Lie algebra cohomology and Macdonald's conjectures

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Introduction

In 1982 Macdonald [21] published his famous article *Some conjectures for root systems*. The questions posed there inspired a lot of new research, and by know most of them are answered. This thesis deals with one of the most persisting conjectures:

> Let R be a root system with exponents $m_1, \dots, m_l$ and take $k \in \mathbb{N} \cup \{\infty\}$. The constant term (depending only on q) of
>
> $$\prod_{\alpha \in R^+} \prod_{i=1}^{k} (1 - q^i e^{\alpha})(1 - q^{i-1} e^{-\alpha}) \quad \text{is} \quad \prod_{i=1}^{l} \begin{bmatrix} k(m_i + 1) \\ k \end{bmatrix}_q .$$

Here $\begin{bmatrix} n \\ r \end{bmatrix}_q$ is the q-binomial coefficient of n and r. It is known that one can prove this if the following conjecture holds:

> Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra with exponents $m_1, \dots, m_l$. The cohomology of $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z]/(z^k)$ is a free exterior algebra with kl generators. For each $1 \leq i \leq l$, there are k generators of cohomology degree $2m_i + 1$, and the z-weight of these generators are the negatives of $0, m_i k + 1, m_i k + 2, \dots, m_i k + k - 1$.

This conjecture occupies a central place in my thesis. Although I did not succeed in finding a complete proof, I strived to explain what appears to be the most promising way to look at it.

Generally, in this thesis I sought to compute the cohomology rings of several classes of Lie algebras, all intimately connected to finite-dimensional semisimple Lie algebras. I aimed to make this accessible for anyone with a general knowledge Lie groups and Lie algebras, describing other background at a more elementary level.

We start with a collection of well-known results for compact Lie groups, reductive Lie algebras and their invariants. In the second chapter all necessary cohomology theory for Lie algebras is developed. Here we prove one new result:

> Let $\mathfrak{g}$ be a Lie algebra over a field a characteristic 0, $\mathfrak{h}$ a subalgebra and V a $\mathfrak{g}$-module, all finite-dimensional. Suppose that $\mathfrak{g}$ and V are completely reducible, both as $\mathfrak{g}$- and as $\mathfrak{h}$-modules. (So in particular $\mathfrak{g}$ is reductive.) Then $H^*(\mathfrak{g}, \mathfrak{h}; V) \cong H^*(\mathfrak{g}, \mathfrak{h}) \otimes V^{\mathfrak{g}}$.

With this machinery we compute the cohomology of a compact Lie group (or equivalently of a reductive Lie algebra) in a more algebraic way than usual.

In chapter 4 we introduce a Lie algebra $\mathfrak{g}[z, s] = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, s]$, where $s^2 = 0$ and $\mathfrak{g}$ is complex, finite-dimensional and reductive. We try to compute the cohomology of this Lie algebra, which unfortunately just falls short. Nevertheless we come across a generalization of Chevalley's restriction, which is probably new:

> Let $\mathfrak{g}$ be a finite-dimensional complex reductive Lie algebra with adjoint group G, $\mathfrak{h}$ a Cartan subalgebra and W the Weyl group. Then the restriction map
> $$S\mathfrak{g}^* \otimes \bigwedge(s\mathfrak{g})^* \to S\mathfrak{h}^* \otimes \bigwedge(s\mathfrak{g})^*$$
> induces an isomorphism
> $$\left(S\mathfrak{g}^* \otimes \bigwedge(s\mathfrak{g})^*\right)^{\mathfrak{g}[s]} \to \left(S\mathfrak{h}^* \otimes \bigwedge(s\mathfrak{h})^*\right)^W.$$

After this we conjecture what the so-called restricted cohomology of $\mathfrak{g}[z, s]$ should look like, and derive the two afore-mentioned conjectures from this.

Chapter 1

Compact Lie groups and reductive Lie algebras

In this first chapter we state some general results on compact Lie groups and reductive Lie algebras. These are mostly well known, and their proofs can be found in most standard books on Lie algebras. We refer especially to Carter [4], Humphreys [15] and Varadarajan [24]. We start with an introduction to representations of Lie groups and Lie algebras objects. Then we study the root space decomposition and the corresponding root system of a reductive Lie algebra. This leads us to the Weyl group, its invariants and the exponents of a reductive Lie algebra. We close with a survey on harmonic polynomials.

1.1 Representations

In this thesis we will encounter many representations of Lie groups and Lie algebras. For convenience we recall the terminology and a couple of examples. After that we state a few standard results. From now on G is a real Lie group and $\mathfrak{g}$ is a real Lie algebra. Almost all the following is also valid for Lie algebras over an arbitrary field, but but we prefer to keep the notation simpler.

Definition 1.1 *Let V be a topological real vector space. A representation of G on V is an analytic group homomorphism $\pi : G \to \mathrm{Aut}\, V$ such that the map $G \times V \to V : (g, v) \to \pi(g)v$ is continuous.*

A representation of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}\, V$.

If the dimension of V is finite it can be shown that every continuous group homomorphism $G \to \mathrm{Aut}\, V$ satisfies these conditions. We will often write $g \cdot v$ and $X \cdot v$ for $\pi(g)v$ and $\rho(X)v$ ($g \in G, X \in \mathfrak{g}, v \in V$) and say that G or $\mathfrak{g}$ acts on the module V.

For any subspace $V' \subset V$, we denote by $G \cdot V'$, respectively $\mathfrak{g} \cdot V'$, the subspace of V spanned by all elements $g \cdot v$ ($g \in G, v \in V'$), respectively $X \cdot v$ ($X \in \mathfrak{g}, v \in V'$).

Since $e \cdot v = v$ if e is the unit element of G, we always have $V' \subset G \cdot V'$. We call V' a G-submodule of V if $G \cdot V' \subset V'$, or equivalently if $G \cdot V' = V'$. Similarly V' is a $\mathfrak{g}$-submodule of V if $\mathfrak{g} \cdot V' \subset V'$. In these cases V/V' is a submodule in a natural way; it is called a quotient module. Note that $\mathfrak{g} \cdot V$ is a submodule of V, and that it is not necessarily equal to V.

The module V is called irreducible if it has exactly two submodules: 0 and V itself. More generally, V is completely reducible if for every submodule V' there is another submodule V'' such that $V = V' \oplus V''$. An equivalent condition is that V must be the direct sum of some of its irreducible submodules.

The elements of V that are fixed by all $\pi(g)$ are called the G-invariants. They form a submodule V^G of V on which G acts as the identity. (Such a group module is called trivial.) The only element of V that is fixed by all $\rho(X)$ is 0, for $\rho(0) = 0$. Yet there is a notion of $\mathfrak{g}$-invariant vectors. Namely, an element v of V is $\mathfrak{g}$-invariant if $\forall X \in \mathfrak{g} : \rho(X)v = 0$. Later on it will become clear why this is a reasonable definition. These invariants also form a $\mathfrak{g}$-submodule $V^{\mathfrak{g}}$ of V on which $\mathfrak{g}$ acts as 0. (This is called a trivial Lie algebra module.)

From one representation we can construct many other representations. Firstly there are the contragredient representations on the dual space V^*. For $f \in V^*$ they are defined by

$$(g \cdot f)(v) = f(g^{-1} \cdot v) \text{ and } (X \cdot f)(v) = -f(X \cdot v) = f(-X \cdot v)$$

Secondly, if $V^{\otimes n}$ is the n-fold tensor product of V with itself (over $\mathbb{R}$), then we have representations of G and $\mathfrak{g}$ on $V^{\otimes n}$. We also denote them by π and ρ, and for decomposable tensors they are given by

$$\begin{aligned} \pi(g)(v_1 \otimes \cdots \otimes v_n) &= \pi(g)v_1 \otimes \cdots \otimes \pi(g)v_n \\ \rho(X)(v_1 \otimes \cdots \otimes v_n) &= \sum_{i=1}^{n} v_1 \otimes \cdots \otimes \rho(X)v_i \otimes \cdots \otimes v_n \end{aligned}$$

These actions send (anti-)symmetric tensors to (anti-)symmetric tensors. Therefore we also get representations of G and $\mathfrak{g}$ on the n-th degree parts S^nV and $\bigwedge^n V$ of the symmetric and exterior algebras of V. For decomposable elements they are given by similar formulas as above:

$$\begin{aligned} \pi(g)(v_1 \cdots v_n) &= \pi(g)v_1 \cdots \pi(g)v_n \\ \rho(X)(v_1 \cdots v_n) &= \sum_{i=1}^{n} v_1 \cdots \rho(X)v_i \cdots v_n \\ \pi(g)(v_1 \wedge \cdots \wedge v_n) &= \pi(g)v_1 \wedge \cdots \wedge \pi(g)v_n \\ \rho(X)(v_1 \wedge \cdots \wedge v_n) &= \sum_{i=1}^{n} v_1 \wedge \cdots \wedge \rho(X)v_i \wedge \cdots \wedge v_n \end{aligned}$$

The zeroth tensor power of V is by definition $\mathbb{R}$, and the standard representations of G and $\mathfrak{g}$ on $\mathbb{R}$ are simply

$$g \cdot v = v \text{ and } X \cdot v = 0$$

These formulas together yield representations on the tensor algebra $TV = \bigoplus_{n \geq 0} V^{\otimes n}$, on the symmetric algebra $SV = \bigoplus_{n \geq 0} S^n V$ and on the exterior algebra $\bigwedge V = \bigoplus_{n \geq 0} \bigwedge^n V$. Observe that $\rho(X)$ is a derivation of these algebras, for all $X \in \mathfrak{g}$.

The third construction combines the other two. Recall that $(V^*)^{\otimes n}$, $S^n V^*$ and $\bigwedge^n V^*$ are naturally isomorphic to the dual spaces of $V^{\otimes n}$, $S^n V$ and $\bigwedge^n V$. For decomposable elements these isomorphisms are given by

$$\begin{aligned} f_1 \otimes \cdots \otimes f_n(v_1 \otimes \cdots \otimes v_n) &= f_1(v_1) \cdots f_n(v_n) \\ f_1 \cdots f_n(v_1 \cdots v_n) &= \sum_{\sigma \in S_n} f_1(v_{\sigma 1}) \cdots f_n(v_{\sigma n}) \\ f_1 \wedge \cdots \wedge f_n(v_1 \wedge \cdots \wedge v_n) &= \sum_{\sigma \in S_n} \epsilon(\sigma) f_1(v_{\sigma 1}) \cdots f_n(v_{\sigma n}) \end{aligned}$$

where S_n is the permutation group on n symbols, with sign function ϵ. This leads to the following representations on $(V^*)^{\otimes n}$:

$$\begin{aligned} g \cdot (f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) &= f_1(\pi(g)^{-1} v_1) \cdots f_n(\pi(g)^{-1} v_n) \\ X \cdot (f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) &= -\sum_{i=1}^{n} f_1(v_1) \cdots f_i(\rho(X) v_i) \cdots f_n(v_n) \end{aligned}$$

and similarly for $S^n V^*$ and $\bigwedge^n V^*$. Consequently we also obtain representations of G and $\mathfrak{g}$ on TV^*, SV^* and $\bigwedge V^*$, and $\mathfrak{g}$ acts by derivations on these algebras.

If W is another module for G and $\mathfrak{g}$, then the direct sum $V \oplus W$ becomes a module if we define for $v \in V, w \in W, g \in G, X \in \mathfrak{g}$:

$$g \cdot (v, w) = (g \cdot v, g \cdot w) \text{ and } X \cdot (v, w) = (X \cdot v, X \cdot w)$$

Moreover we can give the tensor product $V \otimes W$ a module structure by

$$g \cdot v \otimes w = g \cdot v \otimes g \cdot w \text{ and } X \cdot v \otimes w = X \cdot v \otimes w + v \otimes X \cdot w$$

Now the natural map $V^* \otimes W \to \mathrm{Hom}(V, W)$ (which is an isomorphism if either V or W has finite dimension) leads us to the following actions of G and $\mathfrak{g}$ on $\phi \in \mathrm{Hom}(V, W)$:

$$(g \cdot \phi)(v) = g \cdot \big(\phi(g^{-1} \cdot v)\big) \text{ and } (X \cdot \phi)(v) = X \cdot (\phi(v)) - \phi(X \cdot v)$$

Notice that the invariants of this module take a particularly nice form. For if ϕ is G-invariant,

$$\phi(g \cdot v) = (g \cdot \phi)(g \cdot v) = g \cdot (\phi(v))$$

So $\big(\mathrm{Hom}(V,W)\big)^G = \mathrm{Hom}_G(V,W)$. It is even more obvious that $\big(\mathrm{Hom}(V,W)\big)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V,W)$.

The next result is valid for Lie algebras over arbitrary fields.

Theorem 1.2 *If V and W are finite-dimensional and completely reducible $\mathfrak{g}$-modules, then all the above representations of $\mathfrak{g}$ are completely reducible.*

Proof. This is a direct consequence of theorem 3.16.1 of [24]. □

Now suppose that we only have a representation π of G on V, and that $\mathfrak{g}$ is the Lie algebra of G. Since the Lie algebra of Aut V is End V, we have an induced Lie algebra homomorphism $d\pi : \mathfrak{g} \to \mathrm{End}\ V$. This representation of $\mathfrak{g}$ is called the differential of π. It is given explicitly by

$$d\pi(X)v = \frac{\partial}{\partial t}\pi(\exp tX)v\big|_{t=0} \tag{1.1}$$

The representations of G on the spaces constructed from V also induce representations of $\mathfrak{g}$ on these spaces. With the help of equation 1.1 it is easily verified that these $\mathfrak{g}$-representations are just the ones given by above formulas, with ρ substituted by $d\pi$.

The next proposition explains the definition of the invariants of a Lie algebra representation.

Proposition 1.3 *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, π a representation of G on V and $d\pi$ its differential. Then the invariants for G and $\mathfrak{g}$ coincide: $V^G = V^{\mathfrak{g}}$. Moreover every G-submodule of V is also a $\mathfrak{g}$-submodule, and conversely.*

Proof. If $\pi(g)v = v\ \forall g \in G$ then by equation 1.1 $d\pi(X)v = 0\ \forall X \in \mathfrak{g}$. On the other hand suppose that $v \in V^{\mathfrak{g}}$. By our definition of a representation the map $\pi_v : G \to V : g \to \pi(g)v$ is analytic, and all its partial derivatives (at $e \in G$) are 0. Since G is connected, π_v is a constant map and $\pi_v(g) = \pi_v(e) = v\ \forall g \in G$. Therefore $V^G = V^{\mathfrak{g}}$.

To simplify the notation, we drop π and $d\pi$. It is direct consequence of equation 1.1 that for every subspace $V' \subset V$ we have $\mathfrak{g} \cdot V' \subset G \cdot V'$, so every G-submodule is also a $\mathfrak{g}$-submodule.

Suppose that $\mathfrak{g} \cdot V' \subset V' \neq G \cdot V'$. Then we can find $g \in G, v_1 \in V'$ such that $gv_1 \in V \setminus V'$. Since G is connected we can write $g = \exp(X_n) \cdots \exp(X_1)$ and

$$gv_1 = \exp(X_n) \cdots \exp(X_1)v_1 = \ldots = \exp(X_n) \cdots \exp(X_i)v_i = \ldots = v_n$$

There is an i with $v_i \in V'$ but $v_{i+1} = \exp(X_i)v_i \notin V'$. Dropping the subscript i gives us an analytic map $f : [0,1] \to V/V' : t \to \exp(tX)v$ with $f(0) = 0$ and $f(1) \neq 0$. It satisfies

$$f'(t) = \frac{\partial}{\partial s}\exp(sX)v\big|_{s=t} = \frac{\partial}{\partial s}\exp(sX)\exp(tX)v\big|_{s=0} = X(\exp(tX)v) = Xf(t)$$

In particular $f'(0) = Xf(0) = Xv = 0 \in V/V'$. But this is impossible, as one can see by using the Taylor expansion of f. We conclude that $V' = G \cdot V'$ and that every $\mathfrak{g}$-submodule is also a G-submodule. $\square$

The quotient module $V/\mathfrak{g} \cdot V$ is called the space of coinvariants of V under $\mathfrak{g}$. If V is especially complicated, we write $\text{coinv}_{\mathfrak{g}} V$ to avoid cumbersome notation. The name is explained by the next result, in connection with proposition 1.3.

Lemma 1.4 *Let V be a $\mathfrak{g}$-module. Then $(V^*)^{\mathfrak{g}}$ is naturally isomorphic to $(V/\mathfrak{g} \cdot V)^*$. If the dimension of V is finite, also $(V^{\mathfrak{g}})^*$ and $V^*/g \cdot V^*$ are naturally isomorphic.*

Proof.

$$\begin{aligned} (V^*)^{\mathfrak{g}} &= \{f \in V^* : \forall X \in \mathfrak{g}, v \in V \ - f(X \cdot v) = 0\} \\ &= \left\{ f \in V^* : f\big|_{\mathfrak{g} \cdot V} = 0 \right\} \cong (V/\mathfrak{g} \cdot V)^* \end{aligned}$$

If $\dim V < \infty$, we substitute V^* for V in these formulas, and take the dual spaces. Then we obtain

$$(V^{\mathfrak{g}})^* \cong ((V^{**})^{\mathfrak{g}})^* \cong (V^*/\mathfrak{g} \cdot V^*)^{**} \cong V^*/\mathfrak{g} \cdot V^*$$

The composite isomorphism $V^*/\mathfrak{g} \cdot V^* \to (V^{\mathfrak{g}})^*$ is just the restriction of functions on V to $V^{\mathfrak{g}}$, and this is natural. $\square$

1.2 Compact Lie groups and reductive Lie algebras

Now we come to the main subjects, compact Lie groups and reductive Lie algebras. Compact Lie groups are arguably the most pleasant topological groups one can encounter. They are manifolds, have a compatible group structure and (by compactness) a Haar measure. This Haar measure is invariant under left and right multiplication and the measure of the total group is normalized to 1. These observations already imply some useful facts about representations.

Proposition 1.5 *Let G be a compact Lie group and π a representation of G on an inner product space V. There exists an inner product on V such that*

$$\forall v, w \in V, g \in G : \langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$$

Proof. Let $\langle \cdot, \cdot \rangle_0$ be any inner product on V and define

$$\langle v, w \rangle := \int_G \langle \pi(g)v, \pi(g)w \rangle_0 \, dg$$

Of course one should check that this is an inner product, but this is trivial. Furthermore

$$\begin{aligned}\langle \pi(h)v, \pi(h)w\rangle &= \int_G \langle \pi(g)\pi(h)v, \pi(g)\pi(h)w\rangle_0 \, dg \\ &= \int_G \langle \pi(gh)v, \pi(gh)w\rangle_0 \, dg \\ &= \int_G \langle \pi(g)v, \pi(g)w\rangle_0 \, dg \;=\; \langle v, w\rangle\end{aligned}$$

because dg is a Haar measure. □

Corollary 1.6 *Every inner product space is a completely reducible G-module.*

Proof. If W is a submodule the orthoplement $W^\perp$ with respect to $\langle\cdot,\cdot\rangle$ is also a submodule. For if $w \in W, v \in W^\perp, g \in G : \langle w, \pi(g)v\rangle = \langle \pi(g^{-1})w, v\rangle = 0$ since $\pi(g^{-1})w \in W$. □

Lemma 1.7 *Let G be a compact Lie group and $\pi : G \to \operatorname{Aut} V$ a finite-dimensional representation. Then $\dim V^G = \int_G \operatorname{tr} \pi(g) dg$.*

Proof. Put $L := \int_G \pi(g) dg \in \operatorname{End} V$. Now for all $h \in G$:

$$\pi(h)L = \pi(h) \int_G \pi(g)dg = \int_G \pi(hg)dg = \int_G \pi(g)dg = L \tag{1.2}$$

By proposition 1.5 we can decompose $V = V^G \oplus W$, where W is a submodule of G with $W^G = 0$. It is clear from the definition that $L\big|_{V^G} = 1$. If $w \in W$, then $Lw \in W$ and is invariant by formula 1.2. Hence $Lw = 0$ and $L\big|_W = 0$. Now we see

$$\dim V^G = \operatorname{tr} L = \operatorname{tr}\left(\int_G \pi(g)dg\right) = \int_G \operatorname{tr} \pi(g)dg$$

and the lemma is proved. □

A consequence (or special case) of this lemma is the following.

Lemma 1.8 *Let $\pi : G \to \operatorname{Aut} V$ be a representation of G on a finite-dimensional real vector space V, and $\rho : G \to \operatorname{Aut} \bigwedge V$ the continuation of π. Then*

$$\dim(\textstyle\bigwedge V)^G = \displaystyle\int_G \det(1 + \pi(g))dg$$

Proof. Suppose that $L \in \operatorname{End} V_{\mathbb{C}}$ is diagonalizable, with eigenvectors $v_1, \ldots, v_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$. For $I \subset \{1, \ldots, n\}$ let v_I be the wedge product of all v_i with $i \in I$, in increasing order. One of the first results on exterior algebras says that

$\{v_I : I \subset \{1, \ldots, n\}\}$ is a basis of $\bigwedge V_{\mathbb{C}}$. If $\bar{L}$ is the induced endomorphism of $\bigwedge V_{\mathbb{C}}$, then using $Lv_I = \lambda_I v_I$,

$$\operatorname{tr} \bar{L} = \sum_I \lambda_I = \prod_{i=1}^n (1 + \lambda_i)$$

But this is also $\det(1 + L)$, hence for diagonalizable L, tr $\bar{L} = \det(1 + L)$. Now the set of all diagonalizable endomorphisms of $V_{\mathbb{C}}$ is dense in End $V_{\mathbb{C}}$, so this holds for all $L \in \operatorname{End} V \subset \operatorname{End} V_{\mathbb{C}}$.

In particular for all $g \in G : \operatorname{tr} \rho(g) = \det(1 + \pi(g))$. Now

$$\dim(\bigwedge V)^G = \int_G \operatorname{tr} \rho(g) dg = \int_G \det(1 + \pi(g)) dg$$

by lemma 1.7. □

Let us give some characterizations of semisimple and reductive Lie algebras. These are in fact well known but deep results and their proofs can be found in (for example) [24] and [15]. Recall that a simple Lie algebra is not abelian and has exactly two ideals: 0 and itself. A Lie algebra is semisimple if its radical is 0, and reductive if its radical equals its center. The Killing form of $\mathfrak{g}$ is the symmetric bilinear form

$$\kappa(X, Y) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y)$$

This κ is invariant: $\forall g \in G, \forall X, Y, Z \in \mathfrak{g}$

$$\begin{aligned} \kappa(\operatorname{Ad}(g)X, \operatorname{Ad}(g)Y) &= \kappa(X, Y) \\ \kappa(X, [Y, Z]) &= \kappa([X, Y], Z) \end{aligned}$$

Theorem 1.9 *For a finite-dimensional Lie algebra $\mathfrak{g}$ with over a field over characteristic 0 the following statements are equivalent.*

1. $\mathfrak{g}$ *is semisimple*
2. $\mathfrak{g}$ *is the sum of its simple ideals*
3. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$
4. *the Killing form of* $\mathfrak{g}$ *is nondegenerate*
5. *every finite-dimensional* $\mathfrak{g}$*-module is completely reducible*

Theorem 1.10 *For a finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0, the following statements are equivalent.*

1. $\mathfrak{g}$ *is reductive*
2. *the adjoint representation of* $\mathfrak{g}$ *is completely reducible*
3. $\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, *where* $Z(\mathfrak{g})$ *is the center of* $\mathfrak{g}$ *and* $[\mathfrak{g}, \mathfrak{g}]$ *is semisimple.*

For a compact Lie group G with Lie algebra $\mathfrak{g}$ condition 2 of theorem 1.10 is valid. For by proposition 1.5 the representation Ad: $G \to \text{Aut}\,\mathfrak{g}$ is completely reducible, and by proposition 1.3, so is the induced representation $d\text{Ad} = \text{ad} : \mathfrak{g} \to \text{End}\,V$. So $\mathfrak{g}$ is a real reductive Lie algebra. Let us call a real Lie algebra of compact type if there is a compact Lie group with that Lie algebra. It is definitely not true that every finite-dimensional real reductive Lie algebra is of compact type.

Proposition 1.11 *A finite-dimensional real reductive Lie algebra $\mathfrak{g}$ is of compact type if and only if for all $X \in \mathfrak{g}$ $\text{ad}\,X$ is semisimple and has pure imaginary eigenvalues.*

Proof. Suppose that G is a Lie group with Lie algebra $\mathfrak{g}$. From proposition 1.5 we know that $\text{Ad}(G)$ can be considered to be a subgroup of the orthogonal group $O(\mathfrak{g})$ of $\mathfrak{g}$. Therefore every $\text{ad}\,X$ is in the Lie algebra $\mathfrak{o}(\mathfrak{g})$ of $O(\mathfrak{g})$ which consists of all skew-symmetric linear maps $\mathfrak{g} \to \mathfrak{g}$. A standard result form linear algebra says that $\text{ad}\,X$ is semisimple and has pure imaginary eigenvalues. The sufficiency of this condition is much more difficult to prove. See theorem 4.11.7 of [24]. $\square$

Notice that it follows from this proposition that the Killing form of semisimple Lie algebra of compact type is negative definite.

Now let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}$. Condition 2 of theorem 1.10 still applies, so $\mathfrak{g}_{\mathbb{C}}$ is a complex reductive Lie algebra. This is very pleasant, as the structure of complex reductive Lie algebras can be described in high detail. We do this in the next section.

Choose a complex Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Considered as a real Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ cannot meet the compactness condition of proposition 1.11, so $G_{\mathbb{C}}$ is not compact. But still, by theorem 4.11.14 of [24], $G_{\mathbb{C}}$ can be chosen in such a way that G is imbedded in it as the subgroup defined by $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$. So every compact Lie group of real dimension n can be considered as a subgroup of some complex reductive Lie group of complex dimension n.

1.3 Structure theory

It is time to investigate the structure of reductive Lie algebras and compact Lie groups. We start with Cartan subalgebras and the root space decomposition of complex reductive Lie algebras. Then we consider root systems and we describe Chevalley bases. These results are carried over to real Lie algebras of compact type. The material in this section is the natural generalization of the classical structure theory for semisimple Lie algebras. All the missing proofs can be found in Humphreys [15] or Varadarajan [24].

Definition 1.12 *Let $\mathfrak{g}$ be a Lie algebra. A Cartan subalgebra (CSA) of $\mathfrak{g}$ is a nilpotent subalgebra that is its own normalizer in $\mathfrak{g}$. An element X of $\mathfrak{g}$ is called* *semisimple if* $\operatorname{ad} X$ *is a semisimple endomorphism of* $\mathfrak{g}$*, and it is called regular if the dimension of its centralizer is minimal among the dimensions of centralizers of elements of* $\mathfrak{g}$*.*

Assume that $\mathfrak{g}$ has finite dimension. The regularity of an element is equivalent to a certain coefficient of the characteristic polynomial of $\operatorname{ad} X$ not being zero, so the set of regular elements of $\mathfrak{g}$ is Zariski open and in particular dense.

Now let $\mathfrak{g}$ be not only finite-dimensional, but also complex and reductive. Then the CSA's are the maximal abelian subalgebras. Every CSA contains $Z(\mathfrak{g})$ and is of the form $\mathfrak{t} = Z(\mathfrak{g}) \oplus \mathfrak{h}$ where $\mathfrak{h}$ is a CSA of the complex semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. All elements of $\mathfrak{h}$ are semisimple, and every semisimple element of $\mathfrak{g}$ is contained in a CSA. (Since $\mathbb{C}$ is algebraically closed this means that all $\operatorname{ad} H$ ($H \in \mathfrak{h}$) are diagonalizable.) Moreover it is known that the union of all CSA's (the set of all semisimple elements) is Zariski open and dense in $\mathfrak{g}$.

The centralizer of a regular semisimple element is a CSA, and every CSA is of this form. Furthermore every two CSA's of $\mathfrak{g}$ are conjugate under an element of the adjoint group of $\mathfrak{g}$. The principal significance of this statement is that all of the properties of $\mathfrak{g}$ we will deduce do not depend on the choice of the CSA (up to isomorphism). The dimension of any CSA is an important invariant of $\mathfrak{g}$ and is called the rank of $\mathfrak{g}$.

Fix a CSA $\mathfrak{t} = Z(\mathfrak{g}) \oplus \mathfrak{h}$. The set $\{\operatorname{ad} H : H \in \mathfrak{h}\}$ consists of commuting diagonalizable endomorphisms of $\mathfrak{g}$. Hence all these endomorphisms can be diagonalized simultaneously and we obtain a decomposition in eigenspaces $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \forall H \in \mathfrak{h}\ [H, X] = \alpha(H)X\} \tag{1.3}$$

Since $\mathfrak{t}$ is abelian and equals its own normalizer in $\mathfrak{g}$, $\mathfrak{g}_0 = \mathfrak{t} = Z(\mathfrak{g}) \oplus \mathfrak{h}$. Put

$$R := \{\alpha \in \mathfrak{h}^* \setminus 0 : \mathfrak{g}_\alpha \neq 0\} \tag{1.4}$$

The elements of R are called the roots of $(\mathfrak{g}, \mathfrak{t})$. Notice that R is finite because $\mathfrak{g}$ has finite dimension. Now we have the famous root space decomposition

$$\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \tag{1.5}$$

There is an enormous amount of theory on this root space decomposition, but we will only concern ourselves with the most important and relevant results.

It can be proved that $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in R$, and that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in R$. By theorem 1.9 the Killing form κ of $\mathfrak{g}$ is nondegenerate on $[\mathfrak{g}, \mathfrak{g}]$. However if $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\beta \in \mathfrak{g}_\beta$

$$\alpha(H)\kappa(X_\alpha, X_\beta) = \kappa([H, X_\alpha], X_\beta) = -\kappa(X_\alpha, [H, X_\beta]) = -\beta(H)\kappa(X_\alpha, X_\beta)$$

so $\kappa(X_\alpha, X_\beta) = 0$ unless $\alpha = -\beta$. Therefore κ is nondegenerate on $\mathfrak{g}_0 \cap [\mathfrak{g}, \mathfrak{g}] = \mathfrak{h}$.

This means that to every $\alpha \in \mathfrak{h}^*$ there corresponds a unique element $T_\alpha \in \mathfrak{h}$ with $\alpha(H) = \kappa(T_\alpha, H)\ \forall H \in \mathfrak{h}$. Now we can transfer the Killing form to $\mathfrak{h}^*$ by $\langle \alpha, \beta \rangle = \kappa(T_\alpha, T_\beta)$. It is obvious that $\langle \cdot, \cdot \rangle$ is a nondegenerate symmetric bilinear form on $\mathfrak{h}^*$.

The real span $\mathfrak{h}_{\mathbb{R}}$ of $\{T_\alpha : \alpha \in R\}$ is a subspace of $\mathfrak{h}^*$ of real dimension $\dim_{\mathbb{C}} \mathfrak{h}$, and $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$. Its (real) dual space $\mathfrak{h}_{\mathbb{R}}^*$, is the real span of R. The Killing form and $\langle \cdot, \cdot \rangle$ are (real valued) inner products on $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$. Every $\alpha \in \mathfrak{h}_{\mathbb{R}}^* \setminus 0$ defines a reflection σ_α of $\mathfrak{h}^*$ that fixes $\alpha^\perp$:

$$\sigma_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha = \beta - \langle \beta, \alpha^\vee \rangle \alpha$$

Now we can summarize some very important properties of R by saying that it is a root system in $\mathfrak{h}_{\mathbb{R}}^*$. To make the consequences of this clear, we recall the basic theory of root systems. After that we return to Lie algebras and postpone other important facts about root systems to the next section. Now we begin with a few definitions.

Definition 1.13 *Let V be a finite-dimensional real inner product space. A root system R in V is a finite subset of $V \setminus 0$ satisfying the following conditions:*

1. *R spans V*
2. *if $\alpha \in R$, the only scalar multiples of α in R are α and $-\alpha$*
3. *if $\alpha, \beta \in R$ then $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$*
4. *for all $\alpha \in R\ :\ \sigma_\alpha R = R$*

The rank of R is $\dim V$ and is denoted by l. Furthermore R is irreducible if whenever $R = R_1 \cup R_2$ with $R_1 \perp R_2$, $R_1 = \emptyset$ or $R_2 = \emptyset$.

Definition 1.14 *A positive system of roots is a subset P of R such that R is the disjoint union of P and $-P$, and*

$$\alpha, \beta \in P, \alpha + \beta \in R \Longrightarrow \alpha + \beta \in P$$

Definition 1.15 *A basis of R a subset Δ of R such that*

1. *Δ is a basis of V*
2. *every $\beta \in R$ can be written (uniquely) as $\beta = \sum_{\alpha \in \Delta} n_{\alpha\beta}\alpha$,*
   where either all $n_{\alpha\beta} \in \mathbb{N}$ or all $n_{\alpha\beta} \in \mathbb{Z}_{\leq 0}$.

The height of β (relative to Δ) is $h(\beta) = \sum_{\alpha \in \Delta} n_{\alpha\beta}$. Thus we can say that $\alpha > \beta$ if and only if $h(\alpha) > h(\beta)$. The roots that are > 0 clearly constitute a positive system. We can also go back from a positive system to a basis, but this is more difficult. If we have a positive system, we call a positive root indecomposable if it is not the sum of two other positive roots. In this case the indecomposable roots form a basis of R.

Let us call an element v of V regular if for all $\alpha \in R : \langle v, \alpha \rangle \neq 0$. The set of regular elements is the complement of a finite number of hyperplanes, so it is open and dense in V. The connected components of this set are called the chambers of the root system R. With the chamber containing v we associate the positive system of all roots α for which $\langle v, \alpha \rangle > 0$. Conversely, with a basis or a positive system we associate the chamber of all $v \in V$ such that $\langle v, \alpha \rangle > 0$ for all α in Δ or in P.

To summarize: there are natural bijections between the bases, positive systems and chambers of R.

For $v \in V \setminus 0$, put $v^\vee = \frac{2v}{\langle v, v \rangle}$. One can easily check that $R^\vee := \{\alpha^\vee : \alpha \in R\}$ is also a root system in V. It is sometimes called the dual root system of R, but to avoid confusion we will not use this name.

Now we return to Lie algebras and define

$$H_\alpha := T_\alpha^\vee = T_{\alpha^\vee} \in \mathfrak{h}_\mathbb{R}$$

By the above, the sets $\{H_\alpha : \alpha \in R\}$ and $\{T_\alpha : \alpha \in R\}$ are root systems in $\mathfrak{h}_\mathbb{R}$.

We can construct bases of this R in a slightly more natural way than for an arbitrary root system. Recall that an element of $\mathfrak{g}$ is regular if the dimension of its centralizer is minimal. But the centralizer of $H \in \mathfrak{h}$ is $Z(\mathfrak{g}) \oplus \mathfrak{h} \oplus \bigoplus_{\alpha(H)=0} \mathfrak{g}_\alpha$. So the regular elements in $\mathfrak{h}$ are just the ones with $\alpha(H) \neq 0 \ \forall \alpha \in R$ and the regular elements in $\mathfrak{h}_\mathbb{R}$ are exactly the regular elements with respect to the two root systems just above. Fix such a regular element $H_0 \in \mathfrak{h}_\mathbb{R}$. We get positive system by defining $\alpha > \beta$ if and only if $\alpha(H_0) > \beta(H_0)$. In this way we can extend the bijections above (for chambers, positive systems and bases of R) to the chambers in $\mathfrak{h}_\mathbb{R}$. Thus a regular element X of $\mathfrak{g}$ for which $\operatorname{ad} X$ has real eigenvalues determines not only a CSA, but also a basis and a positive system of the corresponding root system.

As promised we give a Chevalley basis of $\mathfrak{g}$.

Theorem 1.16 *Let $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{t} = Z(\mathfrak{g}) \oplus \mathfrak{h}$ a CSA, R the root system of $(\mathfrak{g}, \mathfrak{t})$ and Δ a basis of R. For $\alpha \in R$, let $H_\alpha \in \mathfrak{h}$ be the element corresponding to $\alpha^\vee$. Choose any basis $\{Z_i : 1 \leq i \leq r\}$ of $Z(\mathfrak{g})$. Then it is possible to select for every root α an $X_\alpha \in \mathfrak{g}_\alpha$ such that the following hold:*

1. *$\{Z_i : 1 \leq i \leq r\} \cup \{H_\alpha : \alpha \in \Delta\} \cup \{X_\alpha : \alpha \in R\}$ is a basis of $\mathfrak{g}$.*
2. *$[H_\alpha, X_\beta] = \beta(H_\alpha) X_\beta = \frac{2\kappa(H_\alpha, H_\beta)}{\kappa(H_\beta, H_\beta)} X_\beta = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} X_\beta$ with $\beta(H_\alpha) \in \mathbb{Z}$*
3. *$\forall \alpha \in R : [X_\alpha, X_{-\alpha}] = H_\alpha$ is in the $\mathbb{Z}$-span of $\{H_\alpha : \alpha \in \Delta\}$*
4. *$[H_\alpha, H_\beta] = 0 \ \forall \alpha, \beta \in R$.*
5. *if $\alpha, \beta \in R$ but $\alpha + \beta \notin R$, then $[X_\alpha, X_\beta] = 0$*
6. *if $\alpha, \beta, \alpha + \beta \in R$, then $[X_\alpha, X_\beta] = c_{\alpha,\beta} X_{\alpha+\beta}$ with $c_{\alpha,\beta} \in \mathbb{Z} \setminus 0$*
7. *$c_{\beta,\alpha} = c_{-\alpha,-\beta} = -c_{\alpha\beta} = -c_{-\beta,-\alpha}$*

This theorem describes $\mathfrak{g}$ entirely in terms of the dimension of its center r, its root system R and the structure constants $c_{\alpha,\beta}$. But it turns out that we can choose the X_α so cleverly that even these constants can be deduced from the root system. Therefore $\mathfrak{g}$ is determined (up to isomorphism) by r and R. This leads to the well known bijection between root systems and finite-dimensional complex semisimple Lie algebras, the irreducible root systems corresponding to simple Lie algebras.

Example. Let us see how all this stuff works out in the case $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$. The center $Z(\mathfrak{g})$ of $\mathfrak{g}$ consists of all scalar multiples of the identity matrix I_n, and

$$[\mathfrak{g},\mathfrak{g}] = \mathfrak{sl}(n,\mathbb{C}) = \{A \in \mathfrak{gl}(n,\mathbb{C}) : \operatorname{tr} A = 0\}$$

which is known to be simple. The subalgebra $\mathfrak{d}(n,\mathbb{C})$ of all diagonal matrices is clearly abelian and equals its own normalizer in $\mathfrak{g}$, so it is a CSA of $\mathfrak{g}$. Then $\mathfrak{h} := \mathfrak{d}(n,\mathbb{C}) \cap \mathfrak{sl}(n,\mathbb{C})$ (the subalgebra of diagonal matrices with trace 0) is a CSA of $\mathfrak{sl}(n,\mathbb{C})$. We see that the rank of $\mathfrak{g}$ is n and that the rank of $\mathfrak{sl}(n,\mathbb{C})$ is $n-1$. Let $E_{i,j} \in \mathfrak{g}$ be the matrix with the (i,j)-th entry 1 and the rest 0, and let $\lambda_i \in \mathfrak{g}^*$ be the (i,i)-th coordinate function. Then for any $H \in \mathfrak{h}$ we have

$$[H, E_{i,j}] = (\lambda_i - \lambda_j)(H)E_{i,j}$$

So the root system is

$$R = \{\lambda_i - \lambda_j : 1 \leq i,j \leq n, i \neq j\}$$

and $\mathfrak{g}_{\lambda_i - \lambda_j} = \mathbb{C}E_{i,j}$. Now it is also clear that the Killing form restricted to $\mathfrak{d}(n,\mathbb{C})$ is

$$\kappa(H,H') = \sum_{1\leq i,j\leq n} (\lambda_i - \lambda_j)(H)(\lambda_i - \lambda_j)(H') = 2\sum_{1\leq i<j\leq n} (\lambda_i - \lambda_j)(H)(\lambda_i - \lambda_j)(H')$$

Using this we calculate

$$\kappa(E_{i,i}, E_{j,j}) = \begin{cases} 2(n-1) & \text{if } i = j \\ -2 & \text{if } i \neq j \end{cases} \tag{1.6}$$

$$\kappa(E_{i,i}, H) = 2(n-1)\lambda_i(H) - 2\sum_{j\neq i} \lambda_j(H) = 2n\lambda_i(H) \quad \text{if } H \in \mathfrak{h} \tag{1.7}$$

$$\kappa(E_{i,i} - E_{i+1,i+1}, E_{j,j} - E_{j+1,j+1}) = \begin{cases} 4n & \text{if } j = i \\ -2n & \text{if } j = i+1 \text{ or } j = i-1 \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

So $\lambda_i \in \mathfrak{h}^*$ corresponds to $\frac{1}{2n^2}(nE_{i,i} - I_n) \in \mathfrak{h}$ and for $\alpha = \lambda_i - \lambda_j$ we have

$$T_\alpha = \frac{E_{i,i} - E_{j,j}}{2n},\ H_\alpha = \frac{2T_\alpha}{\kappa(T_\alpha, T_\alpha)} = E_{i,i} - E_{j,j}$$

The set

$$\Delta := \{\alpha_i = \lambda_i - \lambda_{i+1} : 1 \leq i < n\}$$

is a basis of R, with positive system $P = \{\lambda_i - \lambda_j : i > j\}$. Writing H_i for $H_{\alpha_i} = E_{i,i} - E_{i+1,i+1}$, we have a Chevalley basis

$$\{I_n\} \cup \{H_i : 1 \leq i < n\} \cup \{E_{i,j} : 1 \leq i, j \leq n, i \neq j\}$$

of $\mathfrak{g}$. The regular elements in $\mathfrak{h}$ are the diagonal matrices of trace 0 for which all diagonal entries are different, and the Weyl chamber of $\mathfrak{h}_\mathbb{R}$ determined by Δ is

$$\{A \in \mathfrak{d}(n, \mathbb{R}) \cap \mathfrak{sl}(n, \mathbb{R}) : a_{1,1} > a_{2,2} > \cdots > a_{n,n}\}$$

The Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ is a most typical and natural example of a complex reductive Lie algebra, so we will reconsider it a couple of times.

Now we set out to find a compact real form of $\mathfrak{g}$. By this we mean a real subalgebra $\mathfrak{g}_c$ of $\mathfrak{g}$ of compact type such that, as vector spaces, $\mathfrak{g} = \mathfrak{g}_c \oplus i\mathfrak{g}_c$. By proposition 1.11 all the eigenvalues of $\operatorname{ad} X (X \in \mathfrak{g}_c)$ are pure imaginary. Since every root is real valued on $\mathfrak{h}_\mathbb{R}$ it is natural to start with $i\mathfrak{h}_\mathbb{R}$, the real span of $\{iH_\alpha : \alpha \in \Delta\}$. However we cannot multiply with $\beta(iH_\alpha)$ in $\mathfrak{g}_c$, so no element of $\mathfrak{g}_\beta$ can be in $\mathfrak{g}_c$. Instead we take for $\alpha > 0$:

$$Y_\alpha := X_\alpha - X_{-\alpha} \text{ and } Y_{-\alpha} := i(X_\alpha + X_{-\alpha})$$

Let us put $c_{\gamma\delta} = 0$ if $\gamma + \delta \notin R$. Then for $H \in \mathfrak{h}_\mathbb{R}, \alpha, \beta \in R, \alpha > \beta > 0$ we deduce from theorem 1.16

$$\begin{array}{lllllll} [iH, iH_\alpha] & = & 0 & [Y_\alpha, Y_\beta] & = & c_{\alpha,\beta} Y_{\alpha+\beta} - c_{\alpha,-\beta} Y_{\alpha-\beta} \\ [iH, Y_\alpha] & = & \alpha(H) Y_{-\alpha} & [Y_{-\alpha}, Y_{-\beta}] & = & -c_{\alpha,\beta} Y_{\alpha+\beta} - c_{\alpha,-\beta} Y_{\alpha-\beta} \\ [iH, Y_{-\alpha}] & = & -\alpha(H) Y_\alpha & [Y_\alpha, Y_{-\beta}] & = & c_{\alpha,\beta} Y_{-\alpha-\beta} + c_{\alpha,-\beta} Y_{\beta-\alpha} \\ [Y_\alpha, Y_{-\alpha}] & = & 2iH_\alpha & [Y_{-\alpha}, Y_\beta] & = & c_{\alpha,\beta} Y_{-\alpha-\beta} - c_{\alpha,-\beta} Y_{\beta-\alpha} \end{array} \tag{1.9}$$

Now the next theorem says that this gives a more or less standard compact real form of $\mathfrak{g}$.

Theorem 1.17 *Let $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{t}$ a CSA, R the root system and Δ a basis of R and $\{Z_i : 1 \leq i \leq r\}$ any basis of $Z(\mathfrak{g})$. Define H_α and $Y_\alpha (\alpha \in R)$ as above.*
The real span of $\{Z_i : 1 \leq i \leq r\} \cup \{iH_\alpha : \alpha \in \Delta\} \cup \{Y_\alpha : \alpha \in R\}$ is a compact real form $\mathfrak{g}_c$ of $\mathfrak{g}$. The real span $\mathfrak{t}_c$ of $\{Z_i : 1 \leq i \leq r\} \cup \{iH_\alpha : \alpha \in \Delta\}$ is a CSA of $\mathfrak{g}_c$.
In particular every complex reductive Lie algebra is the complexification of a real Lie algebra of compact type.
If $\widetilde{\mathfrak{g}_c}$ is another compact real form of $\mathfrak{g}$ that contains $\mathfrak{t}_c$, then $\exists H \in \mathfrak{h}_\mathbb{R}$ such that $\exp(\operatorname{ad} H)\widetilde{\mathfrak{g}_c} = \mathfrak{g}_c$, so $\widetilde{\mathfrak{g}_c}$ and $\mathfrak{g}_c$ are isomorphic.

Now assume that $\mathfrak{g}$ is a real reductive Lie algebra and $\mathfrak{t}$ is a CSA, and $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{t}_\mathbb{C}$ are the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$. We know that $\mathfrak{g}_\mathbb{C}$ is reductive. Because $\mathfrak{t}$ is nilpotent, so is $\mathfrak{t}_\mathbb{C}$, and the normalizer of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$ is just the complexification of $N_\mathfrak{g}(\mathfrak{t}) = \mathfrak{t}$, which is $\mathfrak{t}_\mathbb{C}$ itself. So $\mathfrak{t}_\mathbb{C}$ is a CSA of $\mathfrak{g}_\mathbb{C}$, and $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ has the Chevalley basis of theorem 1.16. It is absolutely not necessary that $\mathfrak{g}$ is the subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}_\mathbb{C}$ described in theorem 1.17. For example $\mathfrak{g}$ could also be the real span of the Chevalley basis. However, $\mathfrak{g}$ doesn't need to have a root space decomposition, as the semisimple endomorphisms $\operatorname{ad} H$ $(H \in \mathfrak{t})$ are not always diagonalizable.

If $\mathfrak{g}$ is of compact type, theorem 1.17 does assure that $\mathfrak{g}$ is conjugate (under an element of the adjoint group of $\mathfrak{g}_\mathbb{C}$) to $\mathfrak{g}_0$. This allows us to speak of the root system of $(\mathfrak{g}, \mathfrak{t})$. Since all CSA's of $\mathfrak{g}_\mathbb{C}$ lead to isomorphic root systems, all CSA's of $\mathfrak{g}$ also lead to isomorphic root systems.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Just as in the complex case, all CSA's of $\mathfrak{g}$ are conjugate under $\operatorname{Ad} G$. However, now *every* element of $\mathfrak{g}$ is contained in a CSA.

We examine the Lie subgroup T of G defined by a CSA $\mathfrak{t}$. Let A be the centralizer of $\mathfrak{t}$ in G. It is a closed and therefore compact subgroup of G. On the other hand $\mathfrak{t}$ is abelian and equals its own normalizer in $\mathfrak{g}$, so it is also its own centralizer in $\mathfrak{g}$. Consequently, the Lie algebra of A is $\mathfrak{t}$, and the subgroup of G defined by $\mathfrak{t}$ is the identity component of A. So T is a compact connected abelian subgroup of G. Moreover T cannot be contained in an abelian subgroup of higher dimension because $\mathfrak{t}$ is maximal abelian. This amounts to saying that T is a maximal torus of G. (Recall that an n-dimensional torus is a Lie group isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$.)

This also goes the other way round. For suppose that T is a maximal torus of G, with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Then $\mathfrak{t}$ is abelian and we claim that it is not contained in any other abelian subalgebra of $\mathfrak{g}$. Assume the contrary. Then $\mathfrak{t}$ is properly contained in some CSA $\mathfrak{a}$ of $\mathfrak{g}$, and therefore T would be properly contained in the (maximal) torus defined by $\mathfrak{a}$. But this contradicts the maximality of T, so $\mathfrak{t}$ is indeed maximal abelian and a CSA of $\mathfrak{g}$.

So we have a bijection between the maximal tori of G and the CSA's of $\mathfrak{g}$. A deep theorem says that every element of G is contained a maximal torus, and that all maximal tori are conjugate under $\operatorname{Ad} G$. In suitable coordinates, $\exp : \mathfrak{t} \to T$ is just the (surjective) map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. Thus we have the following useful result:

Proposition 1.18 *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Then the exponential map $\exp : \mathfrak{g} \to G$ is surjective.*

Example. If $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ then we can take $G = GL(n, \mathbb{C})$ and

$$\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n, \mathbb{C}) : A^* = -A\}$$

is a compact real form of $\mathfrak{g}$. It has a basis consisting of the elements $Y_{i,j} = E_{i,j} - E_{j,i}$, $Y_{j,i} = i(E_{i,j} + E_{j,i})$ $(1 \leq j < i \leq n)$, $iH_j (1 \leq j < n)$ and iI_n.

$$U(n) = \{A \in GL(n, \mathbb{C}) : A^* = A^{-1}\}$$

is a compact Lie group with Lie algebra $\mathfrak{u}(n)$. The set of all diagonal matrices $\mathfrak{d}(n, \mathbb{C}) \cap \mathfrak{u}(n) = i\mathfrak{d}(n, \mathbb{R})$ is once again a CSA of $\mathfrak{u}(n)$. Since the exponential map

$\mathfrak{u}(n) \to U(n)$ is the normal exponential for matrices, this CSA corresponds to the subgroup

$$D(n,\mathbb{C}) \cap U(n) = \{A \in D(n,\mathbb{C}) : \forall i \ |a_{ii}| = 1\} \cong \mathbb{R}^n/\mathbb{Z}^n$$

of $U(n)$, where $D(n,\mathbb{C}) \subset GL(n,\mathbb{C})$ is the group of all invertible diagonal matrices.

The corresponding compact real form of $[\mathfrak{g},\mathfrak{g}] = \mathfrak{sl}(n,\mathbb{C})$ is $\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n,\mathbb{C})$, with compact Lie group $SU(n) = U(n) \cap SL(n,\mathbb{C})$. Furthermore $\mathfrak{d}(n,\mathbb{C}) \cap \mathfrak{su}(n,\mathbb{C})$ is a CSA of $\mathfrak{su}(n)$ and it corresponds to the subgroup

$$D(n,\mathbb{C}) \cap SU(n) = \{A \in D(n,\mathbb{C}) : \forall i \ |a_{ii}| = 1, a_{11} \cdots a_{nn} = 1\} \cong \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$$

1.4 The Weyl group and invariant polynomials

In definition 1.13 of a root system R in V there figured the reflections $\sigma_\alpha : V \to V$, defined by $\sigma_\alpha(v) = v - \frac{2\langle v,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$. The subgroup W of End V generated by $\{\sigma_\alpha : \alpha \in R\}$ is called the Weyl group of R. Since σ_α leaves R invariant, this holds for all elements of W. By definition R spans V, so an element of W is completely determined by its action on R. Moreover R is finite, so W can be regarded as a subgroup of the *finite* group of all permutations of R. In view of proposition 1.5 we may assume that we have a W-invariant inner product $\langle \cdot, \cdot \rangle$ on V. The Weyl group of R is a typical example of a finite reflection group on V. (This does *not* mean that all elements are reflections, but that is generated by reflections.)

In the previous section we saw that there are natural bijections between the chambers, the positive systems of roots and the bases of R. Theorem 10.3 of [15] asserts that the Weyl group acts faithfully and transitively on these sets:

Theorem 1.19 *Let R be a root system in V with Weyl group W. If C and C' are chambers with corresponding bases Δ and Δ' and positive systems P and P', there is a unique $w \in W$ such that $w(C) = C', w(\Delta) = \Delta'$ and $w(P) = P'$.*

Let $SV^* = \bigoplus_{n \geq 0} S^n V^*$ be the ring of polynomials on V. The Weyl group acts on SV^* by $(w \cdot p)(v) = p(w^{-1}v)$. We are interested in the ring of W-invariant polynomials $(SV^*)^W$. Clearly the homogeneous components of a W-invariant polynomial are W-invariant themselves, so $(SV^*)^W$ is a graded subspace of SV^*. The structure of $(SV^*)^W$ was first discovered by Chevalley. See [4] for a proof.

Theorem 1.20 *Let W be a finite reflection group on V. There exist* $\dim V = l$ *algebraically independent homogeneous polynomials $F_1, \dots, F_l \in SV^*$ such that $(SV^*)^W = \mathbb{R}[F_1, \dots, F_l]$. The F_i (and even their linear span) are not uniquely determined, but if* $\deg F_i = d_i$*, then the set $\{d_i : i = 1, \dots, l\}$ does not depend on the choice of the F_i.*

These F_i and d_i are called primitive invariant polynomials and primitive degrees, of W and of R. Because R spans V, $V^W = 0$ and there are no W-invariant linear functions. In particular $d_i \geq 2$ for all i.

Let $V_{\mathbb{C}}$ be the complexification of V. Every $p \in SV^*$ can be extended to a (complex valued) polynomial $\tilde{p}$ on $V_{\mathbb{C}}$. The Weyl group acts complex linearly on $V_{\mathbb{C}}$ and on $SV_{\mathbb{C}}^*$. Clearly $\tilde{p}$ is W-invariant if p is. On the other hand if $q \in (SV_{\mathbb{C}}^*)^W$ we can write (in a unique way) $q = \tilde{q}_1 + i\tilde{q}_2$ with $q_i \in SV^*$. Now $w\tilde{q}_i = \tilde{p}_i$ for some $p_i \in SV^*$, so $q = wq = \tilde{p}_1 + i\tilde{p}_2$. Because of the uniqueness, we see that q_1 and q_2 must be W-invariant. Thus we deduced

$$(SV_{\mathbb{C}}^*)^W = \mathbb{C}[\tilde{F}_i, \dots, \tilde{F}_l] \tag{1.10}$$

Now we collect more information on the primitive degrees d_i. In the following we regard t both as a formal variable and as a complex number, whatever is more appropriate.

Proposition 1.21 *The Poincaré polynomials of* $(SV^*)^W$ *and* $(SV_{\mathbb{C}}^*)^W$ *are*

$$\prod_{i=1}^{l} (1 - t^{d_i})^{-1} = \frac{1}{|W|} \sum_{w \in W} \det(1 - tw)^{-1} \tag{1.11}$$

Proof. Taken from [4], proposition 9.3.1. Since $(SV^*)^W$ and $(SV_{\mathbb{C}}^*)^W$ are polynomial rings with independent generators of degrees $d_1, \dots, d_l$, a standard result says that their Poincaré polynomials are $P(t) = \prod_{i=1}^{l} (1 - t^{d_i})^{-1}$.
Because W is finite we can diagonalize $w \in \operatorname{End} V_{\mathbb{C}}$, with eigenvalues $\lambda_{w1}, \dots, \lambda_{wl}$. Let $w^{(n)}$ be the induced endomorphism of $S^n V_{\mathbb{C}}^*$, so that the eigenvalues of $w^{(n)}$ are $\{\lambda_{w1}^{n_1} \cdots \lambda_{wl}^{n_l} : n_1 + \dots + n_l = n\}$. A little thought suffices to see that the coefficient of t^n in $\sum_{w \in W} \det(1 - tw)^{-1}$ is the sum of these eigenvalues, i.e. the trace of $w^{(n)}$. So

$$\frac{1}{|W|} \sum_{w \in W} \det(1 - tw)^{-1} = \frac{1}{|W|} \sum_{w \in W} \sum_{n=0}^{\infty} \operatorname{tr} w^{(n)} t^n$$

Applying lemma 1.7 for every n shows that this equals
$\sum_{n=0}^{\infty} \dim (SV_{\mathbb{C}}^*)^W \, t^n = P(t)$. □

Corollary 1.22

$$d_1 d_2 \cdots d_l = |W|$$

Proof. Multiplying equation 1.11 with $(1 - t)^l$ gives

$$\prod_{i=1}^{l} (1 + t + \dots + t^{d_i - 1})^{-1} = \frac{1}{|W|} \sum_{w \in W} \prod_{i=1}^{l} \frac{1 - t}{1 - t\lambda_{wi}}$$

Now we take the limit $t \to 1$. All terms on the right hand side vanish, except those with $w = 1$, so we find $\prod_{i=1}^{l} d_i^{-1} = |W|^{-1}$. □

For the rest of this section we fix a basis Δ of R with corresponding positive system R^+. The length $l(w)$ of an element $w \in W$ is defined as the minimum number of terms in an expression $w = \sigma_{\alpha_1}\sigma_{\alpha_2}\cdots\sigma_{\alpha_n}$, with all $\alpha_i \in \Delta$. (By default, $l(1) = 0$). A standard result says that $l(w)$ is the number of $\alpha > 0$ such that $w\alpha < 0$. This obviously depends on the choice of a basis, but by theorem 1.19 the number of elements of a given length does not.

In the previous section we defined the height of a root $\beta = \sum_{\alpha\in\Delta} n_{\alpha\beta}\alpha$ as $h(\beta) = \sum_{\alpha\in\Delta} n_{\alpha\beta} \in \mathbb{Z}$. Just as above, this depends on the basis, but the number of roots of a given height does not.

Let $L(R)$ be the subgroup of V generated by R; it is a lattice of maximal rank. In order to write $L(R)$ as a multiplicative group, we let $e(L(R))$ be set of all $e(\alpha)$ with $\alpha \in L(R)$. The multiplication on $e(L(R))$ is $e(\alpha)e(\beta) = e(\alpha + \beta)$. It is clear that $e(L(R))$ is an abelian group, isomorphic to $L(R)$. Now we can construct the group algebra $\mathbb{Q}[R]$. It is the set of all finite sums $\sum_{i=1}^{n} c_i e(\alpha_i)$ with $c_i \in \mathbb{Q}$ and $\alpha_i \in R$, and the multiplication on $\mathbb{Q}[R]$ is the linear continuation of the multiplication on $e(L(R))$. By introducing a grading on R one shows that $\mathbb{Q}[R]$ has no zero divisors. Let $\mathbb{Q}(R)$ be its fraction field. The following identities in $\mathbb{Q}(R)(t)$ are due to Weyl, Solomon and Macdonald, while their proofs can all be found in [4].

Theorem 1.23

$$\prod_{i=1}^{l} \frac{t^{d_i} - 1}{t - 1} = \sum_{w\in W} t^{l(w)} = \sum_{w\in W} \prod_{\alpha>0} \frac{te(-w\alpha) - 1}{e(-w\alpha) - 1} = \prod_{\alpha>0} \frac{t^{h(\alpha)+1} - 1}{t^{h(\alpha)} - 1}$$

The last term of this equality is connected to the sequence $(r_n)_{n=1}^{\infty}$, where r_n is the number of roots of height n. Surprisingly, the equality enables us to show that this sequence is weakly decreasing.

Lemma 1.24 $l = r_1 \geq r_2 \geq r_3 \geq \ldots$

Proof. Since any basis of R has $l = \dim V$ elements, $r_1 = l$. Suppose that the sequence is not weakly decreasing. There are only finitely many roots, so we can find N such that $r_{N-1} < r_N$ but $r_{n-1} \geq r_n$ for all $n > N$. Let $P(t)$ be the polynomial of theorem 1.23. It equals

$$\prod_{h(\alpha)<N-1} \left(\frac{t^{h(\alpha)+1} - 1}{t^{h(\alpha)} - 1}\right) \frac{(t^{N+1} - 1)^{r_N}}{(t^{N-1} - 1)^{r_{N-1}}(t^N - 1)^{r_N - r_{N-1}}} \prod_{h(\alpha)>N} \left(\frac{t^{h(\alpha)+1} - 1}{t^{h(\alpha)} - 1}\right)$$

and $Q(t) = (t^{N+1} - 1)^{r_N} \prod_{h(\alpha)>N} \left(\frac{t^{h(\alpha)+1}-1}{t^{h(\alpha)}-1}\right)$ is a product of terms $t^n - 1$. Let ζ be a primitive N-th root of unity. Because $P(t)$ is a polynomial, ζ cannot be a pole of it. But ζ is only a zero of $t^n - 1$ if $n = kN$ for some $k \in \mathbb{N}$. So $Q(t)$ contains some terms (at least $r_N - r_{N-1}$) of this form. Let k_0 be the smallest k for which this happens. Then the primitive k_0N-th roots of unity have a higher multiplicity as zeros of $P(t)$ then ζ. But that is impossible, since $P(t) = \prod_{i=1}^{l} \frac{t^{d_i}-1}{t-1}$. $\square$

So the sequence $(r_n)_{n=1}^{\infty}$ is a partition of $|R^+| = |R|/2$. Let us form the dual partition. It consists of $l = r_1$ nonzero terms $m_i := \#\{n : r_n \geq i\}$. Notice that also $m_1 + \cdots + m_l = |R^+| = |R|/2$.

Definition 1.25 *Let r_n be the number of roots of height n in a root system R. The numbers $m_i = \#\{n : r_n \geq i\}$, $i = 1, \ldots, l$ are called the exponents of R.*

Lemma 1.26 *If $d_1 \geq d_2 \geq \ldots \geq d_l$, then $d_i = m_i + 1$.*

Proof. It is readily seen that

$$\prod_{\alpha>0} \frac{t^{h(\alpha)+1} - 1}{t^{h(\alpha)} - 1} = \prod_{i=1}^{l} \frac{t^{m_i+1} - 1}{t - 1}$$

But by theorem 1.23 this equals $\prod_{i=1}^{l} \frac{t^{d_i}-1}{t-1}$ and obviously $m_i \geq m_{i+1}$. $\square$

Now we reconsider the situation of the previous section. Once again G is a compact Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$ and $G_{\mathbb{C}}$ is a complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Furthermore $\mathfrak{t} = Z(\mathfrak{g}) \oplus \mathfrak{h}$ and $\mathfrak{t}_{\mathbb{C}} = Z(\mathfrak{g}_{\mathbb{C}}) \oplus \mathfrak{h}_{\mathbb{C}}$ are CSA's, T and $T_{\mathbb{C}}$ are the connected Lie subgroups they define, $R \subset i\mathfrak{h}^*$ is the corresponding root system, and finally W is the Weyl group of R. Now R is isomorphic to the root system $R' = \{iT_\alpha : \alpha \in R\}$ in $\mathfrak{h}$. Moreover

$$(iT_\alpha)^\perp = \{H \in \mathfrak{h} : \langle iT_\alpha, H\rangle = 0\} = \{H \in \mathfrak{h} : \alpha(H) = 0\} = \ker \alpha \cap \mathfrak{h}$$

so α and iT_α induce the same reflection of $\mathfrak{h}$. Thus we can identify W with a subgroup of End $\mathfrak{h}$ or of End $\mathfrak{h}_{\mathbb{C}}$. This is a good way of looking at the Weyl group, for now the previous theory applies to polynomials on $\mathfrak{h}$ and those are more natural than polynomials on $\mathfrak{h}^*$. Another advantage is that this makes it possible to realize the Weyl group in another way.

Theorem 1.27 *With the above notation, the adjoint actions of $N_G(T)$ and $N_{G_{\mathbb{C}}}(T_{\mathbb{C}})$ on $\mathfrak{h}$ and on $\mathfrak{h}_{\mathbb{C}}$ induce isomorphisms of $N_G(T)/T$ and $N_{G_{\mathbb{C}}}(T_{\mathbb{C}})/T_{\mathbb{C}}$ with W.*

Proof. Combine theorems 4.9.1 and 4.13.1 of [24] with section 1.3. $\square$

Let us determine the W-invariant polynomials on $\mathfrak{t}$. Put $l = \dim \mathfrak{t} = \text{rank } \mathfrak{g}$ and $r = \dim Z(\mathfrak{g})$, so that $l - r = \dim \mathfrak{h}_{\mathbb{R}} = \text{rank } R$. By theorem 1.20 the ring of W-invariant polynomials on $\mathfrak{h}$ is $\mathbb{R}[F_1, \ldots, F_{l-r}]$ for certain homogeneous polynomials F_i of degree d_i. The Weyl group acts as the identity on $Z(\mathfrak{g})$, so all polynomials on $Z(\mathfrak{g})$ are W-invariant. If $\{F_i : 1 + l - r \leq i \leq l\}$ is any basis of $Z(\mathfrak{g})^*$,

$$(S\mathfrak{t}^*)^W = \mathbb{R}[F_1, \ldots, F_l] \tag{1.12}$$

and similarly in the complex case. These F_i are called primitive invariant polynomials of $(\mathfrak{g}, \mathfrak{t})$. We know from theorem 1.26 that the numbers $d_i - 1$ ($i = 1, \ldots, l - r$) are the exponents of R, so it is natural to make the next definition.

Definition 1.28 *The exponents of* $\mathfrak{g}$ *and of* $\mathfrak{g}_{\mathbb{C}}$ *are the exponents of its root system, plus* $\dim Z(\mathfrak{g})$ *times 0.*

It is interesting to compare the W-invariant polynomials on $\mathfrak{t}$ with the G-invariant polynomials on $\mathfrak{g}$. By proposition 1.3 the latter form the ring $(S\mathfrak{g}^*)^G = (S\mathfrak{g}^*)^{\mathfrak{g}}$. The fact that this ring is defined without any reference to Lie groups, CSA's and Weyl groups makes the following even more remarkable. If $p \in (S\mathfrak{g}^*)^G$, $p\big|_{\mathfrak{t}}$ is invariant under $N_G(T)$, so by theorem 1.27 $p\big|_{\mathfrak{t}} \in (S\mathfrak{t}^*)^W$. Suppose that $p\big|_{\mathfrak{t}} = 0$. Since every element of $\mathfrak{g}$ is contained in a CSA, $p = 0$ on $\mathfrak{g}$. Thus we have an injective map $(S\mathfrak{g}^*)^G \to (S\mathfrak{t}^*)^W$. Chevalley's restriction theorem says that this map is also surjective. See [4], theorem 23.1 for a proof.

Theorem 1.29 *The restriction of polynomials on* $\mathfrak{g}$ *to* $\mathfrak{t}$ *gives an algebra isomorphism* $(S\mathfrak{g}^*)^G = (S\mathfrak{g}^*)^{\mathfrak{g}} \to (S\mathfrak{t}^*)^W$*, and the same in the complex case.*

The elements of $(S\mathfrak{g}^*)^G$ that correspond to the F_i will be called primitive invariant polynomials for $\mathfrak{g}$.

Example. We return to our standard example $\mathfrak{g} = \mathfrak{u}(n)$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$. A root α induces the same reflection of $\mathfrak{h}$ as T_α and in fact

$$\sigma_\alpha(H) = H - \frac{2\kappa(T_\alpha, H)}{\kappa(T_\alpha, T_\alpha)} T_\alpha = H - \alpha(H) H_\alpha$$

In particular if $\alpha = \lambda_i - \lambda_j$, σ_α interchanges $E_{i,i}$ and $E_{j,j}$, and fixes the $E_{k,k}$ with $i \neq k \neq j$. So the Weyl group of $\mathfrak{g}_{\mathbb{C}}$ can be identified with the symmetric group S_n on n symbols, acting in the natural way on the CSA $\mathfrak{t}_{\mathbb{C}} = \mathfrak{d}(n, \mathbb{C})$. Therefore the W-invariant polynomials on $\mathfrak{t} = i\mathfrak{d}(n, \mathbb{R})$ are the symmetric polynomials, and it is well known that these form a ring $\mathbb{R}[F_1, \ldots, F_n]$, where

$$F_m = i^m \sum_{1 \leq i_1 < \cdots < i_m \leq n} \lambda_1 \lambda_2 \cdots \lambda_m \tag{1.13}$$

is the m-th elementary symmetric polynomial. The F_m are also the coefficients of the characteristic polynomial on $\mathfrak{t}$:

$$\det(A + \mu I_n) = \mu^n + \sum_{i=1}^{n} \mu^{n-i} F_i(A) \text{ for } A \in \mathfrak{t}$$

The characteristic polynomial on $\mathfrak{t}_{\mathbb{C}}$ is invariant under conjugation by elements of $G_{\mathbb{C}} = GL(n, \mathbb{C})$, so the extensions of the F_m to $G_{\mathbb{C}}$-invariant polynomials on $\mathfrak{g}_{\mathbb{C}}$ are still the coefficients of this characteristic polynomial. This also holds for $\mathfrak{su}(n)$ and $\mathfrak{sl}(n, \mathbb{C})$, but then the trace function F_1 is identically 0.

With respect to the basis Δ of R, the height of $\lambda_i - \lambda_j$ is $i - j$. Hence, if we look at a matrix, we see that there are $n - k$ roots of height k, for $0 < k \leq n$. So by definition the exponents of $\mathfrak{su}(n)$ and $\mathfrak{sl}(n, \mathbb{C})$ are $m_i = n - i$ for $0 < i < n$. This in accordance with lemma 1.26, as $d_m = \deg F_m = m$ and $\{d_m : 1 < m \leq n\} = \{2, 3, \ldots, n\}$. Note also that the exponents of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ are $\{0, 1, \ldots, n-1\}$.

1.5 Harmonic polynomials

In this section we examine the natural companions to the invariant polynomials, the harmonic polynomials. See Helgason [13] for more background.

The inner product on V gives rise to an isomorphism $\phi : V \to V^*$ by $\phi v(v') = \langle v, v' \rangle$. This isomorphism intertwines the actions of the Weyl group:

$$(w \cdot \phi v)(v') = (\phi v)(w^{-1}v') = \left\langle v, w^{-1}v' \right\rangle = \langle wv, v' \rangle = (\phi(wv))(v')$$

Extend this to an algebra isomorphism $\phi : SV \to SV^*$ and put $D_p = \phi^{-1}p$.
We also have an inner product on SV:

$$\langle v_1 \cdots v_n, v'_1 \cdots v'_m \rangle = \begin{cases} \sum_{\tau \in S_n} \langle v_1, v'_{\tau 1} \rangle \cdots \langle v_n, v'_{\tau n} \rangle & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Now we define an inner product on SV^* and a nondegenerate paring between SV and SV^* by putting for $p, q \in SV^*$

$$\langle p, q \rangle = \langle D_p, q \rangle = \langle D_p, D_q \rangle$$

We can (and will) identify SV with algebra of all differential operators on V, with constant coefficients. This means that $v \in V$ corresponds to the derivation ∂_v of SV^* that extends the linear functional $\lambda \to \lambda(v)$ on V^*. (This can be seen as the partial derivative in the direction of v.) With straightforward calculations one can verify that for all $D \in SV, p \in SV^*, w \in W$:

$$(wD)(wp) = w(Dp) \tag{1.14}$$

$$\langle D, p \rangle = (Dp)(0) \tag{1.15}$$

Just as for W-invariant polynomials, the set of all W-invariant differential operators $(SV)^W$ is a graded subalgebra of SV. However we are not really interested in this subalgebra, as it is merely isomorphic to $(SV^*)^W$ by the isomorphism ϕ. Instead we define the harmonic polynomials, that will be of utmost importance in chapter 3.

Definition 1.30 *For any $\mathbb{Z}$-graded algebra $A = \bigoplus_{n \in \mathbb{Z}} A^n$ put $A^+ = \bigoplus_{n > 0} A^n$.
A polynomial $p \in SV^*$ is called harmonic if for all $D \in (S^+V)^W : \; Dp = 0$.
The set of all harmonic polynomials is denoted by $\mathcal{H}$.*

It is easy to see that $\mathcal{H}$ is a graded subspace of SV^*, and because of equation 1.14 it is also a W-submodule.

Theorem 1.31 *Use the above notation and let J be the ideal of SV^* generated by $(S^+V^*)^W$. Then $SV^* = J \oplus \mathcal{H}$ and the multiplication map $\mu : (SV^*)^W \otimes \mathcal{H} \to SV^*$ is an isomorphism.*

Proof. Suppose that $p \in J$ and $u \in \mathcal{H}$. By theorem 1.20 we can write $p = \sum_{i=1}^{l} p_i F_i$. Now

$$\langle p, u \rangle = \langle D_p, u \rangle = D_p u(0) = \sum_{i=1}^{l} D_{p_i} D_{F_i} u(0) = 0$$

since $D_{F_i} \in (S^+V)^W$; so $\mathcal{H} \perp J$. Contrarily if $u \perp J$ then for all $p \in (S^+V^*)^W$:

$$\langle D_p u, D_p u \rangle = D_{D_p u} D_p u(0) = D_p D_{D_p u} u(0) = \langle p D_p u, u \rangle = 0$$

since $pD_p u \in J$. So $\langle p, u \rangle = D_p u = 0$ and $u \in \mathcal{H}$. Thus we established $\mathcal{H} = J^\perp$ and $SV^* = J \oplus \mathcal{H}$.

Note that $1 \in (SV^*)^W$, so that $\mathcal{H} \subset \operatorname{im} \mu$. Now we prove by induction to n that $S^n V^* \subset \operatorname{im} \mu$. For $n = 0$ we have $\mathbb{R} \subset \mathcal{H} \subset \operatorname{im} \mu$. Let $n > 0$ and $p \in S^n V^*$. Because $SV^* = J \oplus \mathcal{H}$ and $\mathcal{H}$ and J are graded, we can find $u \in \mathcal{H}^n$ such that $p - u \in J^n$. Write $p - u = \sum_{i=1}^{l} p_i F_i$. We may assume that the p_i are homogeneous of degree lower than n, so by the induction hypothesis $p_i \in \operatorname{im} \mu$. Write $p_i = \sum_{j=1}^{n_i} a_{ij} b_{ij}$ with $a_{ij} \in (SV^*)^W$ and $b_{ij} \in \mathcal{H}$. Now

$$p = u + \sum_{i=1}^{l} p_i F_i = \mu \left(1 \otimes u + \sum_{i=1}^{l} \sum_{j=1}^{n_i} F_i a_{ij} \otimes b_{ij} \right)$$

Suppose that μ is not injective. Because J and $\mathcal{H}$ are graded, there exist homogeneous polynomials $a_i \in (SV^*)^W$ and $b_i \in \mathcal{H}$ such that $0 = \mu \left(\sum_i a_i \otimes b_i \right) = \sum_i a_i b_i$. We may even assume that the b_i are linearly independent and that b_1 does not belong to the ideal generated by the other b_i's. But this contradicts lemma 9.2.1 of [4]. $\square$

Remark. We followed the proof of theorem 4.15.28 in [24]. The theorem can be generalized substantially to infinite groups, see Helgason [13] or Kostant [17]. For example if G is a complex reductive Lie group with Lie algebra $\mathfrak{g}$ then it holds with $V = \mathfrak{g}$ and $W = G$.

A as consequence of theorem 1.31, $\mathcal{H}$ inherits an algebra structure from SV^*/J. Note though that this not the normal multiplication of polynomials, as $\mathcal{H}$ is not closed under this multiplication (this will become clear from the next corollary).

Corollary 1.32 *The Poincaré polynomial of* $\mathcal{H}$ *is*

$$P_{\mathcal{H}}(t) = \prod_{i=1}^{l} (1 + t + \cdots + t^{d_i - 1})$$

and $\dim \mathcal{H} = |W|$.

Proof. By theorem 1.31 and proposition 1.21

$$P_{\mathcal{H}}(t) = P_{SV^*}(t) / P_{(SV^*)^W}(t) = (1-t)^{-l} \prod_{i=1}^{l} (1 - t^{d_i}) = \prod_{i=1}^{l} (1 + t + \cdots + t^{d_i - 1})$$

Now

$$\dim \mathcal{H} = P_{\mathcal{H}}(1) = \prod_{i=1}^{l} d_i = |W|$$

by corollary 1.22. $\square$

In exercise 4.70.b of [24] Varadarajan suggested a way to determine the W-module structure of $\mathcal{H}$:

Proposition 1.33 *As a W-module, $\mathcal{H}$ is isomorphic to the regular representation $\mathbb{R}[W]$.*

Proof. Pick a regular element $v \in V$. Since W acts faithfully and transitively on the chambers, the orbit Wv consists of precisely $|W|$ elements. Let us denote by a subscript v the polynomials that vanish in v. Clearly $(SV^*)^W = (S^+V^*)_v^W \oplus \mathbb{R}$. Theorem 1.31 gives

$$SV^* \cong (SV^*)^W \otimes \mathcal{H} = \left((S^+V^*)_v^W \oplus \mathbb{R}\right) \otimes \mathcal{H} \cong \left((S^+V^*)_v^W \otimes \mathcal{H}\right) \oplus \mathcal{H}$$

Now $\mu\left((S^+V^*)_v^W \otimes \mathcal{H}\right) \subset (SV^*)_{Wv}$ and codim $(SV^*)_{Wv} = |Wv| = |W| = \dim \mathcal{H}$. Therefore $SV^* = (SV^*)_{Wv} \oplus \mathcal{H}$ and $\mu\left((S^+V^*)_v^W \otimes \mathcal{H}\right) = (SV^*)_{Wv}$. Consider the map $\alpha : SV^* \to \mathbb{R}[W] : f \to \sum_{w\in W} f(wv)w$. Obviously α is surjective and $\ker \alpha = (SV^*)_{Wv}$. Moreover for $w' \in W$:

$$\alpha(w'f) = \sum_{w\in W} (w'f)(wv)w = \sum_{w\in W} f(w'^{-1}wv)w = w'\left(\sum_{w\in W} f(wv)w\right) = w'(\alpha f)$$

We conclude that $\alpha : \mathcal{H} \to \mathbb{R}[W]$ is a W-module isomorphism. $\square$

Of course there are also complex harmonic polynomials on $V_{\mathbb{C}}$. They are by definition the elements of $SV_{\mathbb{C}}^*$ that are annihilated by $(S^+V_{\mathbb{C}}^*)^W$. With a deduction similar to that after theorem 1.20 one sees that this set is nothing else than the complexification $\mathcal{H}_{\mathbb{C}}$ of $\mathcal{H}$. Consequently everything we say about $\mathcal{H}$ will also be true in the complex case (with some obvious modifications).

Let us consider the polynomial $\pi = \phi\left(\prod_{\alpha>0} \alpha\right)$. Because the Weyl group maps positive systems to positive systems, for all $w \in W : w\pi = \pi$ or $w\pi = -\pi$. In fact $w\pi = \pi$ if $l(w) = \#\{\alpha > 0 : w\alpha < 0\}$ is even and $w\pi = -\pi$ if $l(w)$ is odd. Let $\epsilon(w)$ be the determinant of $w \in \operatorname{End} V$. From det $\sigma_\alpha = -1$ and the definition of the length of w it follows that $w\pi = \epsilon(w)\pi$. We say that π transforms by the sign character ϵ of W or that $\pi \in (SV^*)^\epsilon$.

Suppose that $p \in (SV^*)^\epsilon$. If $\alpha(v) = 0$,

$$p(v) = p(\sigma_\alpha^{-1}v) = (\sigma_\alpha p)(v) = \epsilon(\sigma_\alpha)p(v) = -p(v) \tag{1.16}$$

so $p(v) = 0$. Because p vanishes if $\phi(\alpha)$ does and $\phi(\alpha)$ is an irreducible element of SV^* and, it must divide p. But this is valid for all $\alpha \in R$ and the elements

$\phi(\alpha)$ for $\alpha > 0$ are coprime, so π divides p. Their quotient p/π is W-invariant, so $p \in \pi(SV^*)^W$. On the other hand every element of $\pi(SV^*)^W$ transforms by the sign character of W, so $(SV^*)^\epsilon = \pi(SV^*)^W$.
Now it easy to show that π is harmonic. For all $D \in (S^+V^*)^W$ equation 1.14 gives

$$w(D\pi) = (wD)(w\pi) = D(w\pi) = \epsilon(w)D\pi$$

so $D\pi \in (SV^*)^\epsilon$. But $\deg D\pi < \deg \pi$, so $D\pi = 0$ and $\pi \in \mathcal{H}$. Moreover it follows from corollary 1.32 that the maximal degree of an harmonic polynomial is

$$\sum_{i=1}^{l}(d_i - 1) = \sum_{i=1}^{l} m_i = |R^+| = \deg \pi$$

and that $\dim \mathcal{H}^{|R^+|} = 1$. We conclude that $H^{|R^+|} = \mathbb{R}\pi$.

Remark. It is known that the partial derivatives of π span $\mathcal{H}$. In practice this provides a reasonable way to determine $\mathcal{H}$.

Now we have enough tools to examine yet another kind the of Weyl group invariants. The algebra of differential forms on V with polynomial coefficients is $SV^* \otimes \bigwedge V^*$. In section 1.1 we described how the Weyl group acts on it. The W-invariants of this algebra were first described by Solomon. Our proof comes from Helgason [13], proposition 3.10.

Theorem 1.34 *Let $F_1, \dots F_l$ be primitive polynomial invariants in $(SV^*)^W$, and d the exterior differential on V. For any $I \subset \{1, \dots, l\}$ let dF_I be the wedge product of all dF_i with $i \in I$, in increasing order.*
Then $(SV^* \otimes \bigwedge V^*)^W$ is a free $(SV^*)^W$-module with basis $\{dF_I : I \subset \{1, \dots, l\}\}$.

Proof. The F_i are algebraically independent, so by lemma 3.7 of [13]
$dF_1 \wedge \dots \wedge dF_l \neq 0$. Let $x_1, \dots, x_l$ be a basis of V^*, and put $F = \det \left(\frac{\partial F_i}{\partial x_j}\right)_{i,j}$, so that

$$dF_1 \wedge \dots \wedge dF_l = F dx_1 \wedge \dots \wedge dx_l \tag{1.17}$$

The left hand side 1.17 is W-invariant and $dx_1 \wedge \dots \wedge dx_l$ transforms by the sign character of W, so $F \in (SV^*)^\epsilon$. However $\deg F = \sum_{i=1}^{l}(d_i - 1) = \deg \pi$, so $F = c\pi$ for some $c \in \mathbb{R}^\times$.

Now we want to show that the dF_I are linearly independent over $\mathbb{R}(x_1, \dots, x_l)$, since in that case they are also linearly independent over $\mathbb{R}[x_1, \dots, x_l] = SV^*$. On the contrary, suppose that we have a nonzero relation $\sum_I f_I dF_I = 0$, with $f_I \in \mathbb{R}(x_1, \dots, x_l)$. Take an I such that $F_I \neq 0$ and $\#I$ is minimal. Taking the wedge product with dF_{I^c} ($I^c = \{1, \dots, l\} \setminus I$) gives us

$$f_I dF_{\{1,\dots,l\}} = f_I c\pi dx_1 \wedge \dots \wedge dx_l = 0$$

This contradiction proves the linear independence.

The set $\{dF_I : I \subset \{1, \ldots, l\}\}$ has 2^l elements, and that is exactly the dimension of $\bigwedge V^*$, so it is a basis of $\mathbb{R}(x_1, \ldots, x_l) \otimes \bigwedge V^*$ over $\mathbb{R}(x_1, \ldots, x_l)$.
An arbitrary $\omega \in (SV^* \otimes \bigwedge V^*)^W$ can be written in a unique way as $\omega = \sum_I f_I dF_I$, with $f_I \in \mathbb{R}(x_1, \ldots, x_l)$. We show by induction to the number of nonzero f_I's that $\forall I : f_I \in (SV^*)^W$.

This is clear if $\omega = 0$.

Just as above, find an J such that $f_J \neq 0$ and $\#J$ is minimal. Up to sign, we get

$$\omega \wedge dF_{J^c} = f_J dF_{\{1,\ldots,l\}} = f_J c\pi dx_1 \wedge \cdots \wedge dx_l \in (SV^* \otimes \bigwedge V^*)^W .$$

So f_J is W-invariant and $f_J\pi \in SV^*$. Consequently $f_J\pi \in (SV^*)^\epsilon = \pi(SV^*)^W$ and $f_J \in (SV^*)^W$. Now also $\omega' = \omega - f_J dF_J \in (SV^* \otimes \bigwedge V^*)^W$. But $\omega' = \sum_{I \neq J} f_I dF_I$, so by the induction hypothesis all f_I are in $(SV^*)^W$. $\square$

Reeder [23] observed that this theorem enables us to compute the W-invariants in $\mathcal{H} \otimes \bigwedge V^*$:

Proposition 1.35 *$(SV^*/J \otimes \bigwedge V^*)^W$ is a free exterior algebra with l generators $dF_i \in \left((SV^*/J)^{m_i} \otimes \bigwedge^1 V^*\right)^W$. Likewise $(\mathcal{H} \otimes \bigwedge V^*)^W$ is a free exterior algebra with l generators in degrees $(m_i, 1)$.*

Proof. By theorem 1.31 $SV^*/J \cong \mathcal{H}$. Hence every element of $(SV^*/J \otimes \bigwedge V^*)^W$ has a (unique) representant in $(\mathcal{H} \otimes \bigwedge V^*)^W$ and is the projection of an element of $(SV^* \otimes \bigwedge V^*)^W$. By theorem 1.34, the last space is

$$\bigoplus_{I \subset \{1,\ldots,l\}} (SV^*)^W dF_I = \bigoplus_{I \subset \{1,\ldots,l\}} (S^+V^*)^W dF_I \oplus \mathbb{R} dF_I$$

Since $(S^+V^*)^W \subset J$, the set $\{dF_I : I \subset \{1, \ldots, l\}\}$ spans $(SV^*/J \otimes \bigwedge V^*)^W$ over $\mathbb{R}$. To prove that it is a basis it suffices to show that $\dim (\mathcal{H} \otimes \bigwedge V^*)^W = 2^l$. By proposition 1.33 $\mathcal{H} \otimes \bigwedge V^*$ is the W-module contragredient to

$$\mathbb{R}[W]^* \otimes \bigwedge V \cong \mathrm{Hom}(\mathbb{R}[W], \bigwedge V) := M$$

Fortunately $M^W = \mathrm{Hom}_W(\mathbb{R}[W], \bigwedge V)$ and it is easy to see that the map $M^W \to \bigwedge V : \psi \to \psi(1)$ is a linear bijection. Therefore

$$\dim (\mathcal{H} \otimes \bigwedge V^*)^W = \dim M^W = \dim \bigwedge V = 2^l$$

as we wanted. $\square$

We conclude this chapter by considering the harmonic polynomials on a CSA of a reductive Lie algebra. Let $\mathfrak{g}$ be a real Lie algebra of compact type, $\mathfrak{t} = Z(\mathfrak{g}) \oplus \mathfrak{h}$ a CSA, $R \subset i\mathfrak{h}^*$ the root system and W the Weyl group.

Since every differential operator on $Z(\mathfrak{g})$ is W-invariant, the only harmonic polynomials on $Z(\mathfrak{g})$ are the constants. So the harmonic polynomials on $\mathfrak{t}$ are just the harmonic polynomials on $\mathfrak{h}$, extended in the natural way to $\mathfrak{t}$. In particular this set of harmonic polynomials still affords the regular representation of W. Using equation 1.12 and theorem 1.31 we also still have

$$J \oplus \mathcal{H} = S\mathfrak{t}^* \cong (S\mathfrak{t}^*)^W \otimes \mathcal{H} \tag{1.18}$$

where J is the ideal of $S\mathfrak{t}^*$ generated by $(S^+\mathfrak{t}^*)^W$.

Because the roots have pure imaginary values on $\mathfrak{h}$, we must modify the polynomial π to $\prod_{\alpha>0} i\alpha$. If we do this and substitute V by $\mathfrak{t}$, the proofs of theorem 1.34 and proposition 1.35 go through in the same way. Consequently $(S\mathfrak{t}^*/J \otimes \bigwedge \mathfrak{t}^*)^W$ is a free exterior algebra with l generators $dF_i \in \left((S\mathfrak{t}^*/J)^{m_i} \otimes \bigwedge^1 \mathfrak{t}^*\right)^W$, and $(\mathcal{H} \otimes \bigwedge \mathfrak{t}^*)^W$ is a free exterior algebra with l generators in degrees $(m_i, 1)$. (But it is more difficult to give these generators explicitly.)

These things are also valid in the complex case. The corresponding results are obtained by adding subscripts $\mathbb{C}$ to all appropriate objects and doing everything over $\mathbb{C}$.

Chapter 2

Cohomology theory for Lie algebras

We use the De Rham cohomology of compact connected Lie groups to motivate the definitions of Lie algebra cohomology. Cohomology relative to a subalgebra and with coefficients in a module are also treated. These ideas are based on Chevalley and Eilenberg [6]. We close the chapter with a few isomorphism theorems on reductive (sub-)algebras.

I will use a star (*) for many different purposes, so let me state these explicitly to avoid confusion. For a sequence of abelian groups $(A^p)_p$, $A^* := \bigoplus_p A^p$. For a vector space V over a field $\mathbb{F}$, V^* is the dual space, consisting of all linear maps $V \to \mathbb{F}$. If $\phi : V \to W$ is a linear map between vector spaces, $\phi^* : W^* \to V^*$ is called the dual or transpose map and it is defined by $\phi^* f = f \circ \phi$ for $f \in W^*$. A variation on this is the case of a linear endomorphism ϕ of a vector space V with a nondegenerate symmetric bilinear form. Then the adjoint map ϕ^* is defined by $\langle \phi^* v, w \rangle = \langle v, \phi w \rangle$. And finally for a differentiable map $\psi : M \to N$ between manifolds, $\psi^* : \Omega(N) \to \Omega(M)$ denotes the retraction of differential forms.

A bit of terminology for (co-)homology. A differential complex (in the sense of homology) is a sequence $(C_p)_{p\in\mathbb{Z}}$ of abelian groups with homomorphisms $\partial_p : C_p \to C_{p-1}$ such that $\partial_{p-1} \circ \partial_p = 0$. We define $Z_p = \ker \partial_p$ and $B_p = \text{im } \partial_{p+1}$. Since $B_p \subset Z_p$ we can put $H_p = Z_p/B_p$ and call this the p-th homology group of the complex (C_*, ∂). The elements of C_p are called p-chains, ∂ is the boundary map or differential, Z_p consists of the p-cycles and B_p is group of p-boundaries. A differential complex (in the sense of cohomology) is a sequence $(C^p)_{p\in\mathbb{Z}}$ of abelian groups with homomorphisms $d^p : C^p \to C^{p+1}$ such that $d^p \circ d^{p-1} = 0$. We define $Z^p = \ker d^p$ and $B_p = \text{im } d^{p-1}$. Moreover $H^p := Z^p/B^p$ is the p-cohomology group of the complex (C^*, d). The elements of C^p are the p-cochains, d is called the coboundary map or the differential, Z_p consists of the p-cocycles and B_p is group of p-coboundaries. Usually C^p and C_p are vector spaces, in which case d and ∂ must be linear. Often these spaces are only specified for $p \geq 0$, and then we assume that they are 0 for $p < 0$. In this case $B^0 = 0$ and $H^0 = Z^0$.

2.1 De Rham cohomology and Lie group actions

For Lie groups the natural cohomology theory is that of De Rham, because it uses the manifold structure. As we will see, it also enables us to take advantage of the group structure.

Recall the basic notions of De Rham cohomology for a (C^∞-)manifold M. For $p \in \mathbb{N}$ let $\Omega^p(M)$ denote the real vector space of all smooth p-forms on M, and let $d^p : \Omega^p(M) \to \Omega^{p+1}(M)$ be the exterior differential. We call the elements of $Z^p_{DR}(M) = \ker d_p$ closed p-forms and the elements of $B^p_{DR}(M) = \text{im } d^{p-1}$ exact p-forms. Since $d^p \circ d^{p-1} = 0$ we have $B^p_{DR}(M) \subset Z^p_{DR}(M)$. The p-th De Rham cohomology group of M is defined as

$$H^p_{DR}(M) = Z^p_{DR}(M)/B^p_{DR}(M)$$

We summarize this by saying that $H^*_{DR}(M)$ is computed by the differential complex $(\Omega^*(M), d)$. For $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$ we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$$

which expresses the fact that d is an anti-derivation. This means that the wedge product of two closed forms is closed. Moreover if η is closed and $\omega = d\tau$ is exact

$$d(\tau \wedge \eta) = d\tau \wedge \eta - (-1)^p \tau \wedge d\eta = \omega \wedge \eta$$

so $\omega \wedge \eta$ is exact. Therefore $Z^*_{DR}(M)$ is a ring (even a real algebra) and $B^*(M)$ is an ideal in this ring. It follows that $H^*_{DR}(M)$ is a ring with the wedge product. A glance at the properties of this wedge product shows that $H^*_{DR}(M)$ is in fact an associative anti-commutative graded real algebra.

Assume now that a Lie group G acts smoothly on M. This means that we have a group homomorphism ϕ of G into the group of diffeomorphisms $M \to M$, such that the map $(g, m) \to \phi(g)m$ is C^∞. We have the induced linear maps $\phi(g)^* : \Omega^p(M) \to \Omega^p(M)$ and for every $g \in G$ and $\omega \in \Omega^*(M)$ we put $\omega^g := \phi(g)^*\omega$.

First of all we observe that any smooth path from e (the unit element of G) to g gives a smooth homotopy between id_M and $\phi(g)$. Consequently $\phi(g)^*$ induces the identity map on $H^*_{DR}(M)$. In particular for all g in the connected component of e, ω^g and ω are cohomologous (i.e. $\omega^g - \omega$ is exact).

It is easy to see that $(\omega^{g_1})^{g_2} = \omega^{(g_1 g_2)}$ (this is a right representation of G) and $(\omega \wedge \eta)^g = \omega^g \wedge \eta^g$. We call ω G-invariant if $\forall g \in G : \omega^g = \omega$. Furthermore we denote by $\Omega^p(M)^G$ the vector space of all G-invariant p-forms. Then $d(\Omega^p(M)^G) \subset \Omega^{p+1}(M)^G$ since the maps $\phi(g)^*$ commute with d. Thus $(\Omega^*(M)^G, d)$ is a differential complex, with cohomology $H^*_{DR}(M; G)$. Now $B^p_{DR}(M; G) = B^p(M)^G \subset B^p_{DR}(M)$ so the inclusion of $\Omega(M)^G$ in $\Omega(M)$ induces an algebra homomorphism $\phi_G : H^*_{DR}(M; G) \to H^*_{DR}(M)$.

From now on G will be compact, and dg is the Haar measure of G, such that the measure of G is 1. We can average any differential form $\omega \in \Omega(M)$ over G :

$$(I\omega)_m := \int_G (\omega^g)_m dg = \int_G \phi(g)^*(\omega_{\phi(g)m})dg \tag{2.1}$$

It is not trivial to verify that $I\omega$ is a *smooth* differential form. The crucial point is that taking a partial derivative (in a direction on M) of a function on $M \times G$ commutes with integrating it over G, because G is compact. This also gives

$$d(I\omega) = d\left(\int_G \phi(g)^*\omega dg\right) = \int_G d(\phi(g)^*\omega)dg = \int_G \phi(g)^*(d\omega)dg = I(d\omega) \tag{2.2}$$

The idea behind this averaging is of course that $I\omega$ is G-invariant :

$$(I\omega)^h = \phi(h)^*(I\omega) = \phi(h)^*\left(\int_G \phi(g)^*\omega dg\right) = \int_G \phi(h)^*\phi(g)^*\omega dg$$

Now $\phi(h)^*\phi(g)^* = (\phi(g)\phi(h))^* = \phi(gh)^*$ so the above equals

$$\int_G \phi(gh)^*\omega dg = \int_G \phi(g)^* dg = I\omega$$

Because the measure of G is 1, $I\omega = \omega$ if $\omega \in \Omega(M)^G$.

Lemma 2.1 *If G is a compact Lie group acting smoothly on a manifold M, then $\phi_G : H^*_{DR}(M;G) \to H^*_{DR}(M)$ is injective.*

Proof. Let $\omega \in \Omega(M)^G$ be such that $\omega = d\eta$ for some $\eta \in \Omega(G)$. Then

$$d(I\eta) = I(d\eta) = I\omega = \omega$$

so $\omega \in B^*_{DR}(M)^G$ and ϕ_G is injective. □

In general ϕ_G is not surjective and $H^*_{DR}(M;G)$ is not isomorphic to $H^*_{DR}(M)$, even if M is connected.

Example. Take $M = S^2$ and $G = \{1, -1\}$ with the obvious action ϕ. It is well known that $H^0_{DR}(S^2) = \mathbb{R}$, $H^1_{DR}(S^2) = 0$ and that

$$\omega \to \int_{S^2} \omega \ : \ H^2_{DR}(S^2) \to \mathbb{R}$$

is an isomorphism. Moreover $H^0_{DR}(S^2)$ consists of constant functions on S^2 and these are G-invariant. So also $H^0_{DR}(S^2;G) = Z^0_{DR}(S^2;G) = \mathbb{R}$. However, for a G-invariant 2-form ω :

$$\int_{S^2} \omega = \int_{S^2} \phi(-1)^*\omega = \int_{\phi(-1)_* S^2} \omega = -\int_{S^2} \omega$$

because multiplication by -1 reverses the orientation of S^2. Therefore $\int_{S^2} \omega = 0$ and $\omega \in B^2_{DR}(S^2)$. We conclude from lemma 2.1 that $H^2_{DR}(S^2; G) = 0$.

On the other hand, if G is connected then the following important theorem says that ϕ_G is an isomorphism.

Theorem 2.2 *Let G be a connected compact Lie group acting smoothly on a manifold M. Then the inclusion of $\Omega(M)^G$ in $\Omega(M)$ induces an isomorphism of the graded algebras $H^*_{DR}(M; G)$ and $H^*_{DR}(M)$. That is, $H^*_{DR}(M)$ can be computed using only G-invariant differential forms.*

Remark. We already saw that $\forall g \in G \quad \omega^g - \omega$ is exact. So we would like to pick α_g such that $d\alpha_g = \omega^g - \omega$, then define $\alpha = \int_G d\alpha_g dg$ and conclude that

$$d\alpha = \int_G d\alpha_g dg = \int_G (\omega^g - \omega) dg = I\omega - \omega$$

The problem is that $\int_G d\alpha_g dg$ is not defined, let alone smooth, unless the coordinates of $(g, m) \to (\alpha_g)_m$ are smooth functions on $G \times M$. While it is not too hard to find an α_g, this last condition poses difficult problems. For example in [24], ch. 2, exercise 29, Varadarajan constructed an $\eta_{\exp X}$ with $d\eta_{\exp X} = \omega^{\exp X} - \omega$. This is smooth in $X \in \mathfrak{g}$, but not in $g \in G$ because the exponential map is not invertible.

Proof. Following [6] we make a little trip to singular homology theory. De Rham's theorem (see [22]) states that $H^p_{DR}(M)$ is the dual space of the p-th singular homology group $H_p(M)$ of M. This implies that two closed p-forms $\omega, \eta \in Z^p_{DR}(M)$ are cohomologous if and only if $\int_\sigma \omega = \int_\sigma \eta$ for every smooth closed p-chain σ in M. Since ω^g and ω are cohomologous we have for such σ

$$\int_\sigma I\omega = \int_\sigma \left(\int_G \omega^g dg \right) = \int_G \left(\int_\sigma \omega^g \right) dg = \int_G \left(\int_\sigma \omega \right) dg = \int_\sigma \omega$$

Thus $I\omega$ an ω are cohomologous. Because $I\omega \in Z^p_{DR}(M; G)$ this means that ϕ_G is surjective. Combining this with lemma 2.1 proves that ϕ_G is an isomorphism. □

2.2 The definition of the cohomology of a Lie algebra

In this section we relate the De Rham cohomology of a compact Lie group to its Lie algebra. This leads us to the definition of the cohomology of a Lie algebra.

If G is a (compact) Lie group, G_e the connected component of the unit element and G_x another connected component, then G_e and G_x are diffeomorphic. In fact for any $x \in G_x$ the left multiplication map $l_x : G_e \to G_x$ is a diffeomorphism. Consequently the cohomology rings of G_e and G_x are isomorphic and $H^*_{DR}(G)$ is isomorphic to the direct sum of a finite number of copies of $H^*_{DR}(G_e)$.

Therefore in this section G will always be a connected compact Lie group, with Lie algebra $\mathfrak{g}$. Let us apply theorem 2.2 to the case $M = G$, acting on itself by left multiplication. The G-invariant forms are precisely the left invariant forms. But the map $\omega \to \omega_e$ is an algebra isomorphism between the space of left invariant forms and the exterior algebra $\bigwedge \mathfrak{g}^*$ on the dual space of the Lie algebra $\mathfrak{g}$, so the cohomology of G can be computed from a complex whose spaces are $\bigwedge^p \mathfrak{g}^*$. To determine the coboundary map d on this complex we return to general differential forms. It is known ([25], proposition 2.25) that for $\omega \in \Omega^p(G)$ and smooth vector fields $X_0, \ldots, X_p$ on G:

$$\begin{aligned} d\omega(X_0, \ldots, X_p) &= \sum_{i=0}^{p} (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_p)) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p) \end{aligned} \tag{2.3}$$

If we take ω and all X_i left invariant, then $\omega(X_0, \ldots, \hat{X}_i, \ldots, X_p)$ is a left invariant function, i.e. a constant. Hence $X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_p)) = 0$ and the first sum in 2.3 vanishes. Since $d\omega$ is also left invariant, it is completely determined by the remaining equation. We conclude that, under the identification of the left invariant p-forms with $\bigwedge^p \mathfrak{g}^*$, for $X_0, \ldots, X_p \in \mathfrak{g}$:

$$d\omega(X_0, \ldots, X_p) = \sum_{0 \le i < j \le p} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p) \tag{2.4}$$

By definition $d\big|_{\bigwedge^0 \mathfrak{g}^*} = 0$. This d is called the Koszul differential for $\mathfrak{g}$.

A similar formula was derived by Chevalley and Eilenberg in theorem 9.1 of [6]. They used another isomorphism between $(\bigwedge \mathfrak{g})^*$ and $\bigwedge(\mathfrak{g}^*)$ and the highly unusual definition $[X_i, X_j] := X_j X_i - X_i X_j$, which explains the difference.

We would like to use equation 2.4 as a definition on any Lie algebra $\mathfrak{g}$, but do not yet know whether d squares to zero. To prove this, we must first show that d is an anti-derivation of $\bigwedge \mathfrak{g}^*$.

Lemma 2.3 *Let $\mathfrak{g}$ be a Lie algebra and $d : \bigwedge^p \mathfrak{g}^* \to \bigwedge^{p+1} \mathfrak{g}^*$ the linear map defined by equation 2.4. Then for $\omega \in \bigwedge^p \mathfrak{g}^*, \eta \in \bigwedge^q \mathfrak{g}^*$:*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$$

Proof. Recall that the wedge product is defined by

$$(\omega \wedge \eta)(X_1, \ldots, X_{p+q}) = \sum_{\sigma \in S_{p+q}} \frac{\epsilon(\sigma)}{p!\, q!} \omega(X_{\sigma 1}, \ldots, X_{\sigma p}) \eta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})$$

We prove the lemma by induction to p.
For $p = 0$, ω is a scalar, $d\omega = 0$ and the lemma is obvious.
The difficult part is $p = 1$. Then $\omega \in \mathfrak{g}^*$ and the formula reads

$$d(\omega \wedge \eta) = d\omega \wedge \eta - \omega \wedge d\eta$$

With a little thought one sees that for any $\alpha \in \bigwedge^r \mathfrak{g}^*$

$$d\alpha(Y_0, \ldots, Y_r) = \sum_{\sigma \in S_{r+1}} \frac{\epsilon(\sigma)}{2(r-1)!} \alpha([Y_{\sigma 1}, Y_{\sigma 0}], Y_{\sigma 2}, \ldots, Y_{\sigma r})$$

We apply this to $\omega \wedge \eta$:

$$\begin{aligned} d(\omega \wedge \eta)(X_1, \ldots, X_{q+2}) &= \sum_{\sigma \in S_{q+2}} \frac{\epsilon(\sigma)}{2\, q!} (\omega \wedge \eta)([X_{\sigma 2}, X_{\sigma 1}], X_{\sigma 3}, \ldots, X_{\sigma(q+2)}) \\ &= \sum_{\sigma \in S_{q+2}} \frac{\epsilon(\sigma)}{2\, q!} \omega([X_{\sigma 2}, X_{\sigma 1}]) \eta(X_{\sigma 3}, \ldots, X_{\sigma(q+2)}) \quad + \end{aligned}$$
$$\sum_{\sigma \in S_{q+2}} \sum_{i=3}^{q+2} \frac{(-1)^i \epsilon(\sigma)}{2\, q!} \omega(X_{\sigma i}) \eta([X_{\sigma 2}, X_{\sigma 1}], X_{\sigma 3}, \ldots, \hat{X_{\sigma i}}, \ldots, X_{\sigma(q+2)})$$

Since $d\omega(Y_1, Y_2) = \omega([Y_2, Y_1])$ the second line of this equation is $d\omega \wedge \eta(X_1, \ldots, X_{q+2})$. So we must show that the double sum equals

$$\begin{aligned} -\omega \wedge d\eta(X_1, \ldots, X_{q+2}) &= \sum_{\sigma \in S_{q+2}} \frac{-\epsilon(\sigma)}{(q+1)!} \omega(X_{\sigma 1}) d\eta(X_{\sigma 2}, \ldots, X_{\sigma(q+2)}) \\ &= \sum_{i=1}^{q+2} (-1)^i \omega(X_i) d\eta(X_1, \ldots, \hat{X}_i, \ldots, X_{q+2}) \end{aligned}$$

Let S_q^i be the stabilizer of the element i in S_q. Then the last sum is

$$\sum_{i=1}^{q+2} (-1)^i \omega(X_i) \sum_{\sigma \in S_{q+2}^i} \frac{\epsilon(\sigma)}{2(q-1)!} \eta([X_{\sigma 2}, X_{\sigma 1}], X_{\sigma 3}, \ldots, \hat{X}_i, \ldots, X_{\sigma(q+2)})$$

An inspection of the earlier double sum shows that every term appears q times, and is equal to a term of the last double sum. The difference in the coefficients ($q!$ versus $(q-1)!$) makes that these double sums are equal. This proves the lemma for $p = 1$.

Assume now that formula the lemma holds for all $p < r$, that $\omega_1 \in \mathfrak{g}^*$ and that $\omega_2 \in \bigwedge^{r-1} \mathfrak{g}^*$.

$$\begin{aligned} d(\omega_1 \wedge \omega_2 \wedge \eta) &= d\omega_1 \wedge \omega_2 \wedge \eta - \omega_1 \wedge d(\omega_2 \wedge \eta) \\ &= d\omega_1 \wedge \omega_2 \wedge \eta - \omega_1 \wedge d\omega_2 \wedge \eta + (-1)^p \omega_1 \wedge \omega_2 \wedge d\eta = \\ &= d(\omega_1 \wedge \omega_2) \wedge \eta + (-1)^p \omega_1 \wedge \omega_2 \wedge d\eta \end{aligned}$$

Because $\bigwedge^r \mathfrak{g}^*$ is spanned by elements of the form $\omega_1 \wedge \omega_2$, the lemma is proved. □

Corollary 2.4 $d \circ d = 0$

Proof. We take $\omega \in \bigwedge^p \mathfrak{g}^*$. The corollary is obvious for $p = 0$ while for $p = 1$ and $X, Y, Z \in \mathfrak{g}$:

$$\begin{aligned} d(d\omega)(X,Y,Z) &= d\omega([X,Z],Y) - d\omega([X,Y],Z) - d\omega([Y,Z],X) \\ &= \omega([Y,[X,Z]]) - \omega([Z,[X,Y]]) - \omega([X,[Y,Z]]) \\ &= \omega([Y,[X,Z]] - [Z,[X,Y]] - [X,[Y,Z]]) \ = \ 0 \end{aligned}$$

by the Jacobi identity. Because d is an anti-derivation (lemma 2.3) it follows that $d(d\omega) = 0$ for all $\omega \in \bigwedge \mathfrak{g}^*$. □

Definition 2.5 *Let $\mathfrak{g}$ be any Lie algebra. The (cohomology) Koszul complex for $\mathfrak{g}$ has spaces $C^p(\mathfrak{g}) = \bigwedge^p \mathfrak{g}^*$ and (Koszul) differential d defined by equation 2.4. Its cohomology $H^*(\mathfrak{g})$ is called the Lie algebra cohomology of $\mathfrak{g}$.*

Corollary 2.4 makes this a good a definition. Moreover exactly the same argument as for De Rham cohomology shows that the wedge product is well defined on $H^*(\mathfrak{g})$. This makes $H^*(\mathfrak{g})$ into an associative graded anti-commutative algebra.

A direct consequence of this definition, theorem 2.2 and the work in this section is the following theorem.

Theorem 2.6 *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Then $H^*_{DR}(G)$ and $H^*(\mathfrak{g})$ are isomorphic as graded algebras.*

In view of De Rham's theorem it would be nice if we could define the homology of $\mathfrak{g}$ in such a way that its dual space would be $H^*(\mathfrak{g})$. This is achieved by taking the dual complex of $(C^*(\mathfrak{g}), d)$, i.e. taking the dual spaces of $C^*(\mathfrak{g})$ and the transpose maps.

Definition 2.7 *Let $\mathfrak{g}$ be Lie algebra. The (homology) Koszul complex for $\mathfrak{g}$ has spaces $C_p(\mathfrak{g}) = \bigwedge^p \mathfrak{g}$ and boundary map $\partial : C_p(\mathfrak{g}) \to C_{p-1}(\mathfrak{g})$ defined by*

$$\partial(X_1 \wedge \ldots \wedge X_p) = \sum_{1 \le i < j \le p} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_p \quad (2.5)$$

Its homology $H_*(\mathfrak{g})$ is called the homology of $\mathfrak{g}$.

By definition $\partial\big|_{C_0(\mathfrak{g})} = \partial\big|_{C_1(\mathfrak{g})} = 0$. It follows from $d \circ d = 0$ (or from a direct verification) that $\partial \circ \partial = 0$. Notice that

$$\begin{aligned} 2\partial(X_1 \wedge \ldots \wedge X_p) &= 2 \sum_{1 \le i < j \le p} (-1)^i X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge [X_i, X_j] \wedge \ldots \wedge X_p \\ &= \sum_{i \ne j} (-1)^i X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge [X_i, X_j] \wedge \ldots \wedge X_p \\ &= \sum_{i \ne j} (-1)^i \mathrm{ad}\, X_i (X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_p) \end{aligned} \quad (2.6)$$

However, in general ∂ is not an (anti-)derivation and $H_*(\mathfrak{g})$ has no multiplication. This makes cohomology more convenient than homology.

Example. Let $\mathfrak{g}$ be the two-dimensional Lie algebra (over the field $\mathbb{F}$) with basis $\{X, Y\}$ and commutator $[X, Y] = X$. Then $\partial(Y \wedge X) = X$ while obviously $\partial X = \partial Y = 0$. Hence ∂ is not an (anti-)derivation. Clearly $C_p(\mathfrak{g}) = C^p(\mathfrak{g}) = 0$ for $p > 2$. Now we can easily calculate the homology of $\mathfrak{g}$:

$$\begin{aligned}
H_0(\mathfrak{g}) = Z_0(\mathfrak{g})/B_0(\mathfrak{g}) &= C_0(\mathfrak{g})/0 = \mathbb{F} \\
H_1(\mathfrak{g}) = Z_1(\mathfrak{g})/B_1(\mathfrak{g}) &= C_1(\mathfrak{g})/\mathbb{F}X \cong \mathbb{F}Y \\
H_2(\mathfrak{g}) = Z_2(\mathfrak{g})/B_2(\mathfrak{g}) &= 0/0 = 0
\end{aligned}$$

Let $\{\omega, \eta\}$ be the dual basis of $\mathfrak{g}^*$. We have $d\big|_{\bigwedge^0 \mathfrak{g}^*} = d\big|_{\bigwedge^2 \mathfrak{g}^*} = 0$ and

$$\begin{aligned}
d\omega(Y \wedge X) &= \omega(X) = 1 = (\eta \wedge \omega)(Y \wedge X) \\
d\eta(Y \wedge X) &= \eta(X) = 0
\end{aligned}$$

The cohomology of $\mathfrak{g}$ is thus identified as

$$\begin{aligned}
H^0(\mathfrak{g}) = Z^0(\mathfrak{g})/B^0(\mathfrak{g}) &= C^0(\mathfrak{g})/0 = \mathbb{F} \\
H^1(\mathfrak{g}) = Z^1(\mathfrak{g})/B^1(\mathfrak{g}) &= \mathbb{F}\eta/0 = \mathbb{F}\eta \\
H^2(\mathfrak{g}) = Z^2(\mathfrak{g})/B^2(\mathfrak{g}) &= C^2(\mathfrak{g})/\mathbb{F}Y \wedge X = 0
\end{aligned}$$

So indeed $H_p(\mathfrak{g})$ and $H^p(\mathfrak{g})$ are dual in a natural way.

The (co-)homology of Lie algebras behaves well under direct sums.

Proposition 2.8 *Let $\mathfrak{g}$ and $\mathfrak{h}$ be finite-dimensional Lie algebras over a field $\mathbb{F}$. Then $H^*(\mathfrak{g} \oplus \mathfrak{h}) \cong H^*(\mathfrak{g}) \otimes_{\mathbb{F}} H^*(\mathfrak{h})$ and $H_*(\mathfrak{g} \oplus \mathfrak{h}) \cong H_*(\mathfrak{g}) \otimes_{\mathbb{F}} H_*(\mathfrak{h})$. The degree of $H^p(\mathfrak{g}) \otimes H^q(\mathfrak{h})$ is $p + q$ and similarly for the homology case. Moreover the product on $H^*(\mathfrak{g}) \otimes_{\mathbb{F}} H^*(\mathfrak{h})$ is*

$$(\omega_1 \otimes \eta_1) \wedge (\omega_2 \otimes \eta_2) = (-1)^{pq}(\omega_1 \wedge \omega_2) \otimes (\eta_1 \wedge \eta_2)$$

for $\eta_1 \in H^p(\mathfrak{g})$ and $\omega_2 \in H^q(\mathfrak{h})$.

Proof. Identify $\bigwedge(\mathfrak{g} \oplus \mathfrak{h})$ with $\bigwedge \mathfrak{g} \otimes \bigwedge \mathfrak{h}$ and the same for their dual spaces. Take $a \in C_p(\mathfrak{g})$ and $b \in C_q(\mathfrak{h})$. Because $[\mathfrak{g}, \mathfrak{h}] = 0$ it is easy to see that

$$\partial(a \wedge b) = \partial_g a \wedge b + (-1)^p a \wedge \partial_h b$$

and similarly for cohomology. Now the proposition is reduced to Künneth's formula, which is a standard result from homological algebra. $\square$

Now we return to the Lie group G to find an expression for the cohomology of a reductive Lie algebra that is often useful. Besides the action of left multiplication,

G also acts on itself by right multiplication. (To be precise, one takes the action $\phi(x) = r_{x^{-1}}$.) Then the G-invariant differential forms are the right invariant forms, and these also compute the De Rham cohomology of G. Following this track one can identify the right invariant forms with $\bigwedge \mathfrak{g}^*$ and find another complex on $\mathfrak{g}$ that computes the cohomology of G. There is nothing wrong with this approach, but it is less customary than the left invariant case and only leads to equivalent results.

It is much more interesting to combine these actions. We let the compact connected Lie group $G \times G$ act on G by $\phi(x, y) = l_x r_{y^{-1}}$. The $G \times G$-invariant forms are both left and right invariant, and they are called the invariant forms on G. For a left invariant form $\omega \in \bigwedge^p \mathfrak{g}^*$ the condition $r^*_{x^{-1}}\omega = \omega$ is equivalent to $c^*_x\omega = \omega$, where $c_x : G \to G : y \to xyx^{-1}$ is conjugation by x. Since $dc_x = \mathrm{Ad}(x) \in \mathrm{Aut}(\mathfrak{g})$ we have for $X_1, \dots, X_p \in \mathfrak{g}$:

$$c^*_x\omega(X_1, \dots, X_p) = \omega(\mathrm{Ad}(x)X_1, \dots, \mathrm{Ad}(x)X_p) = \mathrm{Ad}(x)^*\omega(X_1, \dots, X_p) \tag{2.7}$$

So with respect to the representation $x \to \mathrm{Ad}(x^{-1})^*$ the invariant forms can be identified with $(\bigwedge \mathfrak{g}^*)^G$. Because G is connected, the invariants of a G-representation are the invariants of the associated representation of $\mathfrak{g}$. Here the $\mathfrak{g}$-representation is $X \to -(\mathrm{ad}\, X)^*$ and

$$-(\mathrm{ad} X)^*\omega(X_1, \dots, X_p) = \sum_{i=1}^{p} \omega(X_1, \dots, [X_i, X], \dots, X_p) = \omega(-\mathrm{ad} X(X_1 \wedge \dots \wedge X_p))$$

Now we can state the result we are after.

Theorem 2.9 *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Then $H^*_{DR}(G)$ is isomorphic to the ring of invariant forms $(\bigwedge \mathfrak{g}^*)^G = (\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$.*

Proof. In view of theorem 2.2 and the above, $H^*_{DR}(G)$ can be computed from $((\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}, d)$. But for $\omega \in (\bigwedge^p \mathfrak{g}^*)^{\mathfrak{g}}$ and $X_0, \dots, X_p \in \mathfrak{g}$

$$\begin{aligned} 2d\omega(X_0 \wedge \dots \wedge X_p) &= \omega(2\partial(X_0 \wedge \dots \wedge X_p)) \\ &= \sum_{i=0}^{p} (-1)^{p+1}\omega(\mathrm{ad}\, X_i(X_0 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge X_p)) = 0 \end{aligned}$$

So all the coboundary maps are 0 and the complex equals its cohomology. $\square$

One can also prove in a purely algebraic way that for any reductive Lie algebra $\mathfrak{g}$ the space of invariant forms $(\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$ is isomorphic to $H^*(\mathfrak{g})$. For abelian Lie algebras this is clear and for semisimple Lie algebras the proof of theorem 19.1 in [6] goes through in characteristic unequal to zero, with some slight modifications. An application of proposition 2.8 would conclude such a proof.

2.3 Relative Lie algebra cohomology

Searching for a definition of the cohomology of a Lie algebra relative to a subalgebra, we turn our attention to homogeneous spaces. In this section G is a Lie group, H is a closed Lie subgroup, M the homogeneous space G/H and $\pi : G \to M$ the map $g \to gH$. Before we proceed, we must introduce some notations. If a is an element of a vector space V, $\epsilon(a) : \bigwedge V \to \bigwedge V$ will be the linear map $b \to a \wedge b$. The transpose map $\bigwedge V^* \to \bigwedge V^*$ is denoted by $i(a)$. For $f \in \bigwedge^p V^*$ and $b_1, \ldots, b_{p-1} \in V$ we have

$$i(a)f(b_1, \ldots, b_{p-1}) = f(a, b_1, \ldots, b_{p-1})$$

Using the definition of the wedge product, a straightforward calculation shows that $i(a)$ is an anti-derivation of $\bigwedge V^*$.

Lemma 2.10 *The retraction $\pi^* : \Omega(M) \to \Omega(G)$ is an isomorphism between $\Omega(M)$ and the subspace of $\Omega(G)$ consisting of all those ω for which*

$$i(X)\omega = 0 \quad \text{if } d\pi(X) = 0 \tag{2.8}$$

$$r_h^*\omega = \omega \quad \forall h \in H \tag{2.9}$$

Proof. Since π is surjective, π^* is injective. If $\eta \in \Omega(M)$, then $\pi^*\eta$ obviously satisfies condition 2.8. Moreover $\pi \circ r_h = \pi$ so

$$r_h^*\pi^*\eta = (\pi \circ r_h)^*\eta = \pi^*\eta$$

and condition 2.9 is also satisfied.

On the other hand, suppose that $\omega \in \Omega^p(G)$ satisfies conditions 2.8 and 2.9. Define $\eta \in \Omega(M)$ by

$$\eta(Y_1, \ldots, Y_p)(gH) = \omega(X_1, \ldots, X_p)(g) \text{ if } d\pi(X_i) = Y_i$$

Because of 2.8 this does not depend on the choice of the X_i. Using 2.9 and the fact $d\pi(X_i - dr_hX_i) = d\pi X_i - d\pi dr_hX_i = 0$ we obtain

$$\begin{aligned}
\omega(X_1, \ldots, X_p)(gh^{-1}) &= (r_{h^{-1}}^*\omega)(dr_hX_1, \ldots, dr_hX_p)(g) \\
&= \omega(dr_hX_1, \ldots, dr_hX_p)(g) \\
&= \omega(X_1, \ldots, X_p)(g)
\end{aligned}$$

So η is a well defined element of $\Omega(M)$ and $\pi^*\eta = \omega$. □

The next result will not be used, but it is interesting enough to state explicitly because it describes to cohomology of $\mathfrak{g}$ relative to the subgroup H of G.

Proposition 2.11 *Let G be a compact Lie group, H a closed Lie subgroup and $\mathfrak{g}$ and $\mathfrak{h}$ their respective Lie algebras. Then $H^*_{DR}(G/H)$ is isomorphic to the cohomology of the complex with spaces*

$$C^p(\mathfrak{g}, H) = \{\omega \in {\textstyle\bigwedge}^p(\mathfrak{g}/\mathfrak{h})^* \subset {\textstyle\bigwedge}^p\mathfrak{g}^* : \forall h \in H \ \mathrm{Ad}(h)\omega = \omega\}$$

and the Koszul differential (for $\mathfrak{g}$).

Proof. We let G act on M by $\bar{l}_x : gH \to xgH$. Since $\bar{l}_x\pi = \pi l_x$, π^* gives an isomorphism between the G-invariant forms on M and the left invariant on G that satisfy conditions 2.8 and 2.9. For left invariant forms and vector fields, $d\pi$ is just the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Then condition 2.8 means $i(X)\omega = 0$ if $X \in \mathfrak{h}$, i.e. $\omega \in \bigwedge(\mathfrak{g}/\mathfrak{h})^*$. From equation 2.7 we know that condition 2.9 is equivalent with $\mathrm{Ad}(h)^*\omega = \omega\ \forall h \in H$. Now theorem 2.2 completes the proof. $\square$

To express the complex $(C^*(\mathfrak{g}, H), d)$ entirely in terms of $\mathfrak{g}$ and $\mathfrak{h}$ we must assume that H is connected. Then it follows from the same argument as after equation 2.7 that $C^p(\mathfrak{g}, H) = (\bigwedge^p(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$ with respect to the representation $X \to -(\mathrm{ad}\, X)^*$. To generalize this to other Lie algebras we need the following

Lemma 2.12 *Let $\mathfrak{g}$ be any Lie algebra and d the Koszul differential. For all $\omega \in \bigwedge^p \mathfrak{g}^*$ and $Y \in \mathfrak{g}$:*

$$d(i(Y)\omega) + i(Y)d\omega = -(\mathrm{ad}\, Y)^*\omega$$

Proof. Take $X_1, \dots, X_p \in \mathfrak{g}$.

$$\begin{aligned} d(i(Y)\omega)(X_1, \dots, X_p) &= \sum_{1 \le i < j \le p} (-1)^{i+j}\omega(Y, [X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p) \\ d\omega(Y, X_1, \dots, X_p) &= \sum_{1 \le i < j \le p} (-1)^{i+j}\omega([X_i, X_j], Y, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p) \\ &+ \sum_{i=1}^{p} (-1)^i \omega([Y, X_i], X_1, \dots, \hat{X}_i, \dots, X_p) \end{aligned}$$

We see that the double sums over i and j cancel and what remains is

$$\begin{aligned} \left(d(i(Y)\omega) + i(Y)d\omega\right)(X_1, \dots, X_p) &= \sum_{i=1}^{p} (-1)^i \omega([Y, X_i], X_1, \dots, \hat{X}_i, \dots, X_p) \\ &= -\sum_{i=1}^{p} \omega(X_1, \dots, [Y, X_i], \dots, X_p) \\ &= -(\mathrm{ad}\, Y)^*\omega(X_1, \dots, X_p) \end{aligned}$$

This is valid for all $X_i \in \mathfrak{g}$, so we're done. $\square$

Now it is easy to see that

$$\left(\textstyle\bigwedge^p(\mathfrak{g}/\mathfrak{h})^*\right)^{\mathfrak{h}} = \bigcap_{Y \in \mathfrak{h}} \ker i(Y) \cap \ker(\mathrm{ad}\, Y)^*$$

is closed under d. For all $\omega \in \left(\bigwedge^p(\mathfrak{g}/\mathfrak{h})^*\right)^{\mathfrak{h}}$ and $Y \in \mathfrak{h}$:

$$\begin{aligned} i(Y)d\omega &= -d(i(Y)\omega) - (\mathrm{ad}\, Y)^*\omega = 0 && (2.10) \\ -(\mathrm{ad}\, Y)^*d\omega &= d(i(Y)d\omega) = d(-(\mathrm{ad}\, Y^*\omega) = 0 && (2.11) \end{aligned}$$

This justifies the next definition.

Definition 2.13 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subalgebra. The complex $(C^*(\mathfrak{g},\mathfrak{h}),d)$ has spaces $C^p(\mathfrak{g},\mathfrak{h}) = (\bigwedge^p(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \subset \bigwedge^p \mathfrak{g}^*$ and Koszul differential d. Its cohomology $H^*(\mathfrak{g},\mathfrak{h})$ is called the cohomology of $\mathfrak{g}$ relative to $\mathfrak{h}$.*

Since $(\operatorname{ad} Y)^*$ is a derivation and $i(Y)$ is an anti-derivation, $C^*(\mathfrak{g},\mathfrak{h})$ is closed under multiplication. Moreover d is still an anti-derivation, so the wedge product makes $H^*(\mathfrak{g},\mathfrak{h})$ into an anti-commutative graded algebra. Because this wedge product is associative, the algebra is associative. Combining this definition with proposition 2.11 yields

Theorem 2.14 *Let G be a compact connected Lie group, H a closed connected Lie subgroup and $\mathfrak{g}$ and $\mathfrak{h}$ their respective Lie algebras. Then $H^*_{DR}(G/H)$ and $H^*(\mathfrak{g},\mathfrak{h})$ are isomorphic as graded algebras.*

We also want to define relative Lie algebra homology. For this sake we dualize the complex $(C^*(\mathfrak{g},\mathfrak{h}),d)$. By lemma 1.4, $(\bigwedge^p(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$ is the dual space of $\operatorname{coinv}_{\mathfrak{h}} \bigwedge^p(\mathfrak{g}/\mathfrak{h})$, with respect to the representation that extends the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$. The dual map ∂ of d is still given by the equations 2.5 and 2.6. Because d maps $C^p(\mathfrak{g},\mathfrak{h})$ to itself, ∂ is well defined.

Definition 2.15 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subalgebra. Put $C_p(\mathfrak{g},\mathfrak{h}) = \operatorname{coinv}_{\mathfrak{h}} \bigwedge^p(\mathfrak{g}/\mathfrak{h})$ and let $\partial : C_p(\mathfrak{g},\mathfrak{h}) \to C_{p-1}(\mathfrak{g},\mathfrak{h})$ be the linear map defined by equation 2.5. The homology of the complex $(C_*(\mathfrak{g},\mathfrak{h}),\partial)$ is called the homology of $\mathfrak{g}$ relative to $\mathfrak{h}$.*

A few basic properties of relative Lie algebra (co-)homology are collected in the next proposition. The proofs are deferred to the more general proposition 2.24.

Proposition 2.16 *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$, $\mathfrak{h}$ a subalgebra and $\mathbb{K}$ an extension field of $\mathbb{F}$. Then 1 - 5 and the homology analogues of 4 and 5 hold, and the isomorphisms 3 and 4 are natural.*

1. $H^0(\mathfrak{g},\mathfrak{h}) = H_0(\mathfrak{g},\mathfrak{h}) = \mathbb{F}$
2. $H^p(\mathfrak{g},\mathfrak{h}) = H_p(\mathfrak{g},\mathfrak{h}) = 0$ *for* $p > \dim \mathfrak{g}/\mathfrak{h}$
3. $H^p(\mathfrak{g},\mathfrak{h}) \cong H_p(\mathfrak{g},\mathfrak{h})^*$
4. $H^p(\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{K}, \mathfrak{h} \otimes_{\mathbb{F}} \mathbb{K}) \cong H^p(\mathfrak{g},\mathfrak{h}) \otimes_{\mathbb{F}} \mathbb{K}$
5. *If $\mathfrak{h}$ is an ideal, $H^*(\mathfrak{g},\mathfrak{h}) = H^*(\mathfrak{g}/\mathfrak{h})$ and the action of $\mathfrak{g}$ on this space is trivial*

For an infinite-dimensional Lie algebra $\mathfrak{g}$ it is not necessarily true that $H_p(\mathfrak{g},\mathfrak{h}) \cong H^p(\mathfrak{g},\mathfrak{h})^*$ because $(\mathfrak{g}^*)^*$ might not be isomorphic to $\mathfrak{g}$.

We consider the simplest possible example. Take

$$G = SU(2) = \left\{A \in GL(2,\mathbb{C}) : \det A = 1, A^* = A^{-1}\right\}$$

This group acts on $\mathbb{P}^1(\mathbb{C})$ and $H = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}$ is the stabilizer of $(1:0) \in \mathbb{P}^1(\mathbb{C})$. Hence G/H is diffeomorphic with $\mathbb{P}^1(\mathbb{C})$. The Lie algebra of G is

$$\mathfrak{g} = \mathfrak{su}(2) = \{A \in \mathfrak{gl}(2,\mathbb{C}) : \text{tr } A = 0, A^* = -A\}$$

and H corresponds to the subalgebra $\mathfrak{h} = \left\{ \begin{pmatrix} \lambda i & 0 \\ 0 & -\lambda i \end{pmatrix} : \lambda \in \mathbb{R} \right\}$. Notice that $\mathfrak{g}$ is definitely not a complex Lie algebra because $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \notin \mathfrak{g}$. In fact $\mathfrak{g}$ has a real basis consisting of

$$X_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

One computes that

$$[X_0, X_1] = -2X_2 \ , \ [X_0, X_2] = 2X_1 \ , \ [X_1, X_2] = -2X_0$$

Now we identify $\mathfrak{g}/\mathfrak{h}$ with the span of X_1 and X_2. Clearly $\mathfrak{h} \cdot (\mathfrak{g}/\mathfrak{h})$ contains X_1 and X_2, so it equals $\mathfrak{g}/\mathfrak{h}$ and $C_1(\mathfrak{g},\mathfrak{h}) = 0$. Furthermore

$$\text{ad } X_0(X_1 \wedge X_2) = [X_0, X_1] \wedge X_2 + X_1 \wedge [X_0, X_2] = -2X_2 \wedge X_2 + X_1 \wedge 2X_1 = 0$$

Hence $\mathfrak{h} \cdot \bigwedge^2(\mathfrak{g}/\mathfrak{h}) = 0$ and $C_2(\mathfrak{g},\mathfrak{h}) = \bigwedge^2(\mathfrak{g}/\mathfrak{h})$. From this we see that

$$H_0(\mathfrak{g},\mathfrak{h}) = \mathbb{R} \ , \ H_1(\mathfrak{g},\mathfrak{h}) = 0 \ , \ H_2(\mathfrak{g},\mathfrak{h}) = \bigwedge\nolimits^2(\mathfrak{g}/\mathfrak{h}) \cong \mathbb{R}X_1 \wedge X_2$$

Let $\{f_0, f_1, f_2\}$ be the dual basis $\mathfrak{g}^*$, so that $(\mathfrak{g}/\mathfrak{h})^*$ is the span of f_1 and f_2.

$$\begin{array}{rcl} C^1(\mathfrak{g},\mathfrak{h}) & = & ((\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} = \{f \in (\mathfrak{g}/\mathfrak{h})^* : f([X_0, X]) = 0 \ \forall \ X \in \mathfrak{g}/\mathfrak{h}\} \\ & \subset & \{f \in (\mathfrak{g}/\mathfrak{h})^* : f(X_1) = f(X_2) = 0\} \quad = \quad 0 \\ C^2(\mathfrak{g},\mathfrak{h}) & = & \left(\bigwedge^2(\mathfrak{g}/\mathfrak{h})^*\right)^{\mathfrak{h}} = \{\omega \in \bigwedge^2(\mathfrak{g}/\mathfrak{h})^* : (\text{ad } X_0)^*\omega = 0\} = \bigwedge^2(\mathfrak{g}/\mathfrak{h})^* \end{array}$$

since $\omega(\text{ad } X_0(X_1 \wedge X_2)) = \omega(0) = 0$. It follows directly that

$$H^0(\mathfrak{g},\mathfrak{h}) = \mathbb{R} \ , \ H^1(\mathfrak{g},\mathfrak{h}) = 0 \ , \ H^2(\mathfrak{g},\mathfrak{h}) = \bigwedge\nolimits^2(\mathfrak{g}/\mathfrak{h})^* \cong \mathbb{R}f_1 \wedge f_2$$

So $H_p(\mathfrak{g},\mathfrak{h})$ and $H^p(\mathfrak{g},\mathfrak{h})$ are naturally dual. Moreover we computed that the cohomology of $\mathbb{P}^1(\mathbb{C})$ has $\dim H^p_{DR}(\mathbb{P}^1(\mathbb{C})) = 1$ for $p = 0$ or $p = 2$ and 0 otherwise. This is in accordance with well known results from algebraic topology.

2.4 Lie algebra cohomology with coefficients

For a paracompact Hausdorff space M and a principle ideal domain K, there exists a notion of the sheaf cohomology $H^*(M;S)$, for a sheaf of K-modules S over M. It is known ([25]) that if M a differentiable manifold, $K=\mathbb{R}$ and $M\times\mathbb{R}$ is the constant sheaf, then $H^*_{DR}(M)\cong H^*(M;M\times\mathbb{R})$. With this isomorphism one can more or less generalize De Rham cohomology to a cohomology theory with coefficients in real vector spaces.

Bearing this in mind it is natural to look for a sensible definition of Lie algebra (co-)homology with coefficients in a vector space over the right field. Unfortunately sheaf cohomology is not given by such nice formulas as De Rham cohomology, so we cannot use the same procedure as in the last two sections. However if we look more carefully, we can still find a clue.

Once again let G be a Lie group with Lie algebra $\mathfrak{g}$, and identify the tangent space $T_x(G)$ to G at x with $\mathfrak{g}$ by means of the isomorphism dl_x. For $\eta\in\bigwedge^p\mathfrak{g}^*$ and $f\in C^\infty(G)$ we can regard $\eta\otimes f$ as an element of $\Omega^p(G)$ by $(\eta\otimes f)_x=f(x)(dl_x^{-1})^*\eta$. This gives rise to an isomorphism

$$\mathrm{Hom}_{\mathbb{R}}\big(\textstyle\bigwedge^p\mathfrak{g},C^\infty(G)\big)\cong\bigwedge\mathfrak{g}^*\otimes C^\infty(G)\to\Omega^p(G)$$

Now for left invariant vector fields $X_0,\dots,X_p\in\mathfrak{g}$ formula 2.3 becomes

$$\begin{aligned} d(\eta\otimes f) &= \sum_{i=0}^{p}(-1)^i\eta(X_0,\dots,\hat{X}_i,\dots,X_p)X_i\cdot f \\ &+ \sum_{0\le i<j\le p}(-1)^{i+j}\eta([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p)f \end{aligned}$$

This uses not only that $C^\infty(G)$ is a vector space but also that it is a $\mathfrak{g}$-module. Therefore we will only have coefficients in Lie algebra modules.

Let H be a closed connected Lie subgroup of G with Lie algebra $\mathfrak{h}\subset\mathfrak{g}$. What is the image of $\pi^*:\Omega(G/H)\to\bigwedge^p\mathfrak{g}^*\otimes_{\mathbb{R}}C^\infty(G)$? Because of condition 2.8, $\mathrm{im}\,\pi^*\subset\bigwedge^p(\mathfrak{g}/\mathfrak{h})^*\otimes_{\mathbb{R}}C^\infty(G)$. In view of equation 2.7 condition 2.9 becomes

$$\eta\otimes f=r_h^*(\eta\otimes f)=\mathrm{Ad}(h^{-1})^*\eta\otimes f\circ r_h$$

The H-invariants of this last representation are the invariants of the associated representation of $\mathfrak{h}$, which for $Y\in\mathfrak{h}$ is $\eta\otimes f\to\eta\otimes Yf-(\mathrm{ad}\,Y)^*\eta\otimes f$. So the decomposable $\mathfrak{h}$-invariants satisfy $\eta\otimes Yf=(\mathrm{ad}\,Y)^*\eta\otimes f$. We conclude that $\mathrm{im}\,\pi^*=\mathrm{Hom}_{\mathfrak{h}}(\bigwedge^p\mathfrak{g},C^\infty(G))$. This is enough to make a final definition of Lie algebra cohomology, which generalizes the earlier two.

Definition 2.17 *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra and $\rho:\mathfrak{g}\to\mathrm{End}\,V$ a representation of $\mathfrak{g}$. Define $C^p(\mathfrak{g},\mathfrak{h};V)=\mathrm{Hom}_{\mathfrak{h}}(\bigwedge^p\mathfrak{g}/\mathfrak{h},V)$ and for $X_0,\dots,X_p\in$*

$\mathfrak{g}$, $\omega \in C^p(\mathfrak{g}, \mathfrak{h}; V)$

$$\begin{aligned} d\omega(X_0, \ldots, X_p) &= \sum_{i=0}^{p} (-1)^i \rho X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_p)) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p) \end{aligned}$$

Then $(C^*(\mathfrak{g}, \mathfrak{h}; V), d)$ is called the Koszul complex for $\mathfrak{g}, \mathfrak{h}$ and V. Its cohomology $H^*(\mathfrak{g}, \mathfrak{h}; V)$ is the cohomology of $\mathfrak{g}$ relative to $\mathfrak{h}$ with coefficients in V. For $\mathfrak{h} = 0$ we get the complex $(C^*(\mathfrak{g}; V), d)$ and this computes the cohomology $H^*(\mathfrak{g}; V)$ of $\mathfrak{g}$ with coefficients in V.

In fact we only defined d as a linear map $C^p(\mathfrak{g}; V) \to C^{p+1}(\mathfrak{g}; V)$. To justify the definition we must check that d maps $C^p(\mathfrak{g}, \mathfrak{h}; V)$ into $C^{p+1}(\mathfrak{g}, \mathfrak{h}; V)$ and that $d \circ d = 0$.

Lemma 2.18 *Denote by L_Y the representation $1 \otimes \rho Y - (\mathrm{ad}\, Y)^* \otimes 1$ of $\mathfrak{g}$ on $C^p(\mathfrak{g}; V)$. Then $\forall \omega \in C^p(\mathfrak{g}; V)$*

$$\begin{aligned} d(d\omega) &= 0 && (2.12) \\ L_Y \omega &= d(i(Y)\omega) + i(Y) d\omega && (2.13) \\ d(L_Y \omega) &= L_Y(d\omega) \;=\; d(i(Y) d\omega) && (2.14) \end{aligned}$$

Proof. 2.12 is proved by a somewhat tiresome direct calculation. Write $d = d_V + d_g$, where d_V is the first sum in definition 2.17 and d_g is the Koszul differential for $\mathfrak{g}$ (tensored with id_V). We know that $d_g \circ d_g = 0$. Take $X_1, \ldots, X_{p+2} \in \mathfrak{g}$, $\eta \in \bigwedge^p \mathfrak{g}^*$ and $v \in V$. For notational simplicity let X^i denote $X_1, \ldots, \hat{X}_i, \ldots, X_{p+2}$ and similarly for a multi-index I and X^I.

$$\begin{aligned} d_V d_g(\eta \otimes v)(X) &= \sum_{k=1}^{p+2} (-1)^{k+1} X_k \cdot \big(d_g(\eta \otimes v)(X^k)\big) \\ &= \sum_{k=1}^{p+2} \left(\sum_{i<j<k} + \sum_{k<i<j} \right) (-1)^{i+j+k+1} \eta([X_i, X_j], X^{ijk}) X_k \cdot v \\ &+ \sum_{i<k<j} (-1)^{i+j+k} \eta([X_i, X_j], X^{ijk}) X_k \cdot v \end{aligned}$$

$$
\begin{aligned}
d_g d_V(\eta\otimes v)(X) &= \sum_{i<j}(-1)^{i+j} d_V(\eta\otimes v)([X_i,X_j],X^{ij}) \\
&= \sum_{i<j}(-1)^{i+j}\eta(X^{ij})[X_i,X_j]\cdot v \\
&+ \sum_{i<j}\left(\sum_{k<i}+\sum_{k>j}\right)(-1)^{i+j+k}\eta([X_i,X_j],X^{ijk})X_k\cdot v \\
&+ \sum_{i<k<j}(-1)^{i+j+k+1}\eta([X_i,X_j],X^{ijk})X_k\cdot v
\end{aligned}
$$

$$
\begin{aligned}
d_V d_V(\eta\otimes v)(X) &= \sum_{i=1}^{p+2}(-1)^i X_i\cdot d_V(\eta\otimes v)(X^i) \\
&= \sum_{i=1}^{p+2}(-1)^i X_i\cdot\left(\left(\sum_{j<i}-\sum_{i<j}\right)(-1)^j\eta(X^{ij})X_j\cdot v\right) \\
&= \left(\sum_{j<i}-\sum_{i<j}\right)(-1)^{i+j}\eta(X^{ij})X_i\cdot(X_j\cdot v) \\
&= \sum_{j<i}(-1)^{i+j}\eta(X^{ij})(X_iX_j-X_jX_i)\cdot v \\
&= \sum_{j<i}(-1)^{i+j}\eta(X^{ij})[X_i,X_j]\cdot v
\end{aligned}
$$

Now we simply add these three terms and everything cancels. Since elements of the form $\eta\otimes v$ span $C^p(\mathfrak{g};V)$ this proves that $d\circ d=0$.
2.13 is less elaborate. Using the same notation and lemma 2.12 we obtain

$$
\begin{aligned}
d(i(Y)\omega)+i(Y)d\omega &= d_g(i(Y)\omega)+i(Y)d_g\omega+d_V(i(Y)\omega)+i(Y)d_V\omega \\
&= -(\operatorname{ad}Y)^*\omega+d_V(i(Y)\omega)+i(Y)d_V\omega
\end{aligned}
$$

So it suffices to show that $d_V(i(Y)\omega)+i(Y)d_V\omega=\rho Y\omega$, but this is a direct consequence of the definitions.
2.14 follows from 2.12 and 2.13 $\square$

Because $C^p(\mathfrak{g},\mathfrak{h};V)=\bigcap_{Y\in\mathfrak{h}}\ker i(Y)\cap\ker L_Y$, formulas 2.13 and 2.14 directly imply that $dC^p(\mathfrak{g},\mathfrak{h};V)\subset C^{p+1}(\mathfrak{g},\mathfrak{h};V)$.

Notice that if $V=C^\infty(G)$, L_Y is the Lie derivative with respect to Y and formulas 2.13 and 2.14 are known for arbitrary manifolds, differential forms and vector fields, e.g. [25], proposition 2.25.

If $V=\mathbb{F}$ is the trivial module the complex $(C^*(\mathfrak{g},\mathfrak{h};V),d)$ reduces to $(C^*(\mathfrak{g},\mathfrak{h}),d)$, so $H^*(\mathfrak{g},\mathfrak{h};\mathbb{F})=H^*(\mathfrak{g},\mathfrak{h})$. In this situation we have the wedge product on $C^*(\mathfrak{g},\mathfrak{h};V)$

and on $H^*(\mathfrak{g}, \mathfrak{h}; V)$, and d is an anti-derivation. In general there is no such multiplication since we cannot multiply elements of V. To compensate for this we use paired modules, a notion that was introduced in [14].

Definition 2.19 *Two $\mathfrak{g}$-modules V_1 and V_2 are paired to a third module V_3 if there is a bilinear map $\lambda : V_1 \times V_2 \to V_3$ such that*

$$\forall X \in \mathfrak{g}, v_1 \in V_1, v_2 \in V_2 : X \cdot \lambda(v_1, v_2) = \lambda(X \cdot v_1, v_2) + \lambda(v_1, X \cdot v_2)$$

Equivalently we can require that $\lambda : V_1 \otimes V_2 \to V_3$ is an homomorphism of $\mathfrak{g}$-modules.

For example the trivial module $\mathbb{F}$ and an arbitrary module V are paired to V with $\lambda(c, v) = cv$.

For paired modules we can combine λ and the wedge product on $\bigwedge \mathfrak{g}^*$ to a map $\mu : C^*(\mathfrak{g}; V_1) \otimes C^*(\mathfrak{g}; V_2) \to C^*(\mathfrak{g}; V_3)$ by defining

$$\mu(\omega_1 \otimes v_1, \omega_2 \otimes v_2) = \omega_1 \wedge \omega_2 \otimes \lambda(v_1, v_2)$$

and extending linearly. This μ will the role of multiplication, and we will refer to it as the wedge product for paired modules. It is more or less an anti-derivation:

Lemma 2.20 *Let the $\mathfrak{g}$-modules V_1 and V_2 be paired to V_3 and denote by d_i the Koszul differential on $C^*(\mathfrak{g}; V_i)$. Then for all $\omega_1 \in C^p(\mathfrak{g}; V_1), \omega_2 \in C^q(\mathfrak{g}; V_2)$*

$$d_3\mu(\omega_1, \omega_2) = \mu(d_1\omega_1, \omega_2) + (-1)^p\mu(\omega_1, d_2\omega_2)$$

Proof. It is enough to proof the lemma for $\omega_1 = \omega \otimes v$ and $\omega_2 = \eta \otimes w$ with $\omega \in \bigwedge^p \mathfrak{g}^*, \eta \in \bigwedge^q \mathfrak{g}^*$ and $v, w \in V$. Let us write simply vw for $\lambda(v, w)$. Using the notation from lemma 2.18, $d_i = d_{V_i} + d_g$ and we know from lemma 2.3 that d_g is an anti-derivation. So we only have to show that

$$d_{V_3}(\omega \wedge \eta \otimes vw) = \mu\left(d_{V_1}(\omega \otimes v), \eta \otimes w\right) + (-1)^p\mu\left(\omega \otimes v, d_{V_2}(\eta \otimes w)\right) \tag{2.15}$$

Well, for $X_0, \ldots, X_{p+q} \in \mathfrak{g}$ we have

$$\begin{aligned}
&\mu\left(d_{V_1}(\omega \otimes v), \eta \otimes w\right)(X_0, \ldots, X_{p+q}) = \\
&\sum_{\sigma \in S_{p+q+1}} \frac{\epsilon(\sigma)}{(p+1)!q!} d_{V_1}(\omega \otimes v)(X_{\sigma 0}, \ldots, X_{\sigma p})\,(\eta \otimes w)(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}) = \\
&\sum_{\sigma \in S_{p+q+1}} \frac{\epsilon(\sigma)}{p!q!} \omega(X_{\sigma 1}, \ldots, X_{\sigma p}) X_{\sigma 0} \cdot v\ \eta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}) w
\end{aligned}$$

$$\begin{aligned}
&(-1)^p\mu\left(\omega \otimes v, d_{V_2}(\eta \otimes w)\right)(X_0, \ldots, X_{p+q}) = \\
&\sum_{\sigma \in S_{p+q+1}} \frac{(-1)^p\epsilon(\sigma)}{p!(q+1)!} (\omega \otimes v)(X_{\sigma 0}, \ldots, X_{\sigma(p-1)})\, d_{V_2}(\eta \otimes w)(X_{\sigma p}, \ldots, X_{\sigma(p+q)}) = \\
&\sum_{\sigma \in S_{p+q+1}} \frac{\epsilon(\sigma)}{p!(q+1)!} (\omega \otimes v)(X_{\sigma 1}, \ldots, X_{\sigma p})\, d_{V_2}(\eta \otimes w)(X_{\sigma 0}, X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}) = \\
&\sum_{\sigma \in S_{p+q+1}} \frac{\epsilon(\sigma)}{p!q!} \omega(X_{\sigma 1}, \ldots, X_{\sigma p}) v\ \eta(X_{\sigma p}, \ldots, X_{\sigma(p+q)}) X_{\sigma 0} \cdot w
\end{aligned}$$

$$
\begin{aligned}
&d_V(\omega \wedge \eta \otimes vw)(X_0, \ldots, X_{p+q}) = \\
&\sum_{i=0}^{p+q} (-1)^i (\omega \wedge \eta)(X^i) X_i \cdot (vw) = \\
&\sum_{i=0}^{p+q} (-1)^i \sum_{\sigma \in S^i_{p+q+1}} \frac{\epsilon(\sigma)}{p!q!} (\omega \otimes \eta)(X_{\sigma 0}, \ldots, \hat{X}_i, \ldots X_{\sigma(p+q)}) \left((X_i \cdot v)w + v(X_i \cdot w) \right) = \\
&\sum_{\sigma \in S_{p+q+1}} \frac{\epsilon(\sigma)}{p!q!} (\omega \otimes \eta)(X_{\sigma 1}, \ldots X_{\sigma(p+q)}) \left((X_{\sigma 0} \cdot v)w + v(X_{\sigma 0} \cdot w) \right)
\end{aligned}
$$

Comparing the final three expressions establishes equation 2.15. $\square$

This product is well-defined in relative cohomology:

Lemma 2.21 *Let the $\mathfrak{g}$-modules V_1 and V_2 be paired to V_3 and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$.*

$$\mu\big(C^p(\mathfrak{g}, \mathfrak{h}; V_1), C^q(\mathfrak{g}, \mathfrak{h}; V_2)\big) \subset C^{p+q}(\mathfrak{g}, \mathfrak{h}; V_3)$$

Proof. Take arbitrary $\omega_1 \in C^p(\mathfrak{g}, \mathfrak{h}; V_1)$ and $\omega_2 \in C^q(\mathfrak{g}, \mathfrak{h}; V_2)$. Clearly $\mu(\omega_1, \omega_2) \in \mathrm{Hom}(\bigwedge^{p+q} \mathfrak{g}/\mathfrak{h}, V_3)$. Using equation 2.13 and lemma 2.20 we have for all $Y \in \mathfrak{h}$:

$$
\begin{aligned}
L_Y \mu(\omega_1, \omega_2) &= d_3 i(Y) \mu(\omega_1, \omega_2) + i(Y) d_3 \mu(\omega_1, \omega_2) \\
&= i(Y) \mu(d_1 \omega_1, \omega_2) + (-1)^p i(Y) \mu(\omega_1, d_2 \omega_2) = 0
\end{aligned}
$$

since $\omega_i, d_i \omega_i \in C^*(\mathfrak{g}, \mathfrak{h}; V_i)$. We conclude that $\mu(\omega_1, \omega_2) \in \mathrm{Hom}_{\mathfrak{h}}(\bigwedge^{p+q} \mathfrak{g}/\mathfrak{h}, V_3)$. $\square$

Let us consider a typical case of the above situation. If A is an algebra and $\rho : \mathfrak{g} \to \mathrm{End}\, A$ is a representation such that $\rho(X)$ is a derivation for all $X \in g$, then A and A are paired to A by the multiplication of A. Now μ is really a multiplication on $C^*(\mathfrak{g}; A)$ and lemma 2.20 shows that d is an anti-derivation. Consequently the product of a cocycle and a coboundary is a coboundary, and $H^*(\mathfrak{g}; A)$ inherits an algebra structure from $C^*(\mathfrak{g}; A)$.

Notice that if we want these multiplications to be associative, it is necessary and sufficient that A is associative, for the wedge product (on $\bigwedge \mathfrak{g}^*$) is always associative. In this case $C^*(\mathfrak{g}; A)$ and $H^*(\mathfrak{g}; A)$ are graded rings. If on the top of this A is commutative, then μ is anti-commutative. So now $C^*(\mathfrak{g}; A)$ and $H^*(\mathfrak{g}; A)$ are associative anti-commutative graded algebras. We summarize this situation in the next definition.

Definition 2.22 *A $\mathfrak{g}$-module V with representation ρ is called multiplicative (for $\mathfrak{g}$) if it is an associative commutative algebra and $\rho(X)$ is a derivation for all $X \in \mathfrak{g}$.*

Of course there exists also something as Lie algebra homology with coefficients in a module. We deduce it (for the third and last time) by taking duals. If $\mathfrak{g}$ and V happen to be of finite dimension, the space $C^p(\mathfrak{g}, \mathfrak{h}; V^*) = (\bigwedge^p (\mathfrak{g}/\mathfrak{h})^* \otimes V^*)^{\mathfrak{h}}$ is naturally dual to the $\mathfrak{h}$-coinvariants of $\bigwedge^p(\mathfrak{g}/\mathfrak{h}) \otimes V$. The dual map of d is

$$\begin{aligned}\partial(X_1 \wedge \ldots \wedge X_p \otimes v) = &\textstyle\sum_{i=1}^p (-1)^i X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_p \otimes \rho X_i\, v \\ &+ \textstyle\sum_{1\le i<j\le p} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_p\end{aligned} \tag{2.16}$$

This ∂ is well defined and squares to zero precisely because it is the transpose of d.

Definition 2.23 *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra and $\rho : \mathfrak{g} \to \operatorname{End} V$ a representation. The complex $(C_*(\mathfrak{g}, \mathfrak{h}; V), \partial)$ has spaces $C_p(\mathfrak{g}, \mathfrak{h}; V) = \operatorname{coinv}_{\mathfrak{h}}\big(\bigwedge^p(\mathfrak{g}/\mathfrak{h}) \otimes V\big)$ and the linear map $\partial : C_p(\mathfrak{g}, \mathfrak{h}; V) \to C_{p-1}(\mathfrak{g}, \mathfrak{h}; V)$ is defined by equation 2.16. The homology $H_*(\mathfrak{g}, \mathfrak{h}; V)$ of this complex is called the homology of $\mathfrak{g}$ relative to $\mathfrak{h}$ with coefficients in V.*

Observe that if $V = \mathbb{F}$ is the trivial module, all these notions reduce to definition 2.13 of the Lie algebra homology of $\mathfrak{g}$ relative to $\mathfrak{h}$ (without coefficients).

Now we state and prove the generalization of proposition 2.16 to (co-)homology with coefficients:

Proposition 2.24 *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$, $\mathfrak{h}$ a subalgebra, V and V_i $\mathfrak{g}$-modules and $\mathbb{K}$ an extension field of $\mathbb{F}$. Then 1 - 6 and the homology analogues of 4 - 6 hold, and the isomorphisms 3,4 and 6 are natural.*

1. *$H^0(\mathfrak{g}, \mathfrak{h}; V) = V^{\mathfrak{g}}$ and $H_0(\mathfrak{g}, \mathfrak{h}; V) = V/\mathfrak{g} \cdot V$*
2. *$H^p(\mathfrak{g}, \mathfrak{h}; V) = H_p(\mathfrak{g}, \mathfrak{h}; V) = 0$ for $p > \dim \mathfrak{g}/\mathfrak{h}$*
3. *If V or $\mathfrak{g}/\mathfrak{h}$ is finite-dimensional, $H^p(\mathfrak{g}, \mathfrak{h}; V^*) \cong H_p(\mathfrak{g}, \mathfrak{h}; V)^*$*
4. *$H^p(\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{K}, \mathfrak{h} \otimes_{\mathbb{F}} \mathbb{K}; V \otimes_{\mathbb{F}} \mathbb{K}) \cong H^p(\mathfrak{g}, \mathfrak{h}; V) \otimes_{\mathbb{F}} \mathbb{K}$*
5. *If $\mathfrak{h}$ is an ideal and $V = V^{\mathfrak{h}}$, $H^*(\mathfrak{g}, \mathfrak{h}; V) = H^*(\mathfrak{g}/\mathfrak{h}; V)$ and the action of $\mathfrak{g}$ on this space is trivial*
6. *$H^p(\mathfrak{g}, \mathfrak{h}; \bigoplus_i V_i) \cong \bigoplus_i H^p(\mathfrak{g}, \mathfrak{h}; V_i)$*

Proof.

1. For $v \in V = C^0(\mathfrak{g}, \mathfrak{h}; V)$ and $X \in \mathfrak{g}$ we have $dv(X) = X \cdot v$, so $Z^0(\mathfrak{g}, \mathfrak{h}; V) = V^{\mathfrak{g}}$. Also $X \otimes v \in C_1(\mathfrak{g}, \mathfrak{h}; V)$, $\partial(X \otimes v) = X \cdot v$ and such elements span $C_1(\mathfrak{g}, \mathfrak{h}; V)$, so $B_0(\mathfrak{g}, \mathfrak{h}; V) = \mathfrak{g} \cdot V$.
2. For $p > \dim \mathfrak{g}/\mathfrak{h}$, $\bigwedge^p(\mathfrak{g}/\mathfrak{h})$ and $\bigwedge^p(\mathfrak{g}/\mathfrak{h})^*$ are 0, so certainly $H_p(\mathfrak{g}, \mathfrak{h}; V)$ and $H_p(\mathfrak{g}, \mathfrak{h}; V)$ are 0.
3. Under this assumptions, $\bigwedge^p(\mathfrak{g}/\mathfrak{h})^* \otimes V^*$ is naturally isomorphic to the dual space of $\bigwedge^p(\mathfrak{g}/\mathfrak{h}) \otimes V$. So by lemma 1.4 $C^p(\mathfrak{g}, \mathfrak{h}; V^*) = \big(\bigwedge^p(\mathfrak{g}/\mathfrak{h})^* \otimes V^*\big)^{\mathfrak{h}}$ is naturally isomorphic to the dual space of $C_p(\mathfrak{g}, \mathfrak{h}; V) = \operatorname{coinv}_{\mathfrak{h}}\big(\bigwedge^p(\mathfrak{g}/\mathfrak{h}) \otimes V\big)$. Because d is the transpose map of ∂, the statement reduces to a simple property of transpose maps.

4. Obvious from the linearity of d and ∂.

5. Since $\mathfrak{h}$ acts trivially on $\mathfrak{g}/\mathfrak{h}$, the complexes for $\mathfrak{g}, \mathfrak{h}$ and V reduce to the complexes for $\mathfrak{g}/\mathfrak{h}$ and V, so the first statement holds in both the homology and the cohomology case. If $X \in \mathfrak{g}$ and $\omega \in Z^p(\mathfrak{g}, \mathfrak{h}; V)$, $L_X\omega = d(i(X)\omega)$ by formula 2.13. However

$$\forall\, Y \in \mathfrak{h} : L_Y i(X)\omega = \rho(Y)(i(X)\omega) - (\operatorname{ad} Y)^*(i(X)\omega) = 0$$

because $\rho : \mathfrak{g} \to \operatorname{End} V$ is 0 on $\mathfrak{h}$ and because $\mathfrak{h}$ is an ideal. So $i(X)\omega \in C^{p-1}(\mathfrak{g}, \mathfrak{h}; V)$ and $L_X\omega \in B^p(\mathfrak{g}, \mathfrak{h}; V)$. Therefore $\mathfrak{g}$ acts trivially on $H^*(\mathfrak{g}, \mathfrak{h}; V)$, and by duality it also acts trivially on $H_*(\mathfrak{g}, \mathfrak{h}; V)$.

6. For all p there is a natural decomposition $C^p(\mathfrak{g}, \mathfrak{h}; \bigoplus_i V_i) \cong \bigoplus_i C^p(\mathfrak{g}, \mathfrak{h}; V_i)$ and d preserves this decomposition since $\mathfrak{g} \cdot V_i \subset V_i$. The homology case is proved in the same way. □

If V is a $\mathfrak{g}$-module, we can make $\mathfrak{g} \oplus V$ into a Lie algebra with commutator

$$[(X, v), (Y, w)] = ([X, Y], X \cdot w - Y \cdot v)$$

We call this Lie algebra the semidirect product of $\mathfrak{g}$ and V and denote it by $\mathfrak{g} \ltimes V$.

Proposition 2.25 *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra and V a $\mathfrak{g}$-module. There are natural isomorphisms*

$$\begin{aligned} H_n(\mathfrak{g} \ltimes V, \mathfrak{h}) &\cong \bigoplus_{p+q=n} H_p(\mathfrak{g}, \mathfrak{h}; \textstyle\bigwedge^q V) \\ H^n(\mathfrak{g} \ltimes V, \mathfrak{h}) &\cong \bigoplus_{p+q=n} H^p(\mathfrak{g}, \mathfrak{h}; \textstyle\bigwedge^q V^*) \end{aligned}$$

Proof. For the complex that computes $H_*(\mathfrak{g} \ltimes V, \mathfrak{h})$ we have natural isomorphisms

$$\begin{aligned} C_n(\mathfrak{g} \ltimes V, \mathfrak{h}) &= \operatorname{coinv}_{\mathfrak{h}} \textstyle\bigwedge^n (\mathfrak{g} \ltimes V/\mathfrak{h}) \cong \operatorname{coinv}_{\mathfrak{h}} \left(\displaystyle\bigoplus_{p+q=n} \textstyle\bigwedge^p (\mathfrak{g}/\mathfrak{h}) \otimes \bigwedge^q V \right) \\ &\cong \bigoplus_{p+q=n} \operatorname{coinv}_{\mathfrak{h}} \big(\textstyle\bigwedge^p (\mathfrak{g}/\mathfrak{h}) \otimes \bigwedge^q V \big) = \displaystyle\bigoplus_{p+q=n} C_p(\mathfrak{g}, \mathfrak{h}; \textstyle\bigwedge^q V) \end{aligned}$$

Using the notation from the proof of lemma 2.18, for decomposable $X \in \bigwedge^p \mathfrak{g}$ and $v \in \bigwedge^q V$:

$$\begin{aligned} \partial(X \wedge v) &= \sum_{1 \le i < j \le p} (-1)^{i+j} [X_i, X_j] \wedge X^{ij} \wedge v + \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{p+i+j} X_i \cdot v_j \wedge X^i \wedge v^j \\ &= \partial_{\mathfrak{g}}(X) \wedge v + \sum_{i=1}^{p} (-1)^i X^i \wedge X_i \cdot v \;=\; \partial_{\mathfrak{g}}(X) \wedge v + \partial_{\bigwedge^q V}(X \wedge v) \end{aligned}$$

This proves the statement about $H_n(\mathfrak{g} \ltimes V, \mathfrak{h})$. As for the cohomology case,

$$\begin{aligned} C^n(\mathfrak{g} \ltimes V, \mathfrak{h}) &= \left(\bigwedge^n (\mathfrak{g} \ltimes V/\mathfrak{h})^*\right)^{\mathfrak{h}} = \left(\bigwedge^n \left((\mathfrak{g}/\mathfrak{h})^* \oplus V^*\right)\right)^{\mathfrak{h}} \\ &\cong \bigoplus_{p+q=n} \left(\bigwedge^p (\mathfrak{g}/\mathfrak{h})^* \otimes \bigwedge^q V^*\right)^{\mathfrak{h}} = \bigoplus_{p+q=n} C^p(\mathfrak{g}, \mathfrak{h}; \bigwedge^q V^*) \end{aligned}$$

and a computation similar to that above shows that

$$\forall\, \omega \in \bigwedge^p (\mathfrak{g}/\mathfrak{h})^* \otimes \bigwedge^q V^* : \ d_{\mathfrak{g} \ltimes V} \omega = d_{\mathfrak{g}, \bigwedge^q V^*} \omega$$

Thus we get the desired expression for $H^n(\mathfrak{g} \ltimes V, \mathfrak{h})$. □

2.5 Isomorphism theorems

In this section we will use the Hochschild-Serre spectral sequence to derive various isomorphism theorems in Lie algebra cohomology. Most of these theorems stem from the original article by Hochschild and Serre [14], which in turn generalized the results of Koszul [18] to cohomology with coefficients. Throughout $\mathfrak{g}$ is a Lie algebra over a field $\mathbb{F}$ of characteristic 0, $\mathfrak{h}$ is a subalgebra and V is a $\mathfrak{g}$-module.

First we introduce the Hochschild-Serre spectral sequence for $\mathfrak{g}, \mathfrak{h}$ and V. Define

$$F^p C^r(\mathfrak{g}; V) = \{\omega \in C^r(\mathfrak{g}; V) : \forall X_i \in \mathfrak{h}\ i(X_1) \cdots i(X_{r+1-p})\omega = 0\}$$

Clearly $F^p C^*(\mathfrak{g}; V)$ is a graded subspace of $C^*(\mathfrak{g}; V)$ and

$$C^r(\mathfrak{g}; V) = F^0 C^r(\mathfrak{g}; V) \supset F^1 C^r(\mathfrak{g}; V) \supset \cdots \supset F^r C^r(\mathfrak{g}; V) \supset F^{r+1} C^r(\mathfrak{g}; V) = 0$$

It follows directly from the definition of d that $d\left(F^p C^r(\mathfrak{g}; V)\right) \subset F^p C^{r+1}(\mathfrak{g}; V)$ so this is a filtration on $C^*(\mathfrak{g}; V)$. By a general construction (see [2], section 14) there exists a spectral sequence $(E_s^{**}, d_s)_{s=0}^{\infty}$ that converges to $H^*(\mathfrak{g}; V)$.

Let us clarify these abstract statements. We define

$$E_0^{p,q} = F^p C^{p+q}(\mathfrak{g}; V) / F^{p+1} C^{p+q}(\mathfrak{g}; V)$$

and $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ the map induced by the Koszul differential d, for $\mathfrak{g}$ and V. Obviously $d_0 \circ d_0 = 0$, so (E_0^{**}, d_0) is a differential complex with a double grading. We take E_1^{**} to be the cohomology of (E_0^{**}, d_0); it still has a double grading. Now d induces a map $d_1 : E_1^{p,q} \to E_1^{p+1,q}$. We continue in this way and obtain a whole sequence of double graded differential complexes (E_s^{**}, d_s), where $d_s : E_s^{p,q} \to E_s^{p+s,q+1-s}$ is still induced by the Koszul differential. It is not unusual that there exists a $N \in \mathbb{N}$ such that for all $s \geq N$ the maps d_s are 0 and hence all the spaces E_s^{**} with $s \geq N$ coincide. In that case we say that the spectral sequence degenerates at E_N^{**} and call the limit term E_∞^{**}. The filtration of $C^*(\mathfrak{g}; V)$ gives a filtration on $H^*(\mathfrak{g}; V)$ by taking $F^p H^r(\mathfrak{g}; V)$ the subspace of all cohomology classes that contain a cocycle in $F^pC^r(\mathfrak{g};V)$. Now we can construct the double graded vector space associated with this filtration :

$$GF^*(H^*(\mathfrak{g};V)) = \bigoplus_{p,q} F^pH^{p+q}(\mathfrak{g};V)/F^{p+1}H^{p+q}(\mathfrak{g};V)$$

The big idea behind spectral sequences is the following. The convergence means that $E_\infty^{**} \cong GF^*(H^*(\mathfrak{g},\mathfrak{h};V))$. We calculate the first few terms of the spectral sequence (for example E_0 , E_1 and E_2) and derive properties of E_∞ from this. Because E_∞ is closely related to $H^*(\mathfrak{g};V)$, many of these properties will carry over to $H^*(\mathfrak{g};V)$.

We also remark that if V is multiplicative for $\mathfrak{g}$, all these things are multiplicative. For then $F^pC^*(\mathfrak{g};V)$ is an ideal in $C^*(\mathfrak{g};V)$, all d_s are anti-derivations and the wedge product makes each term E_s^{**} into an associative, anti-commutative algebra.

Hochschild and Serre computed the first terms of their spectral sequence in sections 2 and 3 of [14]:

Lemma 2.26 *The first terms of the Hochschild-Serre spectral sequence for $\mathfrak{g}$, $\mathfrak{h}$ and V are*

$$\begin{aligned} E_0^{p,q} &\cong C^q\left(\mathfrak{h};\mathrm{Hom}_{\mathbb{F}}\big(\textstyle\bigwedge^p(\mathfrak{g}/\mathfrak{h}),V\big)\right) \\ E_1^{p,q} &\cong H^q\left(\mathfrak{h};\mathrm{Hom}_{\mathbb{F}}\big(\textstyle\bigwedge^p(\mathfrak{g}/\mathfrak{h}),V\big)\right) \\ E_2^{p,0} &\cong H^p(\mathfrak{g},\mathfrak{h};V) \end{aligned}$$

The isomorphisms are obtained by the restriction of linear functions on $\bigwedge^{p+q}\mathfrak{g}$ to $\bigwedge^q\mathfrak{h}\otimes\bigwedge^p\mathfrak{c}$ where $\mathfrak{c}$ is a vector space complement to $\mathfrak{h}$ in $\mathfrak{g}$. Moreover if $\mathfrak{h}$ is an ideal then

$$\begin{aligned} E_1^{p,q} &\cong C^p\big(\mathfrak{g}/\mathfrak{h};H^q(\mathfrak{h};V)\big) \\ E_2^{p,q} &\cong H^p\big(\mathfrak{g}/\mathfrak{h};H^q(\mathfrak{h};V)\big) \end{aligned}$$

Let us recall two definitions from Koszul [18].

Definition 2.27 *A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is reductive in $\mathfrak{g}$ if the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ is completely reducible.*

The subalgebra $\mathfrak{h}$ is homologous to 0 (in $\mathfrak{g}$) if the natural map $H^*(\mathfrak{g}) \to H^*(\mathfrak{h})$ is not surjective, or equivalently, if the natural map $H_*(\mathfrak{h}) \to H_*(\mathfrak{g})$ is not injective.

For example if $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra and $\mathfrak{h}$ is a CSA, then we see from the root space decomposition that $\mathfrak{h}$ is reductive in $\mathfrak{g}$. It follows from theorem 1.10 that a Lie algebra is reductive if and only if it is reductive in itself, which explains this terminology. In general if $\mathfrak{h}$ is reductive in $\mathfrak{g}$ and $\mathfrak{t}$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, then clearly $\mathfrak{t}$ is also a completely reducible $\mathfrak{h}$-module, so $\mathfrak{h}$ is also reductive in $\mathfrak{t}$.

We are much more interested in subalgebras that are *not* homologous to 0, than in those that are homologous to 0. We already met an example; if $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}$

with $\mathfrak{a}$ an ideal, then $\mathfrak{h}$ is not homologous to 0 in $\mathfrak{g}$. For suppose we have a cycle $x \in Z_p(\mathfrak{h}) \cap B_p(\mathfrak{g})$. Let $\{H_i\}_i$ and $\{A_j\}_j$ be bases of $\mathfrak{h}$ and $\mathfrak{a}$, and take an $y \in C_{p+1}(\mathfrak{g})$ with $dy = x$. Write y in terms of the associated basis of $\bigwedge \mathfrak{g}$, let y_1 be the sum of all terms with only H_i's and let y_2 be the sum of all terms with an A_j. Because $\mathfrak{a}$ is an ideal, dy_2 is a sum of terms that involve an A_j. So necessarily $dy_2 = x - dy_1 = 0$ and $x = dy_1 \in B_p(\mathfrak{h})$. Thus the natural map $H_*(\mathfrak{h}) \to H_*(\mathfrak{g})$ is indeed injective.

The following factorization theorem is the obvious generalization of theorem 13 of [14] to reductive subalgebras; the proofs are almost the same.

Theorem 2.28 *Suppose that $\mathfrak{g}$ and V are finite-dimensional, $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}$ where $\mathfrak{a}$ is an ideal, $\mathfrak{h}$ is reductive in $\mathfrak{g}$ and V is completely reducible as $\mathfrak{h}$-module. Then naturally, as graded vector spaces*

$$H^*(\mathfrak{a}; V)^{\mathfrak{g}} = H^*(\mathfrak{a}; V)^{\mathfrak{h}} \cong H^*(\mathfrak{g}, \mathfrak{h}; V) \tag{2.17}$$

$$H^*(\mathfrak{h}) \otimes H^*(\mathfrak{a}; V)^{\mathfrak{h}} \cong H^*(\mathfrak{g}; V) \tag{2.18}$$

where the second isomorphism is given by inclusion followed by the wedge product for the paired modules $\mathbb{F}$ and V.

Proof. Since $H^*(\mathfrak{a}; V)$ is a trivial $\mathfrak{a}$-module, $H^*(\mathfrak{a}; V)^{\mathfrak{g}} = H^*(\mathfrak{a}; V)^{\mathfrak{h}}$. We know that $H^*(\mathfrak{g}, \mathfrak{h}; V)$ is a trivial $\mathfrak{h}$-module and that it is computed by the complex $\big((\bigwedge(\mathfrak{g}/\mathfrak{h})^* \otimes V)^{\mathfrak{h}}, d\big)$, which is naturally isomorphic to the complex $\big((\bigwedge \mathfrak{a}^* \otimes V)^{\mathfrak{h}}, d\big)$. This gives rise to a (natural) map $\phi : H^*(\mathfrak{g}, \mathfrak{h}; V) \to H^*(\mathfrak{a}; V)^{\mathfrak{h}}$.

The assumptions together with theorem 1.2 imply that $C^p(\mathfrak{a}; V) = \bigwedge^p \mathfrak{a}^* \otimes V$ is a completely reducible $\mathfrak{h}$-module. By formula 2.14 the action of $\mathfrak{h}$ commutes with d, so there is a decomposition of $\mathfrak{h}$-modules $C^p(\mathfrak{a}; V) = d\,(C^{p-1}(\mathfrak{a}; V)) \oplus U$. Every cohomology class in $H^p(\mathfrak{a}; V)$ has exactly one representative cocycle in U. Now we decompose U as $U^{\mathfrak{h}} \oplus U'$, so that every element of $H^p(\mathfrak{a}; V)^{\mathfrak{h}}$ has a representative in $U^{\mathfrak{h}} \subset C^p(\mathfrak{a}; V)^{\mathfrak{h}}$. This means that ϕ is surjective. We also have a $\mathfrak{h}$-module decomposition $C^{p-1}(\mathfrak{a}; V) = Z^{p-1}(\mathfrak{a}; V) \oplus W$ and this implies $B^p(\mathfrak{a}; V) = dW$. If $f \in W$ and $df \in B^p(\mathfrak{a}; V)^{\mathfrak{h}}$ then formula 2.14 tells us that $\forall Y \in \mathfrak{h} : L_Y f \in Z^{p-1}$. But also $L_Y f \in W$ so $L_Y f = 0$ and $f \in W^{\mathfrak{h}}$. It follows that ϕ is also injective and equation 2.17 is proved.

Now consider the Hochschild-Serre spectral sequence for $\mathfrak{g}, \mathfrak{a}$ and V. By lemma 2.26 the second term is $E_2^{p,q} \cong H^p(\mathfrak{h}; H^q(\mathfrak{a}; V))$. By theorem 10 of [14] the right hand side is isomorphic to $H^p(\mathfrak{h}) \otimes H^q(\mathfrak{a}; V)^{\mathfrak{h}}$. Looking again at lemma 2.26 we see that the isomorphism to E_2 is given by the inclusions (induced by the decomposition $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}$) of both factors into $H^*(\mathfrak{g})$ and $H^*(\mathfrak{g}; V)$, followed by the wedge product for the paired modules $\mathbb{F}$ and V. Now we show that this sequence already converges at $E_2^{*,*}$. Let $\omega \otimes \eta \in Z^p(\mathfrak{h}) \otimes Z^q(\mathfrak{g}, \mathfrak{h}; V)$ be a representative of an element of $E_2^{p,q}$. Using the wedge product for paired modules, $\omega \otimes \eta$ can be considered as an element of $C^*(\mathfrak{g}; V)$. By lemma 2.20 and since $\mathfrak{h}$ is complementary to an ideal

$$d(\omega \otimes \eta) = d_g\omega \otimes \eta + (-1)^p \omega \otimes d\eta = d_a\omega \otimes \eta = 0$$

Such elements span $Z^*(\mathfrak{h}) \otimes Z^*(\mathfrak{g}, \mathfrak{h}; V)$, so every element of $E_2^{*,*}$ can be represented by a cocycle in $C^*(\mathfrak{g}; V)$. Consequently $d_2 = 0$ and $E_3 = E_2$. But for E_3 the same is valid, so also $d_3 = 0$ and $E_4 = E_3 = E_2$. With induction we conclude that $E_2 = E_\infty \cong GF^*(H^*(\mathfrak{g}; V))$. Now the image of the multiplication map $H^p(\mathfrak{h}) \otimes H^q(\mathfrak{a}; V)^{\mathfrak{h}} \to E_2$ actually lies in $H^*(\mathfrak{g}; V)$, so it must be an isomorphism onto $H^*(\mathfrak{g}; V)$. □

The major part of sections 6 and 7 of [14] is summarized in the next two results. For the trivial module $V = \mathbb{F}$ they can also be found in [18].

Proposition 2.29 *Let $d^p : C^p(\mathfrak{g}, \mathfrak{h}; V) \to C^{p+1}(\mathfrak{g}, \mathfrak{h}; V)$ be the Koszul differential. Suppose that $\mathfrak{g}$ and V are finite-dimensional, $\mathfrak{h}$ is reductive in $\mathfrak{g}$ and that V is completely reducible for $\mathfrak{h}$. Then*

$$\begin{aligned} E_1^{p,q} &\cong H^q(\mathfrak{h}) \otimes C^p(\mathfrak{g}, \mathfrak{h}; V) \\ d_1^{p,q} &= (-1)^q \otimes d^p \\ E_2^{p,q} &\cong H^q(\mathfrak{h}) \otimes H^p(\mathfrak{g}, \mathfrak{h}; V) \end{aligned}$$

where the isomorphisms are induced by the wedge product for the paired modules $\mathbb{F}$ and V.

Theorem 2.30 *Assume the same as in proposition 2.29, and that $\mathfrak{h}$ is not homologous to 0. There exists an homomorphism of graded algebras $\phi : H^*(\mathfrak{h}) \to H^*(\mathfrak{g})$ such that $r\phi = \mathrm{id}$, where r is induced by the restriction map from $C^*(\mathfrak{g})$ to $C^*(\mathfrak{h})$. For such ϕ*

$$\mu \circ (\phi \otimes 1) : H^*(\mathfrak{h}) \otimes H^*(\mathfrak{g}, \mathfrak{h}; V) \to H^*(\mathfrak{g}; V)$$

is an isomorphism of graded vector spaces.

It is natural to compare this theorem with theorem 2.28. 2.30 is more general since $\mathfrak{h}$ does not have to be complementary to an ideal. On the other hand the isomorphism from 2.28 is natural, and it can be used without relative cohomology. Finally 2.28 can be generalized to some infinite-dimensional Lie algebras and modules (as we will see in chapter 4), while 2.30 cannot.

Proposition 2.29 leads to a generalization a well known theorem:

Theorem 2.31 *Assume the same as in proposition 2.29, and that V is completely reducible as a $\mathfrak{g}$-module.*

$$H^*(\mathfrak{g}, \mathfrak{h}; V) \cong H^*(\mathfrak{g}, \mathfrak{h}) \otimes V^{\mathfrak{g}}$$

Proof. By proposition 2.24.6 it is sufficient to prove that if V is irreducible and nontrivial as a $\mathfrak{g}$-module, then $H^*(\mathfrak{g}, \mathfrak{h}; V) = 0$. For $\mathfrak{h} = 0$ this is theorem 10 of [14]. So the spectral sequence for $\mathfrak{g}$, $\mathfrak{h}$ and V converges to 0. By proposition 2.29 $E_2^{p,q} = H^q(\mathfrak{h}) \otimes H^p(\mathfrak{g}, \mathfrak{h}; V)$. Now we prove by induction to p that the row $E_2^{p,*} = H^*(\mathfrak{h}) \otimes H^p(\mathfrak{g}, \mathfrak{h}; V)$ is 0.

For $p = 0$ this is proposition 2.24.1.

Since $d_s(E_s^{p,0}) \subset E_s^{p+s,1-s} = 0$ for $s \geq 2$, $E_\infty^{p,0} = 0$ is a quotient of $E_2^{p,0} = H^p(\mathfrak{g}, \mathfrak{h}; V)$. In fact it is a quotient of $E_2^{p,q}$ by the images of elements of lower rows. However by the induction hypothesis all these rows are 0, so we must have $E_2^{p,0} = H^p(\mathfrak{g}, \mathfrak{h}; V) = 0$ and $E_2^{p,*} = H^*(\mathfrak{h}) \otimes H^p(\mathfrak{g}, \mathfrak{h}; V) = 0$ □

If $\mathfrak{t}$ is a subalgebra of $\mathfrak{h}$, there exists a spectral sequence (also named after Hochschild and Serre) that converges to $H^*(\mathfrak{g}, \mathfrak{t}; V)$. It constructed by starting with the subcomplex $(C^*(\mathfrak{g}, \mathfrak{t}), d)$ of $(C^*(\mathfrak{g}), d)$ and taking the same steps as we did without $\mathfrak{t}$. The proof of lemma 2.26 in [14] goes through in the same way and yields

Lemma 2.32 *The first terms of the Hochschild-Serre spectral sequence for* $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{t}$ *and* V *are*

$$\begin{aligned} E_0^{p,q} &\cong C^q\left(\mathfrak{h}, \mathfrak{t}; \mathrm{Hom}_{\mathbb{F}}\big(\textstyle\bigwedge^p(\mathfrak{g}/\mathfrak{h}), V\big)\right) \\ E_1^{p,q} &\cong H^q\left(\mathfrak{h}, \mathfrak{t}; \mathrm{Hom}_{\mathbb{F}}\big(\textstyle\bigwedge^p(\mathfrak{g}/\mathfrak{h}), V\big)\right) \\ E_2^{p,0} &\cong H^p(\mathfrak{g}, \mathfrak{h}; V) \end{aligned}$$

The analogue of proposition 2.29 is

Proposition 2.33 *Suppose that* $\mathfrak{g}$ *and* V *are finite-dimensional,* $\mathfrak{h}$ *and* $\mathfrak{t}$ *are reductive in* $\mathfrak{g}$ *and that* V *is completely reducible for* $\mathfrak{h}$ *and* $\mathfrak{t}$*. If* $d^p : C^p(\mathfrak{g}, \mathfrak{h}; V) \to C^{p+1}(\mathfrak{g}, \mathfrak{h}; V)$ *is the Koszul differential then*

$$\begin{aligned} E_1^{p,q} &\cong H^q(\mathfrak{h}, \mathfrak{t}) \otimes C^p(\mathfrak{g}, \mathfrak{h}; V) \\ d_1^{p,q} &= (-1)^q \otimes d^p \\ E_2^{p,q} &\cong H^q(\mathfrak{h}, \mathfrak{t}) \otimes H^p(\mathfrak{g}, \mathfrak{h}; V) \end{aligned}$$

Proof. From lemma 2.32 and theorem 2.31 we get

$$\begin{aligned} E_1^{p,q} &\cong H^q\left(\mathfrak{h}, \mathfrak{t}; \mathrm{Hom}_{\mathbb{F}}\big(\textstyle\bigwedge^p(\mathfrak{g}/\mathfrak{h}), V\big)\right) \cong H^q(\mathfrak{h}, \mathfrak{t}) \otimes \mathrm{Hom}_{\mathfrak{h}}\big(\textstyle\bigwedge^p(\mathfrak{g}/\mathfrak{h}), V\big) \\ &= H^q(\mathfrak{h}, \mathfrak{t}) \otimes C^p(\mathfrak{g}, \mathfrak{h}; V) \end{aligned}$$

The differential $d_1^{p,q}$ is induced by the same $d : C^{p+q}(\mathfrak{g}; V) \to C^{p+q+1}(\mathfrak{g}; V)$ as was $d_1^{p,q}$ in lemma 2.32. Therefore it must also equal $(-1)^q \otimes d^p$. The statement for E_2 is now obvious. □

I do not know whether theorem 2.30 can be generalized to the situation $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{g}$; the proofs are badly suited for that.

Chapter 3

The cohomology of compact Lie groups

In this chapter we will compute the cohomology rings of compact Lie groups in terms of their exponents. As a consequence we also obtain the cohomology rings of all real and complex reductive Lie algebras.

In earlier times this result was not so much formulated with De Rham cohomology, but in terms of the Poincaré polynomial and the Betti numbers of a Lie group. In this form it was first proved by a case by case consideration of all simple Lie groups, where of course the exceptional ones were most problematic. The first general proof is probably due to Chevalley [5], but he refers to discoveries by A. Weil which I could not retrace. Other proofs come from Borel [1], Leray [19] and Reeder [23].

Studying these proofs it struck to me that they used difficult results from algebraic topology and often ignored Lie algebras. Therefore I set out to calculate the cohomology of a finite-dimensional complex reductive Lie algebra in a completely algebraic way, translating the above proofs to the language of Lie algebras. This turned out to be difficult, so I used compact Lie groups a few times.

Throughout this chapter G is a connected compact Lie group with Lie algebra $\mathfrak{g}$, T is a maximal torus of G and $\mathfrak{t} = \mathfrak{h} \oplus Z(\mathfrak{g})$ the corresponding CSA. The root system and the Weyl group are denoted by R and W, and we fix a basis Δ with positive system R^+. Note that $\mathfrak{t}$ is reductive in $\mathfrak{g}$ because T is compact.

We will first compute the cohomology of G/T, then $H^*_{DR}(G) \cong H^*(\mathfrak{g})$ and finally the cohomology of reductive Lie algebras over subfields of $\mathbb{C}$.

3.1 G/T

The homogeneous space G/T is the quotient of compact Lie group by a closed connected subgroup, so it is compact and orientable. By theorem 2.14 $H^*_{DR}(G/T) \cong H^*(\mathfrak{g}, \mathfrak{t})$. First of all we want to find the dimension of this algebra. This can be done using the Bruhat decomposition $G/T = \bigsqcup_{w\in W} X_w$, where each Schubert cell

X_w has dimension $2l(w)$. Then cellular homology theory implies that the Poincaré polynomial of G/T is $\sum_{w\in W} t^{2l(w)}$ and the required dimension is $|W|$. However we use another approach, inspired by lemme 26.1 of Borel[1].

Lemma 3.1 *$H^p(\mathfrak{g},\mathfrak{t}) = 0$ for odd p.*

Proof. We use induction to both $\dim\mathfrak{g}$ and $\operatorname{rank}\mathfrak{g} = \dim\mathfrak{t}$. For every rank, we have the basic situation $\operatorname{rank}\mathfrak{g} = \dim\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{t}$ is abelian and $H^*(\mathfrak{g},\mathfrak{t})$ is computed by the complex $((\bigwedge(\mathfrak{g}/\mathfrak{g})^*)^{\mathfrak{g}}, d)$. Clearly $H^*(\mathfrak{g},\mathfrak{t}) = H^0(\mathfrak{g},\mathfrak{t}) = \mathbb{R}$ in this case.

Now we assume that the lemma is true for all pairs $(\mathfrak{g}',\mathfrak{t}')$ with $\dim\mathfrak{g}' < \dim\mathfrak{g}$ and $\dim\mathfrak{t}' \leq \dim\mathfrak{t}$. Since the computing differential complexes are the same, $H^*(\mathfrak{g},\mathfrak{t}) \cong H^*([\mathfrak{g},\mathfrak{g}],\mathfrak{h})$. If $Z(\mathfrak{g}) \neq 0$ then $H^p([\mathfrak{g},\mathfrak{g}],\mathfrak{h}) = 0$ for odd p by the induction hypothesis. Therefore we assume that $Z(\mathfrak{g}) = 0$ so that $\mathfrak{g}$ is semisimple. By the compactness of G and by theorem 2.13.2 of [24], the kernel of $\mathfrak{t} \to \operatorname{Aut}\mathfrak{g} : X \to \exp(\operatorname{ad} X) = \operatorname{Ad}(\exp X)$ is discrete. So we can choose $X_0 \in \mathfrak{t}$ such that $\sigma := \exp(\operatorname{ad} X_0) \neq \mathrm{id}_{\mathfrak{g}}$ but $\sigma^2 = \exp(\operatorname{ad} 2X_0) = \mathrm{id}_{\mathfrak{g}}$. Let A be (the identity component of) the centralizer of $\exp X_0$ in G. Since $\exp X_0 \notin Z(G)$, A is a proper Lie subgroup of G. Moreover A is closed, hence compact, and so its Lie algebra $\mathfrak{a} = \{X \in \mathfrak{g} : \sigma X = X\}$ is a proper reductive subalgebra of $\mathfrak{g}$. Note that $\mathfrak{t} \subset \mathfrak{a}$.

Let us show that $H^p(\mathfrak{g},\mathfrak{a}) = 0$ for odd p. Since $\mathfrak{a}$ is reductive in $\mathfrak{g}$ we can find a $\mathfrak{a}$-module decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{c}$. (For example we could take for $\mathfrak{c}$ the orthoplement of $\mathfrak{a}$ with respect to the Killing form of $\mathfrak{g}$.) Then the complex $((\bigwedge \mathfrak{c}^*)^{\mathfrak{a}}, d)$ computes $H^*(\mathfrak{g},\mathfrak{a})$. Now $\sigma^2 = \mathrm{id}_{\mathfrak{g}}$, $\sigma\mathfrak{c} \subset \mathfrak{c}$ and $\sigma Y \neq Y$ for $Y \in \mathfrak{c}$, so $\sigma\big|_{\mathfrak{c}} = -\mathrm{id}_{\mathfrak{c}}$. Furthermore σ is in the adjoint group of $\mathfrak{a}$, so every element of $(\bigwedge \mathfrak{c}^*)^{\mathfrak{a}}$ is invariant under σ. We conclude that $(\bigwedge \mathfrak{c}^*)^{\mathfrak{a}}$ and $H^*(\mathfrak{g},\mathfrak{a})$ are 0 in odd degrees.

Now we can apply proposition 2.33 with the trivial module $V = \mathbb{R}$. It says that there is a spectral sequence converging to $H^*(\mathfrak{g},\mathfrak{t})$ with $E_2^{p,q} \cong H^q(\mathfrak{a},\mathfrak{t}) \otimes H^p(\mathfrak{g},\mathfrak{a})$. But by the induction hypothesis $H^q(\mathfrak{a},\mathfrak{t}) = 0$ for odd q, and we just saw that $H^p(\mathfrak{g},\mathfrak{a}) = 0$ for odd p. The limit term $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$, so it can only be nonzero if p and q are even. Since the degree of $E_\infty^{p,q}$ is $p+q$ and the sequence converges to $H^*(\mathfrak{g},\mathfrak{t})$, the lemma is proved. $\square$

Proposition 3.2 *Let χ denote the Euler characteristic.*

$$\dim H^*_{DR}(G/T) = \chi(G/T) = |W|$$

Proof. By theorem 2.14 and lemma 3.1 the Euler characteristic of G/T is

$$\chi(G/T) = \sum_{p\geq 0} (-1)^p \dim H^p_{DR}(G/T) = \sum_{q\geq 0} \dim H^{2q}_{DR}(G/T) = \dim H^*_{DR}(G/T)$$

Recall the Poincaré-Hopf index theorem [2]. It says that if V is a smooth vector field on G/T with only finitely many zeros, then $\chi(G/T)$ is the sum of all the indices of these zeros. A torus is a topologically cyclic group and $\exp : \mathfrak{t} \to T$ is surjective, so we can find $Y \in \mathfrak{t}$ such that the powers of exp Y constitute a dense subgroup of T. There is a natural identification of the tangent space of G/T at the point gT with $\mathfrak{g}/\mathrm{Ad}(g)\mathfrak{t}$. We define our vector field in a particularly simple way, by putting $V(gT) = Y \in \mathfrak{g}/\mathrm{Ad}(g)\mathfrak{t}$ for all $g \in G$. The zeros of V are easily identified:

$$\begin{aligned} V(gT) = 0 &\iff Y \in \mathrm{Ad}(g)\mathfrak{t} \iff \exp Y \in gTg^{-1} \\ &\iff \forall n \in \mathbb{Z} : g^{-1}(\exp Y)^n g \in T \\ &\iff g \in N_G(T) \iff gT \in N_G(T)/T \end{aligned}$$

First show that the index of the zero T is 1, and then that all the indices are equal. Let $\mathfrak{m}$ be the orthoplement of $\mathfrak{t}$ with respect to the Killing form of $\mathfrak{g}$; it is the direct sum of all root spaces $\mathfrak{g}_\alpha$ with $\alpha \neq 0$. It follows from theorem 2.10.1 of [24] that there are neighborhoods U_1 of 0 in $\mathfrak{m}$ and U_2 of T in G/T such that $\overline{\exp} : U_1 \to U_2 : X \to (\exp X)T$ is a diffeomorphism. We trivialize the tangent bundle of U_2 by the isomorphisms

$$\mathrm{Ad}(\exp X) = \exp(\mathrm{ad}\, X) : T_T(G/T) = \mathfrak{g}/\mathfrak{t} \to \mathfrak{g}/\mathrm{Ad}(\exp X)\mathfrak{t} = T_{\overline{\exp}X}(G/T)$$

Now we have for $X \in U_1$:

$$\begin{aligned} V(\overline{\exp}\, X) &= \exp(\mathrm{ad}\ -X)Y = \sum_{i=0}^{\infty} \frac{(\mathrm{ad}\ -X)^i Y}{i!} \\ &= Y - [X,Y] + \text{h.o.t.} = [Y,X] + \text{h.o.t.} \in \mathfrak{g}/\mathfrak{t} \cong \mathfrak{m} \end{aligned}$$

So locally V looks like the vector field $V'(X) = [Y,X]$ on $\mathfrak{m}$. By proposition 1.11 ad $Y \in$ End $\mathfrak{g}$ is semisimple and has pure imaginary eigenvalues. The kernel of $\mathrm{ad}\, Y$ is the Lie algebra of the centralizer of $\exp Y$ in G. So by the above this kernel is simply $\mathfrak{t}$ and $\mathrm{ad}\, Y$ is a nonsingular real linear transformation of $\mathfrak{m}$. Thus the eigenvalues of $\mathrm{ad}\, Y$ come in complex conjugate pairs and $\det \mathrm{ad}\, Y)\big|_{\mathfrak{m}} > 0$. Therefore the indices of V' at $0 \in \mathfrak{m}$ and of V at $T \in G/T$ are 1.

If $wT \in N_G(T)/T$ is another zero of V, the map $gT \to gwT$ is an orientation preserving diffeomorphism from U_2 onto some neighborhood of wT in G/T. Moreover $\mathrm{Ad}(w)(\mathfrak{t}) = \mathfrak{t}$ so $T_{gwT}(G/T)$ and $T_{gT}(G/T)$ are both naturally isomorphic to $\mathfrak{g}/\mathrm{Ad}(g)\mathfrak{t}$. Thus the index of V at wT equals the index of V at T, which we saw is 1.

Now the Poincaré-Hopf index theorem tells us that $\chi(G/T)$ is the number of zeros of V, and by theorem 1.27 there are exactly $|W|$ zeros. □

The Weyl group $W \cong N_G(T)/T$ acts on G/T by $w \cdot gT = gw^{-1}T$. So if $\omega \in \Omega^p(G/T)$ and $X_1, \ldots, X_p$ are vector fields on G/T:

$$w^*\omega(X_1, \ldots, X_p) = \omega(dr_w^{-1}X_1, \ldots, dr_w^{-1}X_p)(gw^{-1}T)$$

If ω and all the X_i are G-invariant, this expression is the same in every point of G/T and in fact equals

$$\omega(dl_w dr_w^{-1}X_1, \ldots, dl_w dr_w^{-1}X_p) = \omega(\mathrm{Ad}(w^{-1})X_1, \ldots, \mathrm{Ad}(w^{-1})X_p)$$

so in this case $w^*\omega = \mathrm{Ad}(w^{-1})^*\omega$. In section 2.3 we proved that we can compute the De Rham cohomology of G/T with the complex $C^*(\mathfrak{g},\mathfrak{t}) = \left(\bigwedge(\mathfrak{g},\mathfrak{t})^*\right)^{\mathfrak{t}} = \left(\bigwedge(\mathfrak{g}/\mathfrak{t})^*\right)^T$ of G-invariant forms. So the Weyl group acts on $H^*_{DR}(G/T)$ and on $C^*(\mathfrak{g}/\mathfrak{t})$ by $\omega \to \mathrm{Ad}(w^{-1})^*\omega$, even though it does not act on $\mathfrak{g}/\mathfrak{t}$ or on $\bigwedge(\mathfrak{g}/\mathfrak{t})^*$.

Let us consider the map

$$\psi : \mathfrak{g}^* \to \bigwedge\nolimits^2 \mathfrak{g}^* : \xi \to d\xi$$

Since $\bigwedge^2 \mathfrak{g}^*$ is in the center of $\bigwedge \mathfrak{g}^*$ we can extend ψ to an algebra homomorphism $\psi : S\mathfrak{g}^* \to \bigwedge \mathfrak{g}^*$. For all $\xi \in \mathfrak{g}^*$, $X, Y \in \mathfrak{g}$, $g \in G$:

$$\begin{aligned}
\mathrm{Ad}(g)^*(\psi\xi)(X,Y) &= \psi\xi(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = \xi\left([\mathrm{Ad}(g^{-1})Y, \mathrm{Ad}(g)X]\right) \\
&= \xi\left(\mathrm{Ad}(g)[Y,X]\right) = \psi(\mathrm{Ad}(g)^*\xi)(X,Y)
\end{aligned}$$

so ψ is a homomorphism of G-modules.

If $H \in \mathfrak{t}$ and $X \in \mathfrak{g}$, $[H,X]$ has no component in $\mathfrak{t}$ (relative to the orthogonal decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$), so for all $\xi \in \mathfrak{t}^*$ and $Y \in \mathfrak{g}$:

$$\begin{aligned}
\psi\xi(H,X) &= \xi([X,H]) \;=\; 0 \\
\psi\xi(\mathrm{ad}\, H \cdot (X \wedge Y)) &= \psi\xi\left([H,X] \wedge Y + X \wedge [H,Y]\right) \\
&= \xi([Y,[H,X]] + [[H,Y],X]) \\
&= \xi([[X,Y],H]) \;=\; 0
\end{aligned}$$

Therefore $\psi(\mathfrak{t}^*) \subset Z^2(\mathfrak{g},\mathfrak{t})$ and $\psi(S\mathfrak{t}^*) \subset Z^*(\mathfrak{g},\mathfrak{t})$. Since T acts trivially on $S\mathfrak{t}^*$ and on $Z^*(\mathfrak{g},\mathfrak{t}) \subset C^*(\mathfrak{g},T)$, $\psi : S\mathfrak{t}^* \to Z^*(\mathfrak{g},\mathfrak{t})$ is a homomorphism of W-modules.

As in section 1.5, let J be the ideal of $S\mathfrak{t}^*$ generated by $(S^+\mathfrak{t}^*)^W$. According to Chevalley [5], the proof of the next lemma stems from A. Weil.

Lemma 3.3 $J \subset \ker\psi$

Proof. Since ψ is an algebra homomorphism and $\psi(Z(\mathfrak{g})^*) = 0$, it suffices to show that $\psi\big((S^+\mathfrak{h}^*)^W\big) = 0$ for semisimple $\mathfrak{g}$. Let $\{\xi_i : 1 \leq i \leq l - r\}$ be a basis of $\mathfrak{h}^*$ and $\{\xi_i : l + 1 - r \leq i \leq n\}$ a basis of $\mathfrak{m}$. Take an arbitrary $f \in (S^p\mathfrak{h}^*)^W$, where $p > 0$. By theorem 1.29 it can be extended in a unique way to a G-invariant polynomial $\bar{f} \in (S^p\mathfrak{g}^*)^G$. Express $\bar{f}$ in terms of these ξ_i, let $\partial_i\bar{f}$ be the partial derivative of $\bar{f}$ with respect to ξ_i and put

$$\eta := \frac{1}{p}\sum_{i=1}^{n} \psi(\partial_i\bar{f})\xi_i \in \bigwedge\nolimits^{2p-1}\mathfrak{g}^*$$

With a routine calculation one verifies that η is G-invariant. Because $\psi(\partial_i\bar{f})$ is closed we have

$$d\eta = \frac{1}{p}\sum_{i=1}^{n} \psi(\partial_i\bar{f})d\xi_i = \frac{1}{p}\psi\left(\sum_{i=1}^{n} \partial_i\bar{f}\xi_i\right) = \psi(\bar{f})$$

However by theorem 2.9 every G-invariant form is closed, so $\psi(\bar{f}) = d\eta = 0$.

Let us carry over the Killing to $\mathfrak{g}^*$ and extend this multilinearly to a nondegenerate bilinear form on $S\mathfrak{g}^*$ (see also chapter 1). Since the restriction to $\mathfrak{g}^*$ is negative definite, the restriction to $S^p\mathfrak{g}^*$ is definite. Clearly $S\mathfrak{h}^*$ is orthogonal to the ideal of all polynomials that are 0 on $\mathfrak{h}$, which is generated by $\mathfrak{m}^*$. Therefore

$$\langle \psi(f), \psi(f) \rangle = \left\langle \psi(f), \psi(f - \bar{f}) \right\rangle = 0$$

and we conclude that $\psi(f) = 0$. $\square$

With these preliminaries we are able to obtain a very satisfactory description of the cohomology of G/T. This is due to Borel [1], but we use the proof of Reeder [23].

Theorem 3.4 *The map ψ induces an algebra isomorphism*

$$\mathcal{H} \cong S\mathfrak{t}^*/J \to H^*(\mathfrak{g}, \mathfrak{t}) \cong H^*_{DR}(G/T)$$

This isomorphism doubles the degrees and intertwines the actions of the Weyl group.

Proof. Let $\bar{\psi} : S\mathfrak{t}^*/J \to H^*(\mathfrak{g}, \mathfrak{t})$ be the induced algebra homomorphism. We already observed that $\psi : S\mathfrak{t}^* \to Z^*(\mathfrak{g}, \mathfrak{t})$ is a homomorphism of W-modules, so this certainly holds for $\bar{\psi}$. By theorem 1.31 $(S\mathfrak{t}^*)/J \cong \mathcal{H}$ as vector spaces and by corollary 1.32 and proposition 3.2

$$\dim(S\mathfrak{t}^*)/J = \dim \mathcal{H} = |W| = \dim H^*(\mathfrak{g}, \mathfrak{t})$$

Thus the theorem is proved once we show that the kernel of $\bar{\psi}$ is precisely J, or equivalently that the restriction of $\bar{\psi}$ to $\mathcal{H}$ is injective. Our map $\bar{\psi}$ doubles the degrees, so it sufficient to prove that $\ker \bar{\psi} \cap \mathcal{H}^p = 0$ or $\ker \bar{\psi} \cap S^p\mathfrak{t}^* \subset J$ for every p. We do this by induction, starting in the top degree $\nu := |R^+|$.

In section 1.5 we showed that $\mathcal{H}^\nu = \mathbb{R}\pi$, where $\pi = \prod_{\alpha>0} i\alpha$. Since G/T is connected, compact and orientable and has dimension $|R| = 2\nu$, $\dim H^{2\nu}_{DR}(G/T) = 1$. By proposition 2.16.5 also $\dim H_{2\nu}(\mathfrak{g}, \mathfrak{t}) = 1$.

Now we use the basis of theorem 1.17. Let $\alpha_1, \ldots, \alpha_\nu$ be an ordering of the positive roots, $H_j = iH_{\alpha_j} \in \mathfrak{h}$, $X_j = Y_{\alpha_j}$ and $X_{i+\nu} = Y_{-\alpha_i}$. It is clear that $X_1 \wedge \cdots \wedge X_{2\nu}$ is a nonzero element of $H_{2\nu}(\mathfrak{g}, \mathfrak{t})$. Again by proposition 2.16.5, if $\psi\pi(X_1 \wedge \cdots \wedge X_{2\nu}) \neq 0$, then $\bar{\psi}\pi \neq 0$. Observe that $[X_j, X_{i+\nu}] \notin \mathfrak{m}$ for $j \neq i$ and $[X_j, X_{j+\nu}] = 2H_i$. Using this and the conventions from section 1.5 we compute

$$\begin{aligned}
\psi\pi(X_1, X_{1+\nu}, \ldots, X_\nu, X_{2\nu}) &= \sum_{\tau \in S_{2\nu}} \epsilon(\tau) i d\alpha_1(X_{\tau 1}, X_{\tau(1+\nu)}) \cdots i d\alpha_\nu(X_{\tau\nu}, X_{\tau(2\nu)}) \\
&= \sum_{\tau \in S_{2\nu}} \epsilon(\tau) i\alpha_1([X_{\tau(1+\nu)}, X_{\tau 1}]) \cdots i\alpha_\nu([X_{\tau(2\nu)}, X_{\tau\nu}]) \\
&= 2^\nu \sum_{\tau \in S_\nu} i\alpha_1(-2H_{\tau 1}) \cdots i\alpha_\nu(-2H_{\tau\nu}) \\
&= 4^\nu \sum_{\tau \in S_\nu} \alpha_1(H_{\tau 1}) \cdots \alpha_\nu(H_{\tau\nu}) \\
&= 4^\nu \partial_{H_1} \cdots \partial_{H_\nu} \pi \;=\; 4^\nu \langle H_1 \cdots H_\nu, D_\pi \rangle
\end{aligned}$$

By definition $D_{i\alpha_j}$ is a nonzero scalar multiple of H_j, so D_π is a nonzero scalar multiple of $H_1 \cdots H_\nu$. Since this bilinear form is positive definite on $S(i\mathfrak{h})$, the above expression is not zero. Therefore $\ker \bar{\psi} \cap \mathcal{H}^\nu = 0$.

Now let $k < \nu$. Since $\bar{\psi}$ is a homomorphism of W-modules, $V := \ker \bar{\psi} \cap \mathcal{H}^k$ is a W-submodule of $\mathcal{H}^k$. Assume that $V \neq 0$. Because the degree of elements of V is too low, they do not transform by the sign character of W. So we can find an $\alpha \in R$ such that σ_α does not act as -1 on V. Because $\sigma_\alpha^2 = 1$ we can decompose $V = V_+ \oplus V_-$ in the σ_α-eigenspaces for 1 and -1, where $V_+ \neq 0$. Pick $f \in V_+ \setminus 0$. Clearly $i\alpha f \in \ker \bar{\psi} \cap S^{k+1}\mathfrak{t}^*$, so by the induction hypothesis $i\alpha f \in J^{k+1}$. We also decompose $\mathcal{H} = \mathcal{H}_+ \oplus H_-$ in σ_α-eigenspaces. Reasoning as in equation 1.16, we see that the elements of $\mathcal{H}_-$ are divisible (in $S\mathfrak{t}^*$) by $i\alpha$. Let $i\alpha p_1, \ldots, i\alpha p_m$ be a basis of $\mathcal{H}_-$. We took f σ_α-invariant, so $\sigma_\alpha(i\alpha f) = -i\alpha f$. It follows from theorem 1.31 that the multiplication map $\mathcal{H} \otimes (S^+\mathfrak{t}^*)^W \to J$ is an isomorphism, so we can write $i\alpha f = \sum_{j=1}^m i\alpha p_j f_j$ with $f_j \in (S^+\mathfrak{t}^*)^W$. Consequently $f = \sum_{j=1}^m p_j f_j \in J$; but this contradicts the choice of $f \in \mathcal{H}^k \setminus 0$. Therefore our assumption that $V \neq 0$ is incorrect and $V = \ker \bar{\psi} \cap \mathcal{H}^k = 0$. □

3.2 G and $\mathfrak{g}$

Now we come to the computation of the cohomology of a compact Lie group G. Just as for G/T, we must first compute the dimension of this cohomology algebra.

Lemma 3.5 $\dim H^*_{DR}(G) = 2^l$ *where* $l = \dim T$ *is the rank of* G.

Proof. By theorem 2.9 $H^*_{DR}(G)$ is isomorphic to the ring of invariant forms $(\bigwedge \mathfrak{g}^*)^G$ and by lemma 1.8

$$\dim(\textstyle\bigwedge\mathfrak{g}^*)^G \;=\; \dim(\textstyle\bigwedge\mathfrak{g})^G = \displaystyle\int_G \det(1 + \mathrm{Ad}(g))dg \tag{3.1}$$

To perform this integration we use Weyl's integration formula (see corollary 4.13.8 of [24]). It says that for a class function f on G

$$\int_G f(g)dg \;=\; \frac{1}{|W|}\int_T f(t)\det(1-\operatorname{Ad} t)\big|_{\mathfrak{m}}dt \tag{3.2}$$

where dt is the Haar measure with total mass 1 on T. Moreover we observe that, identifying T with $\mathbb{R}^l/\mathbb{Z}^l$

$$\int_T f(t^2)dt \;=\; \int_{\mathbb{R}^l/\mathbb{Z}^l} f(t^2)dt = 2^l\int_{[0,1/2]^{\,l}} f(t^2)dt = \int_{[0,1]^{\,l}} f(t)dt = \int_T f(t)dt$$

for any continuous function f on T. Hence

$$\begin{aligned}
\int_G \det(1+\operatorname{Ad}(g))\big|_{\mathfrak{g}}dg &= \frac{1}{|W|}\int_T \det(1+\operatorname{Ad}(g))\big|_{\mathfrak{g}}\det(1-\operatorname{Ad}(g))\big|_{\mathfrak{m}}dt\\
&= \frac{2^l}{|W|}\int_T \det(1-\operatorname{Ad}(t^2))\big|_{\mathfrak{m}}dt\\
&= \frac{2^l}{|W|}\int_T \det(1-\operatorname{Ad}(t))\big|_{\mathfrak{m}}dt\\
&= \int_G 2^l dg = 2^l
\end{aligned}$$

Combining this with equation 3.1 proves the lemma. □

Now we are fully prepared to prove the main theorem of this chapter. It should be viewed in connection with proposition 1.35. Leray [19] proved this by considering the spectral sequence for the fibration $G \to G/T$, and Reeder [23] observed that this spectral sequence already degenerates at the second term. Our proof does the same for Lie algebras.

Theorem 3.6 *Let $\mathcal{H}_{(2)}$ be the space of harmonic polynomials with doubled degrees. As graded algebras*

$$H^*_{DR}(G) \cong H^*(\mathfrak{g}) \cong \left(\mathcal{H}_{(2)} \otimes \textstyle\bigwedge \mathfrak{t}^*\right)^W$$

They are all free exterior algebras with l generators in degrees $2m_i+1$.
If F_i $(i = 1,\dots,l)$ are primitive invariant polynomials of $(\mathfrak{g},\mathfrak{t})$ then, using the notation from section 3.2,

$$\left\{\sum_{i=1}^{l} \bar{\psi}(\partial_i F_j)\wedge \xi_i : 1 \le j \le l\right\}$$

is a set of generators of $H^*(\mathfrak{g})$.

Proof. We consider the Hochschild-Serre spectral sequence for $\mathfrak{g}$ and $\mathfrak{t}$, which we introduced in section 2.5. Since $\mathfrak{t}$ is reductive in $\mathfrak{g}$, proposition 2.29 applies and $E_2^{p,q} \cong H^q(\mathfrak{t}) \otimes H^p(\mathfrak{g}, \mathfrak{t})$. However, $\mathfrak{t}$ is homologous to zero in $\mathfrak{g}$, so the spectral sequence doesn't converge at E_2. The Weyl group acts on $C^*(\mathfrak{t})$ and on $C^*(\mathfrak{g}, \mathfrak{t})$, and these actions commute with the respective Koszul differentials, for they are of the form $\mathrm{Ad}(g)$ for certain $g \in G$. So W acts on every term $E_s^{p,q}$ with $s \geq 2$. (In fact it also acts on $E_1^{p,q}$ but that is irrelevant here.) Because W is finite, we can write $E_s^{p,q} = (E_s^{p,q})^W \oplus D_s^{p,q}$, where $D_s^{p,q}$ is a W-stable subspace of $E_s^{p,q}$. By theorem 2.9 the actions of $\mathfrak{g}$ and G on $H^*(\mathfrak{g})$ are trivial, so in particular $H^*(\mathfrak{g})$ is a trivial W-module. But our sequence converges to $H^*(\mathfrak{g})$, so $E_\infty^{*,*} = (E_\infty^{*,*})^W$. Therefore the spectral subsequence with terms $(E_s^{p,q})^W$ also converges to $H^*(\mathfrak{g})$. By theorem 3.4 and because $H^*(\mathfrak{t}) = \bigwedge \mathfrak{t}^*$ the second term of this subsequence is

$$(E_2^{*,*})^W \cong (H^*(\mathfrak{t}) \otimes H^*(\mathfrak{g}, \mathfrak{t}))^W \cong \left(\mathcal{H}_{(2)} \otimes \bigwedge \mathfrak{t}^*\right)^W$$

Moreover by lemma 3.5 and proposition 1.35

$$\begin{aligned} \dim(E_\infty^{*,*})^W &= \dim E_\infty^{*,*} = \dim H^*(\mathfrak{g}) = 2^l \\ &= \dim\left(\mathcal{H} \otimes \bigwedge \mathfrak{t}^*\right)^W = \dim(E_2^{*,*})^W \end{aligned}$$

Therefore the subsequence of W-invariants already converges at the second term and

$$\left(\mathcal{H}_{(2)} \otimes \bigwedge \mathfrak{t}^*\right)^W \cong GF^*(H^*(\mathfrak{g})) = \bigoplus_{p,q} F^p H^{q+p}(\mathfrak{g}) / F^{p+1} H^{q+p}(\mathfrak{g})$$

Again by proposition 1.35, the left hand side is a free exterior algebra with l *homogeneous* generators $\overline{dF_i}$ in degrees $(2m_i, 1)$. So we can find elements $\omega_i \in H^*(\mathfrak{g})$ that map to $\overline{dF_i}$ under the above isomorphism. Looking carefully, we see that map from left to right is $\bar{\psi} \otimes 1$ composed with the multiplication (induced by the orthogonal decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$) $H^*(\mathfrak{g}, \mathfrak{t}) \otimes H^*(\mathfrak{t}) \to H^*(\mathfrak{g})$ and this is multiplicative. So the 2^l elements w_I, $I \subset \{1, \ldots, l\}$ correspond to the elements $\overline{dF_I}$. These last elements are linearly independent in $\left(\mathcal{H} \otimes \bigwedge \mathfrak{t}^*\right)^W$, so the ω_I are also linearly independent in $GF^*(H^*(\mathfrak{g}))$ and certainly in $H^*(\mathfrak{g})$. But the dimension of the latter is still 2^l so the set $\{\omega_I : I \subset \{1, \ldots, l\}\}$ is a basis of $H^*(\mathfrak{g})$, and$H^*(\mathfrak{g})$ is a free exterior algebra with generators ω_i $(i = 1, \ldots, l)$.

Retracing the above isomorphism shows that $\overline{dF_i}$ goes to $\sum_{i=1}^{l} \bar{\psi}(\partial_i F_j) \wedge \xi_i \in H^{2m_i+1}(\mathfrak{g})$, so we can take this element as ω_i. □

Knowing this it is an easy matter to determine the cohomology of a larger class of reductive Lie algebras.

Corollary 3.7 *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$, $\mathfrak{g}$ a reductive Lie algebra over $\mathbb{F}$ and $m_1, \ldots, m_l$ the exponents of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{F}} \mathbb{C}$. Then $H^*(\mathfrak{g})$ is a free exterior algebra (over $\mathbb{F}$) with l generators in degrees $2m_i + 1$.*

Proof. First consider the case $\mathbb{F} = \mathbb{C}$. By theorem 1.17 $\mathfrak{g}$ has a compact real form $\mathfrak{g}_c$, and theorem 3.6 says that $H^*(\mathfrak{g}_c)$ is a real free exterior algebra with generators in degrees $2m_i + 1$, $(i = 1, \ldots, l)$. But by proposition 2.16.4, $H^*(\mathfrak{g}) \cong H^*(\mathfrak{g}_c) \otimes_{\mathbb{R}} \mathbb{C}$, so $H^*(\mathfrak{g})$ is a free complex exterior algebra with generators of the same degrees.

Now let $\mathbb{F}$ be a subfield of $\mathbb{C}$. Then $\mathfrak{g}_{\mathbb{C}}$ is a complex reductive Lie algebra, of which we just computed the cohomology. Again by proposition 2.16.4, $H^p(\mathfrak{g}_{\mathbb{C}}) \cong H^p(\mathfrak{g}) \otimes_{\mathbb{F}} \mathbb{C}$, so with induction to p we see that $H^*(\mathfrak{g})$ has generators in the same degrees as $H^*(\mathfrak{g}_{\mathbb{C}})$. $\square$

Chapter 4

Some specific Lie algebras and their cohomology

Having computed the cohomologies of all complex reductive Lie algebras, we turn our attention to a wider class of Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional complex reductive Lie algebra, z a normal complex variable and s an odd variable (which means that $s^2 = 0$). Define the following Lie algebras:

$$\begin{aligned}
\mathfrak{g}[z] &:= \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z] \\
\mathfrak{g}[z,s] &:= \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z,s] = \mathfrak{g}[z] \ltimes s\mathfrak{g}[z] \\
\mathfrak{g}[z]/(z^k) &:= \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z]/(z^k) = \mathfrak{g}[z]/z^k\mathfrak{g}[z]
\end{aligned}$$

If we consider only the degree of s, $\mathfrak{g}[z,s]$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra. Since $[s\mathfrak{g}[z], s\mathfrak{g}[z]] = s^2[\mathfrak{g}[z], \mathfrak{g}[z]] = 0$, we can say that besides $[X,Y] = -[Y,X]$ also

$$[X,Y] = (-1)^{1+\deg \mathrm{X} \deg \mathrm{Y}}[Y,X].$$

This means that $\mathfrak{g}[z,s]$ is a so-called super-algebra. (See Fuks [10] for a short introduction to super-algebras.) If we look at proposition 2.25, it is tempting to say that $H_*(\mathfrak{g}[z,s]) = H_*\big(\mathfrak{g}[z]; \bigwedge(s\mathfrak{g}[z])\big)$. However this ignores the fact that $s\mathfrak{g}[z]$ is an odd vector space (all elements have degree 1 in s).

For a vector space V, one possible definition of $\bigwedge V$ is the quotient of TV by the ideal generated by all elements of the form

$$x \otimes y - (-1)^{\deg \mathrm{X} \deg \mathrm{Y}} y \otimes x.$$

But if V is an odd vector space the grading of TV must be adjusted: now the elements of T^1V have degree 2. Consequently all homogeneous elements of TV have even degree. So if we use the above definition of $\bigwedge V$, it is TV divided out by the ideal generated by all elements of the form $x \otimes y - y \otimes x$. But this is just the usual definition of SV.

Using this kind of logic, it is not strange anymore to define

$$\begin{aligned} H_*(\mathfrak{g}[z,s]) &:= H_*\big(\mathfrak{g}[z]; S(s\mathfrak{g}[z])\big) \\ H^*(\mathfrak{g}[z,s]) &:= H^*\big(\mathfrak{g}[z]; S(s\mathfrak{g}[z])^*\big) \end{aligned}$$

The gradings are

$$H_n(\mathfrak{g}[z,s]) = \bigoplus_{q+2p=n} H_q\big(\mathfrak{g}[z]; S^p(s\mathfrak{g}[z])\big)$$

$$H^n(\mathfrak{g}[z,s]) = \prod_{q+2p=n} H^q\big(\mathfrak{g}[z]; S^p(s\mathfrak{g}[z])^*\big)$$

I am not sure whether this last expression describes a grading, but at least it comes close. Observe that $S(s\mathfrak{g}[z])^*$ is multiplicative for $\mathfrak{g}[z]$, so that $H^*(\mathfrak{g}[z,s])$ is a ring.

In this chapter we will compute the cohomologies of the algebras $\mathfrak{g}[z], \mathfrak{g}[z]/(z^k)$ and $\mathfrak{g}[z,s]$, giving the degrees of a free set of generators. Although they are listed here in increasing order of complexity, we will do this in the reverse order.

One might wonder why we consider these algebras. The answer is provided by some conjectures of Macdonald's. This is explained in detail in the next chapter.

4.1 $\mathfrak{g}[z,s]$

The computation of the cohomology of $\mathfrak{g}[z,s]$ was an interesting unsolved problem ever since Feigin [7] related it to $H^*(\mathfrak{g}[z]/(z^k))$. Feigin also claimed that he calculated $H^*(\mathfrak{g}[z,s])$, but the crucial part of his argument turns out to be nonsense. (In the lemma on page 93 of [8], Feigin considers the elements of $\bigwedge^n(\mathfrak{g}[z,s])$ as functions $\mathbb{C}^{2n} \to \bigwedge^n \mathfrak{g}$. Then he implicitly assumes that the Koszul differential corresponds to the restriction of these functions to a certain subset of $\mathbb{C}^{2n}$, but this is not correct.)

Therefore we follow the track of Fishel, Grojnowski and Teleman [9]. Their method is probably correct but highly complicated, both in the technical and in the conceptual sense.

One of the first problems we are confronted with is that $\mathfrak{g}[z]$ is the infinite direct sum of its z-weight spaces, so that $\mathfrak{g}[z]^*$ is an infinite direct product. To overcome this inconvenience, we are not going to compute exactly $H^*(\mathfrak{g}[z,s])$, but something called the restricted cohomology of $\mathfrak{g}[z,s]$. It is not difficult to see that the Koszul differential d preserves the z-grading of $\bigwedge \mathfrak{g}[z]^* \otimes S(s\mathfrak{g}[z])^*$. Therefore d is also well defined on the complex with spaces

$$C^n_{\text{res}}(\mathfrak{g}[z,s]) := \bigoplus_{q+2p=n} \bigoplus_{n_1 \geq 0, n_2 \geq 0} \big(\textstyle\bigwedge^q \mathfrak{g}[z]\big)^*_{n_1} \otimes \big(S^p s\mathfrak{g}[z]\big)^*_{n_2}$$

where the n_i refer to the z-grading of the symmetric and exterior algebras of $\mathfrak{g}[z]$. This can also be described as taking the restricted dual of $\bigwedge^q(\mathfrak{g}[z]) \otimes S^p(s\mathfrak{g}[z])$ with respect the z-weight spaces. The resulting cohomology $H^*_{\text{res}}(\mathfrak{g}[z,s])$ is the direct sum of the z-weight spaces of $H^*(\mathfrak{g}[z,s])$. Because d is still the transpose map of ∂ for $C_*(\mathfrak{g}[z,s])$, $H^n_{\mathrm{res}}(\mathfrak{g}[z,s])$ is the restricted dual of $H_n(\mathfrak{g}[z,s])$, again with respect to the z-weight spaces.

Unless $\mathfrak{g}$ is abelian, the ideal $z\mathfrak{g}[z] \ltimes s\mathfrak{g}[z]$ has no complementary ideal, so $\mathfrak{g}[z,s]$ is not reductive. However $\mathfrak{g}$ is reductive in itself and the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}[z,s]$ preserves the z- and s-weights, so $\mathfrak{g}[z,s]$ is a completely reducible $\mathfrak{g}$-module and $\mathfrak{g}$ is reductive in $\mathfrak{g}[z,s]$. Seeing also that

$$\mathfrak{g}[z,s] = \mathfrak{g} \ltimes (z\mathfrak{g}[z] \ltimes s\mathfrak{g}[z])$$

we are severely tempted to generalize theorem 2.28. This turns out to be possible because we have a $\mathfrak{g}$-invariant grading on $\mathfrak{g}[z,s]$.

Lemma 4.1

$$H^*_{\mathrm{res}}(\mathfrak{g}[z,s]) \cong H^*(\mathfrak{g}) \otimes H^*_{\mathrm{res}}(\mathfrak{g}[z,s],\mathfrak{g}) \cong H^*(\mathfrak{g}) \otimes H^*_{\mathrm{res}}(z\mathfrak{g}[z] \ltimes s\mathfrak{g}[z])^{\mathfrak{g}}$$

Proof. In the proof of theorem 2.28 we used not really that $\mathfrak{g}$ and V were finite-dimensional, but only that $\bigwedge \mathfrak{a}^* \otimes V$ was a direct sum of finite-dimensional irreducible $\mathfrak{h}$-modules. (On the other hand the decomposition $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}$, with $\mathfrak{a}$ an ideal and $\mathfrak{h}$ finite-dimensional and reductive in $\mathfrak{g}$, was essential.)

In the present situation we have a finite-dimensional reductive subalgebra $\mathfrak{g}$, complementary to the ideal $z\mathfrak{g}[z]$ in $\mathfrak{g}[z]$. Using theorem 1.2 several times one sees that $\mathfrak{g}[z], \bigwedge \mathfrak{g}[z]$ and $S\mathfrak{g}[z]$ are direct sums of finite-dimensional irreducible $\mathfrak{g}$-modules. Since the z-weight spaces are finite-dimensional and preserved by $\mathfrak{g}$, this also holds for the restricted dual spaces $\mathfrak{g}[z]^*_{\mathrm{res}}, \bigwedge(\mathfrak{g}[z]^*_{\mathrm{res}}), S(\mathfrak{g}[z]^*_{\mathrm{res}})$ and $\bigwedge(\mathfrak{g}[z]^*_{\mathrm{res}}) \otimes S(s\mathfrak{g}[z]^*_{\mathrm{res}})$. Now theorem 2.28 becomes a statement on the cohomology of the complex $C^*_{\mathrm{res}}(\mathfrak{g}[z,s])$ and what it says is precisely lemma 4.1. $\square$

To make things easier we assume that $\mathfrak{g}$ is semisimple, and let G be the adjoint group of $\mathfrak{g}$. If all computations are done we will plug in the center of a reductive $\mathfrak{g}$ without difficulties.

By theorem 1.16 there exists a basis of $\mathfrak{g}$ whose real span $\mathfrak{g}_{\mathbb{R}}$ is a real semisimple Lie algebra. In particular the Killing form of $\mathfrak{g}_{\mathbb{R}}$ is an inner product and we can find an orthonormal basis $\{\xi_a : a \in A\}$ of $\mathfrak{g}_{\mathbb{R}}$, where $A = \{1,\ldots,\dim\mathfrak{g}\}$. Then $\{z^m\xi_a : a \in A, m \geq 0\}$ is a basis of $\mathfrak{g}[z]$. Let $\{\psi^a(-m) : a \in A, m \geq 0\}$ and $\{\sigma^a(-m) : a \in A, m \geq 0\}$ be the dual bases of $\bigwedge^1(\mathfrak{g}[z]^*_{\mathrm{res}})$ and $S^1(\mathfrak{g}[z]^*_{\mathrm{res}})$. Define a symmetric bilinear form on $\bigwedge^1(z\mathfrak{g}[z]^*_{\mathrm{res}})$ by

$$\left\langle \psi^a(m), \psi^b(n) \right\rangle = \begin{cases} -1/n & \text{if } a = b, m = n < 0 \\ 0 & \text{otherwise} \end{cases}$$

Extend this to a nondegenerate bilinear form on $\bigwedge(z\mathfrak{g}[z]^*_{\mathrm{res}})$ by

$$\langle \psi_1 \wedge \cdots \wedge \psi_n, \psi'_1 \wedge \cdots \wedge \psi'_m \rangle = \begin{cases} \sum_{\tau \in S_n} \epsilon(\tau) \langle \psi_1, \psi'_{\tau 1} \rangle \cdots \langle \psi_n, \psi'_{\tau n} \rangle & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

With respect to this bilinear form

$$\{\psi^{a_i}(m_1) \wedge \cdots \wedge \psi^{a_n}(m_n) : n \geq 0 < m_1 \leq \ldots \leq m_n,\ (m_i = m_j, i < j) \Rightarrow a_i < a_j\}$$

is an orthonormal basis of $\bigwedge(z\mathfrak{g}[z]^*_{\mathrm{res}})$.

Likewise we define a symmetric bilinear form on $S^1(\mathfrak{g}[z]^*_{\mathrm{res}})$ by declaring the basis $\{\sigma^a(-m) : a \in A, m \geq 0\}$ to be orthonormal. We extend this to a nondegenerate bilinear form on $S(\mathfrak{g}[z]^*_{\mathrm{res}})$ by

$$\langle \sigma_1 \cdots \sigma_n, \sigma'_1 \cdots \sigma'_m \rangle = \begin{cases} \sum_{\tau \in S_n} \langle \sigma_1, \sigma'_{\tau 1} \rangle \cdots \langle \sigma_n, \sigma'_{\tau n} \rangle & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Then we have an orthogonal basis

$$\{\sigma^{a_i}(m_1) \cdots \sigma^{a_n}(m_n) : n \geq 0 \leq m_1 \leq \ldots \leq m_n,\ (m_i = m_j, i \leq j) \Rightarrow a_i \leq a_j\}$$

Moreover we get a nondegenerate symmetric bilinear form on $\bigwedge(\mathfrak{g}[z]^*_{\mathrm{res}}) \otimes S(s\mathfrak{g}[z]^*_{\mathrm{res}})$ if we ignore s temporarily and put

$$\langle \omega \otimes p, \eta \otimes q \rangle = \langle \omega, \eta \rangle \langle p, q \rangle$$

The product of the bases of $\bigwedge(z\mathfrak{g}[z]^*_{\mathrm{res}})$ and $S(s\mathfrak{g}[z]^*_{\mathrm{res}})$ is an orthogonal basis of

$$\mathcal{A} := \bigwedge(z\mathfrak{g}[z]^*_{\mathrm{res}}) \otimes S(s\mathfrak{g}[z]^*_{\mathrm{res}}) \tag{4.1}$$

The point of this discussion is that now we also have a nondegenerate bilinear form on the complex

$$C^* := C^*_{\mathrm{res}}(\mathfrak{g}[z,s], \mathfrak{g}) = \left(\bigwedge(z\mathfrak{g}[z]^*_{\mathrm{res}}) \otimes S(s\mathfrak{g}[z]^*_{\mathrm{res}})\right)^{\mathfrak{g}} = \mathcal{A}^{\mathfrak{g}}$$

so we can construct the adjoint map d^* and d. (Recall that this is defined by $\langle d^*x, y \rangle = \langle x, dy \rangle\ \forall x, y \in C^*$.) Since $dd = 0$ also $d^*d^* = 0$.

Let us introduce the Laplacian operator $\Box = dd^* + d^*d = (d + d^*)^2$. The cochains in the kernel of the Laplacian are called harmonic. We denote the set of these by $\mathcal{H}$, but it is to be distinguished form the harmonic polynomials in chapters 1 and 3. It turns out that also this $\mathcal{H}$ is crucial in cohomology, as it is ring-isomorphic to $H^*_{\mathrm{res}}(\mathfrak{g}[z,s], \mathfrak{g})$. Clearly d preserves the z-degree, decreases the exterior degree by one and increases the symmetric by one. Because elements of different degrees are orthogonal with respect to our bilinear form, d^* has precisely the opposite effect on the degrees. Therefore $\Box$ preserves these three gradings, and they also apply to $\mathcal{H}$.

Lemma 4.2 *Every harmonic cochain is a cocycle and every cohomology class has a harmonic representative. As triple graded vector spaces $\mathcal{H} \cong H^*_{\mathrm{res}}(\mathfrak{g}[z,s], \mathfrak{g})$.*

Proof. By some standard properties of transpose mappings:

$$C^* = \ker d \oplus \operatorname{im} d^* = \operatorname{im} d \oplus \ker d^* = \operatorname{im} d \oplus (\ker d \cap \ker d^*) \oplus \operatorname{im} d^*$$

Clearly $\ker d \cap \ker d^* \subset \ker \Box$. On the other hand d^* is injective on im d and d is injective on im d^*, so

$$\begin{aligned} \operatorname{im} d \cap \ker \Box &= \operatorname{im} d \cap \ker dd^* = 0 \\ \operatorname{im} d^* \cap \ker \Box &= \operatorname{im} d^* \cap \ker d^* d = 0 \end{aligned}$$

Consequently $\mathcal{H} = \ker \Box = \ker d \cap \ker d^*$ and

$$C^* = \operatorname{im} d \oplus \mathcal{H} \oplus \operatorname{im} d^* = \ker d \oplus \operatorname{im} d^*$$

The lemma follows directly from this. $\Box$

This is nice, but we want to get a set of generators for $H^*_{\text{res}}(\mathfrak{g}[z,s],\mathfrak{g})$, not just a basis. Therefore we must first prove that $\mathcal{H}$ is closed under multiplication. We do this by showing that $\mathcal{H}$ is the joint kernel of a set of derivations of C^*. The involved calculations are very long, and it is difficult to understand beforehand why they might give the right results. This is one reason why the task of this section is so complicated.

Before we state this result, we must of course first define the derivations we referred to. There is a unique linear bijection $\phi : \mathcal{A} \to \bigwedge(z\mathfrak{g}[z]) \otimes S(s\mathfrak{g}[z])$ satisfying $\langle f, x\rangle = f(\phi x)$. Because our bilinear form is the multilinear extension of a bilinear form on $\mathfrak{g}[z]^*_{\text{res}}$, ϕ is an algebra-isomorphism. Now consider the adjoint representation of $\mathfrak{g}[z,z^{-1},s] = \mathfrak{g} \otimes \mathbb{C}[z,z^{-1},s]$ on itself. This doesn't map $\mathfrak{g}[z,s]$ to itself, but we can still make $\mathfrak{g}[z,z^{-1},s]$ act on $z\mathfrak{g}[z] \ltimes s\mathfrak{g}[z]$. Namely for $X \in \mathfrak{g}[z,z^{-1},s], Y \in z\mathfrak{g}[z] \ltimes s\mathfrak{g}[z]$ write $[X,Y] \in \mathfrak{g}[z,z^{-1},s]$ as a polynomial in z, z^{-1} and s, and truncate all terms with $z^m (m \leq 0)$ or $sz^m (m < 0)$. We denote this action of $\mathfrak{g}[z,z^{-1},s]$ on $\mathfrak{g}[z,s]$ by $\widetilde{\text{ad}}$. (It is only a representation of $\mathfrak{g}[z,s] \subset \mathfrak{g}[z,z^{-1},s]$.) Let ρ be the induced action of $\mathfrak{g}[z,z^{-1},s]$ on $\mathfrak{g}[z]^*_{\text{res}}$ and continue it to a derivation of $\mathcal{A}$. Its transpose equals $\rho(X)^* = -\phi^{-1} \circ \widetilde{\text{ad}}(X) \circ \phi$, where $\widetilde{\text{ad}}(X)$ is extended to a derivation of $\bigwedge(z\mathfrak{g}[z]) \otimes S(s\mathfrak{g}[z])$. Consequently $\rho(X)^*$ is also a derivation.

We give the explicit formulas of the actions of the basis elements, at the same time introducing some new notation. The elements $\psi^a(m)$ satisfy

$$\psi^a(m)(z^i\xi) = \delta_{-m,i}\kappa(\xi,\xi_a)\ \forall \xi \in \mathfrak{g}$$

where κ is the Killing form of $\mathfrak{g}$. Therefore

$$\begin{aligned} \rho(z^n\xi_a)\psi^b(m)(z^i\xi) &= \psi^b(m)[z^i\xi, z^n\xi_a] = \psi^b(m)(z^{n+i}[\xi,\xi_a]) \\ &= \delta_{-m,n+i}\kappa(\xi_b,[\xi,\xi_a]) = \delta_{m+n,-i}\kappa(\xi,[\xi_a,\xi_b]) \end{aligned}$$

If $m+n \geq 0$ the above expression is zero on $z\mathfrak{g}[z]$, as there $i > 0$. For this reason we write

$$\rho(z^n\xi_a)\psi^b(m) = \begin{cases} \psi^{[a,b]}(n+m) & \text{if } m+n < 0 \\ 0 & \text{if } m+n \geq 0 \end{cases} \tag{4.2}$$

For any vector space V and $f \in V^*$ let $\tilde{s}f$ denote the corresponding element of $(sV)^*$. We can make exactly the same computation as above with all ψ's replaced by σ's. The only difference is that we must consider $\tilde{s}\sigma^a(m)$ as a function on $s\mathfrak{g}[z]$ and not on $sz\mathfrak{g}[z]$. So we have

$$\rho(z^n\xi_a)\tilde{s}\sigma^b(m) = \begin{cases} \tilde{s}\sigma^{[a,b]}(n+m) & \text{if } m+n \leq 0 \\ 0 & \text{if } m+n > 0 \end{cases} \tag{4.3}$$

With this information we compute $\rho(z^n\xi_a)^*$:

$$\begin{aligned} \left\langle \rho(z^n\xi_a)^*\tilde{s}\sigma^b(m), \tilde{s}\sigma^c(i)\right\rangle &= \left\langle \tilde{s}\sigma^b(m), \tilde{s}\sigma^{[a,c]}(n+i)\right\rangle = \delta_{m,n+i}\kappa(\xi_b, [\xi_a, \xi_c]) \\ &= \delta_{m-n,i}\kappa([\xi_b, \xi_a], \xi_c) = \left\langle \tilde{s}\sigma^{[b,a]}(m-n), \tilde{s}\sigma^c(i)\right\rangle \end{aligned}$$

$$\rho(z^n\xi_a)^*\tilde{s}\sigma^b(m) = \begin{cases} \tilde{s}\sigma^{[b,a]}(m-n) = -\rho(z^{-n}\xi_a)\tilde{s}\sigma^b(m) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases} \tag{4.4}$$

The analogue for the ψ's is only slightly more complicated:

$$\begin{aligned} \left\langle \rho(z^n\xi_a)^*\psi^b(m), \psi^c(i)\right\rangle &= \left\langle \psi^b(m), \psi^{[a,c]}(n+i)\right\rangle = \frac{\delta_{m,n+i}}{-m}\kappa(\xi_b, [\xi_a, \xi_c]) \\ &= \frac{\delta_{m-n,i}}{-m}\kappa([\xi_b, \xi_a], \xi_c) = \frac{n-m}{-m}\left\langle \psi^{[b,a]}(m-n), \psi^c(i)\right\rangle \end{aligned}$$

$$\rho(z^n\xi_a)^*\psi^b(m) = \begin{cases} \frac{n-m}{-m}\psi^{[b,a]}(m-n) = \frac{n-m}{m}\rho(z^{-n}\xi_a)\psi^b(m) & \text{if } n > m \\ 0 & \text{if } n \leq m \end{cases} \tag{4.5}$$

Next we compute the actions of $sz^n\xi_a$:

$$\begin{aligned} \rho(sz^n\xi_a)\tilde{s}\sigma^b(m)(z^i\xi) &= \tilde{s}\sigma^b(m)([z^i\xi, sz^n\xi_a]) = \delta_{-m,n+i}\kappa(\xi_b, [\xi, \xi_a]) \\ &= \delta_{-i,m+n}\kappa(\xi, [\xi_a, \xi_b]) = \psi^{[a,b]}(m+n)(z^i\xi) \end{aligned}$$

$$\rho(sz^n\xi_a)\tilde{s}\sigma^b(m) = \begin{cases} \psi^{[a,b]}(m+n) & \text{if } m+n < 0 \\ 0 & \text{if } m+n \geq 0 \end{cases} \tag{4.6}$$

$$\begin{aligned} \left\langle \rho(sz^n\xi_a)^*\psi^b(m), \tilde{s}\sigma^c(i)\right\rangle &= \left\langle \psi^b(m), \psi^{[a,c]}(n+i)\right\rangle = \frac{\delta_{m,n+i}}{-m}\kappa(\xi_b, [\xi_a, \xi_c]) \\ &= \frac{\delta_{m-n,i}}{-m}\kappa([\xi_b, \xi_a], \xi_c) = \frac{-1}{m}\left\langle \tilde{s}\sigma^{[b,a]}(m-n), \tilde{s}\sigma^c(i)\right\rangle \end{aligned}$$

$$\rho(sz^n\xi_a)^*\psi^b(m) = \begin{cases} \frac{-1}{m}\tilde{s}\sigma^{[b,a]}(m-n) & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases} \tag{4.7}$$

Moreover it is obvious that

$$\rho(sz^n\xi_a)\psi^b(m) = \rho(sz^n\xi_a)^*\tilde{s}\sigma^b(m) = 0 \tag{4.8}$$

Proposition 4.3

$$\Box = \sum_{m<0,a\in A} \frac{1}{m}\rho(z^m\xi_a)\rho(z^m\xi_a)^* - \rho(sz^m\xi_a)\rho(sz^m\xi_a)^*$$

on C^*. The kernel $\mathcal{H}$ of $\Box$ is the joint kernel of the derivations $\rho(z^m\xi_a)^*$ and $\rho(sz^m\xi_a)^*$ as $m < 0$ and $a \in A$. In particular $\mathcal{H}$ is a subalgebra of C^*.

Proof. The equality for $\Box$ is very tough. Fortunately Fishel, Grojnowsky and Teleman [9] give two proofs, which as far as I can see are both correct.

It is clear from this equality that

$$\bigcap_{m<0,a\in A} \ker\rho(z^m\xi_a)^* \cap \ker\rho(sz^m\xi_a)^* \subset \ker\Box = \mathcal{H} \tag{4.9}$$

To see that the reverse inclusion also holds we confine ourselves to the real span $\mathcal{A}_{\mathbb{R}}$ of the given basis of $\mathcal{A}$. Since $[\xi_a, \xi_b] \in \mathfrak{g}_{\mathbb{R}}$, $\mathcal{A}_{\mathbb{R}}$ is closed under all the maps in this theorem. Moreover our bilinear form becomes an inner product on $\mathcal{A}_{\mathbb{R}}$. Suppose that $X \in \mathcal{A}_{\mathbb{R}}^{\mathfrak{g}}$ and $\Box X = 0$. Then

$$\begin{aligned} 0 = \langle \Box X, X\rangle &= \sum_{m<0,a\in A} \left\langle \frac{1}{m}\rho(z^m\xi_a)\rho(z^m\xi_a)^*X - \rho(sz^m\xi_a)\rho(sz^m\xi_a)^*X, X\right\rangle \\ &= \sum_{m<0,a\in A} \frac{1}{m}||\rho(z^m\xi_a)^*X||^2 - ||\rho(sz^m\xi_a)^*X||^2 \end{aligned}$$

But all these norms are in $[0,\infty)$ and $m < 0$, so the only possibility is that they are all 0. Hence $\rho(z^m\xi_a)^*X = \rho(sz^m\xi_a)^*X = 0\ \forall m < 0, a \in A$ and equation 4.9 is an equality on $\mathcal{A}_{\mathbb{R}}$. But $\mathcal{A} = \mathcal{A}_{\mathbb{R}} + i\mathcal{A}_{\mathbb{R}}$ and all the maps are complex linear so

$$C^* = \mathcal{A}^{\mathfrak{g}} = \mathcal{A}_{\mathbb{R}}^{\mathfrak{g}} + i\mathcal{A}_{\mathbb{R}}^{\mathfrak{g}}.$$

and 4.9 is an equality on C^*. $\Box$

Now that this is settled, we only need to observe that the product on $\mathcal{H}$ is just the product on $\mathcal{A}$, in order to conclude that $\mathcal{H}$ is isomorphic to $H^*_{\text{res}}(\mathfrak{g}[z,s],\mathfrak{g})$ as a graded algebra. To find generators of $\mathcal{H}$ we first bring it in a more pleasant form.

Consider the linear bijection $\frac{d}{dz} : z\mathbb{C}[z] \to \mathbb{C}[z]$. We combine it with multiplication and division by s and the flip between the symmetric and exterior parts of $\mathcal{A}$ to an algebra isomorphism

$$\bigwedge(z\mathfrak{g}[z]) \otimes S(s\mathfrak{g}[z]) \to S(\mathfrak{g}[z]) \otimes \bigwedge(s\mathfrak{g}[z]).$$

Put

$$\Omega := S(\mathfrak{g}[z]^*_{\text{res}}) \otimes \bigwedge(s\mathfrak{g}[z]^*_{\text{res}}) \tag{4.10}$$

and let $\alpha : \Omega \to \mathcal{A}$ be the transpose of the above isomorphism. The continuation of the co-adjoint action of $\mathfrak{g}[z,s]$ on $\mathfrak{g}[z,s]^*_{\mathrm{res}}$ makes Ω into a $\mathfrak{g}[z,s]$-module. The action of $sX \in s\mathfrak{g}[z]$ is given explicitly by

$$\begin{array}{l}(\mathrm{ad}\ -sX)^*\big(P \otimes \tilde{s}\psi^{a_1}(n_1) \wedge \cdots \wedge \tilde{s}\psi^{a_q}(n_q)\big) \ = \\ \sum_{i=1}^q (-1)^i P (\mathrm{ad}\, X)^* \sigma^{a_i}(n_i) \otimes \psi^{a_1}(n_1) \wedge \cdots \wedge \widehat{\psi^{a_i}(n_i)} \wedge \cdots \wedge \tilde{s}\psi^{a_q}(n_q)\end{array}$$

where $P \in S(\mathfrak{g}[z]^*_{\mathrm{res}})$. While $\mathfrak{g}[z]$ acts by derivations on Ω, we have for $\omega, \eta \in \Omega$ of exterior degrees p and q:

$$sX \cdot (\omega \wedge \eta) = (sX \cdot \omega) \wedge \eta + (-1)^{pq} \omega \wedge (sX \cdot \eta)$$

This means in particular that $\Omega^{\mathfrak{g}[z,s]}$ is closed under multiplication. The motivation for this construction comes from section 2.8 of [9]:

Lemma 4.4 $\mathcal{H} = \alpha\, \Omega^{\mathfrak{g}[z,s]}$

Proof. Observe that $\{\tilde{s}\psi^a(m) : a \in A, m \leq 0\}$ is a basis of $\bigwedge^1 (s\mathfrak{g}[z])^*_{\mathrm{res}}$ and that

$$\alpha\big(\tilde{s}\psi^a(m)\big) = \left(\tfrac{d}{dz}\right)^* \big(\psi^a(m)\big) = (1-m)\psi^a(m-1) \tag{4.11}$$

Using equations 4.4, 4.5, 4.7 and 4.11 we obtain

$$\begin{aligned}
\rho(z^n\xi_a)^* \alpha\big(\tilde{s}\psi^b(m)\big) &= (1-m)\rho(z^n\xi_a)^* \psi^b(m-1) \\
&= (n+1-m)\psi^{[b,a]}(m-n-1) \\
&= \alpha\big(\tilde{s}\psi^{[b,a]}(m-n)\big) \\
\rho(z^n\xi_a)^* \alpha\big(\sigma^b(m)\big) &= \rho(z^n\xi_a)^* \big(\tilde{s}\sigma^b(m)\big) \\
&= \tilde{s}\sigma^{[b,a]}(m-n) \\
&= \alpha\big(\sigma^{[b,a]}(m-n)\big) \\
\rho(sz^n\xi_a)^* \alpha\big(\tilde{s}\psi^b(m)\big) &= (1-m)\rho(sz^n\xi_a)^* \psi^b(m-1) \\
&= \tilde{s}\sigma^{[b,a]}(m-n-1) \\
&= \alpha\big(\sigma^{[b,a]}(m-n-1)\big)
\end{aligned}$$

If we compare this with equations 4.2, 4.3 and 4.6 we see that $\rho(z^n\xi_a)^*$ and $\rho(sz^n\xi_a)^*$ correspond to the endomorphisms $\mathrm{ad}(-z^{-n}\xi_a)^*$ and $\mathrm{ad}(-sz^{-n-1}\xi_a)^*$ of Ω. By proposition 4.3 this means that $\alpha^{-1}\mathcal{H}$ is the joint kernel in Ω of the derivations $\mathrm{ad}(z^m\xi_a)^*$ and $\mathrm{ad}(sz^m\xi_a)^*$ for $a \in A, m \geq 0$. But these derivations generate the action of $\mathfrak{g}[z,s]$ on Ω, so $\alpha^{-1}\mathcal{H} = \Omega^{\mathfrak{g}[z,s]}$. □

With this result we reduced the computation of $H^*_{\mathrm{res}}(\mathfrak{g}[z,s], \mathfrak{g})$ to that of $\Omega^{\mathfrak{g}[z,s]}$. Although this more concrete problem is not yet completely solved, there is enough evidence to make a well-founded conjecture.

We identify $s\mathfrak{g}[z]$ with the tangent space of $\mathfrak{g}[z]$, so that the elements of Ω are algebraic differential forms on $\mathfrak{g}[z]$. To support this we let d be the exterior differential of $\mathfrak{g}[z]$; it sends $\sigma^a(m)$ to $\tilde{s}\psi^a(m)$.

Every $F \in S^n\mathfrak{g}^*$ determines a $\mathbb{C}[z]$-linear map

$$\tilde{F} = \sum_{m=0}^{\infty} F(-m)z^m : S^n(\mathfrak{g}[z]) \to \mathbb{C}[z]$$

where $F(-m) \in \Omega$. Similarly $dF \in S^{n-1}\mathfrak{g}^* \otimes \bigwedge^1(s\mathfrak{g})^*$ determines a $\mathbb{C}[z]$-linear map

$$\tilde{dF} = \sum_{m=0}^{\infty} F(-m)z^m : S^{n-1}(\mathfrak{g}[z]) \otimes \bigwedge^1(s\mathfrak{g}[z]) \to \mathbb{C}[z]$$

with $dF(-m) \in \Omega$. Observe that $F(-m)(X) = dF(-m)(X) = 0$ if X is homogeneous of z-degree unequal to m, and that $d(F(-m)) = (dF)(-m)$.

Lemma 4.5 *Take $F \in (S\mathfrak{g}^*)^{\mathfrak{g}}$ and $m \in \mathbb{N}$. Then the m-th coefficients $F(-m)$ and $dF(-m)$ are in $\Omega^{\mathfrak{g}[z,s]}$.*

Proof. Take $n \in \mathbb{N}, Y \in \mathfrak{g}$ and $X = \sum_i X_i \in S(\mathfrak{g}[z]) \otimes \bigwedge(s\mathfrak{g}[z])$, the z-weight of X_i being i. We have

$$\begin{aligned}
(\operatorname{ad} sz^nY)^*F(-m)(X) &= F(-m)(\operatorname{ad}(sz^nY)X) = 0 \\
(\operatorname{ad} z^nY)^*F(-m)(X) &= F(-m)(\operatorname{ad}(z^nY)X) \\
&= F(n-m)(\operatorname{ad}(Y)X) \\
&= F(n-m)(\operatorname{ad}(Y)X'_{m-n}) = 0
\end{aligned}$$

where X'_{m-n} is obtained from X_{m-n} by replacing every z by 1. In exactly the same it follows that $dF(-m)$ is $\mathfrak{g}[z]$-invariant. Note that

$$\begin{array}{lr}
(\operatorname{ad} sz^nY)^* d\big(\sigma^{a_1}(n_1)\cdots\sigma^{a_p}(n_p)\big) & = \\
(\operatorname{ad} sz^nY)^* \sum_{i=1}^p \sigma^{a_1}(n_1)\cdots\widehat{\sigma^{a_i}(n_i)}\cdots\sigma^{a_p}(n_p)\otimes \tilde{s}\psi^{a_i}(n_i) & = \\
\sum_{i=1}^p \sigma^{a_1}(n_1)\cdots\operatorname{ad}(z^nY)\sigma^{a_i}(n_i)\cdots\sigma^{a_p}(n_p) & = \\
\operatorname{ad}(z^nY)^*\big(\sigma^{a_1}(n_1)\cdots\sigma^{a_p}(n_p)\big) &
\end{array}$$

Therefore $\operatorname{ad}(sz^nY)^*dF(-m) = \operatorname{ad}(z^nY)^*F(-m) = 0$. $\square$

As a direct consequence of this lemma, the subalgebra of Ω generated by all the coefficients $F(-m)$ and $dF(-m)$ for $F \in (S\mathfrak{g}^*)^{\mathfrak{g}}$ is $\mathfrak{g}[z,s]$-invariant. Let $F_1, \ldots, F_l$ be primitive invariant polynomials. Clearly the above subalgebra is already generated by the coefficients of these polynomials. Our next result states that they are also free generators, in the sense that all relations among them are consequences of defining relations for symmetric and exterior algebras.

Lemma 4.6 *The polynomials $F_i(-m)$ are algebraically independent and the differential 1-forms $dF_i(-m)$ are linearly independent over (the quotient field of) $S(\mathfrak{g}[z])^*_{\mathrm{res}}$.*

Proof. We show by induction that this holds for all the above elements with $m \leq N$. This proof will mimic that of proposition 3.10 of [13].

First we show that

$$\omega(N) := dF_1(0) \wedge \cdots \wedge dF_l(0) \wedge dF_1(1) \wedge \cdots \wedge dF_l(-N) \neq 0 \tag{4.12}$$

For this purpose we let ρ be the restriction of differential forms on $\mathfrak{g}[z]$ to $\mathfrak{t}[z]$. Let $x_1, \ldots, x_l$ be a basis of $\bigwedge^1(s\mathfrak{t})^*$. For $N = 0$ we saw in the proof of 1.34 that

$$\rho\omega(0) = c\pi x_1 \wedge \cdots \wedge x_l \neq 0$$

where $c \in \mathbb{C}^\times$ and $\pi = \prod_{\alpha>0} \alpha$. By induction we may assume that

$$\rho\omega(N-1) = c^N \pi^N x_1(0) \wedge \cdots \wedge x_l(0) \wedge x_1(1) \wedge \cdots \wedge x_l(1-N)$$

To calculate $\rho\omega(N)$ we only need to consider those terms of $dF_i(-N)$ for which th exterior part is in $\bigwedge^1(sz^N\mathfrak{t})^*$. But the sum of these terms is obtained from dF_i by replacing all x_j's by $x_j(-N)$'s. In the same way as in 1.34 we see that

$$\rho\omega(N) = \omega(N-1) \wedge c\pi x_1(-N) \wedge \cdots \wedge x_l(-N) = (c\pi)^{N+1} x_1(0) \wedge \cdots \wedge x_l(-N) \neq 0$$

Now it follows from lemma 3.7 of [13] that the polynomials $F_i(-m)$ for $m \leq N$ are algebraically independent.

Let $\mathbb{K}$ be quotient field of $S(\mathfrak{g}[z])^*_{\mathrm{res}}$. Suppose that we have a relation

$$\sum_{i=1}^{l} \sum_{m=0}^{N} f_{i,m} dF_i(-m) = 0$$

with $f_{i,m} \in \mathbb{K}$. Take arbitrary j, n with $1 \leq j \leq l$ and $0 \leq n \leq N$, and let η be the wedge product of all $dF_i(-m)$ with $1 \leq i \leq l,\ 0 \leq m \leq N$ and $(i,m) \neq (j,n)$. Then, up to sign

$$0 = \left(\sum_{i=1}^{l} \sum_{m=0}^{N} f_{i,m} dF_i(-m) \right) \wedge \eta = f_{j,m}\omega(N).$$

Hence $f_{j,n} = 0$. But this holds for all j, n, so the differential forms in question are linearly independent over $\mathbb{K}$. □

This should already give a complete description of $\Omega^{\mathfrak{g}[z,s]}$:

Conjecture 4.7 *The set* $\{F_i(m), dF_i(m) : 1 \leq i \leq l, m \leq 0\}$ *freely generates* $\Omega^{\mathfrak{g}[z,s]}$.

We just proved one inclusion and the "free" part of this conjecture. The corresponding statement without z might be of limited value for this conjecture, but it is interesting because it can be regarded as an extended version of Chevalley's restriction theorem (1.29).

Proposition 4.8 *The set $\{F_i, dF_i : 1 \leq i \leq l\}$ freely generates $\big((S\mathfrak{g})^* \otimes (\bigwedge s\mathfrak{g})^*\big)^{\mathfrak{g}[s]}$. The restriction map*

$$\rho : (S\mathfrak{g})^* \otimes (\textstyle\bigwedge s\mathfrak{g})^* \to (S\mathfrak{t})^* \otimes (\textstyle\bigwedge s\mathfrak{g})^*$$

induces an isomorphism

$$\big((S\mathfrak{g})^* \otimes (\textstyle\bigwedge s\mathfrak{g})^*\big)^{\mathfrak{g}[s]} \to \big((S\mathfrak{t})^* \otimes (\textstyle\bigwedge s\mathfrak{t})^*\big)^{W}$$

Proof. Abbreviate $\big((S\mathfrak{g})^* \otimes (\bigwedge s\mathfrak{g})^*\big)^{\mathfrak{g}[s]}$ to M. It is clear that the F_i are $\mathfrak{g}[s]$-invariant. Because the exterior differential d commutes with the action of G and because $\forall X \in \mathfrak{g}$

$$(\mathrm{ad}\, sX)^* dF_i = (\mathrm{ad}\, X)^* F_i = 0,$$

also all the dF_i are $\mathfrak{g}[s]$-invariant. Moreover $\mathfrak{g}[s]$ acts by derivations, so the entire subalgebra generated by the F_i and the dF_i is $\mathfrak{g}[s]$-invariant. Hence by theorem 1.34

$$\big((S\mathfrak{t})^* \otimes (\textstyle\bigwedge s\mathfrak{t})^*\big)^{W} \subset \rho(M). \tag{4.13}$$

By the same theorem the left hand side is freely generated by the set $\{\rho(F_i), \rho(dF_i) : 1 \leq i \leq l\}$. The elements of M are G-invariant polynomial maps $\mathfrak{g} \to (\bigwedge s\mathfrak{g})^*$. Since all CSA's of $\mathfrak{g}$ are conjugate under G and their union is Zariski open, $\rho\big|_M$ is injective. Now the only thing left is to show that the inclusion 4.13 is an equality.

As usual let $\mathfrak{m}$ be the orthoplement of $\mathfrak{t}$ with respect to the Killing form of $\mathfrak{g}$. We introduce a new grading on $(S\mathfrak{t})^* \otimes (\bigwedge s\mathfrak{t})^*$ by giving the elements of $(\bigwedge^1 s\mathfrak{m})^*$ degree 1. It is easily seen that $(\mathrm{ad}\, sX)^*$ decreases this $\mathfrak{m}$-degree by 1, so to find $\rho(M)$ we only have to consider elements that are homogeneous with respect to the symmetric, exterior and $\mathfrak{m}$-gradings. So take a nonzero $\omega \in (S^p\mathfrak{t})^* \otimes (\bigwedge^q s\mathfrak{g})^*$, homogeneous of $\mathfrak{m}$-degree $m > 0$. As a differential form, ω is nonzero on a Zariski open subset of $\mathfrak{t}$, so we can find a regular $H \in \mathfrak{t}$ and $Y_1 \in \mathfrak{m}$, $Y_2, \ldots, Y_q \in \mathfrak{g}$ such that

$$\omega(\underbrace{H \cdots H}_{p \text{ times}} \otimes sY_1 \wedge sY_2 \wedge \cdots \wedge sY_q) \neq 0$$

But then it is also possible to find a $X \in \mathfrak{g}$ with $[X, H] = Y_1$, so that

$$(\mathrm{ad}\, sX)^* \omega(\underbrace{H \cdots H}_{p+1 \text{ times}} \otimes sY_2 \wedge \cdots \wedge sY_q) = (p+1)\omega(\underbrace{H \cdots H}_{p \text{ times}} \otimes [sX, H] \wedge sY_2 \wedge \cdots \wedge sY_q) \neq 0$$

So this ω is not $s\mathfrak{g}$-invariant and $\omega \notin \rho(M)$. Therefore all elements of $\rho(M)$ have $\mathfrak{m}$-degree 0, i.e.

$$\rho(M) \subset (S\mathfrak{t})^* \otimes (\textstyle\bigwedge s\mathfrak{t})^*.$$

Since the elements of M are G-invariant, the elements of $\rho(M)$ are invariant for $N_G(T)$ and in particular for W. We conclude that 4.13 is indeed an equality. $\square$

Remark. Broer [3] studied the restriction map

$$\rho : \big((S\mathfrak{g})^* \otimes V\big)^G \to \big((S\mathfrak{t})^* \otimes V^T\big)^W$$

for a general G-module V. He proved that under certain conditions this ρ is an isomorphism. These conditions do not apply to the case $V = (\bigwedge s\mathfrak{g})^*$, as can be seen by comparing proposition 4.8 with theorem 3.6. We needed the invariance with respect to $s\mathfrak{g}$ to make the restriction into an isomorphism.

Conjecture 4.7 is best understood in the following way. Put $P = \mathbb{C}^l$ and define a G-invariant algebraic morphism

$$q : \mathfrak{g} \to P : X \to (F_1(X), \dots, F_l(X))$$

Identify $P[s] := P \oplus sP$ with the tangent bundle of P. The differential of q gives another G-invariant algebraic morphism

$$q' : \mathfrak{g}[s] \to P[s] : X + sY \to \big(q(X), s\, dq_X(Y)\big)$$

These morphisms have been studied a lot and many remarkable properties are known. For example Kostant [17] proved that q is surjective and that dq_X is surjective if and only if $X \in \mathfrak{g}$ is regular.

Let $p_1, \dots, p_l$ be the standard coordinate functions on P. By construction $q^* p_i = F_i$ and $q^* dp_i = dF_i$. We include z in this picture as follows. Elements of $\mathfrak{g}[z, s]$ are polynomial maps $\mathbb{C} \to \mathfrak{g}[s]$, so composing them with q' gives polynomial maps $\mathbb{C} \to P[s]$. In other words, q' induces a map $Q : \mathfrak{g}[z, s] \to P[z, s]$ satisfying $(QX)(\lambda) = q'(X(\lambda))$ for $X \in \mathfrak{g}[z, s]$ and $\lambda \in \mathbb{C}$.

Define $p_i(-m) \in P[z]^*$ by $p_i(-m)(z^n v) = \delta_{n,m} p_i(v)$. Clearly $\{p_i(m) : 1 \le i \le l, m \le 0\}$ is a basis of $P[z]^*_{\text{res}}$ and $\{\tilde{s}\, dp_i(m) : 1 \le i \le l, m \le 0\}$ is a basis of $\bigwedge^1 (sP[z]^*_{\text{res}})$. Hence the union of these sets freely generates $S(P[z]^*_{\text{res}}) \otimes \bigwedge(sP[z]^*_{\text{res}})$. By construction $Q^*(p_i(m)) = F_i(m)$ and $Q^*(dp_i(m)) = dF_i(m)$. With these notions we can reformulate conjecture 4.7 as follows:

Conjecture 4.9 *The pullback Q^* of Q gives an isomorphism*

$$S(P[z])^*_{\text{res}} \otimes \bigwedge (sP[z])^*_{\text{res}} \to \Omega^{\mathfrak{g}[z,s]}$$

It follows already from lemmas 4.5 and 4.6 that Q^* is injective on this domain and that its image is contained in $\Omega^{\mathfrak{g}[z,s]}$, so what is left is to prove that every $\mathfrak{g}[z, s]$-invariant differential form on $\mathfrak{g}[z]$ is the pull-back of a differential form on $P[z]$. In [9] an interesting attempt is made to prove this, but it is hard to judge whether it is correct, since some definitions and references are missing.

Now apply these conjectures to cohomology algebras we started with. Because the isomorphism α involves s we must be a little careful. Let $\tilde{s}_\wedge$ and $\tilde{s}_S$ be the continuations of $\tilde{s} : \mathfrak{g}[z]^*_{\text{res}} \to (s\mathfrak{g}[z])^*_{\text{res}}$ to the respective exterior and symmetric algebras. Then the elements $dF_i(m)$ of conjecture 4.7 are more accurately described as $\tilde{s}_\wedge dF_i(m)$. Combining conjecture 4.7 with lemmas 4.2, 4.4 and proposition 4.3 yields the next theorem.

Theorem 4.10 *Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra and assume that conjecture 4.7 holds. Then the set*

$$\{\tilde{s}_S F_i(m), \left(\tfrac{d}{dz}\right)^*_\wedge \tilde{s}_S dF_i(m) : 1 \leq i \leq l, m \leq 0\}$$

freely generates $\mathcal{H}$ and $H^*_{\text{res}}(\mathfrak{g}[z,s],\mathfrak{g})$.

This theorem can easily be generalized to reductive Lie algebras:

Corollary 4.11 *Let $\mathfrak{g}$ be a finite-dimensional reductive complex Lie algebra and assume that conjecture 4.7 holds. Then the set*

$$\{\tilde{s}_S F_i(m), \left(\tfrac{d}{dz}\right)^*_\wedge \tilde{s}_S dF_i(m) : 1 \leq i \leq l, m \leq 0\}$$

freely generates $H^*_{\text{res}}(\mathfrak{g}[z,s],\mathfrak{g})$.

Proof. Since $Z(\mathfrak{g})[z,s]$ is the center of $\mathfrak{g}[z,s]$

$$\begin{aligned}
H^*_{\text{res}}(\mathfrak{g}[z,s]) &\cong H^*_{\text{res}}([\mathfrak{g},\mathfrak{g}][z,s]) \otimes H^*_{\text{res}}(Z(\mathfrak{g})[z,s]) \\
&= H^*_{\text{res}}([\mathfrak{g},\mathfrak{g}][z,s]) \otimes \textstyle\bigwedge(Z(\mathfrak{g})[z]^*_{\text{res}}) \otimes S(sZ(\mathfrak{g})[z]^*_{\text{res}}) \\
H^*_{\text{res}}(\mathfrak{g}[z,s],\mathfrak{g}) &\cong H^*_{\text{res}}([\mathfrak{g},\mathfrak{g}][z,s],[\mathfrak{g},\mathfrak{g}]) \otimes H^*_{\text{res}}(Z(\mathfrak{g})[z,s],Z(\mathfrak{g})) \\
&= H^*_{\text{res}}([\mathfrak{g},\mathfrak{g}][z,s],[\mathfrak{g},\mathfrak{g}]) \otimes \textstyle\bigwedge(zZ(\mathfrak{g})[z]^*_{\text{res}}) \otimes S(sZ(\mathfrak{g})[z]^*_{\text{res}})
\end{aligned}$$

Moreover we know from section 1.4 that a set of primitive invariant polynomials for $\mathfrak{g}$ consists of such a set for $[\mathfrak{g},\mathfrak{g}]$, plus a basis of $Z(\mathfrak{g})^*$. Now the statement is really an immediate corollary of theorem 4.10. □

With this corollary, theorem 3.6 and lemma 4.1 one can find explicit generators of $H^*_{\text{res}}(\mathfrak{g}[z,s])$. Then we can also determine the unrestricted cohomology $H^*(\mathfrak{g}[z,s])$, as this is the direct product of the (finite-dimensional) z-weight spaces of $H^*_{\text{res}}(\mathfrak{g}[z,s])$.

4.2 $\mathfrak{g}[z]/(z^k)$

It has been known for a long time that knowledge of the cohomology of $\mathfrak{g}[z]/(z^k)$ for a semisimple complex Lie algebra $\mathfrak{g}$ enables one to proof certain conjectures of Macdonald; see the next chapter. For this reason the supposed description of $H^*(\mathfrak{g}[z]/(z^k))$ is also known as the strong Macdonald conjecture. It turned out that this is really difficult to prove. In fact it is so hard that up to today the most promising approach is to derive from the restricted cohomology of the infinite-dimensional (hence more complicated) Lie algebra $\mathfrak{g}[z,s]$, which we almost computed in the previous section. We show how this works for reductive $\mathfrak{g}$, since this more general setting presents no extra problems.

Introduce derivations ∂_s of $\mathbb{C}[z,s]$, defined by $\partial_s(z^n) = 0$, $\partial_s(sz^n) = z^{n+k}$. We denote the induced derivations of $\mathfrak{g}[z,s]$ and $S(s\mathfrak{g}[z]) \otimes \bigwedge(\mathfrak{g}[z])$ also by ∂_s. If $X_1, \ldots, X_n \in \mathfrak{g}[z]$ and $\omega \in \bigwedge(\mathfrak{g}[z])$ then

$$\begin{aligned}\partial_s^2(sX_1 \cdots sX_n \otimes \omega) \;=\; \partial_s\left(\sum_{i=1}^{n} sX_1 \cdots \widehat{sX_i} \cdots sX_n \otimes z^k X_i \wedge \omega\right) &= \\ \sum_{i \neq j} sX_1 \cdots \widehat{sX_i} \cdots \widehat{sX_j} \cdots sX_n \otimes z^k X_j \wedge z^k X_i \wedge \omega &= \\ \sum_{i<j} sX_1 \cdots \widehat{sX_i} \cdots \widehat{sX_j} \cdots sX_n \otimes (z^k X_j \wedge z^k X_i + z^k X_i \wedge z^k X_j) \wedge \omega &= 0\end{aligned}$$

Hence $\partial_s^2 = 0$ in all these three cases.

The tool for relating the above cohomologies is the double graded complex with spaces

$$C^{p,q} = S^p(s\mathfrak{g}[z]^*_{\text{res}}) \otimes \textstyle\bigwedge^{q-p}(\mathfrak{g}[z]^*_{\text{res}})$$

As maps we have the Koszul differential (for $\mathfrak{g}[z,s]$) $d : C^{p,q} \to C^{p,q+1}$ and the transpose $\partial_s^* : C^{p,q} \to C^{p+1,q}$ of ∂_s. With this double graded complex we associate a single graded differential complex (C^*, D) with $C^n = \bigoplus_{p+q=n} C^{p,q}$ and $D\big|_{C^{p,q}} = (-1)^p d + \partial_s^*$. We write H_D for the resulting cohomology. It has a grading and a double filtration, but in general no double grading. By theorem 14.14 of [2], the filtration $F^i C^{**} = \bigoplus_{p \geq i, q} C^{p,q}$ gives rise to a spectral sequence $(E_r^{p,q}, D_r)$ with $E_1^{p,q} = H_d^{p,q}$, $H_2^{p,q} = H^p_{\partial_s^*} H^q_d$ and $E_\infty^{p,q} = GF^p H_D^{p+q}$.

Lemma 4.12 $H_D^n \cong H^n(\mathfrak{g}[z]/(z^k))$

Proof. According to [9], this lemma stems from [8], but I was unable to find it over there. First we consider the dual complex, which has spaces

$$C_{p,q} = S^p(s\mathfrak{g}[z]) \otimes \textstyle\bigwedge^{q-p}(\mathfrak{g}[z])$$

and maps $\partial_s : C_{p,q} \to C_{p-1,q}$ and $\partial : C_{p,q} \to C_{p,q-1}$. For a decomposable element

$$x = sX_1 \cdots sX_p \otimes z^k Y_1 \wedge \cdots \wedge z^k Y_m \wedge Z_1 \wedge \cdots \wedge Z_n$$

(where the z-degree of Z_i is smaller than k) we put

$$\tilde{F}x = \sum_{i=1}^{m} (-1)^{i+1} sX_1 \cdots sX_p sY_i \otimes z^k Y_1 \wedge \cdots \wedge \widehat{z^k Y_i} \wedge \cdots \wedge z^k Y_m \wedge Z_1 \wedge \cdots \wedge Z_n$$

Writing X^j for $X_1 \cdots \widehat{X^j} \cdots X_p$ we have

$$\begin{aligned}
\partial_s \tilde{F} x &= mx + \sum_{i=1}^{m} \sum_{j=1}^{p} (-1)^{i+1} s X^j s Y_i \otimes z^k X_j \wedge z^k Y^i \wedge Z \\
\tilde{F} \partial_s x &= px + \sum_{i=1}^{m} \sum_{j=1}^{p} (-1)^{i} s X^j s Y_i \otimes z^k X_j \wedge z^k Y^i \wedge Z \\
(\partial_s \tilde{F} + \tilde{F} \partial_s) x &= (m+p)x
\end{aligned}$$

In view of this we define a linear map $F : C_{**} \to C_{**}$ by

$$Fx = \begin{cases} \frac{1}{m+p} \tilde{F} x & \text{if } m+p > 0 \\ 0 & \text{if } m = p = 0 \end{cases}$$

Clearly $(\partial_s F + F \partial_s)x = x$ if $x \in C_{p,q}$ with $p > 0$. On our original complex we have the transpose map F^* and it satisfies $(F^* \partial_s^* + \partial_s^* F^*)y = y$ if $y \in C^{p,q}$ with $p > 0$. Suppose that $y = \sum_{i=0}^{m} y_i \in \bigoplus_{i=0}^{m} C^{i,n-i} \subset C^n$ and $Dy = 0$. If $m > 0$ then $\partial_s^* y_m = 0$, so $y_m = \partial_s^* F^* y_m$ and $y - DF^* y_m \in \bigoplus_{i=0}^{m-1} C^{i,n-i}$. Repeating this a number of times we eventually get an $y' \in C^{0,n}$ that is cohomologous to y. Obviously $dy' = \partial_s^* y' = 0$. Since $\ker \partial_s^* \cap C^{0,n} = \big(C_{0,n}/\partial_s(C_{1,n})\big)^*_{\mathrm{res}}$ we must have $y' \in S^n(\bigoplus_{i=0}^{k-1} z^i \mathfrak{g})^*$. Moreover $d\big|_{C^{0,*}}$ is the Koszul differential for $\mathfrak{g}[z]$ and it preserves the z-grading. Therefore y' can be considered as a cocycle of $\mathfrak{g}[z]/(z^k)$, and y' is a coboundary in C^{**} if and only if it is a coboundary in $C^*(\mathfrak{g}[z]/(z^k))$. Thus we get a natural injection $H_D^n \to H^n(\mathfrak{g}[z]/(z^k))$. On the other hand it is clear that every element of $Z^*(\mathfrak{g}[z]/(z^k))$ is a cocycle in C^{**}, so this map is a bijection. □

Let us assign z-weight k to s. Then ∂_s^* preserves this z-grading, and so does the whole spectral sequence $(E_r^{p,q}, D_r)$. The explicit description of $E_1^{p,q} = H_d^{p,q} = H^*_{\mathrm{res}}(\mathfrak{g}[z,s])$ we obtained in theorem 3.6 and corollary 4.11 allows us to compute $E_2^{p,q}$. For simplicity let $\omega_i(m)$ denote $\left(\frac{d}{dz}\right)^*_\wedge \tilde{s}_S dF_i(m)$.

Lemma 4.13 *If conjecture 4.7 holds, the generators of* $H^*(\mathfrak{g})$ *and the elements* $\omega_i(-m)$ *with* $1 \le i \le l, 0 \le m < k-1$ *freely generate* E_2^{**}.

Proof. I took the following explicit calculation from [9]. Since ∂_s^* is a derivation, it is sufficient to see what it does on the generators of $H^*_{\mathrm{res}}(\mathfrak{g}[z,s])$. Clearly it acts as zero on (the generators of) $H^*(\mathfrak{g})$ and on the elements $\tilde{s}_S F_i(-m)$. Furthermore

$$
\begin{aligned}
\partial_s^* \omega_i(sX_1,\dots,sX_{d_i}) &= \sum_{j=1}^{d_i} \left(\tfrac{d}{dz}\right)^*_\wedge \tilde{s}_S dF_i(z^k X_j \otimes sX_1 \cdots \widehat{sX_j} \cdots sX_{d_i}) \\
&= \sum_{j=1}^{d_i} dF_i\left(\frac{d(z^k X_j)}{dz} \otimes X_1 \cdots \widehat{X_j} \cdots X_{d_i}\right) \\
&= \sum_{j=1}^{d_i} F_i\left(X_1 \cdots \left(kz^{k-1}X_j + z^k \frac{dX_j}{dz}\right) \cdots X_{d_i}\right) \\
&= d_i k z^{k-1} F_i(X_1 \cdots X_{d_i}) + z^k \frac{dF_i}{dz}(X_1 \cdots X_{d_i}) \\
&= \tilde{s}_S\left(d_i k z^{k-1} F_i + z^k \frac{dF_i}{dz}\right)(sX_1 \cdots sX_{d_i})
\end{aligned}
$$

In particular, if CT denotes the constant term with respect to z:

$$
\begin{aligned}
\partial_s^* \omega_i(-m) &= CT\, \tilde{s}_S\left(d_i k z^{k-1-m} F_i + z^{k-m} \frac{dF_i}{dz}\right) \\
&= \tilde{s}_S\big(d_i k F_i(k-1-m) + (m+1-k) F_i(k-1-m)\big) \\
&= (m_i k + m + 1)\tilde{s}_S F_i(k-1-m)
\end{aligned}
$$

This is 0 if $m < k-1$, so all $\tilde{s}_S F_i(-m), m \geq 0$ are in the image of ∂_s^* and E_2^{**} is generated by the generators of $H^*(\mathfrak{g})$ and the elements $\omega_i(-m)$ with $1 \leq i \leq l, 0 \leq m < k-1$. These generators are free because the image and the kernel of ∂_s^* are both generated by some of the free generators of $H^*_{\text{res}}(\mathfrak{g}[z,s])$. □

Now we come to the main theorem of this section. It was first conjectured by Hanlon [12]. Our proof is a more explicit version of that in [9].

Theorem 4.14 *Let $\mathfrak{g}$ be a finite-dimensional complex reductive Lie algebra and assume that conjecture 4.7 holds. Then $H^*(\mathfrak{g}[z]/(z^k))$ is a free exterior algebra with kl generators. For each exponent m_i, there are k generators of cohomology degree $2m_i+1$, and the z-weights of these generators are the negatives of $0, m_i k+1, m_i k+2, \dots, m_i k+k-1$.*

Proof. First we show that the spectral sequence of this section degenerates at E_2. It suffices to show that all d_r $(r \geq 2)$ act as 0 on the generators of E_2. Since $\omega_i(-m) \in E_2^{m_i,d_i}$,

$$d_r \omega_i(-m) \in E_2^{m_i+r, m_i+2-r} = C^{m_i+r, m_i+2-r} = 0$$

By theorem 2.28

$$H^*(\mathfrak{g}[z]/(z^k)) \cong H^*(\mathfrak{g}) \otimes H^*(z\mathfrak{g}[z]/(z^k))$$

Observe that the elements $\left(\frac{d}{dz}\right)^*_\wedge \tilde{s}_S dF_i(-m)$ are 0 when considered as functions on $\mathfrak{g}$. But by lemma 4.12 the spectral sequence converges to $H^*(\mathfrak{g}[z]/(z^k))$, so the subalgebra $H^*(\mathfrak{g})$ of E_2 must survive to E_∞. Therefore indeed $d_r\omega = 0$ for every generator ω of E_2 and $E_2 = E_\infty$.

We declared the z-weight of s to be k, and $dF_i \in S^{m_i}(\mathfrak{g}[z]^*) \otimes \bigwedge^1(\mathfrak{g}[z]^*)$, so the z-degree of $\omega_i(-m)$ is $-1-m-d_ik$. Moreover the homology degree of s was 1, so the cohomology degree of $\omega_i(-m)$ is $2m_i+1$. Consequently E_∞ is the algebra described in the theorem. We only need to see that it is isomorphic to $H^*(\mathfrak{g}[z]/(z^k)) \cong H^*_D$ as graded algebra. But $E^{**}_\infty \cong GF^*H^*_D$ for a filtration on H^*_D that is compatible with z-grading, and we have a set of homogeneous free generators of E^{**}_∞. So just as in the proof of theorem 3.6, these elements also freely generate H^*_D, and they have the same degrees when considered as elements of H^*_D or of E^{**}_∞. $\square$

As noticed by Feigin [7], a limit case of this theorem gives a simple description of $H^*(\mathfrak{g}[z])$.

Corollary 4.15 *Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and assume that conjecture 4.7 holds. The inclusion of $\mathfrak{g}$ in $\mathfrak{g}[z]$ induces an isomorphism between the respective cohomology algebras.*

Proof. Since the Koszul differential of $\mathfrak{g}[z]$ preserves the z-degrees, $\mathfrak{g} \hookrightarrow \mathfrak{g}[z]$ induces an injection in cohomology. Let $\omega = \sum_{i\geq 0}\omega_i$ be any cocycle in $\bigwedge^n(\mathfrak{g}[z])^*$, the z-weight of ω_i being $-i$. Then ω_i is also a cocycle in $\mathfrak{g}[z]/(z^k)$ for every $k \geq i$. However by theorem 4.14 the highest possible nonzero z-weight of a cohomology class of $\mathfrak{g}[z]/(z^k)$ is $-k-1 < i$, so ω_i must be a coboundary for $i > 0$. Hence ω is cohomologous to $\omega_0 \in \bigwedge^n \mathfrak{g}^*$, and it is a coboundary in $\mathfrak{g}[z]$ if and only if it is in $\mathfrak{g}$. We conclude the map $H^*(\mathfrak{g}) \to H^*(\mathfrak{g}[z])$ is also surjective. $\square$

Note that if $\mathfrak{g}$ is reductive, it follows directly from this corollary that

$$H^*(\mathfrak{g}[z]) \cong H^*([\mathfrak{g},\mathfrak{g}]) \otimes \bigwedge(Z(\mathfrak{g})[z]^*)$$

Theorem 4.14 also allows us to compute cohomology of $\mathfrak{g}[z]/(f) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z]/(f)$ for any $f \in \mathbb{C}[z]$, or equivalently of $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z]/I$ for any ideal $I \subset \mathbb{C}[z]$:

Corollary 4.16 *Let $\mathfrak{g}$ be a finite-dimensional complex reductive Lie algebra, assume that conjecture 4.7 holds, and let $f \in \mathbb{C}[z]$ a polynomial of degree $n > 0$. Then $H^*(\mathfrak{g}[z]/(f)) \cong H^*(\mathfrak{g})^{\otimes n}$.*

Proof. Write $f = \lambda_0(z-\lambda_1)^{n_1}\cdots(z-\lambda_t)^{n_t}$, so that

$$\mathbb{C}[z]/(f) \cong \bigoplus_{i=1}^t \mathbb{C}[z]/\left((z-\lambda_i)^{n_i}\right) \cong \bigoplus_{i=1}^t \mathbb{C}[z]/(z^{n_i})$$

If we ignore the z-weights, theorems 3.6 and 4.14 say that $H^*(\mathfrak{g}[z]/(z^k)) \cong H^*(\mathfrak{g})^{\otimes k}$. Now a repeated application of proposition 2.8 gives

$$H^*(\mathfrak{g}[z]/(f)) \cong H^*\left(\bigoplus_{i=1}^{t} \mathfrak{g}[z]/(z^{n_i})\right) \cong H^*(\mathfrak{g})^{n_i} \otimes \cdots \otimes H^*(\mathfrak{g})^{\otimes n_t} = H^*(\mathfrak{g})^{\otimes n}$$

as required. □

Remark. If all the zeros of f were different, this result would be trivial. In general the generators of $H^*(\mathfrak{g}[z]/(f))$ are not homogeneous in z, since the ring isomorphism $\mathbb{C}[z]/((z-\lambda)^k) \to \mathbb{C}[z]/(z^k)$ does not preserve the z-grading.

Chapter 5

Macdonald's conjectures

Finally we come to the famous root system conjectures by Macdonald. They were first stated in 1982 in a remarkable article [21] that contains more questions than answers. Most of these conjectures were already settled some time ago, but the state of the ones we consider here is still unclear. Hanlon [12] related them to theorem 4.14, but the recent (2001) proof by Fishel, Grojnowsky and Teleman [9] of this theorem is not entirely convincing, as we saw in the previous chapter.

First we state these conjectures, and discuss some specializations. After that we give a short introduction to Kac-Moody algebras and discuss the conjectures in relation to affine root systems.

5.1 The original setting

Before we prove Macdonald's conjecture in its most beautiful form, we consider an equivalent statement, which figures as conjecture 3.1' in [21]. The method of the proof is due to Hanlon [12].

Proposition 5.1 *Let G be a connected compact Lie group with normalized Haar measure dg and Lie algebra $\mathfrak{g}$, l the rank of $\mathfrak{g}$ and $m_1, \dots, m_l$ its exponents. Take $k \in \mathbb{Z}_{>0} \cup \{\infty\}$ and let q be a complex variable, with the restriction that $|q| < 1/2$ if $k = \infty$. If conjecture 4.7 holds then*

$$\int_G \prod_{j=1}^{k-1} \det\left(1 - q^j \mathrm{Ad}(g)\right) dg = \prod_{i=1}^{l} \prod_{j=1}^{k-1} \left(1 - q^{km_i + j}\right)$$

Proof. Denote the above integral by P. Using the proof of lemma 1.8 we see that

$$\begin{aligned} \det\left(1 - q^j \mathrm{Ad}(g)\right) &= \sum_{r=0}^{\dim \mathfrak{g}} (-q^j)^r \mathrm{tr}\mathrm{Ad}(g)\big|_{\bigwedge^r \mathfrak{g}} \\ \prod_{j=1}^{k-1} \det\left(1 - q^j \mathrm{Ad}(g)\right) &= \sum_{\mathbf{r}} \prod_{j=1}^{k-1} (-q^j)^{r_j} \mathrm{tr}\mathrm{Ad}(g)\big|_{\bigwedge^{r_1} \mathfrak{g} \otimes \cdots \otimes \bigwedge^{r_{k-1}} \mathfrak{g}} \end{aligned}$$

where the sum is over all $\mathbf{r} = (r_1, \ldots, r_{k-1})$ with $0 \leq \dim \mathfrak{g}$. Now we identify $\bigwedge^{r_1} \mathfrak{g} \otimes \cdots \otimes \bigwedge^{r_{k-1}} \mathfrak{g}$ with $\bigwedge^{r_1} z\mathfrak{g} \otimes \cdots \otimes \bigwedge^{r_{k-1}} z^{k-1}\mathfrak{g} \subset \bigwedge \mathfrak{g}[z]/(z^k)$ and denote it by $\bigwedge^{\mathbf{r}} \mathfrak{g}[z]/(z^k)$. Then lemma 1.7 says

$$P = \sum_{\mathbf{r}} (-1)^{r_1+\cdots+r_{k-1}} q^{r_1+2r_2+\cdots+(k-1)r_{k-1}} \dim \left(\textstyle\bigwedge^{\mathbf{r}} \mathfrak{g}[z]/(z^k)\right)^G \tag{5.1}$$

(Consider this as a formal power series in q if $k = \infty$.) Next we go from G-invariants to $\mathfrak{g}$-invariants, complexify $\mathfrak{g}[z]/(z^k)$ and take the dual space. This does not change the dimensions in equation 5.1, so it becomes

$$P = \sum_{\mathbf{r}} (-1)^{r_1+\cdots+r_{k-1}} q^{r_1+2r_2+\cdots+(k-1)r_{k-1}} \dim \left(\left(\textstyle\bigwedge^{\mathbf{r}} \mathfrak{g}[z]/(z^k)\right)^*\right)^{\mathfrak{g}_{\mathbb{C}}} \tag{5.2}$$

But this is the Euler characteristic of the double graded complex $C^*_{\text{res}}(\mathfrak{g}_{\mathbb{C}}[z]/(z^k), \mathfrak{g}_{\mathbb{C}})$, where q corresponds to a z-weight of -1. By the Euler-Poincaré lemma (see e.g. page 262 of [10]) this equals the Euler characteristic of the complex $H^*_{\text{res}}(\mathfrak{g}_{\mathbb{C}}[z]/(z^k), \mathfrak{g}_{\mathbb{C}})$.

If $k < \infty$ then by theorem 4.14 $H^*_{\text{res}}(\mathfrak{g}_{\mathbb{C}}[z]/(z^k), \mathfrak{g}_{\mathbb{C}})$ is a free exterior algebra with $(k-1)l$ generators. For each exponent m_i there are $k-1$ generators of degree $2m_i + 1$, and they have z-weights $m_i k - 1, m_i k - 2, \ldots, m_i k + 1 - k$. Therefore

$$P = \prod_{i=1}^{l} \prod_{j=1}^{k-1} (1 + (-1)^{2m_i+1} q^{m_i k + j}) = \prod_{i=1}^{l} \prod_{j=1}^{k-1} (1 - q^{m_i k + j}) \tag{5.3}$$

On the other hand if $k = \infty$, by corollary 4.15

$$\begin{aligned} H^*_{\text{res}}(\mathfrak{g}_{\mathbb{C}}[z], \mathfrak{g}_{\mathbb{C}}) &\cong H^*_{\text{res}}([\mathfrak{g},\mathfrak{g}]_{\mathbb{C}}[z], [\mathfrak{g},\mathfrak{g}]_{\mathbb{C}}) \otimes H^*_{\text{res}}(Z(\mathfrak{g}_{\mathbb{C}})[z], Z(\mathfrak{g}_{\mathbb{C}})) \\ &\cong \mathbb{C} \otimes H^*_{\text{res}}(zZ(\mathfrak{g}_{\mathbb{C}})[z]) \;\cong\; \textstyle\bigwedge (zZ(\mathfrak{g}_{\mathbb{C}})[z])^*_{\text{res}} \end{aligned}$$

so $P = \prod_{j=1}^{\infty} (1 - q^j)^{\dim Z(\mathfrak{g})}$. But if $m_i > 0$ then because $|q| < 1/2$

$$\left| \prod_{j=1}^{k-1} (1 - q^{m_i k + j}) - 1 \right| < (2^{k-1} - 1)|q|^{m_i k + 1} < |2q|^k \to 0 \quad \text{if} \quad k \to \infty$$

Therefore

$$\begin{aligned} \lim_{k\to\infty} \prod_{j=1}^{k-1} (1 - q^{m_i k + j}) &= 1 \quad \text{if} \quad m_i > 0 \\ \lim_{k\to\infty} \prod_{i=1}^{l} \prod_{j=1}^{k-1} (1 - q^{m_i k + j}) &= \prod_{i:\, m_i = 0} \prod_{j=1}^{\infty} (1 - q^j) \end{aligned}$$

By definition 0 appears $\dim Z(\mathfrak{g})$ times as an exponent, so this last expression is indeed P. $\square$

The conventions and notations from chapter 1 are not enough to state the conjecture in full generality; we must make a little trip to the land of q-analogues. Of course everybody knows to binomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n+1-r)}{1\cdot 2\cdots r}$$

We define

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n+1-r})}{(1-q)(1-q^2)\cdots(1-q^r)}$$

Dividing both the numerator and the denominator by $(1-q)^r$ shows that

$$\lim_{q\to 1}\begin{bmatrix} n \\ r \end{bmatrix}_q = \lim_{q\to 1}\frac{\prod_{i=n+1-r}^{n}(1+q+\cdots+q^{i-1})}{\prod_{i=1}^{r}(1+q+\cdots+q^{i-1})} = \binom{n}{r}$$

For this reason one also refers to $\begin{bmatrix} n \\ r \end{bmatrix}_q$ as the q-binomial coefficient (of n and r). Most of the well-known identities for binomial coefficients have q-analogues. Now we can state the conjecture of Macdonald's the we're after. Our proof follows the original paper [21], in which Macdonald relates his conjecture to proposition 5.1.

Theorem 5.2 *Assume that conjecture 4.7 holds. Let R be a root system with positive system R^+ and define for $k \in \mathbb{Z}_{>0} \cup \{\infty\}$:*

$$P(R,k) = \prod_{\alpha\in R^+}\prod_{i=1}^{k}(1-q^i e(\alpha))(1-q^{i-1}e(-\alpha))$$

Then for $k > 0$ the constant term, independent of the $e(\alpha)$ with $\alpha \neq 0$, of $P(R,k)$ is $\prod_{i=1}^{l}\begin{bmatrix} kd_i \\ k \end{bmatrix}_q$. For $k=\infty$ this constant term is $\prod_{i=1}^{\infty}(1-q^j)^{-l}$, provided that $|q| < 1/2$.

Proof. Denote the constant term in question by c_k. If we substitute $t = q^k$ in theorem 1.23 we get

$$\prod_{i=1}^{l}\frac{1-q^{kd_i}}{1-q^k} = \sum_{w\in W}\prod_{\alpha\in R^+}\frac{1-q^k e(-w\alpha)}{1-e(-w\alpha)}$$

Multiply this by $\prod_{\alpha\in R}\prod_{i=1}^{k}(1-q^{j-1}e(\alpha))$:

$$
\begin{aligned}
&\prod_{i=1}^{l}\frac{1-q^{kd_i}}{1-q^k}\prod_{\alpha\in R}\prod_{i=1}^{k}(1-q^{j-1}e(\alpha)) &&= \\
&\sum_{w\in W}\prod_{\alpha\in R^+}\frac{1-q^k e(-w\alpha)}{1-e(-w\alpha)}\prod_{\alpha\in R}\prod_{i=1}^{k}(1-q^{j-1}e(\alpha)) &&= \\
&\sum_{w\in W}\prod_{\alpha\in R^+}\frac{1-q^k e(-w\alpha)}{1-e(-w\alpha)}\prod_{\alpha\in R}\prod_{i=1}^{k}(1-q^{j-1}e(-w\alpha)) &&= \\
&\sum_{w\in W}\prod_{\alpha\in R^+}\prod_{j=1}^{k}(1-q^{j}e(-w\alpha))(1-q^{j-1}e(w\alpha))
\end{aligned}
\tag{5.4}
$$

Clearly each term in the last sum has the same constant term c_k. Let $\mathfrak{g}_{\mathbb{C}}$ be a semisimple complex Lie algebra with root system R, $\mathfrak{g}$ a compact real form of $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{t}$ a CSA of $\mathfrak{g}$, G a connected compact Lie group with Lie algebra $\mathfrak{g}$, T the maximal torus corresponding to $\mathfrak{t}$ and dg and dt the normalized Haar measures of G and T. Since the adjoint representation of T on $\mathfrak{g}_\alpha$ is $(\exp H)\cdot X_\alpha = e^{\alpha(H)}X_\alpha$, $e(\alpha)$ can regarded as a character of T. Now we can get the constant term of the polynomials 5.4 by integrating over T:

$$
\begin{aligned}
&\prod_{i=1}^{l}\frac{1-q^{kd_i}}{1-q^k}\int_T\prod_{\alpha\in R}\prod_{i=1}^{k}(1-q^{j-1}e(\alpha)(t))dt &&= \\
&\prod_{i=1}^{l}\frac{1-q^{kd_i}}{1-q^k}\int_T\prod_{j=1}^{k}\det\left(1-q^{j-1}\mathrm{Ad}(t)\right)_{\mathfrak{g}/\mathfrak{t}}dt &&= \\
&\prod_{i=1}^{l}\frac{1-q^{kd_i}}{1-q^k}\int_T\det\left(1-\mathrm{Ad}(t)\right)_{\mathfrak{g}/\mathfrak{t}}\prod_{j=1}^{k-1}\frac{\det\left(1-q^{j-1}\mathrm{Ad}(t)\right)_{\mathfrak{g}}}{(1-q^j)^l}dt
\end{aligned}
\tag{5.5}
$$

By Weyl's integration formula (equation 3.2) and proposition 5.1 this equals

$$
\begin{aligned}
&|W|\prod_{i=1}^{l}\frac{1-q^{kd_i}}{1-q^k}\int_G\prod_{j=1}^{k-1}\frac{\det\left(1-q^{j-1}\mathrm{Ad}(g)\right)_{\mathfrak{g}}}{(1-q^j)^l}dg &&= \\
&|W|\prod_{i=1}^{l}\frac{1-q^{kd_i}}{1-q^k}\prod_{j=1}^{k-1}\frac{1-q^{km_i+j}}{(1-q^j)^l} &&= \\
&|W|\prod_{i=1}^{l}(1-q^j)^{-l}\prod_{j=0}^{k-1}(1-q^{kd_i-j}) &&= |W|\prod_{i=1}^{l}\begin{bmatrix} kd_i \\ k\end{bmatrix}_q
\end{aligned}
\tag{5.6}
$$

We conclude that $|W|c_k = |W|\prod_{i=1}^{l}\begin{bmatrix} kd_i \\ k\end{bmatrix}_q$. This proof also works for $k=\infty$ and $|q|<1/2$. For then the integration is performed by considering the function as a formal power series in q, and by a computation similar to that in the proof of proposition 5.1 one shows that $\lim_{k\to\infty} \prod_{i=1}^{l} \left[\begin{array}{c} kd_i \\ k \end{array} \right]_q = \prod_{j=1}^{\infty} (1-q^j)^{-l}$. $\square$

Notice that it would be sufficient to prove this theorem for simple $\mathfrak{g}$, or equivalently for irreducible R. For if $R = R_1 \cup R_2$ with $R_1 \perp R_2$, then $P(R,k) = P(R_1,k)P(R_2,k)$ and the same for their constant terms (since $\alpha \in R_1, \beta \in R_2 \Longrightarrow \alpha + \beta \neq 0$).

An important specialization arises when we take the limit $q \to 1$.

Corollary 5.3 *Assume that conjecture 4.7 holds. Let R be a root system and $k \in \mathbb{Z}_{>0}$. Then the constant term of $\prod_{\alpha\in R}(1-e(\alpha))^k$ is $\prod_{i=1}^{l} \binom{kd_i}{k}$.*

Contrarily the theorem 5.2, it makes no sense to take the limit $k \to \infty$.

Example. Let us see what the above polynomials and constant terms look like in our example $\mathfrak{g} = \mathfrak{su}(n)$ from chapter 1. Recall that

$$R = \{\lambda_i - \lambda_j : 1 \leq i,j \leq n, i \neq j\},$$

that $\lambda_i - \lambda_j$ is positive if and only if $i > j$ and that the exponents are $1, 2, \dots, n-1$. Writing x_i for $e(\lambda_i), e(\lambda_i - \lambda_j)$ becomes $x_i x_j^{-1}$. Then theorem 5.2 states that

$$CT\left(\prod_{1\leq j<i\leq n} \prod_{m=1}^{k} (1-q^m x_i x_j^{-1})(1-q^{m-1} x_j x_i^{-1}) \right) = \prod_{m=2}^{n} \left[\begin{array}{c} km \\ k \end{array} \right]_q$$

If we take the limit $q \to 1$, we see that

$$CT\left(\prod_{i\neq j} (1-x_i x_j^{-1})^k \right) = \prod_{m=2}^{n} \binom{km}{k} = \prod_{m=1}^{n} \frac{(km)!}{k!((m-1)k)!} = \frac{(nk)!}{(k!)^n}. \qquad (5.7)$$

This last polynomial is much simpler than the ones we encountered so far, so it is not surprising that already in 1962 Dyson conjectured equation 5.7. This was proved not long afterwards, without the use of root systems.

5.2 Affine Lie algebras

We give a rather intuitive description of Kac-Moody algebras and affine root systems. For a complete treatment with proofs see Kac [16]. All Lie algebras in this section will be complex.

If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra with root system R and basis $\Delta = \{\alpha_i : 1 \leq i \leq l\}$, $A = (a_{ij})_{i,j=1}^{n} = \left(\frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} \right)_{i,j=1}^{n}$ is its Cartan matrix. It is determined by $\mathfrak{g}$ up to a renumbering of the indices and satisfies:

1. $a_{ii} = 2 \forall i$

2. $a_{ij} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$

3. $a_{ij} = 0 \Longleftrightarrow a_{ji} = 0$

It was shown by Serre that $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{g}(A)$ with generators e_i, f_i, h_i $(i = 1, \ldots, l)$ and relations

$$\begin{array}{rclrclrcl} [h_i, h_j] & = & 0 & [h_i, e_j] & = & a_{ij} e_j & (\operatorname{ad} e_i)^{1-a_{ij}} e_j & = & 0 \\ [e_i, f_i] & = & \delta_{i,j} h_i & [h_i, f_j] & = & -a_{ij} f_j & (\operatorname{ad} f_i)^{1-a_{ij}} f_j & = & 0 \end{array} \tag{5.8}$$

Now a matrix A that satisfies conditions 1, 2 and 3 is called a (generalized) Cartan matrix, and decomposition of A is a partition of $\{1, \ldots, l\}$ into two nonempty subsets I and J such that $a_{ij} = 0$ if $i \in I,\ j \in J$. With A one associates a Lie algebra $\mathfrak{g}(A)$ in the same way as above, and this is by definition the Kac-Moody algebra associated to A. For indecomposable A there are three types:

finite A is positive definite.

affine A is positive semidefinite of corank 1.

indefinite there exists a vector v with positive coordinates such that Av has negative coordinates.

In the finite case $\mathfrak{g}(A)$ is a finite-dimensional simple Lie algebra, in the other two cases $\mathfrak{g}(A)$ has infinite dimension. The Cartan matrices of finite and affine type are completely classified, but little is known about those of indefinite type. A large part of the theory of finite-dimensional semisimple Lie algebras (see section 1.3) can be generalized to Kac-Moody algebras, but of course there are many complications. Thus the subspace $\mathfrak{h}$ of $\mathfrak{g}(A)$ spanned by the elements h_i is a CSA and there is a set of roots $R \subset \mathfrak{h}^*$. (This is not a root system in the sense of section 1.3.) The roots α_i corresponding to the elements e_i are by definition simple and they form a basis Δ of R. The Weyl group W is the subgroup of $\operatorname{End} \mathfrak{h}^*$ generated by the reflections induced by the α_i. In the finite case R and W are finite and every root is the image of a simple root under an element of the Weyl group, but this is not true in the affine and indefinite cases. Therefore one defines the set of real roots $R_{\mathrm{re}} := W \cdot \Delta$, and the set imaginary roots $R_{\mathrm{im}} := R \setminus R_{\mathrm{re}}$. If α is a real root then $\dim \mathfrak{g}_\alpha = 1$ and $-\alpha$ is the only other multiple of α in R. (So R_{re} has most of the properties of a finite root system.)

It turns out that if $\mathfrak{g}$ is finite-dimensional semisimple Lie algebra, its loop algebra $\mathfrak{g}[z, z^{-1}] = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ is almost a direct sum of affine Lie algebras. To make this precise we assume that $\mathfrak{g}$ is simple. Define a bilinear form on $\mathfrak{g}[z, z^{-1}]$ by

$$\langle X \otimes P, Y \otimes Q \rangle = \kappa(X, Y) \operatorname{res} \left(\frac{dP}{dz} \frac{Q}{2\pi i} \right)$$

where κ is the Killing form of $\mathfrak{g}$ and $\operatorname{res} f$ is the residue at 0 of a holomorphic function f. Let $\mathfrak{g}[z, z^{-1}] \oplus \mathbb{C}c$ be the central extension of $\mathfrak{g}[z, z^{-1}]$ by c, with Lie bracket

$$[x + \lambda c, y + \mu c] = [x, y] + \langle x, y \rangle \, c$$

Now we write d for the derivation $z\frac{d}{dz}$ of $\mathbb{C}[z, z^{-1}]$ and define the Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g}[z, z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

$$\begin{aligned}&[X \otimes z^n + \lambda c + \mu d, Y \otimes z^m + \lambda' c + \mu' d] = \\ &[X, Y] \otimes z^{n+m} + \langle X \otimes z^n, Y \otimes z^m \rangle \, c + \mu Y \otimes m z^m - \mu' X \otimes n z^n\end{aligned}$$

This $\hat{\mathfrak{g}}$ is the affine Lie algebra corresponding to a matrix that contains the Cartan matrix of $\mathfrak{g}$ as a principal minor. With the usual notations for objects associated to $\mathfrak{g}$, $\hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ is a CSA of $\hat{\mathfrak{g}}$. Let $\delta \in \hat{\mathfrak{h}}^*$ be the linear function that is 0 on $\mathfrak{h}$ and at c, and 1 at d. The affine root system and root spaces of $\hat{\mathfrak{g}}$ are

$$\hat{R} = \{n\delta + \alpha : \alpha \in R, n \in \mathbb{Z}\} \cup \{n\delta : n \in \mathbb{Z} \setminus 0\}$$

$$\hat{\mathfrak{g}}_0 = \hat{\mathfrak{h}}, \; \hat{\mathfrak{g}}_{n\delta} = \mathfrak{h} \otimes z^n, \; \hat{\mathfrak{g}}_{n\delta+\alpha} = \mathfrak{g}_\alpha \otimes z^n$$

Moreover if $\theta \in R^+$ is the unique root of maximal height, $\alpha_0 = \delta - \theta$ is a simple root and $\hat{\Delta} = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ is a basis of $\hat{R}$, so that

$$\hat{R}^+ = R^+ \cup \{n\delta + \alpha : \alpha \in R^+ \cup 0, n \in \mathbb{Z}_{>0}\}$$

The Weyl group $\hat{W}$ is generated by W and the reflection

$$\sigma_0 : v \to v + (\langle v, \theta^\vee \rangle - \lambda(2c|\theta|^{-2}))\alpha_0$$

Therefore the roots of the form $n\delta + \alpha$ are real, those of the form $n\delta$ are imaginary and $\hat{W}$ fixes all imaginary roots.

Remark. The name affine is explained as follows. Regard the root lattice $\mathbb{Z}R \subset \mathfrak{h}^*$ as a group T of translations of $\mathfrak{h}^*$. Then $\hat{W}$ is isomorphic to $W \ltimes T$, which is a group of affine transformations of $\mathfrak{h}*$.

Now we are ready to consider theorem 5.2 is this setting. Put $q = e(-\delta)$, so that

$$P(R, k) = \prod_{\alpha \in R^+} \prod_{i=1}^{k} (1 - e(-i\delta + \alpha))(1 - e((i-1)\delta - \alpha)) = \prod_{\alpha \in \hat{R}^+_{\mathrm{re}} : \alpha < k\delta} (1 - e(-\alpha))$$

It is natural to add the imaginary roots $\delta, 2\delta, \ldots, k\delta$ to this product, and because $\dim \hat{\mathfrak{g}}_{n\delta} = l$, we do this with multiplicity l. This gives us

$$\prod_{0 < \alpha \leq k\delta} (1 - e(-\alpha))^{\dim \hat{\mathfrak{g}}_\alpha} := P(\hat{R}, k)$$

According to theorem 5.2 the constant (or rather imaginary) term of $P(\hat{R}, k)$ is

$$\prod_{i=1}^{l} \left[\begin{array}{c} kd_i \\ k \end{array} \right]_q \prod_{j=1}^{k} (1 - q^j)^l = \prod_{i=1}^{l} \prod_{j=0}^{k-1} (1 - q^{kd_i - j}) = \prod_{i=1}^{l} \prod_{j=0}^{k-1} (1 - e((j - kd_i)\delta))$$

The case $k = \infty$ is particularly interesting, as it says that the constant term of $\prod_{\alpha \in \hat{R}^+} (1 - e(-\alpha))$ is simply 1. In other words, there are no terms corresponding to imaginary roots and coefficient at $e(0)$ is 1. However, this was already known before Macdonald stated his conjectures. In fact it is a special case of a much deeper result, which we recall now.

A matrix A is symmetrizable if there exists a symmetric matrix B and an invertible diagonal matrix D such that $A = DB$. It can be shown that all Cartan matrices of finite or affine type are symmetrizable. Let $\mathfrak{g}(A)$ be the Kac-Moody algebra associated to A and use the notation from the start of this section. Take $\rho \in \mathfrak{h}^*$ with $\rho(h_i) = 1 \ \forall i$. These conditions determine ρ completely if $\det A \neq 0$, and $2\rho = \sum_{\alpha \in R^+} \alpha$ if A is of finite type. It is not difficult to see that for any $w \in W$, $\rho - w\rho$ is the sum $s(w)$ of all $\alpha \in R^+$ such that $w^{-1} \in R^-(= -R^+)$ (cf. proposition 2.5 of [11]). Finally let $\epsilon(w)$ denote the determinant of $w \in \operatorname{End} \mathfrak{h}$.

Theorem 5.4 *Let A be a symmetrizable Cartan matrix. With the above notation,*

$$\prod_{\alpha \in R^+} (1 - e(-\alpha))^{\dim \mathfrak{g}(A)_\alpha} = \sum_{w \in W} \epsilon(w) e(w\rho - \rho) = \sum_{w \in W} \epsilon(w) e(-s(w))$$

This polynomial has no terms $e(\alpha)$ with α imaginary, and the constant term is 1.

Proof. The equality is 10.4.4 in Kac [16]. Clearly $\mathrm{id}_{\mathfrak{h}}$ is the only element of W with $s(w) = 0$, so the constant term is 1. Otherwise $s(w) > 0$ and $w^{-1}s(w) < 0$. But by proposition 5.2.a of [16], W permutes the positive imaginary roots among themselves, so $s(w)$ and $-s(w)$ cannot be imaginary. $\square$

Remark. For A of finite type this famous theorem is due to Weyl and for affine A it was first discovered (in a modified form) by Macdonald. The general case was established by Kac. Furthermore Garland and Lepowsky [11] reached the same result with the help of Lie algebra homology and Looijenga [20] gave a nice proof using theta-functions.

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