

# ON THE SPECTRA OF FINITE TYPE ALGEBRAS

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ABSTRACT. We review Morita equivalence for finite type  $k$ -algebras  $A$  and also a weakening of Morita equivalence which we call *stratified equivalence*. The spectrum of  $A$  is the set of equivalence classes of irreducible  $A$ -modules. For any finite type  $k$ -algebra  $A$ , the spectrum of  $A$  is in bijection with the set of primitive ideals of  $A$ . The stratified equivalence relation preserves the spectrum of  $A$  and also preserves the periodic cyclic homology of  $A$ . However, the stratified equivalence relation permits a tearing apart of strata in the primitive ideal space which is not allowed by Morita equivalence. A key example illustrating the distinction between Morita equivalence and stratified equivalence is provided by affine Hecke algebras associated to affine Weyl groups. Stratified equivalences lie at the heart of the ABPS conjecture, and lead to an explicit description of geometric structure in the smooth dual of a connected split reductive  $p$ -adic group.

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## 1. INTRODUCTION

Let  $X$  be a complex affine variety and  $k$  its coordinate algebra. Equivalently,  $k$  is a unital algebra over the complex numbers which is commutative, finitely generated, and nilpotent-free. A  $k$ -algebra is an algebra  $A$  over the complex numbers  $\mathbb{C}$  which is a  $k$ -module (with an evident compatibility between the algebra structure of  $A$  and the  $k$ -module structure of  $A$ ).  $A$  is not required to have a unit.  $A$  is not required to be commutative. A  $k$ -algebra  $A$  is of finite type if as a  $k$ -module  $A$  is finitely generated. This paper will review Morita equivalence for  $k$ -algebras and will then review — for finite type  $k$ -algebras — a weakening of Morita equivalence called *stratified equivalence*.

The spectrum of  $A$  is, by definition, the set of equivalence classes of irreducible  $A$ -modules. For any finite type  $k$ -algebra  $A$ , the spectrum of  $A$  is in bijection with the set of primitive ideals of  $A$ . The stratified equivalence relation preserves the spectrum of  $A$  and also preserves the periodic cyclic homology of  $A$ . However, the stratified equivalence relation permits a tearing apart of strata in the primitive ideal space which is not allowed by Morita equivalence.

The set of primitive ideals of  $A$  will be denoted  $\text{Prim}(A)$ . When  $A$  is unital and commutative, let  $\text{maxSpec}(A)$  denote the set of maximal ideals in  $A$ , and let  $\text{Spec}(A)$  denote the set of prime ideals in  $A$ . In particular, let  $A = \mathcal{O}(X)$ . Then we have

$$\text{Prim}(A) \simeq \text{maxSpec}(A) \subset \text{Spec}(A).$$

The inclusion is strict, for the prime spectrum  $\text{Spec}(A)$  contains a generic point, namely the zero ideal  $\mathbf{0}$  in  $A$ . The set  $\text{Prim}(A)$  consists of the  $\mathbb{C}$ -rational points of the affine variety  $X$ .

So  $\text{Prim}(A)$  is a generalisation, for  $k$ -algebras  $A$  of finite type, of the maximal ideal spectrum of a unital commutative ring, rather than the prime spectrum of a unital commutative ring. For a finite type  $k$ -algebra  $A$ , the zero ideal  $\mathbf{0}$  is a primitive ideal if and only if  $k = \mathbb{C}$  and  $A = M_n(\mathbb{C})$  the algebra of all  $n$  by  $n$  matrices with entries in  $\mathbb{C}$ . Quite generally, any irreducible representation of a finite type  $k$ -algebra  $A$  is a surjection of  $A$  onto  $M_n(\mathbb{C})$  for some  $n$ . This implies that any primitive ideal is maximal.

A key example illustrating the distinction between Morita equivalence and stratified equivalence is provided by affine Hecke algebras associated to affine Weyl groups. Let  $A$  be the group algebra, with coefficients  $\mathbb{C}$ , of an affine Weyl group. For each non-zero complex number  $\zeta$  there is the affine Hecke algebra (with equal parameters)  $\mathcal{H}_\zeta$ . Here  $\mathcal{H}_1 = A$  and  $\mathcal{H}_\zeta \simeq \mathcal{H}_{1/\zeta}$ . Except for  $\zeta$  in a finite set of roots of unity, none of which is 1, the algebras  $\mathcal{H}_\zeta$  are stratified equivalent. In §10, we give examples of affine Hecke algebras  $\mathcal{H}_\zeta$  which are stratified equivalent, but not Morita equivalent, to  $\mathcal{H}_1$ .

Stratified equivalences lie at the heart of the ABPS (Aubert-Baum-Plymen-Solleveld) conjecture, and lead to an explicit description of geometric structure in the smooth dual of a connected split reductive  $p$ -adic group.

The context of this article can be viewed as noncommutative algebraic geometry. In the case of split reductive  $p$ -adic groups, our geometric description of Bernstein components uses the classical language of schemes. In some examples of Bernstein components for non-split reductive  $p$ -adic groups, the classical language of schemes is not sufficient. See the example of  $\mathrm{SL}_5(D)$  in §12.2.

The main result of this paper is

**Theorem 1.1.** *Consider the affine Hecke algebra  $\mathcal{H}_q := \mathcal{H}_q(\mathrm{SL}_3(\mathbb{C}))$ . For  $|q| \neq 1$ ,  $\mathcal{H}_q$  is not Morita equivalent to  $\mathcal{H}_1$ , but is stratified equivalent to  $\mathcal{H}_1$ .*

Here,  $\mathcal{H}_1$  is the group algebra (with coefficients  $\mathbb{C}$ ) of the affine Weyl group of  $\mathrm{SL}_3(\mathbb{C})$  — and for any nonzero complex number  $q$ ,  $\mathcal{H}_q$  is the associated affine Hecke algebra (with equal parameters) determined by  $q$ .

## 2. $k$ -ALGEBRAS

If  $X$  is an affine algebraic variety over the complex numbers  $\mathbb{C}$ , then  $\mathcal{O}(X)$  will denote the coordinate algebra of  $X$ . Set  $k = \mathcal{O}(X)$ . Equivalently,  $k$  is a unital algebra over the complex numbers which is unital, commutative, finitely generated, and nilpotent-free. The Hilbert Nullstellensatz implies that there is an equivalence of categories

$$\left( \begin{array}{l} \text{unital commutative} \\ \text{finitely generated} \\ \text{nilpotent-free } \mathbb{C}\text{-algebras} \end{array} \right) \sim \left( \begin{array}{l} \text{affine complex al-} \\ \text{gebraic varieties} \end{array} \right)^{\mathrm{op}}$$

$$\mathcal{O}(X) \mapsto X$$

Here “op” denotes the opposite category.

**Definition 2.1.** A  $k$ -algebra is a  $\mathbb{C}$ -algebra  $A$  such that  $A$  is a unital (left)  $k$ -module with:

$$\lambda(\omega a) = \omega(\lambda a) = (\lambda \omega) a \quad \forall (\lambda, \omega, a) \in \mathbb{C} \times k \times A$$

and

$$\omega(a_1 a_2) = (\omega a_1) a_2 = a_1 (\omega a_2) \quad \forall (\omega, a_1, a_2) \in k \times A \times A.$$

*Remark 2.2.*  $A$  is not required to have a unit.

The centre of  $A$  will be denoted  $Z(A)$ :

$$Z(A) := \{c \in A \mid ca = ac \ \forall a \in A\}.$$

*Remark 2.3.* Let  $A$  be a unital  $k$ -algebra. Denote the unit of  $A$  by  $1_A$ . The map  $\omega \mapsto \omega \cdot 1_A$  is then a unital morphism of  $\mathbb{C}$ -algebras

$$k \longrightarrow Z(A)$$

i.e. unital  $k$ -algebra = unital  $\mathbb{C}$ -algebra  $A$  with a given unital morphism of  $\mathbb{C}$ -algebras

$$k \longrightarrow Z(A)$$

**Definition 2.4.** Let  $A, B$  be two  $k$ -algebras. A *morphism of  $k$ -algebras* is a morphism of  $\mathbb{C}$ -algebras

$$f: A \rightarrow B$$

which is also a morphism of (left)  $k$ -modules,

$$f(\omega a) = \omega f(a) \quad \forall (\omega, a) \in k \times A.$$

**Definition 2.5.** Let  $A$  be a  $k$ -algebra. A *representation* of  $A$  [or a (left)  $A$ -module] is a  $\mathbb{C}$ -vector space  $V$  with given morphisms of  $\mathbb{C}$ -algebras

$$\begin{aligned} A &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \\ k &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \end{aligned}$$

such that

- (1)  $k \longrightarrow \text{Hom}_{\mathbb{C}}(V, V)$  is unital.
- (2)  $(\omega a)v = \omega(av) = a(\omega v) \quad \forall (\omega, a, v) \in k \times A \times V.$

From now on in this article,  $A$  will denote a  $k$ -algebra.

A representation of  $A$

$$\begin{aligned} A &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \\ k &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \end{aligned}$$

will often be denoted

$$A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

it being understood that the action of  $k$  on  $V$

$$k \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

is part of the given structure.

**Definition 2.6.** A representation  $\varphi: A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$  is *non-degenerate* if  $AV = V$ , i.e. for any  $v \in V$ , there are  $v_1, v_2, \dots, v_r$  in  $V$  and  $a_1, a_2, \dots, a_r$  in  $A$  with

$$v = a_1 v_1 + a_2 v_2 + \dots + a_r v_r.$$

**Definition 2.7.** A representation  $\varphi: A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$  is *irreducible* if  $AV \neq \{0\}$  and there is no sub- $\mathbb{C}$ -vector space  $W$  of  $V$  with:

$$\{0\} \neq W, \quad W \neq V$$

and

$$\omega w \in W \quad \forall (\omega, w) \in k \times W$$

and

$$aw \in W \quad \forall (a, w) \in A \times W$$

**Definition 2.8.** Two representations of the  $k$ -algebra  $A$

$$\varphi_1: A \rightarrow \text{Hom}_{\mathbb{C}}(V_1, V_1)$$

$$\varphi_2: A \rightarrow \text{Hom}_{\mathbb{C}}(V_2, V_2)$$

are *equivalent* if there is an isomorphism of  $\mathbb{C}$ -vector spaces

$$T: V_1 \rightarrow V_2$$

with

$$T(av) = aT(v) \quad \forall (a, v) \in A \times V$$

and

$$T(\omega v) = \omega T(v) \quad \forall (\omega, v) \in k \times V$$

The *spectrum* of  $A$ , also denoted  $\text{Irr}(A)$ , is the set of equivalence classes of irreducible representations of  $A$ .

$$\text{Irr}(A) := \{\text{Irreducible representations of } A\} / \sim .$$

It can happen that the spectrum is empty. For example, let  $A$  comprise all 2 by 2 matrices of the form

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

with  $x \in \mathbb{C}$ . Then  $A$  is a commutative non-unital algebra of finite type (with  $k = \mathbb{C}$ ) such that  $\text{Irr}(A) = \emptyset$ .

### 3. ON THE ACTION OF $k$

For a  $k$ -algebra  $A$ ,  $A_{\mathbb{C}}$  denotes the underlying  $\mathbb{C}$ -algebra of  $A$ . Then  $A_{\mathbb{C}}$  is obtained from  $A$  by forgetting the action of  $k$  on  $A$ . For  $A_{\mathbb{C}}$  there are the usual definitions : A *representation* of  $A_{\mathbb{C}}$  [or a (left)  $A_{\mathbb{C}}$ -*module*] is a  $\mathbb{C}$ -vector space  $V$  with a given morphism of  $\mathbb{C}$ -algebras

$$A_{\mathbb{C}} \longrightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

An  $A_{\mathbb{C}}$ -module  $V$  is *irreducible* if  $A_{\mathbb{C}}V \neq \{0\}$  and there is no sub- $\mathbb{C}$ -vector space  $W$  of  $V$  with:

$$\{0\} \neq W, \quad W \neq V$$

and

$$aw \in W \quad \forall (a, w) \in A_{\mathbb{C}} \times W$$

Two representations of  $A_{\mathbb{C}}$

$$\varphi_1: A \rightarrow \text{Hom}_{\mathbb{C}}(V_1, V_1)$$

$$\varphi_2: A \rightarrow \text{Hom}_{\mathbb{C}}(V_2, V_2)$$

are *equivalent* if there is an isomorphism of  $\mathbb{C}$ -vector spaces

$$T: V_1 \rightarrow V_2$$

with

$$T(av) = aT(v) \quad \forall (a, v) \in A \times V$$

$\text{Irr}(A_{\mathbb{C}}) := \{\text{Irreducible representations of } A_{\mathbb{C}}\} / \sim$ .

An  $A_{\mathbb{C}}$ -module  $V$  for which the following two properties are valid is *strictly non-degenerate*

- $A_{\mathbb{C}}V = V$
- If  $v \in V$  has  $av = 0 \ \forall a \in A_{\mathbb{C}}$ , then  $v = 0$ .

**Lemma 3.1.** *Any irreducible  $A_{\mathbb{C}}$ -module is strictly non-degenerate.*

*Proof.* Let  $V$  be an irreducible  $A_{\mathbb{C}}$ -module. First, consider  $A_{\mathbb{C}}V \subset V$ .  $A_{\mathbb{C}}V$  is preserved by the action of  $A_{\mathbb{C}}$  on  $V$ . Cannot have  $A_{\mathbb{C}}V = \{0\}$  since this would contradict the irreducibility of  $V$ . Therefore  $A_{\mathbb{C}}V = V$ .

Next, set

$$W = \{v \in V \mid av = 0 \ \forall a \in A_{\mathbb{C}}\}$$

$W$  is preserved by the action of  $A_{\mathbb{C}}$  on  $V$ .  $W$  cannot be equal to  $V$  since this would imply  $A_{\mathbb{C}}V = \{0\}$ . Hence  $W = \{0\}$ . □

**Lemma 3.2.** *Let  $A$  be a  $k$ -algebra, and let  $V$  be a strictly non-degenerate  $A_{\mathbb{C}}$ -module. Then there is a unique unital morphism of  $\mathbb{C}$  algebras*

$$k \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

*which makes  $V$  an  $A$ -module.*

*Proof.* Given  $v \in V$ , choose  $v_1, v_2, \dots, v_r \in V$  and  $a_1, a_2, \dots, a_r \in A$  with

$$v = a_1v_1 + a_2v_2 + \dots + a_rv_r$$

For  $\omega \in k$ , define  $\omega v$  by :

$$\omega v = (\omega a_1)v_1 + (\omega a_2)v_2 + \dots + (\omega a_r)v_r$$

The second condition in the definition of strictly non-degenerate implies that  $\omega v$  is well-defined. □

Lemma 3.2 will be referred to as the “ $k$ -action for free lemma”.

If  $V$  is an  $A$ -module,  $V_{\mathbb{C}}$  will denote the underlying  $A_{\mathbb{C}}$ -module.  $V_{\mathbb{C}}$  is obtained from  $V$  by forgetting the action of  $k$  on  $V$ .

**Lemma 3.3.** *If  $V$  is any irreducible  $A$ -module, then  $V_{\mathbb{C}}$  is an irreducible  $A_{\mathbb{C}}$ -module.*

*Proof.* Suppose that  $V_{\mathbb{C}}$  is not an irreducible  $A_{\mathbb{C}}$ -module. Then there is a sub- $\mathbb{C}$ -vector space  $W$  of  $V$  with:

$$0 \neq W, \quad W \neq V$$

and

$$aw \in W \quad \forall (a, w) \in A \times W$$

Consider  $AW \subset W$ .  $AW$  is preserved by both the  $A$ -action on  $V$  and the  $k$ -action on  $V$ . Thus if  $AW \neq \{0\}$ , then  $V$  is not an irreducible  $A$ -module. Hence  $AW = \{0\}$ . Consider  $kW \supset W$ .  $kW$  is preserved by the  $k$ -action on  $V$  and is also preserved by the  $A$ -action on  $V$  because  $A$  annihilates

$kW$ . Since  $A$  annihilates  $kW$ , cannot have  $kW = V$ . Therefore  $\{0\} \neq kW$ ,  $kW \neq V$ , which contradicts the irreducibility of the  $A$ -module  $V$ .  $\square$

A corollary of Lemma 3.2 is :

**Corollary 3.4.** *For any  $k$ -algebra  $A$ , the map*

$$\begin{aligned} \text{Irr}(A) &\rightarrow \text{Irr}(A_{\mathbb{C}}) \\ V &\mapsto V_{\mathbb{C}} \end{aligned}$$

*is a bijection.*

*Proof.* Surjectivity follows from Lemmas 3.1 and 3.2. For injectivity, let  $V, W$  be two irreducible  $A$ -modules such that  $V_{\mathbb{C}}$  and  $W_{\mathbb{C}}$  are equivalent  $A_{\mathbb{C}}$ -modules. Let  $T: V \rightarrow W$  be an isomorphism of  $\mathbb{C}$  vector spaces with

$$T(av) = aT(v) \quad \forall (a, v) \in A \times V$$

Given  $v \in V$  and  $\omega \in k$ , choose  $v_1, v_2, \dots, v_r \in V$  and  $a_1, a_2, \dots, a_r \in A$  with

$$v = a_1v_1 + a_2v_2 + \dots + a_rv_r$$

Then

$$\begin{aligned} T(\omega v) &= T((\omega a_1)v_1 + (\omega a_2)v_2 + \dots + (\omega a_r)v_r) \\ &= (\omega a_1)T v_1 + (\omega a_2)T v_2 + \dots + (\omega a_r)T v_r \\ &= \omega(a_1T v_1 + a_2T v_2 + \dots + a_rT v_r) \\ &= \omega(Tv). \end{aligned}$$

Hence  $T: V \rightarrow W$  intertwines the  $k$ -actions on  $V, W$  and thus  $V, W$  are equivalent  $A$ -modules.  $\square$

#### 4. FINITE TYPE $k$ -ALGEBRAS

An ideal  $I$  in a  $k$ -algebra  $A$  is *primitive* if  $I$  is the null-space of an irreducible representation of  $A$ , i.e. there is an irreducible representation of  $A$   $\varphi: A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$  with

$$I = \{a \in A \mid \varphi(a) = 0\}$$

$\text{Prim}(A)$  denotes the set of all primitive ideals in  $A$ . The evident map

$$\text{Irr}(A) \rightarrow \text{Prim}(A)$$

sends an irreducible representation to its null-space. On  $\text{Prim}(A)$  there is the Jacobson topology. If  $S$  is any subset of  $\text{Prim}(A)$ ,  $S \subset \text{Prim}(A)$ , then the closure  $\overline{S}$  of  $S$  is :

$$\overline{S} := \{I \in \text{Prim}(A) \mid I \supset \bigcap_{L \in S} L\}$$

**Definition 4.1.** A  $k$ -algebra  $A$  is of *finite type* if, as a  $k$ -module,  $A$  is finitely generated.

For any finite type  $k$ -algebra  $A$ , the following three statements are valid :

- If  $\varphi: A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$  is any irreducible representation of  $A$ , then  $V$  is a finite dimensional  $\mathbb{C}$  vector space and  $\varphi: A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$  is surjective.
- The evident map  $\text{Irr}(A) \rightarrow \text{Prim}(A)$  is a bijection.

- Any primitive ideal in  $A$  is a maximal ideal.

Since  $\text{Irr}(A) \rightarrow \text{Prim}(A)$  is a bijection, the Jacobson topology on  $\text{Prim}(A)$  can be transferred to  $\text{Irr}(A)$  and thus  $\text{Irr}(A)$  is topologized. Equivalently,  $\text{Irr}(A)$  is topologized by requiring that  $\text{Irr}(A) \rightarrow \text{Prim}(A)$  be a homeomorphism.

For a finite type  $k$ -algebra  $A$  ( $k = \mathcal{O}(X)$ ), the central character is a map

$$\text{Irr}(A) \longrightarrow X$$

defined as follows. Let  $\varphi$

$$\begin{aligned} A &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \\ k &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \end{aligned}$$

be an irreducible representation of  $A$ .  $I_V$  denotes the identity operator of  $V$

$$I_V(v) = v \quad \forall v \in V.$$

For  $\omega \in k = \mathcal{O}(X)$ , define

$$T_\omega: V \rightarrow V$$

by

$$T_\omega(v) = \omega v \quad \forall v \in V.$$

$T_\omega$  is an intertwining operator for  $A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$ . By Lemma 3.3 plus Schur's Lemma  $T_\omega$  is a scalar multiple of  $I_V$ .

$$T_\omega = \lambda_\omega I_V \quad \lambda_\omega \in \mathbb{C}$$

The map

$$\omega \mapsto \lambda_\omega$$

is a unital morphism of  $\mathbb{C}$  algebras  $\mathcal{O}(X) \rightarrow \mathbb{C}$  and thus is given by evaluation at a unique ( $\mathbb{C}$  rational) point  $p_\varphi$  of  $X$ .

$$\lambda_\omega = \omega(p_\varphi) \quad \forall \omega \in \mathcal{O}(X)$$

The central character  $\text{Irr}(A) \rightarrow X$  is

$$\varphi \mapsto p_\varphi$$

**Remark.** Corollary 3.4 states that  $\text{Irr}(A)$  depends only on the underlying  $\mathbb{C}$  algebra  $A_{\mathbb{C}}$ . The central character  $\text{Irr}(A) \rightarrow X$ , however, does depend on the structure of  $A$  as a  $k$ -module. A change in the action of  $k$  on  $A_{\mathbb{C}}$  will change the central character.

The central character  $\text{Irr}(A) \rightarrow X$  is continuous where  $\text{Irr}(A)$  is topologized as above and  $X$  has the Zariski topology. For a proof of this assertion see [KNS, Lemma 1, p.326]. From a somewhat heuristic non-commutative geometry point of view,  $A_{\mathbb{C}}$  is a non-commutative complex affine variety, and a given action of  $k$  on  $A_{\mathbb{C}}$ , making  $A_{\mathbb{C}}$  into a finite type  $k$ -algebra  $A$ , determines a morphism of algebraic varieties  $A_{\mathbb{C}} \rightarrow X$ .

## 5. MORITA EQUIVALENCE FOR $k$ -ALGEBRAS

**Definition 5.1.** Let  $B$  be a  $k$ -algebra. A *right  $B$ -module* is a  $\mathbb{C}$ -vector space  $V$  with given morphisms of  $\mathbb{C}$ -algebras

$$\begin{aligned} B^{\text{op}} &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \\ k &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \end{aligned}$$



such that:

- (1)  $k \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$  is unital
- (2)  $v(\omega b) = (v\omega)b = (vb)\omega \quad \forall (v, \omega, b) \in V \times k \times B$ .

$B^{\text{op}}$  is the opposite algebra of  $B$ .  $V$  is non-degenerate if  $VB = V$ .

Remark. “Right  $B$ -module” = “Left  $B^{\text{op}}$ -module.”

With  $k$  fixed, let  $A, B$  be two  $k$ -algebras. An  $A$ – $B$  bimodule, denoted  ${}_A V_B$ , is a  $\mathbb{C}$  vector space  $V$  such that :

- (1)  $V$  is a left  $A$ -module.
- (2)  $V$  is a right  $B$ -module.
- (3)  $a(vb) = (av)b \quad \forall (a, v, b) \in A \times V \times B$ .
- (4)  $\omega v = v\omega \quad \forall (\omega, v) \in k \times V$ .

An  $A$ – $B$  bimodule  ${}_A V_B$  is non-degenerate if  $AV = V = VB$ .  $I_V$  is the identity map of  $V$ .  $A$  is an  $A$ – $A$  bimodule in the evident way.

**Definition 5.2.** A  $k$ -algebra  $A$  has *local units* if given any finite set  $\{a_1, a_2, \dots, a_r\}$  of elements of  $A$ , there is an idempotent  $Q \in A$  ( $Q^2 = Q$ ) with

$$Qa_j = a_jQ = a_j \quad j = 1, 2, \dots, r.$$

**Definition 5.3.** Let  $A, B$  be two  $k$ -algebras with local units. A *Morita equivalence* (between  $A$  and  $B$ ) is given by a pair of non-degenerate bimodules

$${}_A V_B \quad {}_B W_A$$

together with isomorphisms of bimodules

$$\alpha: V \otimes_B W \rightarrow A$$

$$\beta: W \otimes_A V \rightarrow B$$

such that there is commutativity in the diagrams:

$$\begin{array}{ccc} V \otimes_B W \otimes_A V & \xrightarrow{I_V \otimes \beta} & V \otimes_B B \\ \downarrow \alpha \otimes I_V & & \downarrow \cong \\ A \otimes_A V & \xrightarrow{\cong} & V \end{array}$$

$$\begin{array}{ccc} W \otimes_A V \otimes_B W & \xrightarrow{I_W \otimes \alpha} & W \otimes_A A \\ \downarrow \beta \otimes I_W & & \downarrow \cong \\ B \otimes_B V & \xrightarrow{\cong} & W \end{array}$$

The linking algebra. Let  $A, B$  two  $k$ -algebras with local units, and suppose given a Morita equivalence

$${}_A V_B \quad {}_B W_A \quad \alpha: V \otimes_B W \rightarrow A \quad \beta: W \otimes_A V \rightarrow B$$

The linking algebra is

$$L({}_A V_B, {}_B W_A) := \begin{pmatrix} A & V \\ W & B \end{pmatrix}$$

i.e.  $L({}_A V_B, {}_B W_A)$  consists of all  $2 \times 2$  matrices having (1, 1) entry in  $A$ , (2, 2) entry in  $B$ , (2, 1) entry in  $W$ , and (1, 2) entry in  $V$ . Addition and multiplication are matrix addition and matrix multiplication. Note that  $\alpha$  and  $\beta$  are used in the matrix multiplication.

$L({}_A V_B, {}_B W_A)$  is a  $k$ -algebra. With  $\omega \in k$ , the action of  $k$  on  $L({}_A V_B, {}_B W_A)$  is given by

$$\omega \begin{pmatrix} a & v \\ w & b \end{pmatrix} = \begin{pmatrix} \omega a & \omega v \\ \omega w & \omega b \end{pmatrix}$$

Remark. A Morita equivalence between  $A$  and  $B$  determines an equivalence of categories between the category of non-degenerate left  $A$ -modules and the category of non-degenerate left  $B$ -modules. Similarly for right modules. Also, a Morita equivalence determines isomorphisms (between  $A$  and  $B$ ) of Hochschild homology, cyclic homology, and periodic cyclic homology.

A Morita equivalence between two finite type  $k$ -algebras  $A, B$  preserves the central character i.e. there is commutativity in the diagram

$$\begin{array}{ccc} \text{Irr}(A) & \longrightarrow & \text{Irr}(B) \\ \downarrow & & \downarrow \\ X & \xrightarrow{I_X} & X \end{array}$$

where the upper horizontal arrow is the bijection determined by the given Morita equivalence, the two vertical arrows are the two central characters, and  $I_X$  is the identity map of  $X$ .

Example. For  $n$  a positive integer, let  $M_n(A)$  be the  $k$ -algebra of all  $n \times n$  matrices with entries in  $A$ . If  $A$  has local units,  $A$  and  $M_n(A)$  are Morita equivalent as follows. For  $m, n$  positive integers, denote by  $M_{m,n}(A)$  the set of all  $m \times n$  (i.e.  $m$  rows and  $n$  columns) matrices with entries in  $A$ . Matrix multiplication then gives a map

$$M_{m,n}(A) \times M_{n,r}(A) \longrightarrow M_{m,r}(A)$$

With this notation,  $M_{n,n}(A) = M_n(A)$  and  $M_{1,1}(A) = M_1(A) = A$ . Hence matrix multiplication gives maps

$$M_{1,n}(A) \times M_n(A) \longrightarrow M_{1,n}(A) \quad M_n(A) \times M_{n,1} \longrightarrow M_{n,1}(A)$$

Thus  $M_{1,n}(A)$  is a right  $M_n(A)$ -module and  $M_{n,1}(A)$  is a left  $M_n(A)$ -module.

Similarly,  $M_{1,n}(A)$  is a left  $A$ -module and  $M_{n,1}(A)$  is a right  $A$ -module.

With  $A = A$  and  $B = M_n(A)$ , the bimodules of the Morita equivalence are

$V = M_{1,n}(A)$  and  $W = M_{n,1}(A)$ .

Note that the required isomorphisms of bimodules

$$\alpha: V \otimes_B W \rightarrow A$$

$$\beta: W \otimes_A V \rightarrow B$$

are obtained by observing that the matrix multiplication maps

$$M_{1,n}(A) \times M_{n,1}(A) \rightarrow A$$

$$M_{n,1}(A) \times M_{1,n}(A) \rightarrow M_n(A)$$

factor through the quotients  $M_{1,n}(A) \otimes_{M_n(A)} M_{n,1}(A)$ ,  $M_{n,1}(A) \otimes_A M_{1,n}(A)$  and so give bimodule isomorphisms

$$\alpha: M_{1,n}(A) \otimes_{M_n(A)} M_{n,1}(A) \rightarrow A$$

$$\beta: M_{n,1}(A) \otimes_A M_{1,n}(A) \rightarrow M_n(A)$$

If  $A$  has local units, then  $\alpha$  and  $\beta$  are isomorphisms. Therefore  $A$  and  $M_n(A)$  are Morita equivalent.

If  $A$  does not have local units, then  $\alpha$  and  $\beta$  can fail to be isomorphisms, and there is no way to prove that  $A$  and  $M_n(A)$  are Morita equivalent. In examples, this already happens with  $n = 1$ , and there is then no way to prove (when  $A$  does not have local units) that  $A$  is Morita equivalent to  $A$ . For more details on this issue see below where the proof is given that in the new equivalence relation  $A$  and  $M_n(A)$  are equivalent even when  $A$  does not have local units.

A finite type  $k$ -algebra  $A$  has local units iff  $A$  is unital.

## 6. SPECTRUM PRESERVING MORPHISMS

Let  $A, B$  two finite type  $k$ -algebras, and let  $f: A \rightarrow B$  be a morphism of  $k$ -algebras.

**Definition 6.1.**  $f$  is *spectrum preserving* if

- (1) Given any primitive ideal  $J \subset B$ , there is a unique primitive ideal  $I \subset A$  with  $I \supset f^{-1}(J)$

and

- (2) The resulting map

$$\text{Prim}(B) \rightarrow \text{Prim}(A)$$

is a bijection.

**Example 6.2.** Let  $A, B$  two unital finite type  $k$ -algebras, and suppose given a Morita equivalence

$${}_A V_B \quad {}_B W_A \quad \alpha: V \otimes_B W \rightarrow A \quad \beta: W \otimes_A V \rightarrow B$$

With the linking algebra  $L({}_A V_B, {}_B W_A)$  as above, the inclusions

$$A \hookrightarrow L({}_A V_B, {}_B W_A) \hookleftarrow B$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \mapsto b$$

are spectrum preserving morphisms of finite type  $k$ -algebras. The bijection  $\text{Prim}(B) \leftrightarrow \text{Prim}(A)$  so obtained is the bijection determined by the given Morita equivalence.

Remark. If  $f: A \rightarrow B$  is a spectrum preserving morphism of finite type  $k$ -algebras, then the resulting bijection

$$\text{Prim}(B) \leftrightarrow \text{Prim}(A)$$

is a homeomorphism. For a proof of this assertion see [BN, Theorem 3, p.342]. Consequently, if  $A, B$  are two unital finite type  $k$ -algebras, and

$${}_A V_B \quad {}_B W_A \quad \alpha: V \otimes_B W \rightarrow A \quad \beta: W \otimes_A V \rightarrow B$$

is a Morita equivalence, then the resulting bijection

$$\text{Prim}(B) \leftrightarrow \text{Prim}(A)$$

is a homeomorphism.

**Definition 6.3.** An ideal  $I$  in a  $k$ -algebra  $A$  is a  $k$ -ideal if  $\omega a \in I \forall (\omega, a) \in k \times I$ .

Remark. Any primitive ideal in a  $k$ -algebra  $A$  is a  $k$ -ideal.

Given  $A, B$  two finite type  $k$ -algebras, let  $f: A \rightarrow B$  be a morphism of  $k$ -algebras.

**Definition 6.4.**  $f$  is spectrum preserving with respect to filtrations if there are  $k$ -ideals

$$0 = I_0 \subset I_1 \subset \dots \subset I_{r-1} \subset I_r = A \quad \text{in } A$$

and  $k$  ideals

$$0 = J_0 \subset J_1 \subset \dots \subset J_{r-1} \subset J_r = B \quad \text{in } B$$

with  $f(I_j) \subset J_j$ , ( $j = 1, 2, \dots, r$ ) and  $I_j/I_{j-1} \rightarrow J_j/J_{j-1}$ , ( $j = 1, 2, \dots, r$ ) is spectrum preserving.

The primitive ideal spaces of the subquotients  $I_j/I_{j-1}$  and  $J_j/J_{j-1}$  are the strata for stratifications of  $\text{Prim}(A)$  and  $\text{Prim}(B)$ . Each stratum of  $\text{Prim}(A)$  is mapped homeomorphically onto the corresponding stratum of  $\text{Prim}(B)$ . However, the map

$$\text{Prim}(A) \rightarrow \text{Prim}(B)$$

might not be a homeomorphism.

## 7. ALGEBRAIC VARIATION OF $k$ -STRUCTURE

Let  $A$  be a unital  $\mathbb{C}$ -algebra, and let

$$\Psi: k \rightarrow Z(A[t, t^{-1}])$$

be a unital morphism of  $\mathbb{C}$ -algebras. Here  $t$  is an indeterminate, so  $A[t, t^{-1}]$  is the algebra of Laurent polynomials with coefficients in  $A$ . As above  $Z$  denotes ‘‘center’’. For  $\zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$ ,  $ev(\zeta)$  denotes the ‘‘evaluation at  $\zeta$ ’’

map:

$$\begin{aligned} \text{ev}(\zeta): A[t, t^{-1}] &\rightarrow A \\ \sum a_j t^j &\mapsto \sum a_j \zeta^j \end{aligned}$$

Consider the composition

$$k \xrightarrow{\Psi} Z(A[t, t^{-1}]) \xrightarrow{\text{ev}(\zeta)} Z(A).$$

Denote the unital  $k$ -algebra so obtained by  $A_\zeta$ .  $\forall \zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$ , the underlying  $\mathbb{C}$ -algebra of  $A_\zeta$  is  $A$ .

$$(A_\zeta)_{\mathbb{C}} = A \quad \forall \zeta \in \mathbb{C}^\times$$

Such a family  $\{A_\zeta\}$ ,  $\zeta \in \mathbb{C}^\times$ , of unital  $k$ -algebras, will be referred to as an *algebraic variation of  $k$ -structure with parameter space  $\mathbb{C}^\times$* .

## 8. STRATIFIED EQUIVALENCE

With  $k$  fixed, consider the collection of all finite type  $k$ -algebras. On this collection, *stratified equivalence* is, by definition, the equivalence relation generated by the two elementary steps :

Elementary Step 1. If there is a morphism of  $k$ -algebras  $f: A \rightarrow B$  which is spectrum preserving with respect to filtrations, then  $A \sim B$ .

Elementary Step 2. If there is  $\{A_\zeta\}$ ,  $\zeta \in \mathbb{C}^\times$ , an algebraic variation of  $k$ -structure with parameter space  $\mathbb{C}^\times$ , such that each  $A_\zeta$  is a unital finite type  $k$ -algebra, then for any  $\zeta, \eta \in \mathbb{C}^\times$ ,  $A_\zeta \sim A_\eta$ .

Thus, two finite type  $k$ -algebras  $A, B$  are equivalent if and only if there is a finite sequence  $A_0, A_1, A_2, \dots, A_r$  of finite type  $k$ -algebras with  $A_0 = A, A_r = B$ , and for each  $j = 0, 1, \dots, r-1$  one of the following three possibilities is valid :

- a morphism of  $k$ -algebras  $A_j \rightarrow A_{j+1}$  is given which is spectrum preserving with respect to filtrations.
- a morphism of  $k$ -algebras  $A_j \leftarrow A_{j+1}$  is given which is spectrum preserving with respect to filtrations.
- $\{A_\zeta\}$ ,  $\zeta \in \mathbb{C}^\times$ , an algebraic variation of  $k$ -structure with parameter space  $\mathbb{C}^\times$ , is given such that each  $A_\zeta$  is a unital finite type  $k$ -algebra, and  $\eta, \tau \in \mathbb{C}^\times$  have been chosen with  $A_j = A_\eta, A_{j+1} = A_\tau$ .

To give a stratified equivalence relating  $A$  and  $B$ , the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spaces and an isomorphism of

periodic cyclic homology [BN] are determined:

$$\text{Prim}(A) \longleftrightarrow \text{Prim}(B) \quad \text{HP}_*(A) \cong \text{HP}_*(B)$$

**Proposition 8.1.** *If two unital finite type  $k$ -algebras  $A, B$  are Morita equivalent (as  $k$ -algebras) then they are stratified equivalent.*

$$A \underset{\text{Morita}}{\sim} B \implies A \sim B$$

*Proof.* Let  $A, B$  two unital finite type  $k$ -algebras, and suppose given a Morita equivalence

$${}_A V_B \quad {}_B W_A \quad \alpha: V \otimes_B W \rightarrow A \quad \beta: W \otimes_A V \rightarrow B$$

The linking algebra is

$$L({}_A V_B, {}_B W_A) := \begin{pmatrix} A & V \\ W & B \end{pmatrix}$$

The inclusions

$$A \hookrightarrow L({}_A V_B, {}_B W_A) \hookleftarrow B$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \mapsto b$$

are spectrum preserving morphisms of finite type  $k$ -algebras. Hence  $A$  and  $B$  are stratified equivalent.  $\square$

According to the above, a Morita equivalence of  $A$  and  $B$  gives a homeomorphism

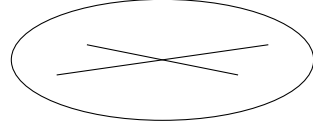
$$\text{Prim}(A) \simeq \text{Prim}(B)$$

However, the bijection

$$\text{Prim}(A) \longleftrightarrow \text{Prim}(B)$$

obtained from a stratified equivalence might not be a homeomorphism, as in the following example.

**Example 8.2.** *Let  $Y$  be a sub-variety of  $X$ .*



We will write  $\mathcal{I}_Y$  for the ideal in  $\mathcal{O}(X)$  determined by  $Y$ , so that

$$\mathcal{I}_Y = \{\omega \in \mathcal{O}(X) \mid \omega(p) = 0 \quad \forall p \in Y\}.$$

Let  $A$  be the algebra of all  $2 \times 2$  matrices whose diagonal entries are in  $\mathcal{O}(X)$  and whose off-diagonal entries are in  $\mathcal{I}_Y$ . Addition and multiplication in  $A$  are matrix addition and matrix multiplication. As a  $k$ -module,  $A$  is the direct sum of  $\mathcal{O}(X) \oplus \mathcal{O}(X)$  with  $\mathcal{I}_Y \oplus \mathcal{I}_Y$ .

$$A = \begin{pmatrix} \mathcal{O}(X) & \mathcal{I}_Y \\ \mathcal{I}_Y & \mathcal{O}(X) \end{pmatrix}$$

Set  $B = \mathcal{O}(X) \oplus \mathcal{O}(Y)$ , so that  $B$  is the coordinate algebra of the disjoint union  $X \sqcup Y$ . We have  $\mathcal{O}(Y) = \mathcal{O}(X)/\mathcal{I}_Y$ . As a  $k = \mathcal{O}(X)$ -module,  $B$  is the direct sum  $\mathcal{O}(X) \oplus (\mathcal{O}(X)/\mathcal{I}_Y)$ .

**Theorem 8.3.** *The algebras  $A$  and  $B$  are stratified equivalent but not Morita equivalent:*

$$A \sim B \quad A \underset{\text{Morita}}{\not\sim} B$$

*Proof.* Let  $M_2(\mathcal{O}(X))$  denote the algebra of all  $2 \times 2$  matrices with entries in  $\mathcal{O}(X)$ . Consider the algebra morphisms

$$A \longrightarrow M_2(\mathcal{O}(X) \oplus \mathcal{O}(Y)) \longleftarrow \mathcal{O}(X) \oplus \mathcal{O}(Y)$$

$$T \mapsto (T, t_{22}|_Y) \quad (T_\omega, \theta) \mapsto (\omega, \theta)$$

where

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \quad T_\omega = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}$$

The filtration of  $A$  is given by

$$\{0\} \subset \begin{pmatrix} \mathcal{O}(X) & \mathcal{I}_Y \\ \mathcal{I}_Y & \mathcal{I}_Y \end{pmatrix} \subset A$$

and the filtration of  $M_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y)$  is given by

$$\{0\} \subset M_2(\mathcal{O}(X) \oplus \{0\}) \subset M_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y).$$

The rightward pointing arrow is spectrum preserving with respect to the indicated filtrations. The leftward pointing arrow is spectrum preserving (no filtrations needed). We infer that

$$A \sim B.$$

Note that

$$\text{Prim}(A) = \text{Diagram showing a circle containing two intersecting red lines representing the spectrum of } A.$$

$$= X \text{ with each point of } Y \text{ replaced by two points}$$

and

$$\begin{aligned} \text{Prim}(B) &= \text{Prim}(\mathcal{O}(X) \oplus \mathcal{O}(Y)) \\ &= X \sqcup Y \end{aligned}$$

The spaces  $\text{Prim}(A)$  and  $\text{Prim}(B)$  are not homeomorphic, and so we have

$$A \underset{\text{Morita}}{\not\sim} B.$$

□

Unlike Morita equivalence, stratified equivalence works well for finite type  $k$ -algebras whether or not the algebras are unital, e.g.  $A$  and  $M_n(A)$  are stratified equivalent even when  $A$  is not unital. See Proposition 8.5 below.

For any  $k$ -algebra  $A$  there is the evident isomorphism of  $k$ -algebras  $M_n(A) \cong A \otimes_{\mathbb{C}} M_n(\mathbb{C})$ . Hence, using this isomorphism, if  $W$  is a representation of  $A$  and  $U$  is a representation of  $M_n(\mathbb{C})$ , then  $W \otimes_{\mathbb{C}} U$  is a representation of  $M_n(A)$ .

As above,  $M_{n,1}(\mathbb{C})$  denotes the  $n \times 1$  matrices with entries in  $\mathbb{C}$ . Matrix multiplication gives the usual action of  $M_n(\mathbb{C})$  on  $M_{n,1}(\mathbb{C})$ .

$$M_n(\mathbb{C}) \times M_{n,1}(\mathbb{C}) \longrightarrow M_{n,1}(\mathbb{C})$$

This is the unique irreducible representation of  $M_n(\mathbb{C})$ . For any  $k$ -algebra  $A$ , if  $W$  is a representation of  $A$ , then  $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$  is a representation of  $M_n(A)$ .

**Lemma 8.4.** *Let  $A$  be a finite type  $k$ -algebra and let  $n$  be a positive integer. Then:*

- (i) *If  $W$  is an irreducible representation of  $A$ ,  $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$  is an irreducible representation of  $M_n(A)$ .*
- (ii) *The resulting map  $\text{Irr}(A) \rightarrow \text{Irr}(M_n(A))$  is a bijection.*

Proof. For (i), suppose given an irreducible representation  $W$  of  $A$ . Let  $J$  be the primitive ideal in  $A$  which is the null space of  $W$ . Then the null space of  $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$  is  $J \otimes_{\mathbb{C}} M_n(\mathbb{C})$ . Consider the quotient algebra  $A \otimes_{\mathbb{C}} M_n(\mathbb{C}) / J \otimes_{\mathbb{C}} M_n(\mathbb{C}) = (A/J) \otimes_{\mathbb{C}} M_n(\mathbb{C})$ . This is isomorphic to  $M_{rn}(\mathbb{C})$  where  $A/J \cong M_r(\mathbb{C})$ , and so  $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$  is irreducible.

**Proposition 8.5.** *Let  $A$  be a finite type  $k$ -algebra and let  $n$  be a positive integer, then  $A$  and  $M_n(A)$  are stratified equivalent.*

*Proof.* Let  $f: A \rightarrow M_n(A)$  be the morphism of  $k$ -algebras which maps  $a \in A$  to the diagonal matrix

$$\begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{bmatrix}$$

It will suffice to prove that  $f: A \rightarrow M_n(A)$  is spectrum preserving.

Let  $J$  be an ideal in  $A$ . Denote by  $J^\diamond$  the ideal in  $M_n(A)$  consisting of all  $[a_{ij}] \in M_n(A)$  such that each  $a_{ij}$  is in  $J$ . Equivalently,  $M_n(A)$  is  $A \otimes_{\mathbb{C}} M_n(\mathbb{C})$  and  $J^\diamond = J \otimes_{\mathbb{C}} M_n(\mathbb{C})$ . It will suffice to prove

- (1) If  $J$  is a primitive ideal in  $A$ , then  $J^\diamond$  is a primitive ideal in  $M_n(A)$ .
- (2) If  $L$  is any primitive ideal in  $M_n(A)$ , then there is a primitive ideal  $J$  in  $A$  with  $L = J^\diamond$ .

For (1),  $J$  primitive  $\implies J^\diamond$  primitive, because the quotient algebra  $M_n(A)/J^\diamond$  is  $(A/J) \otimes_{\mathbb{C}} M_n(\mathbb{C})$  which is (isomorphic to)  $M_{rn}(\mathbb{C})$  where  $A/J \cong M_r(\mathbb{C})$ .

For (2), since  $\mathbb{C}$  is commutative, the action of  $\mathbb{C}$  on  $A$  can be viewed as both a left and right action. Matrix multiplication then gives a left and a right action of  $M_n(\mathbb{C})$  on  $M_n(A)$

$$M_n(\mathbb{C}) \times M_n(A) \rightarrow M_n(A)$$

$$M_n(A) \times M_n(\mathbb{C}) \rightarrow M_n(A)$$

for which the associativity rule

$$(\alpha\theta)\beta = \alpha(\theta\beta) \quad \alpha, \beta \in M_n(A) \quad \theta \in M_n(\mathbb{C})$$



is valid.

If  $V$  is any representation of  $M_n(A)$ , the associativity rule

$$(\alpha\theta)(\beta v) = \alpha[(\theta\beta)v] \quad \alpha, \beta \in M_n(A) \quad \theta \in M_n(\mathbb{C}) \quad v \in V$$

is valid.

Now let  $V$  be an irreducible representation of  $M_n(A)$ , with  $L$  as its null-space. Define a (left) action

$$M_n(\mathbb{C}) \times V \rightarrow V$$

of  $M_n(\mathbb{C})$  on  $V$  by proceeding as in the proof of Lemma 4.2 (the “ $k$ -action for free” lemma) i.e. given  $v \in V$ , choose  $v_1, v_2, \dots, v_r \in V$  and  $\alpha_1, \alpha_2, \dots, \alpha_r \in M_n(A)$  with

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r$$

For  $\theta \in M_n(\mathbb{C})$ , define  $\theta v$  by :

$$\theta v = (\theta\alpha_1)v_1 + (\theta\alpha_2)v_2 + \dots + (\theta\alpha_r)v_r$$

The strict non-degeneracy, Lemmas 4.1 and 4.3, of  $V$  implies that  $\theta v$  is well-defined as follows. Suppose that  $u_1, u_2, \dots, u_s \in V$  and  $\beta_1, \beta_2, \dots, \beta_s \in M_n(A)$  are chosen with

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r = \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_s u_s$$

If  $\alpha$  is any element of  $M_n(A)$ , then

$$\alpha[(\theta\alpha_1)v_1 + (\theta\alpha_2)v_2 + \dots + (\theta\alpha_r)v_r - (\theta\beta_1)u_1 - (\theta\beta_2)u_2 - \dots - (\theta\beta_s)u_s] = (\alpha\theta)[\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r - \beta_1 u_1 - \beta_2 u_2 - \dots - \beta_s u_s] = (\alpha\theta)[v - v] = 0$$

Use  $f: A \rightarrow M_n(A)$  to make  $V$  into an  $A$ -module

$$av := f(a)v \quad a \in A \quad v \in V$$

The actions of  $A$  and  $M_n(\mathbb{C})$  on  $V$  commute. Thus for each  $\theta \in M_n(\mathbb{C})$ ,  $\theta V$  is a sub- $A$ -module of  $V$ , where  $\theta v$  is the image of  $v \mapsto \theta v$ . Denote by  $E_{ij}$  the matrix in  $M_n(\mathbb{C})$  which has 1 for its  $(i, j)$  entry and zero for all its other entries. Then, as an  $A$ -module,  $V$  is the direct sum

$$V = E_{11}V \oplus E_{22}V \oplus \dots \oplus E_{nn}V$$

Moreover, the action of  $E_{ij}$  on  $V$  maps  $E_{jj}V$  isomorphically (as an  $A$ -module) onto  $E_{ii}V$ . Hence as an  $M_n(A) = A \otimes_{\mathbb{C}} M_n(\mathbb{C})$  module,  $V$  is isomorphic to  $(E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$  — where  $\mathbb{C}^n$  is the standard representation of  $M_n(\mathbb{C})$  i.e. is the unique irreducible representation of  $M_n(\mathbb{C})$ .

$$V \cong (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$$

$E_{11}V$  is an irreducible  $A$ -module since if not  $V = (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$  would not be an irreducible  $A \otimes_{\mathbb{C}} M_n(\mathbb{C})$ -module.

If  $J$  is the null space (in  $A$ ) of  $E_{11}V$ , then  $J^\diamond = J \otimes_{\mathbb{C}} M_n(\mathbb{C})$  is the null space of  $V = (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$  and this completes the proof.  $\square$

## 9. AFFINE HECKE ALGEBRAS

Let  $G$  be a connected reductive complex Lie group with maximal torus  $T$ .  $W$  denotes the Weyl group

$$W = N_G(T)/T$$

and  $X^*(T)$  is the character group of  $T$ .  $N_G(T)$  is the normalizer (in  $G$ ) of  $T$ . The semi-direct product  $X^*(T) \rtimes W$  is the affine Weyl group of  $G$ . For

each non-zero complex number  $q$ , there is the affine Hecke algebra  $\mathcal{H}_q(G)$ . This is an affine Hecke algebra with equal parameters and  $\mathcal{H}_1(G)$  is the group algebra of the affine Weyl group:

$$\mathcal{H}_1(G) = \mathbb{C}[X^*(T) \rtimes W] = \mathcal{O}(T) \rtimes W.$$

$\mathcal{H}_q$  is the algebra generated by  $T_x$ ,  $x \in X^*(T) \rtimes W$ , with relations

$$\begin{aligned} T_x T_y &= T_{xy}, & \text{if } \ell(xy) = \ell(x) + \ell(y), \text{ and} \\ (T_s - q)(T_s + 1) &= 0, & \text{if } s \in S. \end{aligned}$$

$\ell$  is the length function on  $X^*(T) \rtimes W$ .

$S$  is the set of order 2 generators of the finite Coxeter group  $W$ .

Using the action of  $W$  on  $T$ , form the quotient variety  $T/W$  and let  $k$  be its coordinate algebra,

$$k = \mathcal{O}(T/W)$$

For all  $q \in \mathbb{C}^\times$ ,  $\mathcal{H}_q(G)$  is a unital finite type  $k$ -algebra.

**Theorem 9.1** (Lusztig). *Except for  $q$  in a finite set of roots of unity, none of which is 1,  $\mathcal{H}_q(G)$  is stratified equivalent to  $\mathcal{H}_1(G)$  :*

$$\mathcal{H}_q(G) \sim \mathcal{H}_1(G).$$

*Proof.* Let  $J$  be Lusztig's asymptotic algebra [Xi, 2.7]. As a  $\mathbb{C}$ -vector space,  $J$  has a basis  $\{T_x : x \in X^*(T) \rtimes W\}$ , and there is a canonical structure of associative  $\mathbb{C}$ -algebra on  $J$ . Except for  $q$  in a finite set of roots of unity (none of which is 1) Lusztig constructs a morphism of  $k$ -algebras

$$\phi_q : \mathcal{H}_q(G) \longrightarrow J$$

which is spectrum preserving with respect to filtrations. The algebra  $\mathcal{H}_q(G)$  is viewed as a  $k$ -algebra via the canonical isomorphism

$$\mathcal{O}(T/W) \cong Z(\mathcal{H}_q(G)).$$

Lusztig's map  $\phi_q$  maps  $Z(\mathcal{H}_q(G))$  to  $Z(J)$  and thus determines a unique  $k$ -structure for  $J$  such that the map  $\phi_q$  is a morphism of  $k$ -algebras.  $J$  with this  $k$ -structure will be denoted  $J_q$ .  $\mathcal{H}_q(G)$  is then stratified equivalent to  $\mathcal{H}_1(G)$  by the three elementary steps

$$\mathcal{H}_q(G) \rightsquigarrow J_q \rightsquigarrow J_1 \rightsquigarrow \mathcal{H}_1(G).$$

The second elementary step (i.e. passing from  $J_q$  to  $J_1$ ) is an algebraic variation of  $k$ -structure with parameter space  $\mathbb{C}^\times$ . The first elementary step uses Lusztig's map  $\phi_q$ , and the third elementary step uses Lusztig's map  $\phi_1$ . Hence (provided  $q$  is not in the exceptional set of roots of unity—none of which is 1)  $\mathcal{H}_q(G)$  is stratified equivalent to

$$\mathcal{H}_1(G) = \mathbb{C}[X^*(T) \rtimes W] = \mathcal{O}(T) \rtimes W.$$

□

As observed in section 11 below,  $\text{Irr}(\mathcal{H}_1(G)) = T//W$ . Thus the stratified equivalence of  $\mathcal{H}_q(G)$  to  $\mathcal{H}_1(G)$  determines a bijection

$$T//W \longleftrightarrow \text{Irr}(\mathcal{H}_q(G))$$

Here  $T//W$  is the extended quotient for the action of  $W$  on  $T$ . See section 11 below.

With  $q = 1$ , there is commutativity in the diagram

$$\begin{array}{ccc}
T//W & \longrightarrow & \text{Irr}(\mathcal{H}_1(G)) \\
\downarrow & & \downarrow \\
T/W & \xrightarrow{I_{T/W}} & T/W
\end{array}$$

where the left vertical arrow is the projection of the extended quotient on the ordinary quotient and the right vertical arrow is the central character for  $\mathcal{H}_1(G) = \mathbb{C}[X^*(T) \rtimes W] = \mathcal{O}(T) \rtimes W$ .

**Theorem 9.2.** *Consider the affine Hecke algebra  $\mathcal{H}_q := \mathcal{H}_q(\text{SL}_3(\mathbb{C}))$ . For  $|q| \neq 1$ ,  $\mathcal{H}_q$  is not Morita equivalent to  $\mathcal{H}_1$ .*

*Proof.* According to [Xi, 11.7],  $\mathcal{H}_q$  is not isomorphic to  $\mathcal{H}_1$  whenever  $q \neq 1$ . We need a much stronger result, and for this we consider Hochschild homology  $\text{HH}_*$ . We note that  $\text{HH}_*$  is Morita invariant. We set

$$\tilde{T} := \{(w, t) \in W \times T : w(t) = t\}.$$

We have the isomorphism in [S2, Theorem 2 (a)] onto the  $W$ -invariant algebraic differential forms on  $\tilde{T}$ :

$$(1) \quad \text{HH}_*(\mathcal{H}_q) \cong \Omega^*(\tilde{T})^W$$

The right-hand-side of (1) is independent of  $q$ .

For every  $q$  under consideration there is a canonical isomorphism  $Z(\mathcal{H}_q) \cong \mathcal{O}(T)^W$ , and the resulting action of  $\mathcal{O}(T)^W$  on (1) does depend on  $q$ . To be precise, the action on  $\Omega(T_i^w)$  is the same as the action via the embedding

$$T_i^w \rightarrow T : t \mapsto c_{w,i}t$$

where  $T_i^w$  is a connected component of  $T^w \cong (w, T^w) \subset \tilde{T}$  and

$$c_{w,i} : X^*(T) \rightarrow \{q^n : n \in \mathbb{Z}\}$$

is defined in [S2, Theorem 1 (c)].

Since any Morita equivalence preserves the center of an algebra, its Hochschild homology and the action of the center on the latter, we can deduce a necessary condition for Morita equivalence between  $\mathcal{H}_q$  and  $\mathcal{H}_{q'}$ . Namely, there must exist an automorphism of  $T/W$  that sends every sub-variety  $c_{w,i}T_i^w/Z_W(w)$  to a sub-variety  $c_{w',i'}T_{i'}^{w'}/Z_W(w')$ .

For the affine Hecke algebra of type  $\tilde{A}_2$  it was shown in [Yan, §3] that this condition is only fulfilled if  $q' = q$  or  $q' = 1/q$ .  $\square$

It appears that the above condition on subvarieties of  $T/W$  is rather strong, at least when  $R$  is not a direct product of root systems  $A_1$ . On this basis we conjecture that Theorem 9.2 holds for any affine Hecke algebra whose root system contains non-perpendicular roots.

## 10. EXTENDED QUOTIENT

Let  $\Gamma$  be a finite group acting as automorphisms of a complex affine variety  $X$ .

$$\Gamma \times X \rightarrow X.$$

For  $x \in X$ ,  $\Gamma_x$  denotes the stabilizer group of  $x$ :

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

Let  $\text{Irr}(\Gamma_x)$  be the set of (equivalence classes of) irreducible representations of  $\Gamma_x$ . These representations are on finite dimensional vector spaces over the complex numbers  $\mathbb{C}$ .

The *extended quotient*, denoted  $X//\Gamma$ , is constructed by replacing the orbit of  $x$  (for the given action of  $\Gamma$  on  $X$ ) by  $\text{Irr}(\Gamma_x)$ . This is done as follows :

Set  $\tilde{X} = \{(x, \tau) \mid x \in X \text{ and } \tau \in \text{Irr}(\Gamma_x)\}$ . Endowed with the topology that sees only the first coordinate, this is an algebraic variety in the sense of [H], although it is usually not separated. Then  $\Gamma$  acts on  $\tilde{X}$  by

$$\Gamma \times \tilde{X} \rightarrow \tilde{X},$$

$$\gamma(x, \tau) = (\gamma x, \gamma_* \tau),$$

where  $\gamma_*: \text{Irr}(\Gamma_x) \rightarrow \text{Irr}(\Gamma_{\gamma x})$ .  $X//\Gamma$  is defined by :

$$X//\Gamma := \tilde{X}/\Gamma,$$

i.e.  $X//\Gamma$  is the usual quotient for the action of  $\Gamma$  on  $\tilde{X}$ .

The projection  $\tilde{X} \rightarrow X$   $(x, \tau) \mapsto x$  is  $\Gamma$ -equivariant and so passes to quotient spaces to give the projection of  $X//\Gamma$  onto  $X/\Gamma$ .

$$\pi_1: X//\Gamma \longrightarrow X/\Gamma$$

Denote by  $\text{triv}_x$  the trivial one-dimensional representation of  $\Gamma_x$ . The inclusion

$$X \hookrightarrow \tilde{X}$$

$$x \mapsto (x, \text{triv}_x)$$

is  $\Gamma$ -equivariant and so passes to quotient spaces to give an inclusion

$$X/\Gamma \hookrightarrow X//\Gamma$$

This will be referred to as the inclusion of the ordinary quotient in the extended quotient.

Let  $\mathcal{O}(X)$  be the coordinate algebra of the complex affine variety  $X$  and let  $\mathcal{O}(X) \rtimes \Gamma$  be the crossed-product algebra for the action of  $\Gamma$  on  $\mathcal{O}(X)$ .

There are canonical bijections

$$\text{Irr}(\mathcal{O}(X) \rtimes \Gamma) \longleftrightarrow \text{Prim}(\mathcal{O}(X) \rtimes \Gamma) \longleftrightarrow (X//\Gamma)$$

where  $\text{Prim}(\mathcal{O}(X) \rtimes \Gamma)$  is the set of primitive ideals in  $\mathcal{O}(X) \rtimes \Gamma$  and  $\text{Irr}(\mathcal{O}(X) \rtimes \Gamma)$  is the set of (equivalence classes of) irreducible representations of  $\mathcal{O}(X) \rtimes \Gamma$ . The irreducible representation of  $\mathcal{O}(X) \rtimes \Gamma$  associated to  $(x, \tau) \in X//\Gamma$  is

$$\text{Ind}_{\mathcal{O}(X) \rtimes \Gamma_x}^{\mathcal{O}(X) \rtimes \Gamma} (\mathbb{C}_x \otimes \tau).$$

Here  $\mathbb{C}_x: \mathcal{O}(X) \rightarrow \mathbb{C}$  is the irreducible representation of  $\mathcal{O}(X)$  given by evaluation at  $x \in X$  and  $\text{Ind}_{\mathcal{O}(X) \rtimes \Gamma_x}^{\mathcal{O}(X) \rtimes \Gamma}$  denotes induction from  $\mathcal{O}(X) \rtimes \Gamma_x$  to  $\mathcal{O}(X) \rtimes \Gamma$ .

## 11. SCHEMES

**Definition 11.1.** [EGA, 0, 4.4.1 and 5.5.1] A ringed space is a pair  $(X, \mathcal{O}_X)$  consisting of a topological space  $X$  and a sheaf of rings  $\mathcal{O}_X$  on  $X$ . The ringed

space  $(X, \mathcal{O}_X)$  is a locally ringed space if for each point  $P \in X$ , the stalk  $\mathcal{O}_{X,P}$  is a local ring, i.e.  $\mathcal{O}_{X,P}$  has exactly one maximal ideal.

Let now  $A$  be a unital commutative ring. Associated to  $A$  we have the prime spectrum  $X := \text{Spec}(A)$ , which is a quasi-compact space in the Zariski topology, and the sheaf of rings  $\mathcal{O}_X$ . For each point  $P \in X$ , the stalk  $\mathcal{O}_{X,P}$  is a local ring. See [EGA, 1, 1.1.1].

**Definition 11.2.** [EGA, 1, 1.7.1] An affine scheme is a locally ringed space isomorphic to the spectrum of a commutative ring.

Given a locally ringed space  $(X, \mathcal{O}_X)$ , one says that an open subset  $V$  is an open affine subset if the locally ringed space  $(V, \mathcal{O}_X|_V)$  is an affine scheme.

**Definition 11.3.** [EGA, 2.1.2] A scheme is a locally ringed space  $(X, \mathcal{O}_X)$  such that each point of  $X$  admits an open affine neighbourhood.

Each scheme is obtained by glueing affine schemes, see [EGA, 2.3.1]. The glueing construction (*recollement*) is elementary and fundamental, see [EGA, 0, 4.1.7].

Just as a smooth manifold is locally Euclidean, a scheme is locally affine.

Many of our schemes will be non-separated. Here is a classic example of a non-separated scheme, which arises as an example of glueing.

**Example 11.4.** [H, 2.3.6] Let  $X_1 = X_2 = \mathbb{A}_{\mathbb{C}}^1$ , let  $U_1 = U_2 = \mathbb{A}_{\mathbb{C}}^1 - P$  where  $P$  is the point corresponding to the maximal ideal  $(x)$ , and let  $\varphi : U_1 \rightarrow U_2$  be the identity map. Let  $X$  be obtained by glueing  $X_1$  and  $X_2$  along  $U_1$  and  $U_2$  via  $\varphi$ . We get an “affine line with the point  $P$  doubled”.

Let  $X$  be as in Example 11.4. Then  $X$  is not separated over  $\mathbb{C}$ . Indeed,  $X \times_{\mathbb{C}} X$  is the affine plane with doubled axes and four origins. The image of the diagonal morphism  $\Delta : X \times_{\mathbb{C}} X \rightarrow X$  is the usual diagonal, with two of those origins. This is not closed, because all four origins are in the closure of  $\Delta(X)$ . See [H, 4.0.1] and [EGA, 1, 5.5.1].

We now return to Example 8.2. Let  $X_1 = X_2 = X$ , let  $U_1 = U_2 = X - Y$ , and let  $\varphi : U_1 \rightarrow U_2$  be the identity map. Then  $\text{Prim}(A)$  is obtained by glueing  $X_1$  and  $X_2$  along  $U_1$  and  $U_2$  via  $\varphi$ . So  $\text{Prim}(A)$  is a non-separated scheme.

## 12. SMOOTH DUALS

Let  $F$  be a non-archimedean local field, and let  $G$  be the  $F$ -points of a connected reductive algebraic group over  $F$ . Let  $\mathfrak{s}$  be a point in the Bernstein spectrum  $\mathfrak{B}(G)$  of  $G$ . Attached to  $\mathfrak{s}$  there is a complex torus  $T_{\mathfrak{s}}$  and a finite group  $W_{\mathfrak{s}}$  acting on  $T_{\mathfrak{s}}$ .

We denote the space of (equivalence classes of) irreducible smooth complex  $G$ -representations by  $\text{Irr}(G)$ . We have the Bernstein decomposition

$$\mathrm{Irr}(G) = \bigsqcup_{s \in \mathfrak{B}(G)} \mathrm{Irr}^s(G)$$

and the (restriction of) the cuspidal support map

$$\mathbf{Sc} : \mathrm{Irr}^s(G) \rightarrow T_s/W_s$$

see [Ren, VI.7.1.1]. The map  $\mathbf{Sc}$  is finite-to-one and the quotient  $T_s/W_s$  has the structure of a complex affine algebraic variety.

Bernstein constructs a finite type  $k^s$ -algebra  $A^s$  with the property that  $\mathrm{Irr}(A^s)$  is in bijection with the Bernstein component  $\mathrm{Irr}^s(G)$ . Here  $k^s$  is the coordinate algebra of  $T^s/W^s$ :

$$k^s = \mathcal{O}(T^s/W^s)$$

The classical theory leaves open the geometric structure of each component  $\mathrm{Irr}^s(G)$ . With this in mind, we recall the ABPS conjecture for split groups.

**Conjecture 12.1.** *Let  $G$  be the  $F$ -points of a connected reductive split algebraic group over  $F$ . For each Bernstein component in the smooth dual of  $G$ , Bernstein's finite type  $k^s$ -algebra  $A^s$  is stratified equivalent to the crossed-product algebra  $\mathcal{O}(T^s) \rtimes W^s$ . In addition, the stratified equivalence between  $A^s$  and  $\mathcal{O}(T^s) \rtimes W^s$  can be chosen such that the resulting bijection*

$$\mathrm{Irr}^s(G) \longleftrightarrow T^s // W^s$$

*satisfies a number of conditions itemized in [ABPS].*

Moussaoui [Mou], building on the work of many mathematicians, notably [A], has verified the ABPS conjecture (without the stratified equivalence) for all the split classical  $p$ -adic groups. A different proof of the ABPS conjecture (without the stratified equivalence) for split classical groups was first obtained by Solleveld [S]. Solleveld's approach is more general since it works as soon as we know that the algebra  $A^s$  is an extended affine Hecke algebra. The stratified equivalence is established for  $\mathrm{GL}_n(F)$  in [BP], and for the principal series of the exceptional group  $G_2$  in [ABP]. A more involved version of the stratified equivalence is proved in our paper on the inner forms of  $\mathrm{SL}_n(F)$ , see [ABPS2].

An essential feature of Moussaoui's work [Mou] is the compatibility of the extended quotient structure (for each Bernstein component) with the local Langlands correspondence (LLC). Moussaoui proves that, *independently* of what is happening in the smooth dual, an extended quotient structure is present in the enhanced Langlands parameters — and that the LLC consists of isomorphisms of extended quotients. This phenomenon was first observed in the special case of  $\mathrm{GL}_n$  in [BP].

**12.1. Geometric structure for split  $p$ -adic groups.** The method in [ABPS, p.124] produces a finite morphism  $\theta_q : T^s // W^s \rightarrow T^s/W^s$ . Let  $C^s$  be any irreducible component in  $T^s // W^s$  except  $T^s/W^s$ . Let

$$X_1 = X_2 = T^s/W^s$$

$$U_1 = U_2 = T^5/W^5 - \theta_q(C^5)$$

and let

$$\varphi : U_1 \rightarrow U_2$$

be the identity map. Let  $X$  be obtained by glueing  $X_1$  and  $X_2$  along  $U_1$  and  $U_2$  via  $\varphi$ . Repeat this procedure for every component  $C^5$  except  $T^5/W^5$ . This will create a highly non-separated scheme  $\mathfrak{X}^5$ . Then  $\mathfrak{X}^5$  is the correct model for  $\text{Irr}^5(G)$  in its Jacobson topology.

For example, let  $G = \text{GL}_2$ , let  $T$  be the diagonal subgroup of  $G$  and let  $\mathfrak{i}$  denote the Iwahori point in  $\mathfrak{B}(G)$ , so that  $T^{\mathfrak{i}}/W^{\mathfrak{i}} = T/W$ . The extended quotient  $T//W$  has two irreducible components, namely  $T/W$  and

$$\{\text{diag}(z, z) : z \in \mathbb{C}^\times\}.$$

The map  $\theta_q$  sends  $(z, z)$  to the unordered pair  $\{q^{-1/2}z, q^{1/2}z\}$ .

Let  $C$  be the image in  $T/W$  of the algebraic curve

$$\{\text{diag}(q^{-1/2}z, q^{1/2}z) : z \in \mathbb{C}^\times\} \subset T.$$

We have to glue  $T/W$  and  $T/W$  along  $U_1 := T/W - C$  and  $U_2 : T/W - C$  via the identity map  $\varphi : U_1 \rightarrow U_2$ . Then  $\mathfrak{X}^{\mathfrak{i}}$  is the quotient variety  $T/W$  with the curve  $C$  doubled. This is a good model of  $\text{Irr}^{\mathfrak{i}}(\text{GL}_2)$ .

We now return to §10 and reconsider the bijective map

$$X//\Gamma \rightarrow \text{Prim}(\mathcal{O}(X) \rtimes \Gamma), \quad (x, \tau) \mapsto \text{Ind}_{\mathcal{O}(X) \rtimes \Gamma_x}^{\mathcal{O}(X) \rtimes \Gamma} (\mathbb{C}_x \otimes \tau).$$

Let  $\mathfrak{o}$  be a  $\Gamma$ -orbit in  $X$  for which the isotropy group  $\Gamma_x$ , with  $x \in \mathfrak{o}$ , is non-trivial. Then the cardinality

$$\delta_x := |\text{Irr } \Gamma_x|$$

of the dual of  $\Gamma_x$  is greater than or equal to 2. Let

$$\begin{aligned} X_1 &= X_2 = X \\ U_1 &= U_2 = X - \mathfrak{o} \end{aligned}$$

and let

$$\varphi : U_1 \rightarrow U_2$$

be the identity map. Glue  $X_1$  and  $X_2$  along  $U_1$  and  $U_2$  via  $\varphi$ . Do this  $\delta_x - 1$  times. Repeat this procedure for the other  $\Gamma$ -orbits in  $X$  with non-trivial isotropy group.

This will create on  $X$  the structure of a non-separated scheme  $\mathfrak{X}$ . Then the underlying topological space  $\text{sp}(\mathfrak{X})$  is a good model of the primitive ideal space  $\text{Prim}(\mathcal{O}(X) \rtimes \Gamma)$ .

## 12.2. Geometric structure for the non-split $p$ -adic group $\text{SL}_5(D)$ .

Let  $D$  be a 4-dimensional noncommutative division algebra over the  $p$ -adic field  $F$ . Let  $\text{SL}_n(D)$  be the kernel of the reduced norm map  $\text{GL}_n(D) \rightarrow F^\times$ . Then  $\text{SL}_n(D)$  is a non-split inner form of  $\text{SL}_{2n}(F)$ .

We consider [ABPS1, Example 5.5]. Let  $G = \text{SL}_5(D)$ . There an inertial equivalence class  $\mathfrak{s} \in \mathfrak{B}(\text{SL}_5(D))$  with remarkable properties is exhibited. The point  $\mathfrak{s}$  in the Bernstein spectrum is given by

$$\mathfrak{s} = [L, \sigma]_G$$

with

$$L = (D^\times)^5 \cap \mathrm{SL}_5(D)$$

In order to define the supercuspidal representation  $\sigma$ , we proceed as follows.

Let  $V_4 \cong (\mathbb{Z}/2\mathbb{Z})^2$  denote the non-cyclic group of order 4. Enumerate the elements as  $1, \epsilon_1, \epsilon_2, \epsilon_3$  and define  $\rho$  as follows:

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(\epsilon_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(\epsilon_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(\epsilon_3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Then  $\rho$  is a *projective* 2-dimensional representation of  $V_4$ . The associated 2-cocycle takes the values  $\pm 1$ . We have an action on  $M_2(\mathbb{C})$ :

$$(2) \quad g \cdot A = \rho(g)A\rho(g)^{-1}$$

for all  $g \in V_4$ , and also a homomorphism

$$(3) \quad V_4 \rightarrow \mathrm{PSL}_2(\mathbb{C})$$

Given a local non-archimedean field  $F$ , we always have

$$(4) \quad W_F \rightarrow F^\times \rightarrow F^\times/(F^\times)^2 \rightarrow V_4$$

since  $F^\times/(F^\times)^2$  is always a product of at least two copies of  $\mathbb{Z}/2\mathbb{Z}$ , so we can always project onto  $V_4$ .

When we combine (3) with (4) we get an  $L$ -parameter

$$\phi: W_F \rightarrow \mathrm{PSL}_2(\mathbb{C})$$

Note that  $\mathrm{PSL}_2(\mathbb{C})$  is the Langlands dual group of  $\mathrm{SL}_2(F)$ . Now the centraliser of the image of  $\phi$  is again the non-cyclic group of order 4, which this time we denote by  $S_\phi$ . When we pull back  $S_\phi$  from  $\mathrm{PSL}_2(\mathbb{C})$  to its universal cover  $\mathrm{SL}_2(\mathbb{C})$  we get the short exact sequence

$$1 \rightarrow \mathcal{Z}_\phi \rightarrow \mathcal{S}_\phi \rightarrow S_\phi \rightarrow 1$$

in which  $\mathcal{S}_\phi$  is isomorphic to the group  $U_8$  of unit quaternions  $\{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ . Now  $U_8$  admits four 1-dimensional representations with trivial central character, and one irreducible representation with nontrivial central character: this representation has dimension 2. The 2-dimensional representation of  $U_8$  parametrises an irreducible representation of the inner form  $\mathrm{SL}_1(D)$ . We will denote this representation by  $\tau$ .

We now consider the group of characters  $\chi$  for which  $\chi\tau \cong \tau$ . This group is isomorphic to the non-cyclic group of order four and comprises the four characters  $\{1, \gamma, \eta, \gamma\eta\}$ , where  $\gamma, \eta$  are quadratic characters. We now define the supercuspidal representation  $\sigma$  as follows:

$$\sigma = \tau \otimes 1 \otimes \gamma \otimes \eta \otimes \gamma\eta \in \mathrm{Irr}(L)$$

Let  $x = (x_1, x_2, x_3, x_4, x_5) \in L$ . Consider what happens when we twist  $\sigma$  by the character

$$\gamma(x) := \gamma(x_1) \otimes \gamma(x_2) \otimes \gamma(x_3) \otimes \gamma(x_4) \otimes \gamma(x_5)$$

Note that

$$\gamma(x_1)\gamma(x_2)\gamma(x_3)\gamma(x_4)\gamma(x_5) := \gamma(\mathrm{Nrd}(x_1x_2x_3x_4x_5)) = \gamma(1) = 1$$

since  $x \in \mathrm{SL}_5(D)$ . Similarly for the characters  $\eta, \gamma\eta$ . So we have

$$\begin{aligned} \gamma\sigma &= \gamma\tau \otimes \gamma \otimes 1 \otimes \gamma\eta \otimes \eta \\ &= \tau \otimes \gamma \otimes 1 \otimes \gamma\eta \otimes \eta \end{aligned}$$

which corresponds to the permutation (12)(34). Twisting by  $\eta$  corresponds to the permutation (13)(24), twisting by  $\gamma\eta$  corresponds to the permutation



(14)(23). The conclusion at this point is that the Bernstein finite group  $W^s$  is isomorphic to  $V_4$ . The corresponding Bernstein variety is the quotient  $T^s/V_4$ , where  $T^s$  has the structure of a complex torus of dimension 4. See the reference [ABPS1] for more details.

The associated direct summand of the Hecke algebra of  $\mathrm{SL}_5(D)$  is Morita equivalent to a finite type algebra of the form

$$(5) \quad (\mathcal{O}(T^s) \otimes M_2(\mathbb{C})) \rtimes V_4.$$

Here  $V_4$  acts on  $T^s \cong (\mathbb{C}^\times)^4$  by permutation of coordinates. The elements of order 2 in  $V_4$  act as (12)(34), (13)(24) and (14)(23). The group  $V_4$  acts on  $M_2(\mathbb{C})$  via (2). In (5) the group  $V_4$  acts diagonally on  $\mathcal{O}(T^s) \otimes M_2(\mathbb{C})$ . Since  $\rho$  is not a linear representation, the algebra (5) is not Morita equivalent to the crossed product  $\mathcal{O}(T^s) \rtimes V_4$ .

The space  $\mathrm{Irr}^s(\mathrm{SL}_5(D))$ , which is isomorphic to the spectrum of (5), can be described explicitly, via the action of the centre  $\mathcal{O}(T^s)^{V_4} = \mathcal{O}(T^s/V_4)$ .

- Let  $t = (z_1, z_2, z_3, z_4) \in T^s$  such that some  $z_i$  appears only once (i.e. some  $z_i$  is not repeated) among the coordinates of  $t$ . Let  $U$  be the open subset of all such  $t$ . If  $t \in U$  then  $(V_4)_t = 1$  and the orbit  $V_4 \cdot t$  produces via parabolic induction just one irreducible representation of  $\mathrm{SL}_5(D)$ .
- Let us define three complex tori of dimension 2, which are sub-tori of  $T^s$ :

$$T_1 = \{(z, w, z, w) : z, w \in \mathbb{C}^\times\}$$

$$T_2 = \{(z, z, w, w) : z, w \in \mathbb{C}^\times\}$$

$$T_3 = \{(z, w, w, z) : z, w \in \mathbb{C}^\times\}$$

The union of these three tori – with the intersection removed – is precisely the locus of points in  $T^s$  where the isotropy group has cardinality 2.

Let  $U_j$  be the open subset of  $T_j$  on which  $z \neq w$ . If

$$t \in U_1 \cup U_2 \cup U_3$$

then  $|(V_4)_t| = 2$ , the orbit  $V_4 \cdot t$  has cardinality 2 and produces via parabolic induction precisely two distinct (i.e. non-equivalent) irreducible representations.

- Define a complex torus of dimension 1, a sub-torus of  $T^s$ :

$$T_4 := \{(z, z, z, z) : z \in \mathbb{C}^\times\}$$

Note that

$$T_4 = T_1 \cap T_2 \cap T_3 = T_1 \cap T_2 = T_1 \cap T_3 = T_2 \cap T_3$$

If  $t \in T_4$  then the isotropy group is  $V_4$ . The representations of (5) on which  $\mathcal{O}(T^s)$  acts via  $t$  can be identified with the representations of  $M_2(\mathbb{C}) \rtimes V_4$ . There is only one irreducible such representation (it has dimension 4).

Thus  $\mathrm{Irr}^s(\mathrm{SL}_5(D))$  is  $T^s/V_4$  with doubling on the image of  $U_1 \cup U_2 \cup U_3$  in  $T^s/V_4$ . Apparently, this space cannot be obtained by recollement (as in Section 11) from affine schemes.

Note that in this example, the locus of reducibility, i.e.  $T_1 \cup T_2 \cup T_3$ , in the quotient  $T^5/V_4$  is a sub-variety of  $T^5/V_4$ . On  $T_1 \cap T_2 \cap T_3$  each representation obtained by parabolic induction is reducible, but its two irreducible constituents are equivalent. Thus, each point in  $T_1 \cap T_2 \cap T_3$  contributes only one point to  $\text{Irr}^s(\text{SL}_5(D))$ .

The algebra (2) above is Morita equivalent to the twisted crossed product

$$\mathcal{O}(T^5) \rtimes_{\mathfrak{h}} V_4$$

where  $\mathfrak{h}$  is the 2-cocycle associated to the above projective representation of  $V_4$ . Hence, (since Morita equivalent algebras are stratified equivalent according to Proposition 8.1) this is a confirming example for the ABPS conjecture. Following [ABPS3],  $\text{Irr}(\mathcal{O}(T^5) \rtimes_{\mathfrak{h}} V_4)$  is the twisted extended quotient  $(T^5//V_4)_{\mathfrak{h}}$  so we have the finite morphism

$$(T^5//V_4)_{\mathfrak{h}} \longrightarrow T^5/V_4$$

See [ABPS3] for further developments involving the ABPS conjecture for non-split  $p$ -adic groups.

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