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Nowhere-Zero 6-Flows

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We prove that every graph with no isthmus has a nowhere-zero 6-flow, that is, a circulation in which the value of the flow through each edge is one of ± 1, $\pm 2, \ldots, \pm 5$. This improves Jaeger's 8-flow theorem, and approaches Tutte's 5-flow conjecture.

1. INTRODUCTION

Graphs in this paper may have loops or multiple edges. A *cut* of the graph $G = (V, E)$ is the set of edges with one end in X_1 and one in X_2, where (X_1, X_2) is a non-trivial partition of V. An *isthmus* is an edge e such that $\{e\}$ is a cut. A graph is *k-edge-connected* if it has no cut of cardinality less than k, and *k-connected* if no graph obtained by deleting at most $k-1$ vertices has more than one component. (Thus complete graphs are k-connected for all k.) When $G = (V, E)$ is a directed graph and $v \in V$, $\partial^+(v)$ is the set of non-loop edges with tail v, and $\partial^-(v)$ the set with head v. A *circulation* in a directed graph $G = (V, E)$ is a real-valued function ϕ on E such that for each $v \in V$,

$$\sum_{e \in \partial^-(v)} \phi(e) = \sum_{e \in \partial^+(v)} \phi(e).$$

When $k > 1$ is an integer, a circulation ϕ is a *k-flow* if for each edge e, $\phi(e)$ is an integer and $-k < \phi(e) < k$. The *support* $S(\phi)$ of ϕ is defined to be $\{e \in E: \phi(e) \neq 0\}$. A *nowhere-zero* k-flow is a k-flow ϕ with $S(\phi) = E$.

A number $\kappa(G)$ of particular interest here is the least integer k such that G has a nowhere-zero k-flow. If G has an isthmus then it cannot have a nowhere-zero k-flow for any k and we set $\kappa(G) = \infty$, but if not then $\kappa(G)$ is defined and finite.

It is easy to see that if G has a nowhere-zero k-flow under some directing of its edges, then it has one under every directing, and so the number $\kappa(G)$ is a function of the underlying undirected graph.

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Tutte [5] observed that when G is a planar graph drawn in the plane, there is a natural correspondence between k-colourings of the faces of the map defined by this drawing and the nowhere-zero k-flows of G. In particular, $\kappa(G)$ is the chromatic number of the map. The use of k-flows thus enables us to extend familiar map-colouring problems to non-planar graphs.

There is of course another way to do this; by using planar duality, map colouring problems become vertex colouring problems, and we can study vertex colouring problems for any graph, planar or not. However, the chromatic number of non-planar graphs can become arbitrarily large. One interesting feature of the number $\kappa(G)$ we have defined is that even for non-planar graphs it stays small.

The four-colour theorem says that $\kappa(G) \leqslant 4$ for planar graphs G without isthmuses. This is not true for all graphs—the Petersen graph has $\kappa(G) = 5$—and so the four-colour theorem cannot be generalized to all graphs in this way. However, Tutte [5] conjectured that the five-colour theorem could be.

(1.1) CONJECTURE. *Every graph with no isthmus has a nowhere-zero 5-flow.*

Jaeger [2, 3] greatly increased the plausibility of (1.1) by proving that there was a universal upper bound for $\kappa(G)$, in the following.

(1.2) *Every graph with no isthmus has a nowhere-zero 8-flow.*

Our main result is that 8 may be replaced by 6 in (1.2). We prove this theorem in Section 3, using a graph-theoretic lemma which we prove in Section 4. Section 2 contains some preliminary results about k-flows.

2. k-FLOWS

The first preliminary is the following. The result is well-known in folk-lore, but no complete proof appears anywhere as far as I know.

(2.1) *For $k > 2$, if $G = (V, E)$ is a graph with no isthmus but with $\kappa(G) > k$, and G has $|V| + |E|$ minimum, then G is simple, cubic and 3-connected.*

Proof. It is clear that G is loopless, 2-connected, and 2-edge-connected. Suppose that $\{e_1, e_2\}$ is a cut of cardinality 2. Then the result of contracting e_1 has no isthmus and has no nowhere-zero k-flow, contrary to the choice of G. Thus G is 3-edge-connected. Suppose next that some vertex v has valency greater than 3. Then by a theorem of Fleischner [1] there are two edges e_1, e_2 incident with v, with other ends u_1, u_2, say, so that if we delete e_1, e_2 and join u_1, u_2 the resulting graph G' has no isthmus; but G' cannot have a nowhere-zero k-flow (because if it did we could find one for G), contradicting the choice of G. Thus no vertices have valency >3, and yet G is 3-edge-connected, and $|V| \geqslant 3$ (since $k > 2$). If follows that G is simple, cubic, and 3-connected, as required.

Our second preliminary is the following, due to Tutte [4–6].

(2.2) *If $k > 0$ and ϕ' is any integer-valued circulation in $G = (V, E)$, then there is a k-flow ϕ such that for each $e \in E$,*

$$\phi(e) \equiv \phi'(e) \qquad (\text{mod } k).$$

3. Proof of the Theorem

When $G = (V, E)$, $X \subseteq E$ and $k > 0$, we define $\langle X \rangle_k$ to be the smallest set $Y \subseteq E$ with the following properties:

(i) $X \subseteq Y$;

(ii) there is no circuit C of G with $0 < |C - Y| \leqslant k$.

[It is convenient to regard a circuit simply as a set of edges rather than as a subgraph.] It is easy to see that if Y_1, Y_2 both satisfy (i) and (ii) then so does $Y_1 \cap Y_2$, and so $\langle X \rangle_k$ is uniquely defined.

It is easy to see that $X \to \langle X \rangle_k$ is a closure operator, that is:

$$X \subseteq \langle X \rangle_k; \qquad \langle \langle X \rangle_k \rangle_k = \langle X \rangle_k; \qquad X \subseteq Y \to \langle X \rangle_k \subseteq \langle Y \rangle_k.$$

(3.1) *Let $G = (V, E)$ be directed, let $k > 1$, and let $X \subseteq E$ have $\langle X \rangle_{k-1} = E$. Then G has a k-flow ϕ with $E - X \subseteq S(\phi)$.*

Proof. We proceed by induction on $|E - X|$. If this is zero the result is trivial, and we assume not. Then $\langle X \rangle_{k-1} \neq X$, and so there is a circuit C with $0 < |C - X| \leqslant k - 1$. Certainly $\langle X \cup C \rangle_{k-1} = E$, and so by induction there is a k-flow ϕ with $E - (X \cup C) \subseteq S(\phi)$. Take a circulation ψ so that $S(\psi) = C$ and $\psi(e) = 0$ or ± 1 $(e \in E)$. Choose an integer n such that $0 \leqslant n \leqslant k - 1$ and $n \not\equiv -\phi(e)/\psi(e)$ (mod k) for each $e \in C - X$ (this is possible since $|C - X| \leqslant k - 1$). Put $\phi' = \phi + n\psi$. Then for $e \in E - (X \cup C)$, $\phi'(e) = \phi(e)$, and for $e \in C - X$, $\phi'(e) = \phi(e) + n\psi(e) \not\equiv 0$ (mod k) by choice of n. Thus for each $e \in E - X$, $\phi'(e) \not\equiv 0$ (mod k). The result follows from (2.2), applied to ϕ'.

Remark. If G is a 3-edge-connected planar graph then $\langle \varnothing \rangle_5 = E$ by the usual argument from Euler's formula, and so $\kappa(G) \leqslant 6$. This is just a disguised version of the usual proof of the six-colour theorem.

Our main theorem follows from (3.1) and the following lemma, proved in the next section.

(3.2) *If* $G=(V, E)$ *is a simple* 3*-connected graph and* $|V| \geqslant 3$, *there are vertex-disjoint circuits* $C_1, \ldots, C_r$ *such that* $\langle C_1 \cup \cdots \cup C_r \rangle_2 = E$.

The main result is

(3.3) *If* $G=(V, E)$ *has no isthmus, then* G *has a nowhere-zero* 6*-flow*.

Proof. By (2.1), it is enough to prove (3.3) for graphs $G=(V, E)$ which are simple, cubic, and 3-connected. In that case, from (3.2), there exists $X \subseteq E$, expressible as a disjoint union of circuits, so that $\langle X \rangle_2 = E$. Let ϕ_1 be a 3-flow with $E - X \subseteq S(\phi_1)$; this exists by (3.1). Let ϕ_2 be a 2-flow with $S(\phi_2) = X$. Put $\phi = \phi_1 + 3\phi_2$. Then for $e \in E - X$, $|\phi(e)| = 1$ or 2, and for $e \in X$, $\phi(e) = \phi_1(e) \pm 3$. In either case, $|\phi(e)| < 6$ and $\phi(e) \neq 0$, and so ϕ is a nowhere-zero 6-flow, as required.

4. Proof of the Lemma

We require the following elementary result.

(4.1) *Let* H *be a non-null simple graph in which all vertices have valency at least* 2. *Then there is a subgraph* B *of* H *with at least three vertices, so that* B *is* 2*-connected and at most one vertex of* B *is adjacent in* H *to vertices of* H *not in* B.

Proof. Let $\mathscr{B}$ be the collection of *blocks* of H (that is, maximal 2-connected subgraphs). Let $\mathscr{A}$ be the collection of vertices which are in more than one block. We define a bipartite graph Γ on $\mathscr{A} \cup \mathscr{B}$ by saying that $A \in \mathscr{A}$, $B \in \mathscr{B}$ are adjacent if A is a vertex of B. Now Γ has no circuits, for if it has a circuit, then it has an induced circuit $A_1 B_1 \cdots A_r B_r A_1$, say, where $r \geqslant 2$; but then the "union" of the blocks $B_1 \cdots B_r$ is 2-connected in H, contrary to the maximality of block B_1. But H is non-null, and so has an edge, and every edge is in a block; and so $\mathscr{B} \neq \varnothing$, and Γ is non-null. It follows that Γ has a vertex of valency at most 1. But each $A \in \mathscr{A}$ has valency $\geqslant 2$ in Γ, and so H has a block B with at most one vertex in other blocks. Now H has no isolated vertices, and so B has at least two vertices. Every edge of H is in a block and so at most one vertex of B is adjacent in H to vertices not in B. Let u be a vertex of B not adjacent to any vertices not in B. Since H is simple and u has valency at least 2, it follows that B has at least 3 vertices. This completes the proof.

Proof of (3.2). We say that $X \subseteq E$ is *connected* if the subgraph of G consisting of X and vertices incident with edges in X is connected. Let

$G=(V, E)$ be as in (3.2). Choose a circuit C. Then $\langle C\rangle_2$ is connected, since G is simple, and so we can choose $r \geqslant 1$ maximum such that there are vertex-disjoint circuits $C_1, \ldots, C_r$ with $\langle C_1 \cup \cdots \cup C_r\rangle_2$ connected. Put $C_1 \cup \cdots \cup C_r = X$, and $\langle X\rangle_2 = Y$. Let U be the set of vertices of G incident with edges in Y, and let H be the graph obtained by deleting U from G. Suppose for a contradiction that H is non-null. No vertex v of H is adjacent in G to distinct $u_1, u_2 \in U$: for otherwise, since Y is connected, there would be a path P connecting u_1 and u_2 using only edges in Y, and then P together with v would give a circuit contradicting the definition of $\langle X\rangle_2$. Thus every vertex of H has valency at least 2 in H. From (4.1), there is a 2-connected subgraph B of H with at least three vertices, and with at most one vertex adjacent in H to vertices of H not in B. Since $U \neq \varnothing$, G is 3-connected and B has at least three vertices, there are at least three vertices of B adjacent in G to vertices of G not in B. Hence there are distinct vertices, b_1, b_2 of B, both adjacent in G to vertices in U. Since B is 2-connected and has at least three vertices, there is a circuit C_{r+1}, say, of B using both b_1 and b_2. Then $C_1, \ldots, C_{r+1}$ are vertex-disjoint; and if e_1, e_2 are edges joining b_1, b_2, respectively, to U, then $\{e_1, e_2\} \subseteq \langle C_1 \cup \cdots \cup C_{r+1}\rangle_2$, and so $\langle C_1 \cup \cdots \cup C_{r+1}\rangle_2$ is connected. This is contrary to the maximality of r.

It follows that H is null and $U = V$. Hence $\langle Y\rangle_2 = E$, since Y is connected; but $Y = \langle X\rangle_2$, and so $\langle X\rangle_2 = E$, as required.

5. A Strengthening

Jaeger proved his theorem by observing that in a 3-edge-connected graph $G=(V, E)$, there is a partition $E=(X_1, X_2, X_3)$ so that

$$\langle E - X_i\rangle_1 = E \qquad (i = 1, 2, 3)$$

and then manipulating statements like (3.1) and (2.2). Our method is not quite the same; we find a partition (X_1, X_2) so that X_1 is a cycle and $\langle X_1\rangle_2 = E$.

But it is possible to preserve the symmetry of Jaeger's method, at the cost of considerably complicating the proof. One can show the following, from which (3.2) (in the relevant cubic case) and hence (3.3) both follow.

(5.1) *Let $G=(V, E)$ be a 3-connected cubic graph. Then there is a partition $E=(X_1, X_2)$ so that*

$$\langle X_1\rangle_1 = E, \qquad \langle X_2\rangle_2 = E.$$

Sketch proof. Choose a partition (A, B, C) of E with C minimal so that

(i) $A \cup B$ is 2-edge-connected;

(ii) C is connected;

(iii) $A \cup C$ includes a spanning tree of G;

(iv) $A \subseteq \langle B \rangle_2$.

If $C = \varnothing$ we are finished, because we may take $X_1 = A$, $X_2 = B$. Suppose for a contradiction that $C \neq \varnothing$. Then there is a subgraph P of G with the following properties:

(i) $E(P) \subseteq C$;

(ii) if $A \cup B = \varnothing$, P is a circuit, and if $A \cup B \neq \varnothing$ then P is a path with both end vertices incident with edges in $A \cup B$;

(iii) $C - E(P)$ is connected.

(Prove!) Let X be the set of end edges of P; put $A' = A \cup X$, $B' = B \cup (E(P) - X)$, $C' = C - E(P)$. Then (A', B', C') contradicts the choice of (A, B, C).

References

1. H. Fleischner, Eine gemeinsame Basis für die Theorie der Eulerschen Graphen und der Satz von Petersen, *Monat. Math.* **81** (1976), 267–278.
2. F. Jaeger, Flows and generalized coloring theorems in graphs, *J. Combinatorial Theory Ser. B* **26** (1979), 205–216.
3. F. Jaeger, On nowhere-zero flows in multigraphs, in "Proceedings, Fifth British Combinatorial Conference, Aberdeen, 1975," Congressus Numerantium XV, Utilitas Mathematica Winnipeg, 373–378.
4. W. T. Tutte, A class of Abelian groups, *Canad. J. Math.* **8** (1956), 13–18.
5. W. T. Tutte, A contribution to the theory of chromatic polynomials, *Canad. J. Math.* **6** (1954), 80–91.
6. W.T. Tutte, On the imbedding of linear graphs in surfaces, *Proc. London Math. Soc. Ser.* 2 **51** (1950), 474–483.