Results related to integer flows and cycle covers are presented. A cycle cover of a graph $G$ is a collection $\mathcal{C}$ of cycles of $G$ which covers all edges of $G$; $\mathcal{C}$ is called a cycle $m$-cover of $G$ if each edge of $G$ is covered exactly $m$ times by the members of $\mathcal{C}$. By using Seymour's nowhere-zero 6-flow theorem, we prove that every bridgeless graph has a cycle 6-cover associated to covering of the edges by 10 even subgraphs (an even graph is one in which each vertex is of even degree). This result together with the cycle 4-cover theorem implies that every bridgeless graph has a cycle $m$-cover for any even number $m \geq 4$. We also prove that every graph with a nowhere-zero 4-flow has a cycle cover $\mathcal{C}$ such that the sum of lengths of the cycles in $\mathcal{C}$ is at most $|E(G)| + |V(G)| - 2$, unless $G$ belongs to a very special class of graphs.

1. INTRODUCTION

The graphs we consider are finite, but may contain loops and multiple edges. A graph is simple if it contains no loops or multiple edges. For a graph $G$, $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$, respectively. An edge is said to be contracted if it is removed and its ends are identified. A subset $S$ of $E(G)$ is a cut of $G$ if its removal leaves a graph with more components and no proper subset of $S$ has this property; $S$ is called a $k$-cut if $|S| = k$. A $k$-cut is called an odd cut if $k$ is odd. A 1-cut is also called a bridge. An even graph is one in which every vertex is of even degree; an eulerian graph is a connected even graph. A graph with every vertex of degree 3 is called a cubic graph. Sometimes, we treat a subgraph as a subset of edges. For instance the symmetric difference of two even subgraphs $Z_1$ and $Z_2$, denoted by $Z_1 \oplus Z_2$, is the even subgraph $(Z_1 \cup Z_2) \setminus (Z_1 \cap Z_2)$. Let $G$ be a graph. A cover of $G$ is a collection $\mathcal{H}$ of subgraphs of $G$ which covers all edges of $G$; $\mathcal{H}$ is called an $m$-cover of $G$ if each edge of $G$ is covered exactly $m$ times by the subgraphs in $\mathcal{H}$. In this paper, we consider the case where each member of $\mathcal{H}$ is an even subgraph.

* Present address: Department of Mathematics, Arizona State University, Tempe, Arizona 85287.
(the empty set \( \emptyset \) is regarded as an even subgraph of every graph). It is clear that a graph is even if and only if it has a decomposition into edge-disjoint cycles. For simplicity, a cover \((m\text{-cover})\) by even subgraphs is also called a cycle cover \((cycle \ m\text{-cover})\).

The following problem was considered by Szekeres [17] for bridgeless cubic graphs, and independently, was formulated as a conjecture by Seymour [15] for bridgeless graphs. It is now known as the "Cycle Double Cover Conjecture."

**Conjecture 1.A.** Every bridgeless graph has a cycle 2-cover.

A stronger form of this conjecture was proposed by Celmius [3].

**Conjecture 1.B.** Every bridgeless graph has a 2-cover by 5 even subgraphs.

Bermond, Jackson, and Jaeger [2] established

**Theorem 1.A.** Every bridgeless graph has a 4-cover by 7 even subgraphs.

Goddyn [8] conjectured that every bridgeless graph has a cycle 6-cover and proved that for each bridgeless graph \( G \) there exists an integer \( k \) (depending on \(|E(G)|\)) such that \( G \) has a cycle \((4k+2)\)-cover. If \( G \) is a cubic graph, it follows from a simple computation that any cycle \(2k\)-cover of \( G \) requires at least \(3k\) even subgraphs, with equality if and only if each of the \(3k\) even subgraphs is a 2-factor of \( G \). In the edge-set of a cubic graph, complementation defines a 1–1 correspondence between the perfect matchings and the 2-factors. Hence, for a cubic graph \( G \), the following two statements are equivalent.

(a) \( G \) has a \( k \)-cover by \( 3k \) perfect matchings.

(b) \( G \) has a \( 2k \)-cover by \( 3k \) even subgraphs.

Thus, the matching polytope theorem of Edmonds [4] implies that for each bridgeless cubic graph \( G \) there exists an integer \( k \) such that \( G \) has a \( 2k \)-cover by \( 3k \) even subgraphs \((2\text{-factors})\). What is the smallest integer \( k \) for which every bridgeless cubic graph has a \( 2k \)-cover by \( 3k \) even subgraphs? This problem, in the perfect matching version, has been studied in great detail by Seymour [14]. Fulkerson [7] (see also [14]) conjectured that every bridgeless cubic graph has a 2-cover by 6 perfect matchings. This is equivalent to

**Conjecture 1.C.** Every bridgeless cubic graph has a 4-cover by 6 even subgraphs.

It is well known that every bridgeless graph is a contraction of some bridgeless cubic graph ("split" each vertex into a cycle). Hence, the following conjecture is equivalent to Conjecture 1.C. 
Conjecture 1.C'. Every bridgeless graph has a 4-cover by 6 even subgraphs.

A weaker form of this conjecture, namely, every bridgeless graph has a 8-cover by 12 even subgraphs, has been suggested by Goddyn (personal communication). Seymour [14] has shown that if $k$ is odd, the Petersen graph has no $k$-cover by $3k$ perfect matchings, equivalently, no $2k$-cover by $3k$ even subgraphs. In particular, it has no 6-cover by 9 even subgraphs, and so neither has any graph that can be contracted to the Petersen graph. Is it possible to find a 6-cover by 10 even subgraphs? In this paper, we shall prove

**Theorem 1.1.** Every bridgeless graph has a 6-cover by 10 even subgraphs.

Clearly, Theorem 1.A implies that every bridgeless graph has a cycle $m$-cover for any $m \equiv 0 \pmod{4}$. If $m$ is odd, a graph $G$ has a cycle $m$-cover if and only if $G$ is even. (Proof: If $G$ is even, then $m$ copies of $G$ give the required cover. Conversely, suppose that $\{H_1, H_2, \ldots, H_m\}$ is a cycle $m$-cover of $G$ and $m$ is odd. Then $G = H_1 \oplus H_2 \oplus \cdots \oplus H_m$ and so $G$ is even.) Theorem 1.1 together with Theorem 1.A yields

**Theorem 1.2.** Every bridgeless graph has a cycle $m$-cover for any even number $m \geq 4$.

## 2. Preliminaries

An orientation $D$ of an undirected graph $G$ is an assignment of a direction to each edge $e \in E(G)$. Let $G$ be a graph with orientation $D$. For each vertex $v \in V(G)$, $E^+(v)$ is the set of non-loop edges with tail $v$, and $E^-(v)$ the set of non-loop edges with head $v$. A flow in $G$ under orientation $D$ is an integer-valued function $\phi$ on $E(G)$ such that

$$\sum_{e \in E^+(v)} \phi(e) = \sum_{e \in E^-(v)} \phi(e) \quad \text{for each} \quad v \in V(G).$$

The support of $\phi$ is defined by

$$S(\phi) = \{ e \in E(G) : \phi(e) \neq 0 \}.$$

For a positive integer $k$, if $-k < \phi(e) < k$ for every $e \in E(G)$, then $\phi$ is called a $k$-flow, and furthermore, if $S(\phi) = E(G)$, then $\phi$ is called a nowhere-zero $k$-flow. A well-known result on integer flows is the following one by Tutte
((6.3) of [19]), an alternative proof of which has been obtained by Younger [20].

**Lemma 2.A.** If $G$ has a flow $\phi$ then, for any integer $k > 1$, $G$ has a $k$-flow $\phi'$ such that $\phi'(e) \equiv \phi(e) \pmod{k}$ for every $e \in E(G)$.

If $\phi_1$ and $\phi_2$ are two flows in $G$ under the same orientation $D$, let $l$ and $m$ be two integers. Then the sum $\phi = l\phi_1 + m\phi_2$ is a flow in $G$ under $D$ with $\phi(e) = l\phi_1(e) + m\phi_2(e)$ for each $e \in E(G)$. Let $\phi$ be a flow in $G$ under a given orientation and $e \in E(G)$. If we reverse the direction of $e$ and change $\phi(e)$ to $-\phi(e)$, we still have a flow in $G$ under the new orientation. This means that

**Proposition 2.1.** If $G$ has a flow $\phi$ under some orientation $D$, then, for any orientation $D'$, $G$ has a flow $\phi'$ under $D'$ with $|\phi'(e)| = |\phi(e)|$ for every edge $e$.

From now on, a flow in $G$ is always associated with some orientation of $G$, and whenever necessary, we can make two flows in $G$ be associated with the same orientation. The following proposition follows easily from the definition.

**Proposition 2.2.** Let $\phi$ be a flow in $G$ and $Z = \{e \in E(G) : \phi(e) \text{ is odd}\}$. Then $Z$ is an even subgraph of $G$.

By the definition, a $k$-flow is a $k'$-flow for any $k' > k$. It is easy to check that the complete graph on 4 vertices has no nowhere-zero 3-flows. Tutte [18] has proved that the Petersen graph has no nowhere-zero 4-flows. Clearly, if $\phi$ is a flow in $G$, then $\phi$ also defines by restriction a flow in the graph obtained by contracting an edge of $G$. The following famous conjectures are due to Tutte (see [20]).

1. Every bridgeless graph without 3-cuts has a nowhere-zero 3-flow.
2. Every bridgeless graph containing no subgraph contractible to the Petersen graph has a nowhere-zero 4-flow.
3. Every bridgeless graph has a nowhere-zero 5-flow.

Towards a proof of (3), Jaeger [13] established that every bridgeless graph has a nowhere-zero 8-flow. This was improved by Seymour [16] in the following famous theorem.

**Theorem 2.A.** Every bridgeless graph has a nowhere-zero 6-flow.
By a result derived from Tutte's work by Jaeger [11] (or see [13]), every graph with a nowhere-zero 4-flow can be covered by two even subgraphs, and these two even subgraphs, together with their symmetric difference, form a 2-cover by three even subgraphs. That is,

**Lemma 2.B.** Every graph with a nowhere-zero 4-flow has a 2-cover by 3 even subgraphs.

The following result was proved by Jaeger [12] by algebraic methods. We present below a constructive proof.

**Lemma 2.C.** Let $G$ be a graph and $H \subseteq E(G)$. $H$ is contained in an even subgraph of $G$ if and only if $H$ contains no odd cut of $G$.

**Proof.** Contract all edges of $E(G) - H$ and denote by $G^*$ the resulting graph. Then $E(G^*) = H$, and $H$ contains no odd cut of $G$ if and only if $G^*$ is even. We show now that $G^*$ is even if and only if $H$ is contained in an even subgraph of $G$. If $G^*$ is even, then it is the union of a set of edge-disjoint cycles. Clearly, each such cycle can be extended, by adding paths in $E(G) - H$, to a cycle of $G$ and the symmetric difference of all these extended cycles is an even subgraph of $G$ which contains $H$. Conversely, if $H$ is contained in an even subgraph of $G$, then $G^*$ is the contraction of this even subgraph and so is even. This proves the lemma.

3. **Proof of Theorem 1.1**

We first prove the following lemma by applying Seymour's 6-flow theorem (Theorem 2.A).

**Lemma 3.1.** For any bridgeless graph $G$, there is a partition of $E(G): E(G) = \bigcup_{i=1}^{6} A_i$, where $A_i \cap A_j = \emptyset$ if $i \neq j$, such that $G - A_i$ has a nowhere-zero 3-flow if $1 \leq i \leq 3$ and a nowhere-zero 4-flow if $4 \leq i \leq 6$, and moreover, $A_1 \cup A_2 \cup A_3$ is an even subgraph of $G$.

**Proof.** By Theorem 2.A, $G$ has a nowhere-zero 6-flow $\phi$. Set

$$Z = \{ e \in E(G) : \phi(e) \text{ is odd} \}.$$  

By Proposition 2.2, $Z$ is an even subgraph and hence has a nowhere-zero 2-flow. We shall choose an orientation $D$ of $G$ under which $G$ has a 2-flow $\phi'$ such that

$$\phi'(e) = \begin{cases} 0, & e \notin Z \\ 1, & e \in Z. \end{cases}$$
Moreover, by Proposition 2.1, $G$ has a nowhere-zero 6-flow $\phi''$ under $D$ with $|\phi''(e)| = |\phi(e)|$ for all edges $e$. Setting

$$E_i = \{ e \in E(G) : \phi''(e) = i \},$$

we have that

$$Z = \bigcup \{ E_i : i \in \{ \pm 1, \pm 3, \pm 5 \} \}.$$

Let

$$\phi_i = \phi'' + i\phi', \quad i \in \{ \pm 1, \pm 3, \pm 5 \}. \quad (3.1)$$

Then $\phi_i$ is a flow in $G$ (not necessarily nowhere-zero). As an illustration, we list the values of $\phi_{-1}$ and $\phi_1$.

$$\phi_{-1}(e) = \begin{cases} 0, & e \in E_1 \\ -6, & e \in E_{-5} \end{cases} \quad \text{and} \quad \phi_1(e) = \begin{cases} 0, & e \in E_{-1} \\ 6, & e \in E_5 \\ \pm 2, \pm 4, & \text{otherwise} \end{cases}.$$

By Lemma 2.A, $\phi_{-1}$ yields a 3-flow in $G$ with support $E(G) - (E_1 \cup E_{-5})$. Similarly, $\phi_1$ yields a 3-flow with support $E(G) - (E_{-1} \cup E_3)$ and $\phi_3$ (or $\phi''$) gives a 3-flow with support $E(G) - (E_{-3} \cup E_3)$. Set $A_1 = (E_1 \cup E_{-3})$, $A_2 = (E_{-1} \cup E_5)$, $A_3 = (E_{-3} \cup E_3)$. Then $A_1 \cup A_2 \cup A_3 = Z$ is an even subgraph of $G$. Furthermore, let $\overline{Z} = E(G) - Z$. If we contract all edges of $Z$, then $(1/2)\phi$ (or $(1/2)\phi''$) defines by restriction a nowhere-zero 6-flow in the resulting graph $G^*$ with $E(G^*) = \overline{Z}$. By Lemma 2.B, $G^*$ has a 2-cover by 3 even subgraphs, say $Z_j$ ($1 \leq j \leq 3$). Let $A_4 = Z_1 \cap Z_2$, $A_5 = Z_1 \cap Z_3$, and $A_6 = Z_2 \cap Z_3$. Then $Z = A_4 \cup A_5 \cup A_6$ and for each $i$, $4 \leq i \leq 6$, $Z - A_i$ is an even subgraph of $G^*$. Consider first $\overline{Z} - A_4$. Extend $\overline{Z} - A_4$, as explained in the proof of Lemma 2C, to an even subgraph of $G$, say $F$, such that $\overline{Z} - A_4 \cup F \subseteq G - A_4$. Then $\{F, Z\}$ covers all edges of $E(G) - A_4$. Let $f_1$ and $f_2$ be two 2-flows in $G$ with $S(f_1) = F$ and $S(f_2) = Z$. Then $f_1 + 2f_2$ is a 4-flow in $G$ with support $E(G) - A_4$. The same arguments can be applied to $\overline{Z} - A_5$ and $\overline{Z} - A_6$. This completes the proof of Lemma 3.1.

Proof of Theorem 1.1. By Lemma 3.1, $G$ has an even subgraph $Z$ with a partition $E(Z) = A_1 \cup A_2 \cup A_3$ such that $G - A_i$ has a nowhere-zero 3-flow, $1 \leq i \leq 3$. By Lemma 2.B, $G - A_i$ has a 2-cover by 3 even subgraphs, say $\{Z_{i1}, Z_{i2}, Z_{i3}\}$. Thus, the 9 even subgraphs $\{Z_{ij} : 1 \leq i, j \leq 3\}$ together cover each edge in $E(G) - Z$ exactly 6 times and each edge in $Z$ exactly 4 times. Consequently, $\{Z_{ij} \oplus Z : 1 \leq i, j \leq 3\}$ covers each edge in $E(G) - Z$ exactly 6 times and each edge in $Z$ exactly 5 times. Therefore, $\{Z ; Z_{ij} \oplus Z : 1 \leq i, j \leq 3\}$ is a 6-cover of $G$ consisting of 10 even subgraphs, as required.
4. GRAPHS WITH A NOWHERE-ZERO 4-FLOW

**Definition 4.1.** A graph is called a *multi-tree* if it is obtained from a tree by replacing each edge by at least two edges with the same ends. Clearly, a multi-tree is 2-edge-connected and every cycle consists of exactly two edges.

**Definition 4.2.** An *odd-multi-tree* is a multi-tree in which each pair of adjacent vertices is joined by an odd number of edges.

**Definition 4.3.** Let \( \phi \) be a flow in a graph \( G \). Define \( E_{\text{even}}(\phi) = \{ e \in E(G) : \phi(e) \text{ is even} \} \).

**Lemma 4.1.** If \( G \) has a nowhere-zero 2m-flow, let \( \phi \) be a nowhere-zero 2m-flow in \( G \) with \( |E_{\text{even}}(\phi)| \) minimum, then \( E_{\text{even}}(\phi) \) contains no cycle, and hence \( |E_{\text{even}}(\phi)| \leq |V(G)| - 1 \). Moreover equality holds if and only if \( G \) is an odd-multi-tree plus some loops or \( |V(G)| = 1 \).

**Proof.** If \( G \) is even, then \( G \) has a nowhere-zero 2-flow, say \( \phi_1 \). By the choice of \( \phi \), \( |E_{\text{even}}(\phi)| = |E_{\text{even}}(\phi_1)| = 0 \). So \( E_{\text{even}}(\phi) = \emptyset \) and hence \( |E_{\text{even}}(\phi)| \leq |V(G)| - 1 \) with equality if and only if \( |V(G)| = 1 \). Let us assume now that \( G \) is not even and so \( |V(G)| > 1 \) and \( m > 1 \).

If there is a cycle \( C \subseteq E_{\text{even}}(\phi) \), let \( \phi_1 \) be a 2-flow with \( S(\phi_1) = C \) under the same orientation as \( \phi \). Then \( \phi + \phi_1 \) is a nowhere-zero 2m-flow with fewer edges of even value than \( \phi \) contradicting the choice of \( \phi \). This proves the first part of the theorem. For the second part, we consider two cases:

**Case 1.** \( G \) is not a multi-tree plus some loops. If \( |E_{\text{even}}(\phi)| = |V(G)| - 1 \), then \( E_{\text{even}}(\phi) \) is a spanning tree and there is a \( e \in E(G) - E_{\text{even}}(\phi) \) such that \( E_{\text{even}}(\phi) \cup \{ e \} \) has a unique cycle \( C_e \) with \( |C_e| \geq 3 \). As before, let \( \phi_1 \) be a 2-flow with \( S(\phi_1) = C_e \). Since \( m > 1 \), we may choose \( \theta \) in \( \{ 1, -1 \} \) such that \( 0 < |\phi(e) + \theta\phi_1(e)| < 2k \). Then \( \phi' = \phi + \theta\phi_1 \) is a nowhere-zero 2m-flow in which \( \phi'(e) \) is odd for every \( e \in C_e - \{ a \} \). So \( \phi' \) has fewer edges of even value than \( \phi \). This contradicts the choice of \( \phi \) and so \( |E_{\text{even}}(\phi)| \neq |V(G)| - 1 \).

**Case 2.** \( G \) is a multi-tree plus some loops. Let \( T \) be a spanning tree of \( G \). For each edge \( e \in T \), denote by \( B_e \) the subgraph induced by all edges parallel to \( e \) in \( G \). It is easy to see that \( B_e \) has a nowhere-zero 2-flow if \( |B_e| \) is even, and a nowhere-zero 3-flow with exactly one edge of value \( \pm 2 \) if \( |B_e| \) is odd. The sum of these flows, together with nowhere-zero 2-flows on loops, yields a nowhere-zero 3-flow with at most \( |E(T)| \) edges of value \( \pm 2 \). This implies, from the choice of \( \phi \), that \( |E_{\text{even}}(\phi)| = |E(T)| \) with equality only if \( |B_e| \) is odd for every \( e \in T \), namely, only if \( G \) is an odd-multi-tree.
plus some loops. On the other hand, if \( G \) is an odd-multi-tree plus some loops, then each block \( B_e \) is an edge cut of \( G \), and thus, for any nowhere-zero flow of \( G \), \( B_e \) must contain at least one edge of even value. So, 
\[ |E_{\text{even}}(\phi)| \geq |E(T)|, \]
and therefore 
\[ |E_{\text{even}}(\phi)| = |E(T)| = |V(G)| - 1. \]
This proves the second part of the lemma, and completes the proof.

In [2], Bermond, Jackson, and Jaeger proved (see, also, Alon and Tarsi [1]) that every bridgeless graph \( G \) can be covered by even subgraphs of total size at most \( \min\{(5/3)|E(G)|, |E(G)| + (7/3)(|V(G)| - 1)\} \). An alternative proof of the first upper bound, and a generalization to weighted graphs, was given in [5]. The second upper bound was improved by Fraisse [6] to \( |E(G)| + (5/4)(|V(G)| - 1) \). Itai and Rodeh [10] proved that if \( G \) has two edge-disjoint spanning trees, then \( G \) can be covered by two even subgraphs of total size at most \( |E(G)| + |V(G)| - 1 \). Jaeger [13] has show that every graph with two edge-disjoint spanning trees has a nowhere-zero 4-flow. The following theorem generalizes the above result of Itai and Rodeh, and completely characterizes the extremal graphs.

**Theorem 4.1.** If \( G \) has a nowhere-zero 4-flow, then the minimum total size of two even subgraphs which together cover \( G \) is at most 
\[ |E(G)| + |V(G)| - 1, \]
with equality if and only if \( G \) is an odd-multi-tree plus some loops or \( |V(G)| = 1 \).

**Proof.** Let \( \phi \) be a nowhere-zero 4-flow of \( G \) with \( |E_{\text{even}}(\phi)| \) minimum

By Lemma 4.1, 
\[ |E_{\text{even}}(\phi)| \leq |V(G)| - 1, \]
with equality only if \( G \) is an odd-multi-tree plus some loops or \( |V(G)| = 1 \), where 
\[ E_{\text{even}}(\phi) = \{e \in E(G) : \phi(e) = 2\}. \]
For simplicity, set \( E_2 = E_{\text{even}}(\phi) \). It follows from the definition of a flow that \( E_2 \) contains no odd cut of \( G \). This together with Lemma 2.C implies that there is an even subgraph \( Z \) such that \( E_2 \subseteq Z \). Note that \( E(G) - E_2 \) is an even subgraph by Proposition 2.2. We see that 
\[ \{(E(G) - E_2) \oplus Z, Z\} \]
is a cover of \( G \) of total size \( |E(G)| + |E_2| \leq |E(G)| + |V(G)| - 1 \), with equality only if \( G \) is an odd-multi-tree plus some loops or \( |V(G)| = 1 \). Therefore, the minimum total size of two even subgraphs which together cover \( G \) is at most \( |E(G)| + |V(G)| - 1 \), with equality only if \( G \) is an odd-multi-tree plus some loops or \( |V(G)| = 1 \). Conversely, if \( G \) is an odd-multi-tree plus some loops or \( |V(G)| = 1 \), any cycle cover of \( G \) must cover at least one edge more than once between each pair of adjacent vertices of \( G \), and so has total size at least \( |E(G)| + |V(G)| - 1 \). This completes the proof.

It was proved in [9] (also see [2]) that every 2-edge-connected planar graph has a cycle cover of total size equal to an optimal solution of the Chinese Postman Problem. This implies that every bridgeless planar graph \( G \) has a cycle cover of total size at most \( |E(G)| + |V(G)| - 1 \); a different
proof has been obtained by Fraisse [6]. This result is sharpened by the following corollary.

**Corollary 4.1.** Every bridgeless planar graph G can be covered by two even subgraphs of total size at most \(|E(G)| + |V(G)| - 1\), this being best possible if and only if G is an odd-multi-tree plus some loops or \(|V(G)| = 1\).

**Proof.** By the Four-Colour-Theorem, G has a nowhere-zero 4-flow. The result follows from Theorem 4.1.

If we call a graph with just one vertex *trivial*, then Corollary 4.1 implies that every nontrivial simple planar graph G can be covered by two even subgraphs of total size at most \(|E(G)| + |V(G)| - 2\). Does every bridgeless graph G have a cycle cover of total size at most \(|E(G)| + |V(G)| - 1\)? This problem, raised by Itai and Rodeh [10], is still open. Consider the complete bipartite graph G with two parts X and Y, where \(X = \{x_0, x_1, x_2\}\) and \(Y = \{y_1, y_2, ..., y_m\}, m \geq 1\). Let D be the orientation such that every edge of G has tail in X and head in Y. Set \(\phi(e) = -2\) if \(e = x_i y_{im+j}\), where \(0 \leq i \leq 2\) and \(1 \leq j \leq m\), and \(\phi(e) = 1\) otherwise. Then \(\phi\) is a nowhere-zero 3-flow in G. It is clear that any cycle cover of G needs at least \(|E(G)| + |V(G)| - 3\) edges. This shows that the upper bound in the following conjecture, if true, is sharp.

**Conjecture 4.1.** If G is a nontrivial simple graph with a nowhere-zero 3-flow, then G can be covered by two even subgraphs of total size at most \(|E(G)| + |V(G)| - 3\).

It was proved in [2] that every 4-edge-connected graph G can be covered by two even subgraphs of total size at most \((4/3)|E(G)|\). We conclude this paper with the following conjecture and remark.

**Conjecture 4.2.** Every 4-edge graph G can be covered by two even subgraphs of total size at most \((6/5)|E(G)|\).

**Remark 4.1.** The upper bound \((6/5)|E(G)|\) in the above conjecture is best possible, in view of the complete bipartite graph with two parts X and Y such that \(|Y| \geq |X| = 5\).

*Remark added in proof.* Conjecture 4.1 has been proved by A. Raspaud and C. Q. Zhang, independently.

**Acknowledgments**

I thank J. A. Bondy and the referees for their helpful comments and suggestions. Especially, I am in debt to the referee who pointed out errors in the original presentation of Lemma 4.1.
and Theorem 4.1, which has resulted in modifications in the proofs. I have had valuable discussions on the topic in Section 1 with L. Goddyn, who drew my attention to Fulkerson's conjecture.

REFERENCES

12. F. JAEGE, A note on sub-eulerian graphs, J. Graph Theory 3 (1979), 91-93.