine building $\mathcal{B}(\mathcal{G}, F)$. Unfortunately, it is not even clear that $N_{G^+}(S_\emptyset)$ acts naturally on

$$X_*(S_\emptyset) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{or} \quad \mathcal{B}(\mathcal{G}/\mathcal{D}(\mathcal{G})\mathcal{R}_{u,F}(\mathcal{G}), F) = X_*(\mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

We can avoid this problem by assuming

$$(101) \quad \begin{array}{l} \text{the action of } G \text{ on } \mathcal{B}(\mathcal{G}/\mathcal{D}(\mathcal{G})\mathcal{R}_{u,F}(\mathcal{G}), F) \text{ by translations} \\ \text{extends to an isometric action of } G^+. \end{array}$$

Let us discuss which parts of Section 3 are valid for G^+ under this assumption. The conjugation action of G^+ on G induces, via Proposition 3.10.c, a canonical isometric action of G^+ on $\mathcal{BT}(\mathcal{G}, F)$. Hence the action of G on

$$\mathcal{B}(\mathcal{G}, F) = \mathcal{BT}(\mathcal{G}, F) \times \mathcal{B}(\mathcal{G}/\mathcal{D}(\mathcal{G})\mathcal{R}_{u,F}(\mathcal{G}), F)$$

extends to an isometric action of G^+ . This G^+ -action is proper because G has finite index in G^+ and the G -action is proper (Theorem 3.8.e). The proof of Proposition 3.9 also works for G^+ . In particular it shows that the G^+ -stabilizer of any vertex of $\mathcal{B}(\mathcal{G}, F)$ is a maximal compact subgroup of G^+ . Theorem 3.11 is valid for G^+ with the same subgroups K_n . We note that to make them normal in $G_{x_s}^+$ we have to take care that in Lemma 3.3.c (which holds just as well for G^+) we select subgroups of $Z_Q(S_\emptyset)$ that are normal in $N_{G^+}(S_\emptyset)$.

Using (99), one sees easily that Theorems 3.12 and 3.14 (except part (d) of the latter) for \mathcal{G} imply the same statements for \mathcal{G}^+ . The Iwasawa and Cartan decompositions do not generalize automatically to \mathcal{G}^+ , for isotropy groups of special vertices are not always good as maximal compact subgroups of G^+ .

Almost everything in Paragraph 4.1 generalizes to G^+ , even without (101). Indeed, these are statements about representations of totally disconnected locally compact groups, for which connectedness as algebraic groups plays only a minor role. Since $\mathcal{R}_{u_s, F}(\mathcal{P}^+) = \mathcal{R}_{u_s, F}(\mathcal{P}^+ \cap \mathcal{G})$, the functors $r_{\mathcal{P}}^{\mathcal{G}}$ and $r_{\mathcal{P}^+}^{G^+}$ do exactly the same on the vector space underlying a G^+ -representation.

We note that for Theorem 4.3 we need the Bruhat decomposition for possibly disconnected linear algebraic groups, as established in Theorem 5.3. This entails that we must restrict Theorem 4.3 to the cases $r_{\mathcal{P}'}^{G^+} i_{\mathcal{P}^+}^{G^+}(\pi, V)$ where \mathcal{P}^+ is a good pseudo-parabolic F -subgroup of \mathcal{G}^+ . In particular the important case (89) is valid for G^+ .

For Theorem 4.5 and most of Paragraph 4.2 we need the Cartan decomposition. To ensure that our arguments for these results hold for G^+ , we need to assume (101) and that G^+ possesses a good maximal compact subgroup in the sense of (77). Unfortunately, some disconnected groups do not possess any good maximal compact subgroup. Surprisingly, that can fail already for groups with a simple neutral component.

Example 5.4. Consider the group $SL_4(F)$. It is simply connected, so its maximal compact subgroups are precisely the stabilizers of special vertices in its Bruhat–Tits building [BrTi1, §4.4.6]. It is known that all vertices of $\mathcal{BT}(SL_4, F)$ are special and that they form 4 orbits under $SL_4(F)$.

Let τ be the inverse transpose mapping, an outer automorphism of SL_4 of order 2. Write $w^+ = \text{Ad} \left(\begin{array}{ccc} & & 1 \\ & 1 & \\ \omega_F & & \end{array} \right) \circ \tau$, an automorphism of $SL_4(F)$ whose square

is inner. Then $SL_4^+ := SL_4 \cup w^+SL_4$ is a disconnected reductive F -group with $W(SL_4^+(F), \mathcal{S}) \cong \{\text{id}, \tau\} \times S_4$ (where \mathcal{S} denotes the group of diagonal matrices in SL_4).

One checks that the natural action of w^+ on $\mathcal{BT}(SL_4, F)$ from Proposition 3.10.c sends every orbit of vertices to another $SL_4(F)$ -orbit. Hence, for any $g \in SL_4(F)$, w^+g does not fix any vertex of $\mathcal{BT}(SL_4, F)$. In other words, there does not exist any vertex of $\mathcal{BT}(SL_4, F)$ whose $SL_4^+(F)$ -stabilizer contains a representative for $w^+ \in W(SL_4^+(F), \mathcal{S})$, and $SL_4^+(F)$ does not possess any good maximal compact subgroup.

To alleviate this inconvenience we formulate two variations on the Iwasawa and Cartan decompositions for G^+ . Unfortunately, part (c) still does not seem sufficient as a substitute for the Cartan decomposition in the proofs of Theorems 4.5 and 4.9.

Theorem 5.5. *Let \mathcal{G}^+ be a linear algebraic F -group whose connected component \mathcal{G} is quasi-reductive over F .*

- (a) *Suppose that (101) holds and that G^+ possesses a good maximal compact subgroup K^+ , in the sense of (77). Then G^+ admits Iwasawa and Cartan decompositions with respect to K^+ , just as in Theorems 3.13 and 3.14.d.*
- (b) *Let K be a good maximal compact subgroup of G . Then*

$$G^+ = K\mathcal{P}^+(F) = \mathcal{P}^+(F)K$$

for every good pseudo-parabolic F -subgroup \mathcal{P}^+ of \mathcal{G}^+ .

- (c) *Let $x, y \in \mathbb{A}_\emptyset$ be such that G_x and G_x are good maximal compact subgroups of $\mathcal{G}(F)$. Using Lemma 5.2 we choose a set of representatives $N^+ \subset N_{G^+}(\mathcal{P}_\emptyset, S_\emptyset)$ for $N_{G^+}(S_\emptyset)/N_G(S_\emptyset)$.*

Then for every $n \in N^+$ there exists a cone in $Z_G(S_\emptyset)/Z_G(S_\emptyset)_{\text{cpt}}$, represented by a subset $A_n \subset Z_G(S_\emptyset)$, such that

$$G_x \backslash G^+ / G_y = \bigsqcup_{n \in N^+} \bigsqcup_{a \in A_n} G_x a n G_y.$$

Proof. (a) Under these conditions the proofs of Theorems 3.13 and 3.14.d are valid for $\mathcal{G}^+(F)$.

(b) By the Iwasawa decomposition for G :

$$K\mathcal{P}^+(F) = K(\mathcal{P}^+ \cap \mathcal{G})(F)\mathcal{P}^+(F) = \mathcal{G}(F)\mathcal{P}^+(F).$$

The definition of “good” for pseudo-parabolic F -subgroups of \mathcal{G}^+ says that the right hand side equals $\mathcal{G}^+(F)$.

(c) First we note that by (94)

$$(102) \quad G_x \backslash G^+ / G_y = \bigsqcup_{n \in N^+} G_x \backslash G n / G_y \cong \bigsqcup_{n \in N^+} G_x \backslash G / (n G_y n^{-1}) \cdot n.$$

Conjugation with $n \in N^+$ is an automorphism of $\mathcal{G}(F)$, so $n G_y n^{-1}$ is again a good maximal compact subgroup of G . By Proposition 3.10.a it fixes a unique point of $\mathcal{BT}(\mathcal{G}, F)$, which lies in the apartment associated to the maximal F -split torus $n S_\emptyset n^{-1} = S_\emptyset$. By the Bruhat–Tits fixed point theorem $n G_y n^{-1}$ equals the G -stabilizer of a point of \mathbb{A}_\emptyset , say $n(y)$. Choose A_n as in (83), for G, x and $n(y)$. By Theorem 3.14.d $G = \bigsqcup_{a \in A_n} G_x a G_{n(y)}$. Now the result follows from (102). \square

To apply some work of Harish–Chandra, Waldspurger [Wal] and Lafforgue [Laf] to quasi-reductive F -groups, we want to exhibit a well-behaved generalization of

the Harish-Chandra's Ξ -function [Wal, §II.1]. Some results in Paragraph 4.1 were selected to facilitate its construction and to show its properties. Let $K \subset G$ be a good maximal compact subgroup which fixes a point $x' \in \mathbb{A}_\emptyset$. (It exists since every G -orbit in $\mathcal{B}(\mathcal{G}, F)$ intersects every apartment.)

Recall the maximal F -split torus \mathcal{S}_\emptyset and the minimal good pseudo-parabolic F -subgroup \mathcal{P}_\emptyset^+ of \mathcal{G}^+ from (98). With Theorem 5.5.b we extend $\delta_{\mathcal{P}_\emptyset^+}$ to a function $G \rightarrow \mathbb{R}_{>0}$ which is right- K -invariant. Let μ_K be the Haar measure on K with $\mu_K(K) = 1$. We define

$$(103) \quad \Xi(g) = \int_K \delta_{\mathcal{P}_\emptyset^+}(kg)^{1/2} d\mu_K(k) \quad g \in G^+.$$

By the compactness of K , this is a continuous function $G^+ \rightarrow \mathbb{R}_{>0}$.

Let $(\pi, V) = i_{\mathcal{P}_\emptyset^+}^{G^+}(\text{triv}, \mathbb{C})$ be the normalized parabolic induction of the trivial representation of $N_{G^+}(\mathcal{P}_\emptyset, \mathcal{S}_\emptyset) \cong \mathcal{P}_\emptyset^+(F)/\mathcal{R}_{us,F}(\mathcal{P}_\emptyset^+)(F)$. We saw in (86) that it is a unitary smooth G^+ -representation. Let $f_K \in V$ be the unique K -invariant function with $f_K(1) = 1$. The uniqueness follows from the Iwasawa decomposition $G^+ = \mathcal{P}_\emptyset^+K$, as in Theorem 5.5.b. The definition of the inner product on V [Cas, Proposition 3.1.3] works out to

$$(104) \quad \Xi(g) = \langle \pi(g)f_K, f_K \rangle \quad g \in G^+.$$

The unitarity of π implies that, for all $g \in G^+$:

$$(105) \quad \Xi(g) \leq \Xi(1) = \langle f_K, f_K \rangle = \int_K f_K^2(k) d\mu_K(k) = \int_K d\mu_K(k) = 1.$$

By (104) and the uniqueness of f_K , the proof of [Wal, Lemme II.1.3] (which was meant for connected reductive groups) also applies here. It says that

$$(106) \quad \int_K \Xi(g_1kg_2) = \Xi(g_1)\Xi(g_2) \quad g_1, g_2 \in G^+.$$

Let us recall the construction of a length function on G^+ from [Wal, p. 242]. Choose an embedding of F -groups $\tau : \mathcal{G}^+ \rightarrow GL_n$. Since every maximal compact subgroup of $GL_n(F)$ is conjugate to $GL_n(\mathcal{O}_F)$, we may assume that $\tau(K) \subset GL_n(\mathcal{O}_F)$. For $g \in G^+$, let $\text{mc}(g, g^{-1})$ be the collection of all matrix coefficients of $\tau(g)$ and $\tau(g)^{-1}$. Let ν_F be the discrete valuation of F , and define

$$\ell(g) = \sup\{-\nu_F(x) : x \in \text{mc}(g, g^{-1})\}.$$

This is a K -biinvariant length function $G^+ \rightarrow \mathbb{R}_{\geq 0}$.

Lemma 5.6. *The following integral converges for sufficiently large $t \in \mathbb{R}_{>0}$:*

$$\int_{G^+} \Xi(g)^2 (1 + \ell(g))^{-t} d\mu_{G^+}(g).$$

Proof. We can use the proof of [Wal, Lemme II.1.5] (which is the desired statement, but only for connected reductive F -groups), when we can show that all the ingredients are also valid for $G^+ = \mathcal{G}^+(F)$.

The first ingredient is an estimate on the volume of double cosets KaK [Wal, p. 241] with $a \in N_{G^+}(\mathcal{P}_\emptyset, \mathcal{S}_\emptyset)$. This holds for G^+ because of the Cartan decomposition (Theorem 5.5.c) and the existence of compact open subgroups with an Iwahori decomposition (Proposition 2.3 and Theorem 3.11). Notice that elements of $N_{G^+}(\mathcal{P}_\emptyset, \mathcal{S}_\emptyset)$ normalize U and hence also \bar{U} .

The most involved step in Waldspurger’s argument is an estimate on $\Xi(g)$ for $g \in N_{G^+}(P_\emptyset, S_\emptyset)$, namely [Wal, Lemme II.1.1]. An essential ingredient of the proof of the latter is the normalized Jacquet restriction $r_{P_\emptyset^+}^{G^+}(\pi) = r_{P_\emptyset^+}^{G^+} i_{P_\emptyset^+}^{G^+}(\text{triv})$. By the special case (89) of Theorem 4.3, the trivial representation of $N_{G^+}(P_\emptyset, S_\emptyset)$ is the only irreducible constituent of $r_{P_\emptyset^+}^{G^+} i_{P_\emptyset^+}^{G^+}(\text{triv})$, and it appears with multiplicity $|W(\mathcal{G}, \mathcal{S}_\emptyset)|$. Furthermore the argument for [Wal, Lemme II.1.1] uses Theorem 4.1 and that $N_{G^+}(P_\emptyset, S_\emptyset) = S_\emptyset C$ for some compact subset C , which follows from Lemma 2.1.b (and its analogue Lemma 3.3.a for quasi-reductive F -groups). With all this at hand, Waldspurger’s proof goes through, and provides bounds on $\Xi(g)$ in terms of the length function ℓ on G^+ .

Waldspurger uses the above estimates to reduce the integral (15) to a sum over the set

$$W(\mathcal{G}, \mathcal{S}_\emptyset) \backslash N_{G^+}(P_\emptyset, S_\emptyset) / Z_G(S_\emptyset)_{\text{cpt}} \cong \bigsqcup_{n \in N^+} A_n n$$

appearing in the Cartan decomposition. In Lemmas 2.6 and 3.3.b we showed that $Z_G(S_\emptyset) / Z_G(S_\emptyset)_{\text{cpt}}$ is a finitely generated free abelian group, just as in the Cartan decomposition for connected reductive p -adic groups. It is normal and of finite index in $N_{G^+}(P_\emptyset, S_\emptyset) / Z_G(S_\emptyset)_{\text{cpt}}$. Consequently the same estimates as used in [Wal, Lemme II.1.5] work in our case, as long as we take t sufficiently large compared to $\dim(\mathbb{A}_\emptyset)$. \square

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