JOURNAL OF COMBINATORIAL THEORY, Series B 54, 113–122 (1992)

Integer Flows and Cycle Covers

GENGHUA FAN*

Department of Combinatorics and Optimization, University of Waterloo, Ontario, Canada N2L 3G1

Communicated by the Editors

Received October 27, 1988

Results related to integer flows and cycle covers are presented. A cycle cover of a graph G is a collection $\mathscr{C}$ of cycles of G which covers all edges of G; $\mathscr{C}$ is called a cycle m-cover of G if each edge of G is covered exactly m times by the members of $\mathscr{C}$. By using Seymour's nowhere-zero 6-flow theorem, we prove that every bridgeless graph has a cycle 6-cover associated to covering of the edges by 10 even subgraphs (an even graph is one in which each vertex is of even degree). This result together with the cycle 4-cover theorem implies that every bridgeless graph has a cycle m-cover for any even number $m \geqslant 4$. We also prove that every graph with a nowhere-zero 4-flow has a cycle cover $\mathscr{C}$ such that the sum of lengths of the cycles in $\mathscr{C}$ is at most $|E(G)| + |V(G)| - 2$, unless G belongs to a very special class of graphs.

1. INTRODUCTION

The graphs we consider are finite, but may contain loops and multiple edges. A graph is *simple* if it contains no loops or multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. An edge is said to be *contracted* if it is removed and its ends are identified. A subset S of $E(G)$ is a *cut* of G if its removal leaves a graph with more components and no proper subset of S has this property; S is called a *k-cut* if $|S| = k$. A k-cut is called an *odd cut* if k is odd. A 1-cut is also called a *bridge*. An *even graph* is one in which every vertex is of even degree; an *eulerian graph* is a connected even graph. A graph with every vertex of degree 3 is called a *cubic graph*. Sometimes, we treat a subgraph as a subset of edges. For instance the *symmetric difference* of two even subgraphs Z_1 and Z_2, denoted by $Z_1 \oplus Z_2$, is the even subgraph $(Z_1 \cup Z_2) \setminus (Z_1 \cap Z_2)$. Let G be a graph. A *cover* of G is a collection $\mathscr{H}$ of subgraphs of G which covers all edges of G; $\mathscr{H}$ is called an *m-cover* of G if each edge of G is covered exactly m times by the subgraphs in $\mathscr{H}$. In this paper, we consider the case where each member of $\mathscr{H}$ is an even subgraph

* Present address: Department of Mathematics, Arizona State University, Tempe, Arizona 85287.

0095-8956/92 $3.00

(the empty set $\varnothing$ is regarded as an even subgraph of every graph). It is clear that a graph is even if and only if it has a decomposition into edge-disjoint cycles. For simplicity, a cover (m-cover) by even subgraphs is also called a *cycle cover* (*cycle m-cover*).

The following problem was considered by Szekeres [17] for bridgeless cubic graphs, and independently, was formulated as a conjecture by Seymour [15] for bridgeless graphs. It is now known as the "Cycle Double Cover Conjecture."

Conjecture 1.A. Every bridgeless graph has a cycle 2-cover.

A stronger form of this conjecture was proposed by Celmius [3].

Conjecture 1.B. Every bridgeless graph has a 2-cover by 5 even subgraphs.

Bermond, Jackson, and Jaeger [2] established

THEOREM 1.A. *Every bridgeless graph has a* 4-*cover by* 7 *even subgraphs.*

Goddyn [8] conjectured that every bridgeless graph has a cycle 6-cover and proved that for each bridgeless graph G there exists an integer k (depending on $|E(G)|$) such that G has a cycle $(4k+2)$-cover. If G is a cubic graph, it follows from a simple computation that any cycle $2k$-cover of G requires at least $3k$ even subgraphs, with equality if and only if each of the $3k$ even subgraphs is a 2-factor of G. In the edge-set of a cubic graph, complementation defines a $1-1$ correspondence between the perfect matchings and the 2-factors. Hence, for a cubic graph G, the following two statements are equivalent.

(a) G has a k-cover by $3k$ perfect matchings.

(b) G has a $2k$-cover by $3k$ even subgraphs.

Thus, the matching polytope theorem of Edmonds [4] implies that for each bridgeless cubic graph G there exists an integer k such that G has a $2k$-cover by $3k$ even subgraphs (2-factors). What is the smallest integer k for which every bridgeless cubic graph has a $2k$-cover by $3k$ even subgraphs? This problem, in the perfect matching version, has been studied in great detail by Seymour [14]. Fulkerson [7] (see also [14]) conjectured that every bridgeless cubic graph has a 2-cover by 6 perfect matchings. This is equivalent to

Conjecture 1.C. Every bridgeless cubic graph has a 4-cover by 6 even subgraphs.

It is well known that every bridgeless graph is a contraction of some bridgeless cubic graph ("split" each vertex into a cycle). Hence, the following conjecture is equivalent to Conjecture 1.C.

Conjecture 1.C′. Every bridgeless graph has a 4-cover by 6 even subgraphs.

A weaker form of this conjecture, namely, every bridgeless graph has a 8-cover by 12 even subgraphs, has been suggested by Goddyn (personal communication). Seymour [14] has shown that if k is odd, the Petersen graph has no k-cover by $3k$ perfect mathcings, equivalently, no $2k$-cover by $3k$ even subgraphs. In particular, it has no 6-cover by 9 even subgraphs, and so neither has any graph that can be contracted to the Petersen graph. Is it possible to find a 6-cover by 10 even subgraphs? In this paper, we shall prove

THEOREM 1.1. *Every bridgeless graph has a* 6-*cover by* 10 *even subgraphs.*

Clearly, Theorem 1.A implies that every bridgeless graph has a cycle m-cover for any $m \equiv 0 \pmod 4$. If m is odd, a graph G has a cycle m-cover if and only if G is even. (*Proof.* If G is even, then m copies of G give the required cover. Conversely, suppose that $\{H_1, H_2, ..., H_l\}$ is a cycle m-cover of G and m is odd. Then $G = H_1 \oplus H_2 \oplus \cdots \oplus H_l$ and so G is even.) Theorem 1.1 together with Theorem 1.A yields

THEOREM 1.2. *Every bridgeless graph has a cycle m-cover for any even number $m \geq 4$.*

2. PRELIMINARIES

An *orientation* D of an undirected graph G is an assignment of a direction to each edge $e \in E(G)$. Let G be a graph with orientation D. For each vertex $v \in V(G)$, $E^+(v)$ is the set of non-loop edges with tail v, and $E^-(v)$ the set of non-loop edges with head v. A *flow* in G under orientation D is an integer-valued function ϕ on $E(G)$ such that

$$\sum_{e \in E^+(v)} \phi(e) = \sum_{e \in E^-(v)} \phi(e) \qquad \text{for each} \quad v \in V(G).$$

The *support* of ϕ is defined by

$$S(\phi) = \{e \in E(G) : \phi(e) \neq 0\}.$$

For a positive integer k, if $-k < \phi(e) < k$ for every $e \in E(G)$, then ϕ is called a *k-flow*, and furthermore, if $S(\phi) = E(G)$, then ϕ is called a *nowhere-zero k-flow*. A well-known result on integer flows is the following one by Tutte

((6.3) of [19]), an alternative proof of which has been obtained by Younger [20].

LEMMA 2.A. *If G has a flow ϕ then, for any integer $k > 1$, G has a k-flow ϕ' such that $\phi'(e) \equiv \phi(e) \pmod{k}$ for every $e \in E(G)$.*

If ϕ_1 and ϕ_2 are two flows in G under the same orientation D, let l and m be two integers. Then the sum $\phi = l\phi_1 + m\phi_2$ is a flow in G under D with $\phi(e) = l\phi_1(e) + m\phi_2(e)$ for each $e \in E(G)$. Let ϕ be a flow in G under a given orientation and $e \in E(G)$. If we reverse the direction of e and change $\phi(e)$ to $-\phi(e)$, we still have a flow in G under the new orientation. This means that

PROPOSITION 2.1. *If G has a flow ϕ under some orientation D, then, for any orientation D', G has a flow ϕ' under D' with $|\phi'(e)| = |\phi(e)|$ for every edge e.*

From now on, a flow in G is always associated with some orientation of G, and whenever necessary, we can make two flows in G be associated with the same orientation. The following proposition follows easily from the definition.

PROPOSITION 2.2. *Let ϕ be a flow in G and $Z = \{e \in E(G) : \phi(e)$ is odd$\}$. Then Z is an even subgraph of G.*

By the definition, a k-flow is a k'-flow for any $k' > k$. It is easy to check that the complete graph on 4 vertices has no nowhere-zero 3-flows. Tutte [18] has proved that the Petersen graph has no nowhere-zero 4-flows. Clearly, if ϕ is a flow in G, then ϕ also defines by restriction a flow in the graph obtained by contracting an edge of G. The following famous conjectures are due to Tutte (see [20]).

(1) Every bridgeless graph without 3-cuts has a nowhere-zero 3-flow.

(2) Every bridgeless graph containing no subgraph contractible to the Petersen graph has a nowhere-zero 4-flow.

(3) Every bridgeless graph has a nowhere-zero 5-flow.

Towards a proof of (3), Jaeger [13] established that every bridgeless graph has a nowhere-zero 8-flow. This was improved by Seymour [16] in the following famous theorem.

THEOREM 2.A. *Every bridgeless graph has a nowhere-zero 6-flow.*

By a result derived from Tutte's work by Jaeger [11] (or see [13]), every graph with a nowhere-zero 4-flow can be covered by two even subgraphs, and these two even subgraphs, together with their symmetric difference, form a 2-cover by three even subgraphs. That is,

LEMMA 2.B. *Every graph with a nowhere-zero* 4-*flow has a* 2-*cover by* 3 *even subgraphs.*

The following result was proved by Jaeger [12] by algebraic methods. We present below a constructive proof.

LEMMA 2.C. *Let G be a graph and $H \subseteq E(G)$. H is contained in an even subgraph of G if and only if H contains no odd cut of G.*

Proof. Contract all edges of $E(G) - H$ and denote by G^* the resulting graph. Then $E(G^*) = H$, and H contains no odd cut of G if and only if G^* is even. We show now that G^* is even if and only if H is contained in an even subgraph of G. If G^* is even, then it is the union of a set of edge-disjoint cycles. Clearly, each such cycle can be extended, by adding paths in $E(G) - H$, to a cycle of G and the smmetric difference of all these extended cycles is an even subgraph of G which contains H. Conversely, if H is contained in an even subgraph of G, then G^* is the contraction of this even subgraph and so is even. This proves the lemma. ▮

3. PROOF OF THEOREM 1.1

We first prove the following lemma by applying Seymour's 6-flow theorem (Theorem 2.A).

LEMMA 3.1. *For any bridgeless graph G, there is a partition of $E(G): E(G) = \bigcup_{i=1}^{6} A_i$, where $A_i \cap A_j = \varnothing$ if $i \neq j$, such that $G - A_i$ has a nowhere-zero* 3-*flow if $1 \leqslant i \leqslant 3$ and a nowhere-zero* 4-*flow if $4 \leqslant i \leqslant 6$, and moreover, $A_1 \cup A_2 \cup A_3$ is an even subgraph of G.*

Proof. By Theorem 2.A, G has a nowhere-zero 6-flow ϕ. Set

$$Z = \{e \in E(G) : \phi(e) \text{ is odd}\}.$$

By Proposition 2.2, Z is an even subgraph and hence has a nowhere-zero 2-flow. We shall choose an orientation D of G under which G has a 2-flow ϕ' such that

$$\phi'(e) = \begin{cases} 0, & e \notin Z \\ 1, & e \in Z. \end{cases}$$

Moreover, by Proposition 2.1, G has a nowhere-zero 6-flow ϕ'' under D with $|\phi''(e)| = |\phi(e)|$ for all edges e. Setting

$$E_i = \{e \in E(G) : \phi''(e) = i\},$$

we have that

$$Z = \bigcup \{E_i : i \in \{\pm 1, \pm 3, \pm 5\}\}.$$

Let

$$\phi_i = \phi'' + i\phi', \qquad i \in \{\pm 1, \pm 3, \pm 5\}. \tag{3.1}$$

Then ϕ_i is a flow in G (not necessarily nowhere-zero). As an illustration, we list the vaues of ϕ_{-1} and ϕ_1.

$$\phi_{-1}(e) = \begin{cases} 0, & e \in E_1 \\ -6, & e \in E_{-5} \\ \pm 2, \pm 4, & \text{otherwise} \end{cases} \quad \text{and} \quad \phi_1(e) = \begin{cases} 0, & e \in E_{-1} \\ 6, & e \in E_5 \\ \pm 2, \pm 4, & \text{otherwise.} \end{cases}$$

By Lemma 2.A, ϕ_{-1} yields a 3-flow in G with support $E(G) - (E_1 \cup E_{-5})$. Similarly, ϕ_1 yields a 3-flow with support $E(G) - (E_{-1} \cup E_5)$ and ϕ_3 (or ϕ'') gives a 3-flow with support $E(G) - (E_{-3} \cup E_3)$. Set $A_1 = (E_1 \cup E_{-5})$, $A_2 = (E_{-1} \cup E_5)$, $A_3 = (E_{-3} \cup E_3)$. Then $A_1 \cup A_2 \cup A_3 = Z$ is an even subgraph of G. Furthermore, let $\bar{Z} = E(G) - Z$. If we contract all edges of Z, then $(1/2)\phi$ (or $(1/2)\phi''$) defines by restriction a nowhere-zero 3-flow in the resulting graph G^* with $E(G^*) = \bar{Z}$. By Lemma 2.B, G^* has a 2-cover by 3 even subgraphs, say $\bar{Z}_j$ $(1 \leqslant j \leqslant 3)$. Let $A_4 = \bar{Z}_1 \cap \bar{Z}_2$, $A_5 = \bar{Z}_1 \cap \bar{Z}_3$, and $A_6 \cap \bar{Z}_2 \cap \bar{Z}_3$. Then $\bar{Z} = A_4 \cup A_5 \cup A_6$ and for each i, $4 \leqslant i \leqslant 6$, $\bar{Z} - A_i$ is an even subgraph of G^*. Consider first $\bar{Z} - A_4$. Extend $\bar{Z} - A_4$, as explained in the proof of Lemma 2C, to an even subgraph of G, say F, such that $\bar{Z} - A_4 \subseteq F \subseteq G - A_4$. Then $\{F, Z\}$ covers all edges of $E(G) - A_4$. Let f_1 and f_2 be two 2-flows in G with $S(f_1) = F$ and $S(f_2) = Z$. Then $f_1 + 2f_2$ is a 4-flow in G with support $E(G) - A_4$. The same arguments can be applied to $\bar{Z} - A_5$ and $\bar{Z} - A_6$. This completes the proof of Lemma 3.1. ∎

Proof of Theorem 1.1. By Lemma 3.1, G has an even subgraph Z with a partition $E(Z) = A_1 \cup A_2 \cup A_3$ such that $G - A_i$ has a nowhere-zero 3-flow, $1 \leqslant i \leqslant 3$. By Lemma 2.B, $G - A_i$ has a 2-cover by 3 even subgraphs, say $\{Z_{i1}, Z_{i1}, Z_{i3}\}$. Thus, the 9 even subgraphs $\{Z_{ij} : 1 \leqslant i, j \leqslant 3\}$ together cover each edge in $E(G) - Z$ exactly 6 times and each edge in Z exactly 4 times. Consequently, $\{Z_{ij} \oplus Z : 1 \leqslant i, j \leqslant 3\}$ covers each edge in $E(G) - Z$ exactly 6 times and each edge in Z exactly 5 times. Therefore, $\{Z; Z_{ij} \oplus Z : 1 \leqslant i, j \leqslant 3\}$ is a 6-cover of G consisting of 10 even subgraphs, as required. ∎

4. Graphs with a Nowhere-Zero 4-Flow

Definition 4.1. A graph is called a *multi-tree* if it is obtained from a tree by replacing each edge by at least two edges with the same ends. Clearly, a multi-tree is 2-edge-connected and every cycle consists of exactly two edges.

Definition 4.2. An *odd-multi-tree* is a multi-tree in which each pair of adjacent vertices is joined by an odd number of edges.

Definition 4.3. Let ϕ be a flow in a graph G. Define $E_{\text{even}}(\phi) = \{e \in E(G) : \phi(e) \text{ is even}\}$.

Lemma 4.1. *If G has a nowhere-zero $2m$-flow, let ϕ be a nowhere-zero $2m$-flow in G with $|E_{\text{even}}(\phi)|$ minimum, then $E_{\text{even}}(\phi)$ contains no cycle, and hence $|E_{\text{even}}(\phi)| \leqslant |V(G)| - 1$. Moreover equality holds if and only if G is an odd-multi-tree plus some loops or $|V(G)| = 1$.*

Proof. If G is even, then G has a nowhere-zero 2-flow, say ϕ_1. By the choice of ϕ, $|E_{\text{even}}(\phi)| \leqslant |E_{\text{even}}(\phi_1)| = 0$. So $E_{\text{even}}(\phi) = \varnothing$ and hence $|E_{\text{even}}(\phi)| \leqslant |V(G)| - 1$ with equality if and only if $|V(G)| = 1$. Let us assume now that G is not even and so $|V(G)| > 1$ and $m > 1$.

If there is a cycle $C \subseteq E_{\text{even}}(\phi)$, let ϕ_1 be a 2-flow with $S(\phi_1) = C$ under the same orientation as ϕ. Then $\phi + \phi_1$ is a nowhere-zero $2m$-flow with fewer edges of even value than ϕ contradicting the choice of ϕ. This proves the first part of the theorem. For the second part, we consider two cases:

Case 1. G is not a multi-tree plus some loops. If $|E_{\text{even}}(\phi)| = |V(G)| - 1$, then $E_{\text{even}}(\phi)$ is a spanning tree and there is $a \in E(G) - E_{\text{even}}(\phi)$ such that $E_{\text{even}}(\phi) \cup \{a\}$ has a unique cycle C_a with $|C_a| \geqslant 3$. As before, let ϕ_1 be a 2-flow with $S(\phi_1) = C_a$. Since $m > 1$, we may choose θ in $\{1, -1\}$ such that $0 < |\phi(a) + \theta\phi_1(a)| < 2k$. Then $\phi' = \phi + \theta\phi_1$ is a nowhere-zero $2m$-flow in which $\phi'(e)$ is odd for every $e \in C_a - \{a\}$. So ϕ' has fewer edges of even value than ϕ. This contradicts the choice of ϕ and so $|E_{\text{even}}(\phi)| \neq V(G)| - 1$.

Case 2. G is a multi-tree plus some loops. Let T be a spanning tree of G. For each edge $e \in T$, denote by B_e the subgraph induced by all edges parallel to e in G. It is easy to see that B_c has a nowhere-zero 2-flow if $|B_c|$ is even, and a nowhere-zero 3-flow with exactly one edge of value ± 2 if $|B_e|$ is odd. The sum of these flows, together with nowhere-zero 2-flows on loops, yields a nowhere-zero 3-flow with at most $|E(T)|$ edges of value ± 2. This implies, from the choice of ϕ, that $|E_{\text{even}}(\phi)| \leqslant |E(T)|$ with equality only if $|B_e|$ is odd for every $e \in T$, namely, only if G is an odd-multi-tree plus some loops. On the other hand, if G is an odd-multi-tree plus some loops, then each block B_e is an edge cut of G, and thus, for any nowhere-zero flow of G, B_e must contain at least one edge of even value. So, $|E_{\text{even}}(\phi)| \geqslant |E(T)|$, and therefore $|E_{\text{even}}(\phi)| = |E(T)| = |V(G)| - 1$. This proves the second part of the lemma, and completes the proof. ∎

In [2], Bermond, Jackson, and Jaeger proved (see, also, Alon and Tarsi [1]) that every bridgeless graph G can be covered by even subgraphs of total size at most $\min\{(5/3)\,|E(G)|, |E(G)| + (7/3)(|V(G)| - 1)\}$. An alternative proof of the first upper bound, and a generalization to weighted graphs, was given in [5]. The second upper bound was improved by Fraisse [6] to $|E(G)| + (5/4)(|V(G)| - 1)$. Itai and Rodeh [10] proved that if G has two edge-disjoint spanning trees, then G can be covered by two even subgraphs of total size at most $|E(G) + |V(G)| - 1$. Jaeger [13] has show that every graph with two edge-disjoint spanning trees has a nowhere-zero 4-flow. The following theorem generalizes the above result of Itai and Rodeh, and completely characterizes the extremal graphs.

THEOREM 4.1. *If G has a nowhere-zero 4-flow, then the minimum total size of two even subgraphs which together cover G is at most $|E(G)| + |V(G)| - 1$, with equality if and only if G is an odd-multi-tree plus some loops or $|V(G)| = 1$.*

Proof. Let ϕ be a nowhere-zero 4-flow of G with $|E_{\text{even}}(\phi)|$ minimum By Lemma 4.1, $|E_{\text{even}}(\phi)| \leqslant |V(G) - 1$, with equality only if G is an odd-multi-tree plus some loops or $|V(G)| = 1$, where $E_{\text{even}}(\phi) = \{e \in E(G) : |\phi(e)| = 2\}$. For simplicity, set $E_2 = E_{\text{even}}(\phi)$. It follows from the definition of a flow that E_2 contains no odd cut of G. This together with Lemma 2.C implies that there is an even sugraph Z such that $E_2 \subseteq Z$. Note that $E(G) - E_2$ is an even subgraph by Proposition 2.2. We see that $\{(E(G) - E_2) \oplus Z, Z\}$ is a cover of G of total size $|E(G)| + |E_2| \leqslant |E(G)| + |V(G)| - 1$, with equality only if G is an odd-multi-tree plus some loops or $|V(G)| = 1$. Therefore, the minimum total size of two even subgraphs which together cover G is at most $|E(G)| + |V(G)| - 1$, with equality only if G is an odd-multi-tree plus some loops or $|V(G)| = 1$. Conversely, if G is an odd-multi-tree plus some loops or $|V(G)| = 1$, any cycle cover of G must cover at least one edge more than once between each pair of adjacent vertices of G, and so has total size at least $|E(G)| + |V(G)| - 1$. This completes the proof. ∎

It was proved in [9] (also see [2]) that every 2-edge-connected planar graph has a cycle cover of total size equal to an optimal solution of the Chinese Postman Problem. This implies that every bridgeless planar graph G has a cycle cover of total size at most $|E(G)| + |V(G)| - 1$; a different proof has been obtained by Fraisse [6]. This result is sharpened by the following corollary.

COROLLARY 4.1. *Every bridgeless planar graph G can be cofered by two even subgraphs of total size at most $|E(G)| + |V(G)| - 1$, this being best possible if and only if G is an odd-multi-tree plus some loops or $|V(G)| = 1$.*

Proof. By the Four-Colour-Theorem, G has a nowhere-zero 4-flow. The result follows from Theorem 4.1.

If we call a graph with just one vertex *trivial*, then Corollary 4.1 implies that every nontrivial simple planar graph G can be covered by two even subgraphs of total size at most $|E(G)| + |V(G)| - 2$. Does every bridgeless graph G have a cycle cover of total size at most $|E(G)| + |V(G)| - 1$? This problem, raised by Itai and Rodeh [10], is still open. Consider the complete bipartite graph G with two parts X and Y, where $X = \{x_0, x_1, x_2\}$ and $Y = \{y_1, y_2, ..., y_{3m}\}$, $m \geqslant 1$. Let D be the orientation such that every edge of G has tail in X and head in Y. Set $\phi(e) = -2$ if $e = x_i y_{im+j}$, where $0 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant m$, and $\phi(e) = 1$ otherwise. Then ϕ is a nowhere-zero 3-flow in G. It is clear that any cycle cover of G needs at least $|E(G)| + |V(G) - 3$ edges. This shows that the upper bound in the following conjecture, if true, is sharp.

Conjecture 4.1. If G is a nontrivial simple graph with a nowhere-zero 3-flow, then G can be covered by two even subgraphs of total size at most $|E(G)| + |V(G)| - 3$.

It was proved in [2] that every 4-edge-connected graph G can be covered by two even subgraphs of total size at most $(4/3)\,|E(G)|$. We conclude this paper with the following conjecture and remark.

Conjecture 4.2. Every 4-edge graph G can be covered by two even subgraphs of total size at most $(6/5)\,|E(G)|$.

Remark 4.1. The upper bound $(6/5)\,|E(G)|$ in the above conjecture is best possible, in view of the complete bipartite graph with two parts X and Y such that $|Y| \geqslant |X| = 5$.

Remark added in proof. Conjecture 4.1 has been proved by A. Raspaud and C. Q. Zhang, independently.

ACKNOWLEDGMENTS

I thank J. A. Bondy and the referees for their helpful comments and suggestions. Especially, I am in debt to the referee who pointed out errors in the original presentation of Lemma 4.1

and Theorem 4.1, which has resulted in modifications in the proofs. I have had valuable discussions on the topic in Section 1 with L. Goddyn, who drew my attention to Fulkerson's conjecture.

REFERENCES

1. N. ALON AND M. TARSI, Covering multigraphs by simple circuits, *SIAM J. Discrete Math.* **6** (1985), 345–350.
2. J. C. BERMOND, B, JACKSON, AND F. JAEGER, Shortest covering of graphs with cycles, *J. Combin. Theory Ser. B* **35** (1983), 297–308.
3. U. A. CELMINS, "On Cubic Graphs That Do Not Have an Edge-3-Coloring," Ph.D. thesis, University of Waterloo, Waterloo, 1984.
4. J. EDMONDS, Maximum matching and a polyhedron with 0, 1-vertices, *J. Res. Nat. Bur. Standards Sect. B* **69** (1965), 125–130.
5. G. FAN, Covering weighted graphs by even subgraphs, *J. Combin. Theory Ser. B* **49** (1990), 137–141.
6. P. FRAISSE, Cycle covering in bridgeless graphs, *J. Combin. Theory Ser. B* **39** (1985), 146–152.
7. D. R. FULKERSON, Blocking and antiblocking pairs of polyhedra, *Math. Programming* **1** (1971), 168–194.
8. L. GODDYN, Talk, Workshop on cycles and rays, Montréal, 1987.
9. M. GUAN AND H. FLEISCHNER, On the minimum weighted cycle covering problem for planar graphs, *Ars. Combin.* **20** (1985), 61–68.
10. A. ITAI AND M. RODEH, Covering a graph by circuits, *in* "Automata, Languages and Programming," pp. 289–299, Lecture Notes in Computer Science, Vol. 62, Springer-Verlag, Berlin, (1978).
11. F. JAEGER, On nowhere-zero flows in multigraphs, *Congr. Numer.* **15** (1975), 373–378.
12. F. JAEGER, A note on sub-eulerian graphs, *J. Graph Theory* **3** (1979), 91–93.
13. F. JAEGER, Flows and generalized coloring theorems in graphs, *J. Combin. Theory Ser. B* **26** (1979), 205–216.
14. P. D. SEYMOUR, On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte, *Proc. London Math. Soc. (3)* **38** (1979), 423–460.
15. P. D. SEYMOUR, Sums of circuits, *in* "Graph Theory and Related Topics" (J. A. Bondy and U. S. R. Murty, Eds.), pp. 341–355, Academic Press, New York, 1979.
16. P. D. SEYMOUR, Nowhere-zero 6-flows, *J. Combin. Theory Ser. B* **30** (1981), 130–135.
17. G. SZEKERES, Polyhedral decompositions of cubic graphs, *Bull. Austral. Math. Soc.* **8** (1973), 367–387.
18. W. T. TUTTE, A contribution to the theory of chromatic polynomials, *Canad. J. Math.* **6** (1954), 80–91.
19. W. T. TUTTE, A class of abelian groups, *Canad. J. Math.* **8** (1956), 13–28.
20. D. H. YOUNGER, Integer flows, *J. Graph Theory* **7** (1983), 349–357.