Squaring a Tournament: A Proof of Dean's Conjecture

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ABSTRACT

Let the square of a tournament be the digraph on the same nodes with arcs where the directed distance in the tournament is at most two. This paper verifies Dean's conjecture: any tournament has a node whose outdegree is at least doubled in its square. © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION

In a “round robin tournament,” each team plays every other team exactly once. Assuming no ties, for each pair of teams $i$ and $j$, either $i$ beats $j$, or $j$ beats $i$, but not both (see the table below). Team $i$ “sort-of-beats” team $j$ if either $i$ beats $j$, or $i$ beats some team which beat $j$. Instead of looking at the teams they beat, a team with a modest record might prefer to look at the usually larger set they sort-of-beam.

The results can be recorded with a “tournament”. A digraph $D$ is a set of nodes $V(D) = \{v_1, v_2, \ldots, v_n\}$ and a set of ordered pair of nodes $R(D)$ called arcs. If $(v_i,v_j) \in R(D)$, we will say $v_i$ beats $v_j$ in $D$. Let the outset $O_D(v_i)$ be the nodes that $v_i$ beats and let the inset $I_D(v_i)$ be the nodes that beat $v_i$. Let $d_D(v_i) = |O_D(v_i)|$ be the outdegree of $v_i$. A tournament $T$ is a digraph where either $(v_i,v_j) \in T$ or $(v_j,v_i) \in T$, but not both. Figure 1(a) shows the tournament $T$ which records the results from the table. If team $i$ beats team $j$ in the table, then $v_i$ beats $v_j$ in $T$ depicted with an arrow pointing from $v_i$ to $v_j$.

The relation “sort-of-beats” is shown with the “square” of a digraph. The square of a digraph $D$ (notated $D^2$) is a digraph with $V(D^2) = V(D)$ and with $(v_i,v_j) \in R(D^2)$ if either $(v_i,v_j) \in R(D)$ or there is a $k$ with $(v_i,v_k) \in R(D)$ and $(v_k,v_j) \in R(D)$. Figure 1(b) shows the square of the tournament in Figure 1(a). It has an arc from $v_i$ to $v_j$ if the directed distance from $v_i$ to $v_j$ in the tournament is at most two.
Dean conjectured that any tournament has a node whose outdegree at least doubles in its square. Here there are three such nodes: $v_2$, $v_3$, and $v_5$.

Dean's conjecture is that any tournament $T$ has a node $v_i$ with $d_{T^2}(v_i) \geq 2d_T(v_i)$. It is clearly true if $d_T(v_i) = 0$ for some node $v_i$, because then $d_{T^2}(v_i) = 0$. If the minimum outdegree of $T$ is one, some node $v_i$ with $d_T(v_i) = 1$ beats $v_j$ which must beat another node $v_k$. Thus $v_i$ beats both $v_j$ and $v_k$ in $T^2$ giving $d_{T^2}(v_i) \geq 2$. Dean and Latka [1] continued this to verify the conjecture for tournaments whose minimum outdegree is five or less. They also verified the conjecture for regular and almost regular tournaments (where outdegrees differ by at most one).

The problem with an approach based on the minimum outdegree of a tournament is that the outdegree of nodes with minimum outdegree may not double in its square. Figure 2 gives an example of this. Instead this paper uses a “losing” probability density to verify Dean's conjecture. Section 2 defines a losing density and shows it exists for every digraph. Section 3 shows that when a node is picked from a losing density on a tournament $T$, its expected outdegree in $T^2$ is at least twice what it is in $T$.

Dean's conjecture is a special case of another conjecture. An oriented graph is a loopless digraph with at most one arc between each pair of nodes. Seymour (quoted from [1]) conjectured that an oriented graph $D$ has a node $v_i$ with $d_D(v_i) \geq 2d_D(v_i)$. The more general conjecture remains unresolved. To highlight why the approach here does not extend to oriented graphs, results are proved to the greatest extent possible for digraphs.

### TABLE I

A fictitious round-robin tournament. A team “sort-of-beats” another if it either beats that team, or beats a team which beat that team. For example, team 6 sort-of-beats teams 1, 2, 3, 4 and 5, because it beats 2, 3 and 5, it beats 3 which beats 4, and it beats 5 which beats 1.

<table>
<thead>
<tr>
<th>Team</th>
<th>Teams they beat</th>
<th>Teams they “sort-of-beat”</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2, 3, 4, 6, 8</td>
<td>2, 3, 4, 5, 6, 7, 8</td>
</tr>
<tr>
<td>2</td>
<td>3, 5</td>
<td>1, 3, 4, 5</td>
</tr>
<tr>
<td>3</td>
<td>4, 5</td>
<td>1, 2, 4, 5, 6</td>
</tr>
<tr>
<td>4</td>
<td>2, 5, 6</td>
<td>1, 2, 3, 5, 6</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1, 2, 3, 4, 6, 8</td>
</tr>
<tr>
<td>6</td>
<td>2, 3, 5</td>
<td>1, 2, 3, 4, 6</td>
</tr>
<tr>
<td>7</td>
<td>1, 2, 3, 4, 5, 6</td>
<td>1, 2, 3, 4, 5, 6, 8</td>
</tr>
<tr>
<td>8</td>
<td>2, 3, 4, 5, 6, 7</td>
<td>1, 2, 3, 4, 5, 6, 7</td>
</tr>
</tbody>
</table>
FIGURE 2. An Example. The center node in this tournament has outdegree five, while all other nodes have outdegree at least six. However the center node beats only nine nodes in $T^2$. Thus, it is not always true that a node of minimum outdegree beats at least twice as many nodes in $T^2$ as it beats in $T$ (adapted from [1]).

2. WINNING AND LOSING DENSITIES

If a node of a digraph never loses, it is a winner. What if there are no winners? Fisher and Ryan [2] and Laslier, Laffond, and Le Breton [3] independently developed the idea of a "winning density" of a tournament. Here this idea is extended to digraphs.

A (probability) density $f$ on a digraph $D$ gives each node a nonnegative value with $f(V(D)) = 1$ (let the probability of a set be the sum of the probabilities of its members). A density $w$ is winning if $w(I_D(v_i)) \geq w(O_D(v_i))$ for all nodes $v_i$. In other words for any node $v_i$, a random node picked from a winning density is at least as likely to beat $v_i$ as it is to lose to $v_i$ (see Figure 3(a)).

If a digraph has a winner, picking that node with probability 1 (and any other node with probability 0) gives a winning density. So a winner can be thought of as a winning density, but digraphs without winners can have winning densities as seen in Figure 3. Thus winning densities are a generalization of winners.

Do all digraphs have winning densities? Fisher and Ryan [2] and Laslier, Laffond, and Le Breton [3] showed the answer is yes for tournaments (they also showed that a tournament has only one winning density, and it gives positive probabilities to an odd number of nodes). Theorem 1 uses Farkas's Lemma (see for example, Solow [4, p. 279]) to extend this. Let $\mathbf{0}$ and $\mathbf{1}$ be vectors of all zeros and ones, respectively. For vectors $x$ and $y$, let $x \geq y$ mean that $x_i \geq y_i$ for all $i$.

**Farkas's Lemma.** Given a matrix $M$ and a vector $b$, exactly one of these systems has a solution: (a) $Mx = b$ with $x \geq 0$; or (b) $MTy \geq 0$ with $b^Ty < 0$.

Let the adjacency matrix $A(D)$ of a digraph $D$ with $n$ nodes be the $n \times n$ matrix with $a_{ij} = 1$ if $v_i$ beats $v_j$ and $a_{ij} = 0$ otherwise. Let $K(D) = A(D) - A(D)^T$. For a density $w$,
let the associated vector be \( w = (w(v_1), w(v_2), \ldots, w(v_n))^T \). Then \( w \geq 0 \) and \( 1^T w = 1 \).

Further \( (K(D)w)_i = w(O_T(v_i)) - w(I_T(v_i)) \). So \( w \) is winning if \( K(D)w \leq 0 \).

**Theorem 1.** Any digraph \( D \) has a winning density. Further for a winning density \( w \), if \( w(v_i) > 0 \), then \( w(O_D(v_i)) = w(I_D(v_i)) \).

**Proof.** Suppose \( D \) has no winning density. Then this system has no solutions (\( I \) is the identity matrix):

\[
\begin{bmatrix}
K(D) & I \\
1^T & 0^T
\end{bmatrix}
\begin{bmatrix}
w \\
z
\end{bmatrix}
= \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\text{ with } \begin{bmatrix} w \\ z \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Since \( K(D)^T = -K(D) \), Farkas’s Lemma shows this system has a solution:

\[
\begin{bmatrix}
-K(D) & 1 \\
I & 0
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
\geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\text{ with } (0^T \ 1) \begin{bmatrix} u \\ v \end{bmatrix} < 0.
\]

Thus \( u \geq 0 \) and \( K(D)u \leq v1 \) with \( v < 0 \). So \( K(D)u < 0 \) and hence \((1^T u)^{-1} u\) is the associated vector of a winning density, a contradiction. Therefore \( D \) has a winning density.

Now let \( w \) be a winning density on \( D \) with associated vector \( w \). Then \( K(D)w \leq 0 \) and \( w \geq 0 \) and hence \( (w)_i(K(D)w)_i \leq 0 \) for all \( i \). Since \( K(D) \) is skew-symmetric, \( w^T K(D)w = 0 \). Thus \( (w)_i(K(D)w)_i = 0 \) for all \( i \). Therefore if \( (w)_i = w(v_i) > 0 \), then

\[ 0 = (K(D)w)_i = w(O_T(v_i)) - w(I_T(v_i)). \]

So if \( w(v_i) > 0 \), then \( w(O_D(v_i)) = w(I_D(v_i)) \).

A density \( l \) on a digraph \( D \) is losing if \( l(I_D(v_i)) \leq l(O_D(v_i)) \) for all nodes \( v_i \) (see Figure 3(b)). Since a losing density is a winning density on the digraph formed by reversing its arcs, Theorem 1 has a counterpart for losing densities.

**Corollary.** Any digraph \( D \) has a losing density. Further for a losing density \( l \), if \( l(v_i) > 0 \), then \( l(O_D(v_i)) = l(I_D(v_i)) \).
3. AN ANALYTICAL PROOF

Given a real value function \( r \) on the nodes of a digraph \( D \), the expected value of \( r \) on a random node picked from density \( f \) is

\[
E_f(r) = \sum_{i=1}^{n} f(v_i)r(v_i).
\]

Lemma 1 gives an indirect way to find the expected outdegree. For the density in Figure 3(b), the expected outdegree is \( 4 \cdot \frac{1}{9} \cdot 2 + \frac{2}{9} \cdot 2 + \frac{1}{3} \cdot 3 + \frac{1}{3} \cdot 1 = \frac{27}{9} \). Lemma 1 shows this equals the sum of the probabilities of the insets: \( \frac{1}{3} + \frac{4}{9} + \frac{2}{3} + \frac{1}{3} + \frac{2}{9} + 0 + \frac{1}{3} = \frac{27}{9} \).

**Lemma 1.** Let \( D \) be a digraph with density \( f \). Then

\[
E_f(d_D) = \sum_{j=1}^{n} f(I_D(v_j)).
\]

**Proof.** Since \( v_j \in O_D(v_i) \) if and only if \( v_i \in I_D(v_j) \), we have

\[
E_f(d_D) = \sum_{i=1}^{n} f(v_i)d_D(v_i) = \sum_{i=1}^{n} \sum_{v_j \in O(v_i)} f(v_i) = \sum_{j=1}^{n} f(v_i) = \sum_{j=1}^{n} f(I_D(v_j)).
\]

Lemma 2 only applies to a losing density \( l \) on a tournament \( T \). It shows that given a node \( v_i \), the probability that a node picked from \( l \) will beat \( v_i \) is at least twice as much in \( T^2 \) as in \( T \). For example, the probability a node picked from the density in Figure 3(b) will beat \( v_6 \) is \( \frac{1}{9} \) in \( T \) and \( \frac{1}{5} \) in \( T^2 \).

**Lemma 2.** Let \( l \) be a losing density on a tournament \( T \). Then \( l(I_{T^2}(v_i)) \geq 2l(I_T(v_i)) \) for all nodes \( v_i \).

**Proof.** Since \( l \) is a losing density, \( l(I_T(v_i)) \leq \frac{1}{2} \). If \( l(I_{T^2}(v_i)) = 1 \), we are done. Otherwise let \( Q \) be the subtournament \( V(T) - I_{T^2}(v_i) \). Within \( Q \), we have

\[
\sum_{v_j \in V(Q)} l(v_j)l(I_Q(v_j)) = \sum_{v_j \in V(Q)} \sum_{v_k \in I_Q(v_j)} l(v_j)l(v_k) = \sum_{v_k \in V(Q)} \sum_{v_j \in O_Q(v_k)} l(v_j)l(v_k) = \sum_{v_k \in V(Q)} l(v_k)l(O_Q(v_k)).
\]

Since \( l(I_{T^2}(v_i)) < 1 \) and hence \( l(V(Q)) > 0 \), we have \( l(I_Q(v_h)) \geq l(O_Q(v_h)) \) for some \( v_h \in V(Q) \) with \( l(v_h) > 0 \).

Since \( v_h \notin I_{T^2}(v_i) \), no node of \( I_T(v_i) \) can lose to \( v_h \). Then each node of \( I_T(v_i) \) beats \( v_h \) because \( T \) is a tournament. So \( I_T(v_i) \subseteq I_T(v_h) \). Since \( I_T(v_i) \subseteq I_{T^2}(v_i) \), we then get \( I_T(v_i) \subseteq I_T(v_h) \cap I_{T^2}(v_i) \). Since \( Q \) is the subtournament \( V(T) - I_{T^2}(v_i) \), we have

\[
l(I_T(v_h)) = l(I_Q(v_h)) + l(I_T(v_h) \cap I_{T^2}(v_i)) \geq l(I_Q(v_h)) + l(I_T(v_i)).
\]
Similarly, $O_T(v_h) \subseteq O_T(v_i)$ and hence $O_T(v_h) \cap I_{T^2}(v_i) \subseteq O_T(v_i) \cap I_{T^2}(v_i) = I_{T^2}(v_i) - I_T(v_i)$. Therefore

$$l(O_T(v_h)) = l(O_Q(v_h)) + l(O_T(v_h) \cap I_{T^2}(v_i)) \leq l(O_Q(v_h)) + l(I_{T^2}(v_i) - I_T(v_i)).$$

Since $l(v_h) > 0$, the corollary in Section 2 gives $l(O_T(v_h)) = l(I_T(v_h))$. Thus

$$l(O_Q(v_h)) + l(I_{T^2}(v_i) - I_T(v_i)) \geq l(I_Q(v_h)) + l(I_T(v_i)).$$

Since $l(O_Q(v_h)) \geq l(I_Q(v_h))$, we then have $l(I_{T^2}(v_i) - I_T(v_i)) \geq l(I_T(v_i))$. The result then follows because $I_T(v_h) \subseteq I_{T^2}(v_h)$.

Theorem 2 shows that for a node picked from a losing density on a tournament $T$, its expected outdegree in $T^2$ is at least twice what it is in $T$. So for the density in Figure 3(b), the expected outdegree in $T^2$ is $\frac{1}{3} \cdot 7 + \frac{1}{5} \cdot 4 + \frac{1}{5} \cdot 5 + \frac{1}{5} \cdot 5 + \frac{1}{3} \cdot 6 = 5\frac{5}{9}$ which is more than double the expected outdegree in $T$ (calculated above to be $2\frac{2}{5}$).

**Theorem 2.** Let $l$ be a losing density on a tournament $T$. Then $E_l(d_{T^2}) \geq 2E_l(d_T)$.

**Proof.** Lemma 1 (applied to both $T$ and $T^2$) and Lemma 2 give

$$E_l(d_{T^2}) = \sum_{j=1}^{n} l(I_{T^2}(v_j)) \geq \sum_{j=1}^{n} 2l(I_T(v_j)) = 2E_l(d_T).$$

**Corollary.** In any tournament $T$, there is a node $v_i$ with $d_{T^2}(v_i) \geq 2d_T(v_i)$.

**References**


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