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IMPROVED BOUNDS FOR THE CROSSING NUMBERS OF $K_{m,n}$ AND K_n*

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Abstract. It has been long conjectured that the crossing number $\mathrm{cr}(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ equals the Zarankiewicz number $Z(m,n) := \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{m}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor$. Another longstanding conjecture states that the crossing number $\mathrm{cr}(K_n)$ of the complete graph K_n equals $Z(n) := \frac{1}{4} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$. In this paper we show the following improved bounds on the asymptotic ratios of these crossing numbers and their conjectured values:

(i) for each fixed $m \geq 9$, $\lim_{n\to\infty} \mathrm{cr}(K_{m,n})/Z(m,n) \geq 0.83m/(m-1)$;
(ii) $\lim_{n\to\infty} \mathrm{cr}(K_{n,n})/Z(n,n) \geq 0.83$; and
(iii) $\lim_{n\to\infty} \mathrm{cr}(K_n)/Z(n) \geq 0.83$.

The previous best known lower bounds were $0.8m/(m-1)$, 0.8, and 0.8, respectively. These improved bounds are obtained as a consequence of the new bound $\mathrm{cr}(K_{7,n}) \geq 2.1796n^2 - 4.5n$. To obtain this improved lower bound for $\mathrm{cr}(K_{7,n})$, we use some elementary topological facts on drawings of $K_{2,7}$ to set up a quadratic program on 6! variables whose minimum p satisfies $\mathrm{cr}(K_{7,n}) \geq (p/2)n^2 - 4.5n$, and then use state-of-the-art quadratic optimization techniques combined with a bit of invariant theory of permutation groups to show that $p \geq 4.3593$.

Key words. crossing number, semidefinite programming, copositive cone, invariants and centralizer rings of permutation groups

AMS subject classifications. 05C10, 05C62, 90C22, 90C25, 57M15, 68R10

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1. Introduction.

In the earliest known instance of a crossing number question, Turán raised the problem of calculating the crossing number of the complete bipartite graphs $K_{m,n}$. Turán's interesting account of the origin of this problem can be found in [27].

We recall that in a *drawing* of a graph in the plane, different vertices are drawn as different points, and each edge is drawn as a simple arc whose endpoints coincide with the drawings of the endvertices of the edge. Furthermore, the interior of the arc for an edge is disjoint from all the vertex points. We often make no distinction between a graph object, such as a vertex, edge, or cycle, and the subset of the plane that represents it in a drawing of the graph.

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise intersections of edges (at a point other than a vertex) in a drawing of G in the plane.

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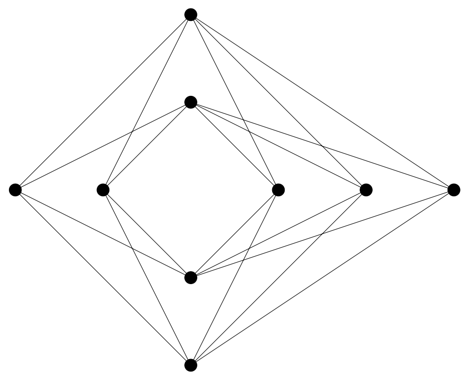

FIG. 1. *A drawing of* $K_{4,5}$ *with* 8 *crossings. A similar strategy can be used to construct drawings of* $K_{m,n}$ *with exactly* $Z(m,n)$ *crossings.*

Exact crossing numbers of graphs are in general very difficult to compute. Longstanding conjectures involve the crossing numbers of interesting families of graphs, such as $K_{m,n}$ and K_n. On a positive note, it was recently proved by Glebsky and Salazar [9] that the crossing number of the Cartesian product $C_m \times C_n$ of the cycles of sizes m and n equals its long conjectured value, namely $(m-2)n$, at least for $n \geq m(m+1)$. For recent surveys of crossing number results, see [23] or [26].

Zarankiewicz published a paper [29] in which he claimed that $\mathrm{cr}(K_{m,n}) = Z(m,n)$ for all positive integers m, n, where

$$Z(m,n) = \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor. \tag{1}$$

However, several years later Ringel and Kainen independently found a hiatus in Zarankiewicz's argument. A comprehensive account of the history of the problem, including a discussion of the gap in Zarankiewicz's argument, is given by Guy [11].

Figure 1 shows a drawing of $K_{4,5}$ with 8 crossings. As Zarankiewicz observed, such a drawing strategy can be naturally generalized to construct, for any positive integers m, n, drawings of $K_{m,n}$ with exactly $Z(m,n)$ crossings. This observation implies the following well-known upper bound for $\mathrm{cr}(K_{m,n})$:

$$\mathrm{cr}(K_{m,n}) \leq Z(m,n).$$

No one has yet exhibited a drawing of any $K_{m,n}$ with fewer than $Z(m,n)$ crossings. In allusion to Zarankiewicz's failed attempt to prove that this is the crossing number of $K_{m,n}$, the following is commonly known as *Zarankiewicz's crossing–number conjecture*:

$$\mathrm{cr}(K_{m,n}) \stackrel{?}{=} Z(m,n) \qquad \text{for all positive integers } m, n.$$

In 1973, Guy and Erdős [6] wrote, "Almost all questions that one can ask about crossing numbers remain unsolved." More than three decades later, despite some definite progress in our understanding of this elusive parameter, most of the fundamental and more important questions about crossing numbers remain open. Zarankiewicz's conjecture has been verified by Kleitman [13] for $\min\{m,n\} \leq 6$ and by Woodall [28] for the special cases $7 \leq m \leq 8$, $7 \leq n \leq 10$.

Since the crossing number of $K_{m,n}$ is unknown for all other values of m and n, it is natural to ask what are the best general lower bounds known for $\mathrm{cr}(K_{m,n})$. A standard counting argument, together with the fact that $\mathrm{cr}(K_{5,n})$ is as conjectured, yields the best general lower bound (2) known for $\mathrm{cr}(K_{m,n})$. It goes as follows: Suppose we know a lower bound c_r on $\mathrm{cr}(K_{r,n})$ for $2 < r < m \leq n$. Each crossing in the embedding of $K_{m,n}$ lies in $\binom{m-2}{r-2}$ distinct $K_{r,n} \subset K_{m,n}$. As there are in total $\binom{m}{r}$ distinct $K_{r,n}$'s, one obtains

$$\mathrm{cr}(K_{m,n}) \geq \frac{c_r\binom{m}{r}}{\binom{m-2}{r-2}}; \quad \text{for } r = 5 \text{ one derives} \quad \mathrm{cr}(K_{m,n}) \geq 0.8\, Z(m,n). \tag{2}$$

A small improvement on the 0.8 factor (roughly to something around 0.8001) was recently reported by Nahas [18].

Zarankiewicz's conjecture for $K_{7,n}$ states that

$$\mathrm{cr}(K_{7,n}) \stackrel{?}{=} 9\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor = \begin{cases} 2.25n^2 - 4.5n + 2.25, & n \text{ odd}, \quad n \geq 7, \\ 2.25n^2 - 4.5n, & n \text{ even}, \quad n \geq 8. \end{cases}$$

As we observed above, this has been verified only for $n = 7, 8, 9$, and 10. Using $\mathrm{cr}(K_{7,10}) = 180$, a standard counting argument gives the best known lower bounds for $\mathrm{cr}(K_{7,n})$ for $11 \leq n \leq 22$. However, for $n \geq 23$, the best known lower bounds for $\mathrm{cr}(K_{7,n})$ are obtained by the same counting argument, but using the known value of $\mathrm{cr}(K_{5,n})$ instead of $\mathrm{cr}(K_{7,10})$. Summarizing, previous to this paper, the best known lower bounds for $\mathrm{cr}(7,n)$ were

$$\mathrm{cr}(K_{7,n}) \geq \begin{cases} 2n(n-1), & 11 \leq n \leq 22, \\ 2.1n^2 - 4.2n + 2.1, & \text{odd } n \geq 23, \\ 2.1n^2 - 4.2n, & \text{even } n \geq 24. \end{cases} \tag{3}$$

In this paper we prove the following theorem.

THEOREM 1. *For all integers n,*

$$\mathrm{cr}(K_{7,n}) > 2.1796n^2 - 4.5n.$$

An elementary calculation shows that this is an improvement, for all $n \geq 23$, on the bounds for $\mathrm{cr}(K_{7,n})$ given in (3).

The strategy of the proof can be briefly outlined as follows. Let (A, B) be the bipartition of the vertex set of $K_{7,n}$, where $|A| = 7$ and $|B| = n \geq 2$. Let b, b' be vertices in B. In any drawing $\mathcal{D}$ of $K_{7,n}$, the number of crossings that involve an edge incident with b and an edge incident with b' is bounded from below by a function of the cyclic rotation schemes of b and b'. This elementary topological observation on drawings of $K_{2,7}$ naturally yields a standard quadratic (minimization) program whose minimum p satisfies $\mathrm{cr}(K_{7,n}) \geq (p/2)n^2 - 4.5n$ (see Lemma 2). We then use state-of-the-art quadratic programming techniques to show that $p \geq 4.3593$ (see Proposition 3), thus implying Theorem 1.

The rest of this paper is organized as follows. In section 2, we review some elementary topological observations about drawings of $K_{2,n}$ and use these facts to set up the quadratic program mentioned in the previous paragraph. The bound for $\mathrm{cr}(K_{7,n})$ in terms of the minimum of this quadratic program is the content of Lemma 2. In section 3 we prove Proposition 3, which gives a lower bound for the quadratic program. As we observe at the end of section 3, Theorem 1 is an obvious consequence of Lemma 2 and Proposition 3. In section 4 we discuss consequences of Theorem 1: The improved bound for $\mathrm{cr}(K_{7,n})$ implies improved asymptotic bounds for the crossing numbers of $\mathrm{cr}(K_{m,n})$ and $\mathrm{cr}(K_n)$.

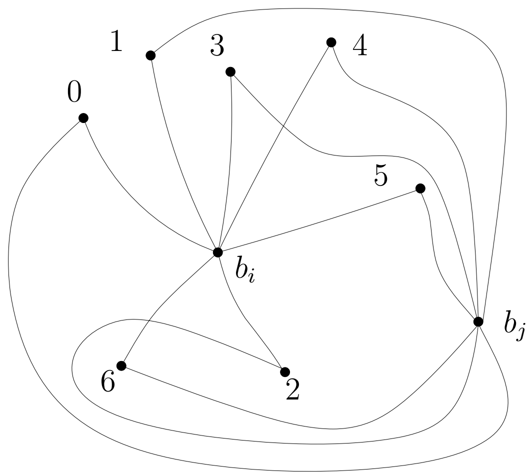

FIG. 2. *Here $m = 7$. Vertices b_i and b_j have cyclic orderings* $(0\,1\,3\,4\,5\,2\,6)$ *and* $(0\,2\,6\,5\,3\,4\,1)$*, respectively (we write i for a_i for the sake of brevity). It is easy to check that the minimum number of interchanges among adjacent elements in* $(0\,1\,3\,4\,5\,2\,6)$ *required to obtain* $(0\,2\,6\,5\,3\,4\,1)^{-1}$ *(namely* $(0\,1\,4\,3\,5\,6\,2)$*) is* 2*. Thus,* $Q((0\,1\,3\,4\,5\,2\,6), (0\,1\,3\,4\,5\,2\,6)) = 2$*. Therefore, there must be at least* 2 *crossings (as is indeed the case in the drawing above) that involve edges incident with b_i and b_j.*

2. Quadratic optimization problem yielding a lower bound for cr($K_{m,n}$). Our goal in this section is to establish Lemma 2, a statement that gives a lower bound for $\mathrm{cr}(K_{m,n})$ for $m \le n$ (and thus for $\mathrm{cr}(K_{7,n})$) in terms of the solution of a quadratic minimization problem on $(m-1)!$ variables.

Let $n \ge m$ be fixed. Let V denote the vertex set of $K_{m,n}$, and let (A, B) denote the bipartition of V such that each vertex of $A = \{a_0, a_1, \ldots, a_{m-1}\}$ is adjacent to each vertex of $B = \{b_0, b_1, \ldots, b_{n-1}\}$.

Consider a fixed drawing $\mathcal{D}$ of $K_{m,n}$. To each vertex b_i we associate a cyclic ordering $\pi_{\mathcal{D}}(b_i)$ of the elements in A, defined by the (clockwise) cyclic order in which the edges incident with b_i leave b_i toward the vertices in A (see Figure 2). Let Π denote the set of all cyclic orderings of $\{a_0, a_1, \ldots, a_{m-1}\}$. Note that $|\Pi| = m!/m = (m-1)!$.

Following Kleitman [13], let $\mathrm{cr}_{\mathcal{D}}(b_i, b_j)$ denote the number of crossings in $\mathcal{D}$ that involve an edge incident with b_i and an edge incident with b_j. Further, let $\rho_1, \rho_2 \in \Pi$ and $Q(\rho_1, \rho_2)$ be the minimum number of interchanges of adjacent elements of ρ_1 required to produce ρ_2^{-1}. Then, for all b_i, b_j with $b_i \ne b_j$,

$$\mathrm{cr}_{\mathcal{D}}(b_i, b_j) \ge Q(\pi_{\mathcal{D}}(b_i), \pi_{\mathcal{D}}(b_j)). \tag{4}$$

This inequality is stated in [13] and proved in [28]. This observation alone yields a lower bound for $\mathrm{cr}(K_{m,n})$, as follows. Fix any drawing $\mathcal{D}$ of $K_{m,n}$. For each $\rho \in \Pi$, let

$$x_\rho := \frac{1}{n} |\{b_i \in B \mid \pi_{\mathcal{D}}(b_i) = \rho\}|.$$

The matrix Q can be viewed as the matrix of quadratic form $Q(\cdot, \cdot)$ on the space $\mathbb{R}^{|\Pi|}$.

It follows from (4) that

$$\begin{aligned}\operatorname{cr}(\mathcal{D}) &\geq \sum_{\substack{\rho,\rho'\in\Pi\\ \rho\neq\rho'}} Q(\rho,\rho')(x_\rho n)(x_{\rho'} n) + \sum_{\rho\in\Pi} Q(\rho,\rho)\binom{x_\rho n}{2}\\ &= \frac{n}{2}\left(n\sum_{\rho,\rho'\in\Pi} Q(\rho,\rho')x_\rho x_{\rho'} - \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\right),\end{aligned}$$

using the (easily verifiable; see, e.g., [28]) fact that $Q(\rho,\rho) = \lfloor m/2\rfloor\lfloor(m-1)/2\rfloor$ for every $\rho\in\Pi$.

Since the drawing $\mathcal{D}$ was arbitrary, we have proved the following lemma.

LEMMA 2. *Let Q be the $(m-1)!\times(m-1)!$ matrix of the form $Q(\cdot,\cdot)$, and let e denote the all ones vector. Then, for every integer $n\geq m\geq 2$,*

$$\operatorname{cr}(K_{m,n}) \geq \frac{n}{2}\left(n\min\{x^TQx \mid x\in\mathbb{R}_+^{(m-1)!}, e^Tx=1\} - \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\right),$$

$$\operatorname{cr}(K_{7,n}) \geq \frac{n}{2}(n\min\{x^TQx \mid x\in\mathbb{R}_+^{6!}, e^Tx=1\} - 9).$$

Remark. In this paper we focus on the case $m=7$. For obvious reasons (for $m=7$, Q is a 720×720 matrix) we do not include in this paper the matrix Q in table form. As we mentioned above, $Q(\rho,\rho)=9$ for every $\rho\in\Pi$, and therefore all the diagonal entries of Q are 9. It is not difficult to show that $Q(\rho,\rho')\leq 8$ if $\rho\neq\rho'$, so every nondiagonal entry of Q is at most 8. The calculation of the entries of Q, using the definition of $Q(\cdot,\cdot)$ and taking its symmetries into account (see section 3.2), takes only a few seconds of computer time.

3. Finding a lower bound for the optimization problem. Our aim in this section is to find a (reasonably good) lower bound for the quadratic programming problem with $m=7$ given in Lemma 2, in order to obtain a (reasonably good) lower bound for $\operatorname{cr}(K_{7,n})$. The main result in this section is the following.

PROPOSITION 3. *Let Q be the $6!\times 6!$ matrix of the quadratic form $Q(\cdot,\cdot)$. Then*

$$\min\{x^TQx \mid x\in\mathbb{R}_+^{6!}, e^Tx=1\} \geq 4.3593.$$

We devote this section to the proof of Proposition 3. It involves computer calculations; more details on this are given in section 3.8.

3.1. The standard quadratic programming problem. The problem we have formulated is known as *standard quadratic optimization problem.* The standard quadratic optimization problem (standard QP) is to find the global minimizers of a quadratic form over the standard simplex; i.e., we consider the global optimization problem

$$\underline{p} := \min_{x\in\Delta} x^TQx, \tag{5}$$

where Q is an arbitrary symmetric $d\times d$ matrix, e is the all ones vector, and Δ is the standard simplex in $\mathbb{R}^d$,

$$\Delta = \{x\in\mathbb{R}_+^d : e^Tx = 1\}.$$

We will now reformulate the standard QP as a convex optimization problem in conic form. First, we will review the relevant convex cones as well as the duality theory of conic optimization. We define the following convex cones:

- the $d \times d$ symmetric matrices:

$$\mathcal{S}_d = \{X \in \mathbb{R}^d \times \mathbb{R}^d,\ X = X^T\};$$

- the $d \times d$ symmetric positive semidefinite matrices:

$$\mathcal{S}_d^+ = \{X \in \mathcal{S}_d,\ y^T X y \geq 0\ \ \forall y \in \mathbb{R}^d\};$$

- the $d \times d$ symmetric copositive matrices:

$$\mathcal{C}_d = \{X \in \mathcal{S}_d,\ y^T X y \geq 0\ \ \forall y \in \mathbb{R}^d,\ y \geq 0\};$$

- the $d \times d$ symmetric completely positive matrices:

$$\mathcal{C}_d^* = \left\{ X = \sum_{i=1}^{k} y_i y_i^T,\ y_i \in \mathbb{R}^d,\ y_i \geq 0\ (i = 1, \ldots, k) \right\};$$

- the $d \times d$ symmetric nonnegative matrices:

$$\mathcal{N}_d = \{X \in \mathcal{S}_d,\ X_{ij} \geq 0\ \ (i, j = 1, \ldots, d)\}.$$

Recall that the completely positive cone is the dual of the copositive cone [12], and that the nonnegative and semidefinite cones are self-dual for the inner product $\langle X, Y \rangle := \mathrm{Tr}(XY)$, where "Tr" denotes the trace operator.

For a given cone $\mathcal{K}_d$ and its dual cone $\mathcal{K}_d^*$ we define the primal and dual pair of conic linear programs:

$$\text{(P)} \qquad p^* := \inf_{X \in \mathcal{K}_d} \left\{ \mathrm{Tr}(CX) \,|\, \mathrm{Tr}(A_i X) = b_i\ (i = 1, \ldots, M) \right\},$$

$$\text{(D)} \qquad d^* := \sup_{y \in \mathbb{R}^m} \left\{ b^T y \,\middle|\, \sum_{i=1}^{M} y_i A_i + S = C,\ S \in \mathcal{K}_d^* \right\}.$$

If $\mathcal{K}_d = \mathcal{S}_d^+$, we refer to semidefinite programming; if $\mathcal{K}_d = \mathcal{N}_d$, to linear programming; and if $\mathcal{K}_d = \mathcal{C}_d$, to copositive programming.

The well-known conic duality theorem (see, e.g., Renegar [20]) gives the duality relations between (P) and (D).

THEOREM 4 (conic duality theorem). *If there exists an interior feasible solution* $X^0 \in \mathrm{int}(\mathcal{K}_d)$ *of* (P) *and a feasible solution of* (D)*, then* $p^* = d^*$ *and the supremum in* (D) *is attained. Similarly, if there exist feasible* y^0, S^0 *for* (D)*, where* $S^0 \in int(\mathcal{K}_d^*)$*, and a feasible solution of* (P)*, then* $p^* = d^*$ *and the infimum in* (P) *is attained.*

Optimization over the cones $\mathcal{S}_d^+$ and $\mathcal{N}_d$ can be done in polynomial time (to compute an ϵ-optimal solution), but some NP-hard problems can be formulated as copositive programs; see, e.g., de Klerk and Pasechnik [14].

3.1.1. Convex reformulation of the standard QP. We rewrite problem (5) in the following way:

$$\underline{p} := \min_{x \in \Delta} \mathrm{Tr}(Qxx^T).$$

Now we define the cone of matrices

$$\mathcal{K} = \left\{ X \in \mathcal{S}_d \ : \ X = xx^T, \ x \geq 0 \right\}.$$

Note that the requirement $x \in \Delta$ corresponds to $X \in \mathcal{K}$ with $\mathrm{Tr}(ee^T X) = 1$.

We arrive at the following reformulation of problem (5):

$$\underline{p} = \min \left\{ \mathrm{Tr}(QX) \ : \ \mathrm{Tr}(ee^T X) = 1, \ X \in \mathcal{K} \right\}. \tag{6}$$

The last step is to replace the cone $\mathcal{K}$ by its convex hull, which is simply the cone of completely positive matrices, i.e.,

$$\mathrm{conv}\,(\mathcal{K}) = \mathcal{C}_d^* = \left\{ X = \sum_{i=1}^{k} y_i y_i^T, \ y_i \in \mathbb{R}^n, \ y_i \geq 0 \ (i = 1, \ldots, k) \right\}.$$

Replacing the feasible set by its convex hull does not change the optimal value of problem (6), since its objective function is linear. Thus we obtain the well-known convex reformulation

$$\underline{p} = \min \left\{ \mathrm{Tr}(QX) \mid \mathrm{Tr}\big(ee^T X\big) = 1, \ X \in \mathcal{C}_d^* \right\}. \tag{7}$$

The dual problem takes the form

$$\underline{p} = \max \left\{ t \mid Q - tee^T \in \mathcal{C}_d \right\}, \tag{8}$$

where $\mathcal{C}_d$ is the cone of copositive matrices, as before. Note that both problems have the same optimal value, in view of the conic duality theorem.

3.2. Exploiting group symmetries. We can reduce considerably the number of variables in the optimization problems in (7), (8) by exploiting the invariance properties of the quadratic function $x^T Q x$. This will also prove to be computationally necessary for the problems we intend to solve.

Consider the situation where the matrix Q is invariant under the action of a group G of order $k = |G|$ of permutation matrices $P \in G$, in the sense that

$$Q = P^T Q P \quad \forall \ P \in G.$$

Then we have

$$\begin{aligned}
\underline{p} &= \min \left\{ \mathrm{Tr}(QX) \mid \mathrm{Tr}\big(ee^T X\big) = 1, \ X \in \mathcal{C}_d^* \right\} \\
&= \min \left\{ \mathrm{Tr}\big(P^T Q P X\big) \mid \mathrm{Tr}\big(Pee^T P X\big) = 1, \ X \in \mathcal{C}_d^* \right\} \text{ for any } P \in G \\
&= \min \left\{ \mathrm{Tr}\big(Q P^T X P\big) \mid \mathrm{Tr}\big(ee^T P^T X P\big) = 1, \ X \in \mathcal{C}_d^* \right\} \text{ for any } P \in G \\
&= \min \left\{ \mathrm{Tr}\left(Q \frac{1}{k} \left[\sum_{P \in G} P^T X P \right] \right) \middle| \mathrm{Tr}\left(ee^T \left[\frac{1}{k} \sum_{P \in G} P^T X P \right] \right) = 1, \ X \in \mathcal{C}_d^* \right\}.
\end{aligned}$$

We can therefore restrict the optimization to the subset of the feasible set obtained by replacing each feasible X by the *group average* $\frac{1}{k} \sum_{P \in G} P^T X P$, i.e., replacing X by its image under what is known in invariant theory as the Reynolds operator. Note that if $X \in \mathcal{C}_d^*$, then so is its image under the group average.

In particular, we wish to compute a basis for the so-called *fixed point subspace*

$$\mathcal{A} := \left\{ Y \in \mathcal{S}_d \mid Y = \frac{1}{k} \sum_{P \in G} P^T X P,\ X \in \mathcal{S}_d \right\}.$$

Note that Q and ee^T are elements of $\mathcal{A}$ (set $X = Q$, respectively, $X = ee^T$). Hence $Q - tee^T \in \mathcal{A}$ for any t, and

$$\underline{p} = \max \left\{ t \mid Q - tee^T \in \mathcal{C}_d \right\} = \max \left\{ t \mid Q - tee^T \in \mathcal{C}_d \cap \mathcal{A} \right\}.$$

The right-hand side here is the dual of the primal problem when it is restricted to $\mathcal{A}$ as above.

The next step is to compute a basis for the subspace $\mathcal{A}$.

3.3. Computing a basis for the fixed point subspace. We assume for simplicity that G acts transitively as a permutation group on the standard basis vectors. (This holds in our setting. A more general, and computationally less efficient, setting can be found in Gatermann and Parrilo [8].) The theory here is well known and goes back to Burnside, Schur, and Wielandt. See, e.g., Cameron [5] for details. Although we need a basis of $\mathcal{A}$, the subspace of *symmetric* matrices fixed by G, it is more natural to compute the basis $\mathcal{X}$ of the subspace $\mathcal{B}$ of *all* matrices fixed by G and then pass on to $\mathcal{A}$.

The dimension of $\mathcal{B}$ equals the number r of orbits of G on the Cartesian square of the standard basis. The set of the latter orbits, also known as 2-orbits, naturally corresponds to certain set $\mathcal{X}$ of $d \times d$ zero-one matrices. Namely, for each $X \in \mathcal{X}$ one has $X_{ij} = 1$ if and only if $X_{P(i),P(j)} = 1$ for all $P \in G$ and all $1 \le i \le j \le |\Pi|$. As G is transitive on the standard basis vectors, the identity matrix I belongs to $\mathcal{X}$. We also have $\sum_{X \in \mathcal{X}} X = ee^T$.

As $\mathcal{X}$ is closed under the matrix transposition, i.e., $X^T \in \mathcal{X}$ for any $X \in \mathcal{X}$,

$$\mathcal{X}_{\mathcal{A}} = \{A_1, \ldots, A_M\} = \{X \mid X = X^T \in \mathcal{X}\} \cup \{X + X^T \mid X \in \mathcal{X},\ X \neq X^T\}$$

is a basis of $\mathcal{A}$. Each $A \in \mathcal{X}_{\mathcal{A}}$ is a symmetric zero-one matrix, and $\sum_{A \in \mathcal{X}_{\mathcal{A}}} A = ee^T$. Moreover,

$$\left\{ Y \in \mathcal{S}_d \mid Y = \sum_{i=1}^{M} y_i A_i \right\} = \mathcal{A} \equiv \left\{ Y \in \mathcal{S}_d \mid Y = \frac{1}{k} \sum_{P \in G} P^T X P,\ X \in \mathcal{S}_d \right\}.$$

Since $Q \in \mathcal{A}$, we will write $Q = \sum_{i=1}^{M} b_i A_i$.

It is worth mentioning that algebraically the vector space $\mathcal{B}$ behaves very nicely: it is closed under multiplication. In other words, $\mathcal{B}$ is a matrix algebra of dimension r, also known as the *centralizer ring* of the permutation group G.

We proceed to describe G and $\mathcal{B}$ in our case. For us G is isomorphic to the direct product $\mathrm{Sym}(m) \times \mathrm{Sym}(2)$ of symmetric groups $\mathrm{Sym}(m)$ and $\mathrm{Sym}(2)$, where $\mathrm{Sym}(m)$ acts (as a permutation group) by conjugation on the $d = (m-1)!$ elements of Π, and $\mathrm{Sym}(2)$ acts (as a permutation group) on Π by switching $\pi \in \Pi$ with $\pi^{-1} \in \Pi$.

Computing $\mathcal{X}$ is an elementary combinatorial procedure, which can be found in one form or another in many computer algebra systems, so one does not have to program this again. First, the permutations that generate $\mathrm{Sym}(m) \times \mathrm{Sym}(2)$ in its action on Π are computed. The action of $\mathrm{Sym}(2)$ is already known, and is described by the permutation g_0, say. In its usual action on m symbols, $\mathrm{Sym}(m)$ is generated by $h_1 = (0, 1, \ldots, m-1)$ and $h_2 = (0, 1)$. These h_i (for $i = 1, 2$) act on Π by mapping each $\pi \in \Pi$ to $h_i \pi h_i^{-1}$. Denote by g_i (for $i = 1, 2$) the permutations of Π that realize these actions.

Next, one computes the orbits of the permutation group $\mathrm{Sym}(m) \times \mathrm{Sym}(2) = \langle g_0, g_1, g_2 \rangle$ on the Cartesian square $\Pi \times \Pi$ of Π, by "spinning" $(\pi_i, \pi_j) \in \Pi \times \Pi$: Begin with $S_{ij} = \{(\pi_i, \pi_j)\}$ and apply the generators g_i, $0 \le i \le 2$, in a loop until S_{ij} stops growing. Then one sets $\Pi := \Pi - S_{ij}$ and repeats until Π is exhausted.

When $m = 7$, one has $r = 78$ and $M = 56$. Note that here the algebra $\mathcal{B}$ is not commutative.

When $m = 5$, one has $r = M = 6$, and $\mathcal{B}$ is commutative.

3.4. Reformulation of the optimization problem. We can now reformulate the dual problem by using the basis of $\mathcal{A}$ to obtain

$$\underline{p} = \max\left\{t \mid Q - tee^T \in \mathcal{C}_d \cap \mathcal{A}\right\} = \max\left\{t \,\middle|\, \sum_{i=1}^{M} (b_i - t) A_i \in \mathcal{C}_d\right\}.$$

We will now proceed to derive a lower bound on $\underline{p}$ by solving the dual problem approximately.

3.5. Approximations of the copositive cone. The problem of determining whether a matrix is not copositive is NP-complete, as shown by Murty and Kabadi [17]. We therefore wish to replace the copositive cone $\mathcal{C}_d$ by a conic subset, in such a way that the resulting optimization problem becomes tractable. We can represent the copositivity requirement for a $d \times d$ symmetric matrix S as

$$P(x) := (x \circ x)^T S (x \circ x) = \sum_{i,j=1}^{d} S_{ij} x_i^2 x_j^2 \ge 0 \quad \forall x \in \mathbb{R}^d, \tag{9}$$

where "$\circ$" indicates the componentwise (Hadamard) product. We therefore wish to know whether the polynomial $P(x)$ is nonnegative for all $x \in \mathbb{R}^d$. Although one apparently cannot answer this question in polynomial time in general, as it is an NP-hard problem, one can decide using semidefinite programming whether $P(x)$ can be written as a sum of squares.

Parrilo [19] showed that $P(x)$ in (9) allows a sum of squares decomposition if and only if $S \in \mathcal{S}_d^+ + \mathcal{N}_d$, which is a well-known sufficient condition for copositivity. Set $\mathcal{K}_d^0$ to be the convex cone $\mathcal{K}_d^0 = \mathcal{S}_d^+ + \mathcal{N}_d$.

Higher order sufficient conditions can be derived by considering the polynomial

$$P^{(\ell)}(x) = P(x)\left(\sum_{i=1}^{d} x_i^2\right)^{\ell} = \left(\sum_{i,j=1}^{d} S_{ij} x_i^2 x_j^2\right)\left(\sum_{i=1}^{d} x_i^2\right)^{\ell}, \tag{10}$$

and asking whether $P^{(\ell)}(x)$—which is a homogeneous polynomial of degree $2(\ell+2)$—has a sum of squares decomposition, or whether it has only nonnegative coefficients.

For $\ell = 1$, Parrilo [19] showed that a sum of squares decomposition exists if and only if[1] the following system of linear matrix inequalities has a solution:

$$S - S^{(i)} \in \mathcal{S}_d^+, \quad i = 1, \dots, d, \tag{11}$$

$$S_{ii}^{(i)} = 0, \quad i = 1, \dots, d, \tag{12}$$

$$S_{jj}^{(i)} + 2S_{ij}^{(j)} = 0, \quad i \neq j, \tag{13}$$

$$S_{jk}^{(i)} + S_{ik}^{(j)} + S_{ij}^{(k)} \geq 0, \quad i < j < k, \tag{14}$$

where $S^{(i)}$ $(i = 1, \dots, d)$ are symmetric matrices. Similar to the $\ell = 0$ case, we define $\mathcal{K}_d^1$ as the (convex) cone of matrices S for which the above system has a solution.

We will consider the lower bounds we get by replacing the copositive cone by either $\mathcal{K}_d^0$ or $\mathcal{K}_d^1$:

$$\underline{p} \geq p_\ell := \max \left\{ t \mid Q - tee^T \in \mathcal{K}_d^\ell \right\}, \qquad \ell \in \{0, 1\}. \tag{15}$$

3.6. Approximations (relaxations) of the copositive cone. We will now study the relaxation obtained by replacing the copositive cone by its proper subset $\mathcal{K}_d^0$. In other words, we study the relaxation

$$\begin{aligned} \underline{p} &= \max \left\{ t \,\middle|\, \sum_{i=1}^M (b_i - t) A_i \in \mathcal{C}_d \right\} \\ &\geq p_0 := \max \left\{ t \,\middle|\, \sum_{i=1}^M (b_i - t) A_i \in \mathcal{K}_d^0 = \mathcal{S}_d^+ + \mathcal{N}_d \right\}. \end{aligned}$$

We rewrite $\sum_{i=1}^M (b_i - t) A_i \in \mathcal{K}_d^0$ as

$$\sum_{i=1}^M (b_i - t) A_i = \sum_{i=1}^M y_i A_i + \sum_{i=1}^M z_i A_i,$$

where $\sum_{i=1}^M y_i A_i \in \mathcal{S}_d^+$ and $\sum_{i=1}^M z_i A_i \in \mathcal{N}_d$.

Note that, since the A_i's are zero-one matrices that sum to ee^T, it follows that $z_i \geq 0$. Moreover,

$$b_i - t = y_i + z_i \qquad \text{implies} \qquad b_i - t - y_i \geq 0.$$

We obtain the relaxation

$$p_0 = \max \left\{ t \mid b_i - t - y_i \geq 0 \ (i = 1, \dots, M), \ \sum_{i=1}^M y_i A_i \in \mathcal{S}_d^+ \right\}. \tag{16}$$

3.7. Block factorization. The next step in reducing the problem size is to perform a similarity transformation that simultaneously block-diagonalizes the matrices $A_1, \dots, A_M$. In particular, we want to find an orthogonal matrix V such that the matrices

$$\tilde{A}_i := V A_i V^{-1}, \quad i = 1, \dots, M,$$

[1] In fact, Parrilo [19] proved only the "if" part; the converse is proved in Bomze and de Klerk [4].

all have the same block-diagonal structure, and the maximum block size is as small as possible. Note that the conjugation preserves spectra, and orthogonality of V preserves symmetry.

This will further reduce the size of the relaxation (16) via

$$\begin{aligned} p_0 &= \max\left\{ t \mid b_i - t - y_i \geq 0 \ (i = 1, \dots, M), \ \sum_{i=1}^{M} y_i A_i \in \mathcal{S}_d^+ \right\} \\ &= \max\left\{ t \mid b_i - t - y_i \geq 0 \ (i = 1, \dots, M), \ \sum_{i=1}^{M} y_i V A_i V^{-1} \in \mathcal{S}_d^+ \right\} \\ &= \max\left\{ t \mid b_i - t - y_i \geq 0 \ (i = 1, \dots, M), \ \sum_{i=1}^{M} y_i \tilde{A}_i \in \mathcal{S}_d^+ \right\}. \end{aligned}$$

The necessity to restrict to orthogonal V's lies in the fact that there is currently no software (or algorithms) available that would be able to deal with nonsymmetric $\tilde{A}_i$'s.

Computing the finest possible block decomposition (this would mean finding explicitly the orthogonal bases for the irreducible submodules of the natural module of G in its action by the matrices P) is computationally not easy, especially due to the orthogonality requirement on V. We restricted ourselves to decomposing into two blocks of equal size $\frac{d}{2} \times \frac{d}{2}$. Namely, each row corresponds to a cyclic permutation $g \in \Pi$, and the natural pairing (g, g^{-1}) can be used to construct $V = \frac{\sqrt{2}}{2} V'$ as follows:

- the first half of the rows of V' are characteristic vectors of the 2-subsets $\{g, g^{-1}\}$, $g \in \Pi$;
- the second half of the rows of V' consists of "twisted" rows from the first half: namely, one of the two 1's is replaced by -1.

It is obvious that $V' V'^T = 2I$ and thus V is orthogonal.

Remark. It is worth mentioning that in [22] Schrijver essentially dealt, in a different context, with a similar setup, except that in his case the elements of the basis $\mathcal{X}$ of $\mathcal{B}$ were symmetric and (hence) the algebra $\mathcal{B}$ commutative. In such a situation the elements of $\mathcal{X}$ can be simultaneously diagonalized, and the corresponding optimization problem becomes a linear programming problem.

3.8. Computational results: Proof of Theorem 3. The combinatorial/group theoretic part of the computations, namely of the A_i's, V, and $Q = \sum_i b_i A_i$, was performed using a computer algebra system GAP [7], version 4.3, and its shared package GRAPE by Soicher [24]. Semidefinite programs (SDPs) were solved by Sturm [25] using SeDuMi, version 1.05 under MATLAB 6.5. The biggest SDP took about 10 minutes of CPU time of a Pentium 4 with 1 GB of RAM.

In addition, the results were verified using MAPLE. Namely, for $t = p_0$ and y, the variables computed upon solving (16), we checked that the corresponding (matrix and scalar) inequalities in (16) hold. As p_0 is a lower bound on $\underline{p}$, we thus validated the computed value of p_0 independently of the SDP solver used.

For the test case of $K_{5,n}$ we solved the relaxed problem (15) with $\ell = 1$ to obtain

$$p_1 \approx 1.9544, \quad \text{that is,} \quad \operatorname{cr}(K_{5,n}) \geq \frac{1}{2}(1.9544)n^2 = 0.9772n^2,$$

asymptotically. The correct asymptotic value is known to be $\operatorname{cr}(K_{5,n}) = n^2$, which shows the quality of the bound. In fact, we could show that $p_1 \approx 1.9544$ corresponds to the optimal value of the first optimization problem in Lemma 2 for $m = 5$. This shows that the optimal value of this optimization problem is a strict lower bound of the crossing number of $K_{m,n}$, even for $m = 5$.

The weaker bound for $\ell = 0$ in (15) yields, still quite tight,

$$p_0 \approx 1.94721, \quad \text{that is,} \quad \mathrm{cr}(K_{5,n}) \geq \frac{1}{2}(1.94721)n^2 = 0.973605n^2.$$

For the case $K_{7,n}$ we solved the relaxed problem (15) with $\ell = 0$ to obtain

$$p_0 \approx 4.3593, \quad \text{that is,} \quad \mathrm{cr}(K_{7,n}) \geq \frac{1}{2}(4.3593)n^2 = 2.1796n^2,$$

asymptotically.

Proof of Theorem 1. For the sake of completeness, we close this section with the observation that Theorem 1 has been proved. It follows from Lemma 2 and Proposition 3. □

4. Improved bounds for the crossing numbers of $K_{m,n}$ and K_n. Perhaps the most appealing consequence of our improved bound for $\mathrm{cr}(K_{7,n})$ is that it also allows us to give improved lower bounds for the crossing numbers of $K_{m,n}$ and K_n. The quality of the new bounds is perhaps best appreciated in terms of the following asymptotic parameters:

$$A(m) := \lim_{n\to\infty} \frac{\mathrm{cr}(K_{m,n})}{Z(m,n)}, \qquad B := \lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{Z(n,n)},$$

(see Richter and Thomassen [21]). These natural parameters give us a good idea of our current standing with respect to Zarankiewicz's conjecture. It is not difficult to show that $A(m)$ (for every integer $m \geq 3$) and B both exist [21].

Previous to the new bound we report in Theorem 1, the best known lower bounds for $A(m)$ and B were $A(m) \geq 0.8\frac{m}{m-1}$ and (consequently) $B \geq 0.8$. Both bounds were obtained by using the known value of $\mathrm{cr}(K_{5,n})$ and applying a standard counting argument.

By applying the same counting argument but instead using the bound given by Theorem 1, we improve these asymptotic quotients to $A(m) > 0.83\frac{m}{m-1}$ and $B > 0.83$.

The improved lower bound for B has an additional, important application. It has been long conjectured that $\mathrm{cr}(K_n) = Z(n)$, where

$$Z(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor,$$

but this has been verified only for $n \leq 10$ (see, for instance, [6]). As we did with $K_{m,n}$, it is natural to inquire about the asymptotic parameter

$$C := \lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{Z(n)}.$$

In [21] it is proved that C exists, and, moreover, that $C \geq B$. In view of this, our improved lower bound for B yields $C > 0.83$.

We summarize these results in the following statement.

THEOREM 5. *With $Z(m,n)$ and $Z(n)$ as above,*

$$\lim_{n\to\infty}\frac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \ge 0.83\frac{m}{m-1}, \quad \lim_{n\to\infty}\frac{\mathrm{cr}(K_{n,n})}{Z(n,n)} \ge 0.83,$$

$$\text{and} \quad \lim_{n\to\infty}\frac{\mathrm{cr}(K_n)}{Z(n)} \ge 0.83. \qquad \square$$

Recall that these results followed from an improved lower bound on $\mathrm{cr}(K_{7,n})$ obtained by solving the optimization problem (15) for $m = 7$. The results can be further improved by solving (15) for larger values of m. After the first submission of the present work, the optimization problem was successfully solved for $m = 9$ by de Klerk, Pasechnik, and Schrijver [15], by using a more sophisticated way of exploiting the algebraic symmetry. In particular, the constant 0.83 in Theorem 5 could thus be improved to 0.859.

We close this section with a few words on some important recent developments involving the *rectilinear* crossing number of K_n.

The *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane, with the additional restriction that all edges of G must be drawn as straight segments.

It is known that $\overline{\mathrm{cr}}(K_n)$ and $\mathrm{cr}(K_n)$ may be different (for instance, $\overline{\mathrm{cr}}(K_8) = 19$, whereas $\mathrm{cr}(K_8) = 18$; see [10]). While we have a (nonrectilinear) way of drawing K_n that shows $\mathrm{cr}(K_n) \le Z(n)$ (equality is conjectured to hold, as we observed above), good upper bounds for $\overline{\mathrm{cr}}(K_n)$ are notoriously difficult to obtain. Currently, the best upper bound known is $\overline{\mathrm{cr}}(K_n) \le 0.3807\binom{n}{4}$ (see Aichholzer, Aurenhammer, and Krasser [2]).

For many years the best lower bounds known for $\overline{\mathrm{cr}}(K_n)$ were considerably smaller (around $0.32\binom{n}{4}$) than the best upper bounds available (currently around $0.3807\binom{n}{4}$). However, remarkably better lower bounds have been recently proved independently by Ábrego and Fernández–Merchant [1] and Lovász et al. [16], and refined by Balogh and Salazar [3]. In [1], the technique of allowable sequences was used to show that $\overline{\mathrm{cr}}(K_n) \ge 0.375\binom{n}{4}$. Lovász et al. used similar methods to prove $\overline{\mathrm{cr}}(K_n) > 0.37501\binom{n}{4} + O(n^3)$. Recently, Balogh and Salazar improved this to $\overline{\mathrm{cr}}(K_n) > 0.37553\binom{n}{4} + O(n^3)$ [3]. The importance of establishing that $\overline{\mathrm{cr}}(K_n)$ is strictly greater than $0.375\binom{n}{4} + O(n^3)$ is that it effectively shows that the ordinary and the rectilinear crossing numbers of K_n are different in the asymptotically relevant term, namely n^4.

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