Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Combinatorial Theory, Series B 96 (2006) 455–471
Journal of Combinatorial Theory Series B
www.elsevier.com/locate/jctb

Crossing number is hard for cubic graphs ☆

Petr Hliněný [a,b,1]

[a] *Department of Computer Science, Technical University of Ostrava, 17, Listopadu 15, 708 33 Ostrava, Czech Republic*
[b] *Faculty of Informatics, Masaryk University, Botanická 68a, 602 00 Brno, Czech Republic*

Received 7 June 2004

Available online 2 November 2005

Abstract

It was proved by [M.R. Garey, D.S. Johnson, Crossing number is NP-complete, SIAM J. Algebraic Discrete Methods 4 (1983) 312–316] that computing the crossing number of a graph is an NP-hard problem. Their reduction, however, used parallel edges and vertices of very high degrees. We prove here that it is NP-hard to determine the crossing number of a simple 3-connected cubic graph. In particular, this implies that the minor-monotone version of the crossing number problem is also NP-hard, which has been open till now.

Keywords: Crossing number; Cubic graph; NP-completeness

1. Background on crossing number

We assume that the reader is familiar with basic terms of graph theory. In this paper we consider finite simple graphs, unless we specifically speak about multigraphs. A graph is cubic if it has all vertices of degree 3.

In a (*proper*) *drawing* of a graph G in the plane the vertices of G are points and the edges are simple curves joining their endvertices. Moreover, it is required that no edge passes through a vertex (except at its ends), and that no three edges intersect in a common point which is not a vertex. An *edge crossing* is an intersection point of two edges-curves in the drawing which is not a vertex. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings in a proper drawing of G in the plane (thus, a graph is planar if and only if its crossing number is 0).

☆ An extended abstract published in: Proceedings MFCS'04, in: Lecture Notes in Comput. Sci., vol. 3153, Springer, 2004, pp. 772–782.

E-mail address: hlineny@member.ams.org.

[1] Supported by Czech research grant GAČR 201/05/0050.

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2005.09.009

A proper drawing of G with $\mathrm{cr}(G)$ crossings is called *optimal*. We remark that there are several possible definitions of crossing number which are similar or even looking equivalent with each other [11], but one of them has, surprisingly, turned out recently to be different [12].

Crossing number problems were introduced by Turán, whose work in a brick factory during the Second World War led him to inquire about the crossing number of the complete bipartite graphs $K_{m,n}$. Turán devised a natural drawing of $K_{m,n}$ with $\lfloor m/2\rfloor\lfloor (m-1)/2\rfloor\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor$ crossings, but the conjecture of Zarankiewic that such a drawing is the best possible, is still wide open. (Look at an interesting story of a false "proof" of the conjecture [8].) Not surprisingly, exact crossing numbers are in general very difficult to compute. As an example of another graph family whose crossing number has been deeply studied, we mention the Cartesian products of cycles $C_m \times C_n$—their crossing number $m(n-2)$ for $m \geqslant n$ was conjectured in [9]. There has been a number of particular results on this difficult problem (such as [10] for example), and, remarkably, the problem is almost solved now [6]. That is one of only a few nontrivial exact crossing numbers known today.

The algorithmic problem CROSSINGNUMBER is given as follows:

Input: A multigraph G and an integer k.
Question: Is it true that $\mathrm{cr}(G) \leqslant k$?

Computing the crossing number has important applications in, for example, VLSI design, or in graph visualization. The problem is in NP since one could guess the optimal drawing, replace the crossings in it with new (simultaneously subdividing) vertices, and verify planarity of the resulting graph. It has been proved by Garey and Johnson [4] that CROSSINGNUMBER is an NP-complete problem for k on the input.

Since then, a new significant complexity result about graph crossing number has appeared only recently—a paper by Grohe [7] presenting a quadratic-time (FPT) algorithm for CROSSINGNUMBER(k) with constant k. To illustrate algorithmic difficulty of the crossing number problem in general, we remark that it is quite nontrivial even to approximate the crossing number of special projective graphs [5]. There is also a long-standing open question, originally asked by Seese (cf. [14]): What is the complexity of CROSSINGNUMBER for graphs of fixed tree-width? (Here we leave aside other results dealing with various restricted versions of the crossing number problem appearing in connection with VLSI design or with graph drawing, such as the "layered" or "rectilinear" crossing numbers, etc.)

Before the above mentioned FPT algorithm of Grohe for crossing number appeared; Fellows [2] had observed that there are finitely many excluded minors for the cubic graphs of crossing number at most k, which implied a (non-constructive) algorithm for CROSSINGNUMBER(k) with constant k over cubic graphs. That observation might still suggest that CROSSINGNUMBER was easier to solve over cubic graphs than in general. However, that is not so, as we show in this paper.

2. Crossing number and OLA

We first recall another classical NP-complete combinatorial problem [3] called OPTIMALLINEARARRANGEMENT, which is given as follows:

Input: An n-vertex graph G and an integer a.
Question: Is there a bijection $\alpha : V(G) \to \{1, \dots, n\}$ (a *linear arrangement* of vertices) such that the following holds:

$$\sum_{uv \in E(G)} \left|\alpha(u) - \alpha(v)\right| \leqslant a? \tag{1}$$

The sum on the left of (1) is called the *weight* of α.

The above mentioned paper [4] actually reduces CROSSINGNUMBER from OPTIMALLINEARARRANGEMENT. We, however, consider that reduction "unrealistic" in the following sense: The reduction in [4] creates many large classes of parallel edges, and it uses vertices of very high degrees. (There seems to be no easy modification avoiding those.) So we consider it natural to ask what can be said about the crossing number problem on simple graphs with small vertex degrees [13].

It might be tempting to construct a "nicer" polynomial reduction for CROSSINGNUMBER from another NP-complete problem called Planar-SAT (a version of the satisfiability problem with a planar incidence graph). There have been, to our knowledge, a few attempts in this direction, so far unsuccessful. We consider this phenomenon remarkable since Planar-SAT seems to be much closer to crossing-number problems than the Linear Arrangement is.

Still, we have found another construction reducing CROSSINGNUMBER from OPTIMALLINEARARRANGEMENT, which produces cubic graphs. The basic idea of our construction is similar to [4], but the restriction to degree-3 vertices brings many more difficulties to the proofs. The construction establishes our main result which reads:

Theorem 2.1. CROSSINGNUMBER *is NP-complete for simple* 3*-connected cubic graphs.*

Let us, moreover, define the *minor-monotone crossing number* $\mathrm{mcr}(G)$: A *minor* F of a graph G is a graph obtained from a subgraph of G by contractions of edges. Then $\mathrm{mcr}(G)$ as the smallest crossing number $\mathrm{cr}(H)$ over all graphs H having G as a minor. The traditional versions of crossing number do not behave well with respect to taking minors; one may find graphs G such that $\mathrm{cr}(G) = 1$ but $\mathrm{cr}(G')$ is arbitrarily large for a minor G' of G. On the other hand, $\mathrm{mcr}(G') \leqslant \mathrm{mcr}(G)$ for a minor G' of G by definition. We refer to [1] for a closer discussion of the properties of minor-monotone crossing number.

The algorithmic problem MM-CROSSINGNUMBER (from "Minor-Monotone") is defined as follows:

Input: A multigraph G and an integer k.
Question: Is it true that $\mathrm{mcr}(G) \leqslant k$?

Our main result immediately extends to a proof that also $\mathrm{mcr}(G)$ is NP-hard to compute, which has been an open question till now.

Corollary 2.2. MM-CROSSINGNUMBER *is NP-complete.*

Observation. Let a cubic graph G be a minor of a multigraph H. Then some subdivision of G is contained as a subgraph in H. Hence $\mathrm{cr}(G) \leqslant \mathrm{cr}(H)$.

Thus $\mathrm{cr}(G) = \mathrm{mcr}(G)$ for cubic graphs, and the corollary follows directly from Theorem 2.1.

3. The cubic reduction

Let us call a *cubic grid* the graph illustrated in Fig. 1 (looking like a "brick wall"). We say that the cubic-grid *height* equals the number of the "horizontal" paths, and the *length* equals the number of edges on the "top-most" horizontal path. (The positions are referred to as in Fig. 1.) Formally, the cubic grid of even height h and length ℓ, denoted by $\mathcal{C}'_{h,\ell}$, is defined

$$\begin{aligned}
V(\mathcal{C}'_{h,\ell}) &= \{v_{i,j}\colon\ i=1,2,\ldots,h;\ j=0,1,\ldots,\ell\}\\
&\quad \cup \{w_{i,j}\colon\ i=2,3,\ldots,h-1;\ j=1,2,\ldots,\ell\},\\
E(\mathcal{C}'_{h,\ell}) &= \{v_{2i-1,j}v_{2i,j}\colon\ i=1,2,\ldots,h/2;\ j=0,1,\ldots,\ell\}\\
&\quad \cup \{w_{2i,j}w_{2i+1,j}\colon\ i=1,2,\ldots,h/2-1;\ j=1,2,\ldots,\ell\}\\
&\quad \cup \{v_{i,j-1}w_{i,j},\, w_{i,j}v_{i,j}\colon\ i=2,3,\ldots,h-1;\ j=1,2,\ldots,\ell\}\\
&\quad \cup \{v_{i,j-1}v_{i,j}\colon\ i=1,h;\ j=1,2,\ldots,\ell\}.
\end{aligned}$$

Suppose we now identify the "left-most" vertices in the grid $\mathcal{C}'_{h,\ell}$ with the "right-most" ones, formally $v_{i,0}=v_{i,\ell}$ for $i=1,2,\ldots,h$, and simplify the resulting graph. Then we obtain the *cyclic cubic grid* $\mathcal{C}_{h,\ell}$ (which is, indeed, a cubic graph).

Let us have a cubic grid $\mathcal{C}'_{h,\ell}$ or $\mathcal{C}_{h,\ell}$ as above. We say that an edge f is *attached to the grid at low position* j if the edge $v_{1,j-1}v_{1,j}$ is subdivided with a vertex x_f, where x_f is an endvertex of f as well. We say that f is *attached at high position* j if an analogous construction is done for the edge $v_{h,j-1}v_{h,j}$. This is illustrated on a detailed picture in Fig. 2. Notice that the new vertex x_f introduced when attaching an edge f has degree 3, and that the degrees of other vertices are unchanged. Similarly, a vertex x is *attached to the grid at position* j if two new edges f, f' with a common endvertex x are attached via their other endvertices at low and high positions j, respectively, to our cubic grid.

In a cyclic cubic grid $\mathcal{C}_{h,\ell}$, the cycles M^i on vertices $v_{i,0}w_{i,1}v_{i,1}w_{i,2}\ldots v_{i,\ell-1}w_{i,\ell}$ for $i=2,3,\ldots,h-1$ and on vertices $v_{i,0}v_{i,1}\ldots v_{i,\ell-1}$ for $i=1,h$ are called the *main cycles* of the grid $\mathcal{C}_{h,\ell}$. M^1 and M^h are also referred to as the *outer* main cycles. We use the same names, main

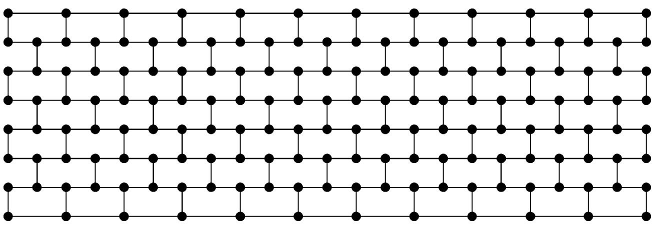

Fig. 1. An illustration of a cubic grid (a fragment of length 11 and height 8).

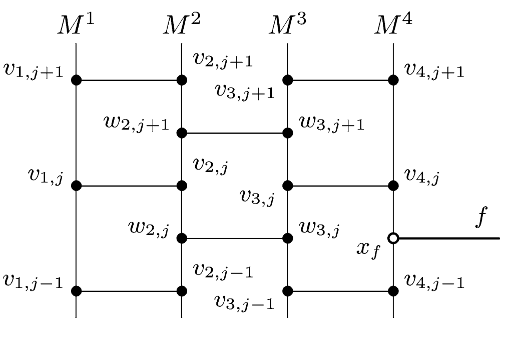

Fig. 2. A detail of the cyclic cubic grid $\mathcal{C}_{4,\ell}$, with an edge f attached at high position j.

cycles, for the subdivisions of the cycles M^i in graphs created from the grid $\mathcal{C}_{h,\ell}$ by attaching edges.

Assume now that we are given a graph G on n vertices. In order to prove Theorem 2.1, we are going to construct a cubic graph H_G depending on G. (Although our graph H_G is huge, it has polynomial size in G.) We show then how one can compute the weight of an optimal linear arrangement for G from the crossing number $\mathrm{cr}(H_G)$, and vice versa. Our construction uses several size parameters defined next:

$$\begin{aligned}
&n = |V(G)|, \qquad m = |E(G)|,\\
&t = 2mn,\\
&r = t^2 = 4m^2n^2,\\
&s = m^3 r = 4m^5n^2,\\
&q = (m^3+n+1)r = 4m^5n^2 + 4m^2n^3 + 4m^2n^2,\\
&z = 2\big((s+rn)nt + r\big) = 16m^6n^4 + 16m^3n^5 + 8m^2n^2. \qquad (2)
\end{aligned}$$

Without loss of generality we may assume that the graph G is sufficiently large, say

$$m > n > 100. \qquad (3)$$

We start with two copies B_1, B_2 of the cyclic cubic grid $\mathcal{C}_{z,q}$, called here the *boulders* (for their huge size that keeps the rest of our graph "in place"). Then we make n disjoint copies $R_1, \dots, R_n$ of the cyclic cubic grid $\mathcal{C}_{t,q}$, called here the *rings*. An intermediate step in the construction—our graph $H_{m,n}$ is obtained by the following operations:

- Start with the disjoint union $B_1 \cup B_2 \cup R_1 \cup \cdots \cup R_n$ of the two boulders and the n rings.
- For every pair of integers $0 \leqslant i < m^3$ and $0 \leqslant j < r$, take a new edge κ_{i+jm^3}, and attach κ_{i+jm^3} at low positions $i + j(m^3+n+1) < q$ to the boulder B_1 via one end, and to B_2 via the other end. These s new edges $\kappa_0, \dots, \kappa_{s-1}$ are called the *free spokes* in $H_{m,n}$.
- For every pair of integers $1 \leqslant i \leqslant n$ and $0 \leqslant j < r$, set $p = i - 1 + m^3 + j(m^3+n+1) < q$, and take two new vertices $\nu_{i,j}$ and $\nu'_{i,j}$ connected by an edge $\mu^3_{i,j}$. Then attach a new edge $\mu^1_{i,j}$ with one end $\nu_{i,j}$ (new edge $\mu^5_{i,j}$ with one end $\nu'_{i,j}$) to the boulder B_1 (boulder B_2) at low position p via the other end. Finally, attach a new edge $\mu^2_{i,j}$ with one end $\nu_{i,j}$ (new edge $\mu^4_{i,j}$ with one end $\nu'_{i,j}$) to the ring R_i at low (high) position p via the other end. The path formed by three edges $\mu^1_{i,j}, \mu^3_{i,j}, \mu^5_{i,j}$ is called the *jth ring spoke* of R_i in $H_{m,n}$.

We remark that the above construction attaches only one edge at the same position of each of the boulders and rings, and so the operations are well defined. (Fig. 3.) This remark applies also to further constructions on the graph H_G.

To simplify our notation, the above names of the boulders B_1, B_2 and the rings R_i are inherited to the subdivisions of those boulders and rings created in the construction of $H_{m,n}$. The same simplified notation is used further for the graph H_G, too.

So far, the constructed graph $H_{m,n}$ does not depend on a particular structure of G, but only on its size and our choice of the parameters (2). One may say that $H_{m,n}$ acts as a skeleton in the forthcoming construction, in which the rings of $H_{m,n}$ shall model the vertices of G, and the order the rings are drawn in shall correspond to a linear arrangement of vertices of G. The following

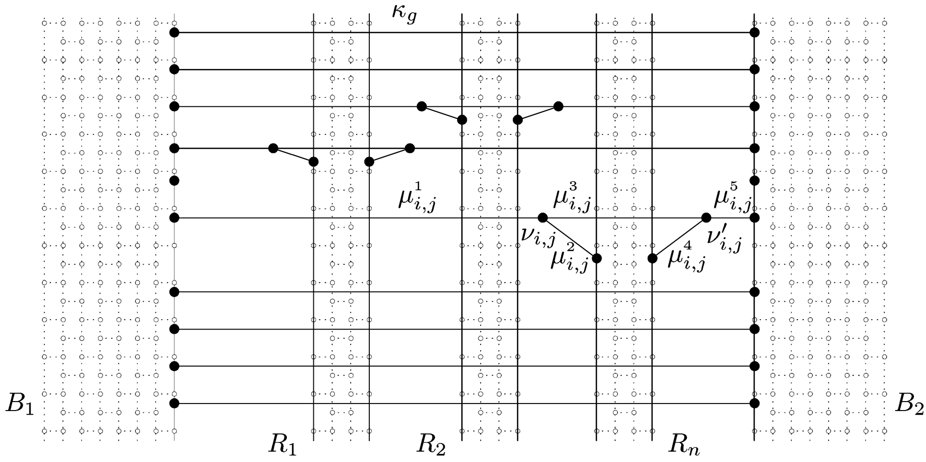

Fig. 3. How to attach free and ring spokes in the graph $H_{m,n}$.

simple lemma shows necessary "flexibility" of drawings of $H_{m,n}$ with any order of the rings. (Actually, the number of crossings in the lemma is optimal, as we implicitly show in Section 5.)

Lemma 3.1. *For any permutation π of the set $\{1, 2, \ldots, n\}$, there is a drawing of the graph $H_{m,n}$ with $(s + rn)nt$ crossings conforming to the following: The subdrawings of all the rings are pairwise disjoint, each ring separates the two boulders in $H_{m,n}$ from each other, and any free spoke in the drawing intersects all the rings in order $R_{\pi(1)}, \ldots, R_{\pi(n)}$ from B_1 to B_2.*

Proof. We start with the unique planar embedding of the boulders and the free and ring spokes of $H_{m,n}$. Then we draw each ring R_i of $H_{m,n}$ so that R_i separates the boulders from each other in the drawing, and that the rings are nested into each other in the required order $R_{\pi(1)}, \ldots, R_{\pi(n)}$. So each of the s free spokes, and each of the rn ring spokes, has t crossings with each ring (one with every main cycle), summing to a total of $(s + rn)nt$ crossings. We finally attach, in a suitable drawing, each of the ring spokes to its ring by the edges $\mu^2_{i,j}$ and $\mu^4_{i,j}$ with no additional crossings. See Fig. 3 for an illustration. □

Finally, the particular graph H_G needed for our polynomial reduction from G is constructed as follows:

- Start with the graph $H_{m,n}$, for $n = |V(G)|$ and $m = |E(G)|$. Number the vertices $V(G) = \{1, 2, \ldots, n\}$.
- For every ordered pair $0 < i, j \leqslant n$ such that $\{i, j\} \in E(G)$, set $p = (i - 1 + jn - n)4m^2(m^3 + n + 1) + m^3 + n < q$. In the graph $H_{m,n}$, attach new vertices χ_{ij}, χ'_{ij} to the rings R_i, R_j, respectively, at positions p, and add a new edge $\chi_{ij}\chi'_{ij}$. The subgraph $X_{i,j}$ induced on the five new edges incident with χ_{ij}, χ'_{ij} is called a *handle* of the edge ij in H_G (Fig. 4).

That is, the rings in H_G model the vertices of G, and the handles model the edges of G. As we show later, an optimal drawing of H_G uniquely determines an ordering of the rings of $H_{m,n}$, and

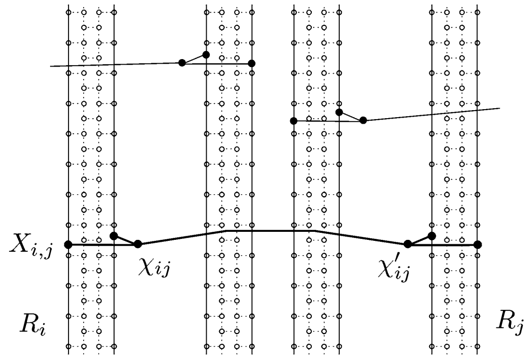

Fig. 4. How to attach handles of the edges of G in the graph H_G.

hence the weight of an optimal linear arrangement of G corresponds to the number of crossings between the rings and the handles in an optimal drawing of the graph H_G.

We conclude with an upper bound on the crossing number of our constructed graph, which naturally follows from the drawings introduced in Lemma 3.1.

Proposition 3.2. *Let us, for a given graph G, construct the graph H_G as described above. If G has a linear arrangement of weight A, then the crossing number of H_G is*

$$\mathrm{cr}(H_G) \leqslant (s+rn)nt + 2(A+m)t - 4m,$$

where the weight of a linear arrangement is defined by (1) *in Section* 2*, and m, n, r, s, t are given by* (2) *in Section* 3.

Proof. Let α be the linear arrangement of G of weight A. We draw the graph $H_{m,n} \subset H_G$ by Lemma 3.1 with $(s+rn)nt$ edge crossings, such that the rings are ordered as $R_{\alpha^{-1}(1)}, \ldots, R_{\alpha^{-1}(n)}$ from B_1 to B_2. Then we draw the handles in H_G for all edges of G in the natural (shortest) way, as illustrated in Fig. 4.

Now for $0 < i, j \leqslant n$ such that $\{i, j\} \in E(G)$, the handle of ij in H_G has $t-1$ crossings with the main cycles of the ring R_i and $t-1$ crossings with those of R_j. Moreover, the handle has $t \cdot |\alpha(i) - \alpha(j)| - t$ crossings with the rings "between R_i and R_j." Keeping in mind that each edge of G actually makes two handles of ij and of ji, we sum the crossings of the handles:

$$\sum_{ij \in E(G)} 2\bigl(t \cdot \bigl|\alpha(i) - \alpha(j)\bigr| - t + 2t - 2\bigr)$$
$$= 2t \sum_{ij \in E(G)} \bigl|\alpha(i) - \alpha(j)\bigr| + 2mt - 4m = 2At + 2mt - 4m.$$

Altogether, the described drawing of H_G has $(s+rn)nt + 2(A+m)t - 4m$ crossings. □

Using the obvious inequality $A \leqslant m(n-1)$, it is easy to conclude:

Corollary 3.3. *For any G conforming to* (3)*, $\mathrm{cr}(H_G) < z/2 = (s+rn)nt + r$.*

Now, to prove correctness of our reduction, we have to prove a lower bound on the crossing number of our graph H_G, depending on weight of the optimal linear arrangement of G. We achieve this goal in the next sections by showing that an optimal drawing of H_G has to look (almost) like the drawing described in the proof of Proposition 3.2. Here we give a brief outline of the main steps: We argue that the boulders of H_G have to be drawn without crossings at all, and that each ring has to separate the two boulders from each other. Such a configuration already forces the number of crossings of $H_{m,n}$ as in Lemma 3.1. Then we identify a linear ordering of the rings, and show that every edge handle in H_G generates at least as many additional crossings as expected from the ordering of rings.

4. Assorted topological lemmas

We need to be a bit more formal in this section. A *curve* γ is a continuous function mapping the interval [0, 1] to a topological space. A curve γ is a *closed* curve if $\gamma(0) = \gamma(1)$. A closed curve γ is *contractible* in a topological space if γ can be continuously deformed to a single point there. We call a *cylinder* the topological space obtained from the unit square by identifying one pair of opposite edges in the same direction. (A cylinder has two disjoint closed curves as the boundary.)

We are going to deal with collections of curves having somehow special structure. A set Γ of curves is called *nice* if all of the following are true:

- No three curves in Γ have a common intersection.
- If x is a self-intersection point of a curve $\gamma \in \Gamma$, i.e. $x = \gamma(a) = \gamma(b)$ for distinct $a, b \in [0, 1]$, then no other curve in Γ passes through x.
- If x in an intersection point of $\gamma, \gamma' \in \Gamma$, then in a sufficiently small neighbourhood U of x, the curves γ, γ' are otherwise disjoint, and they intersect the boundary of U in a cyclic order of $\gamma, \gamma', \gamma, \gamma'$ (they "properly cross").

A subset of a nice set of curves is nice as well by the definition. Naturally, we call a *crossing* of curves the intersection point of two curves in a nice set. This obviously corresponds with the notion of an edge crossing in a topological graph.

Lemma 4.1. *Let k, ℓ be positive integers, let $p = k(\ell + 1)$, and let Π be a cylinder with two closed boundary curves π_1, π_2. Let $X_1, \dots, X_p$ be distinct points on π_1 in this cyclic order, and let $Y_1, \dots, Y_p$ be distinct points on π_2 in the corresponding cyclic order. Suppose that $\mathcal{S} = \{\sigma_i\colon i = 1, \dots, p\}$ is a nice set of p curves on Π such that each σ_i has ends X_i and Y_i, and that τ is a contractible closed curve on Π disjoint from π_1, π_2. Moreover, assume that τ intersects each one of the curves in $\mathcal{S}_0 = \{\sigma_{i(\ell+1)}\colon i = 1, \dots, k\} \subset \mathcal{S}$, and that $(\mathcal{S} \setminus \mathcal{S}_0) \cup \{\tau\}$ also forms a nice set of curves. Then at least one of the two cases happens*:

(i) *τ crosses twice at least $\frac{3}{5}k\ell$ of the curves in $\mathcal{S} \setminus \mathcal{S}_0$, or*
(ii) *there are at least $(\frac{2}{25}k^2 - \frac{1}{5}k)\ell$ crossings of curves in $\mathcal{S}$.*

Proof. First notice that since τ is a contractible closed curve, it divides Π into two connected regions, one of them containing both π_1, π_2. So if a curve $\sigma \in \mathcal{S} \setminus \mathcal{S}_0$ intersects τ (and, recall $\{\sigma, \tau\}$ is nice), then σ has (at lest) two crossings with τ by the Jordan Curve Theorem. Hence, let us assume that more than $\frac{2}{5}k\ell$ of the curves in $\mathcal{S} \setminus \mathcal{S}_0$ are disjoint from τ, and denote their subset by $\mathcal{S}_1 \subseteq \mathcal{S} \setminus \mathcal{S}_0$.

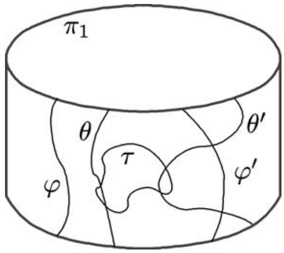

Fig. 5. An illustration to Claim 4.2.

Claim 4.2. *For any two $\varphi, \varphi' \in \mathcal{S}_1$ and two $\theta, \theta' \in \mathcal{S}_0$ such that the ends of φ, φ' on π_1 separate the ends of θ, θ' there, one of θ or θ' has at least two crossings with $\varphi \cup \varphi'$.*

To see that the claim holds true; observe that one of the connected components of the topological space $\Pi \setminus (\varphi \cup \varphi')$ contains the whole curve τ by our choice of φ, φ', and no such component may contain an end of θ and an end of θ' at the same time. See Fig. 5.

For the rest of the proof of Lemma 4.1 we focus on the collection of curves $\mathcal{S}_0 \cup \mathcal{S}_1$, and consider their cyclic ordering determined by their ends on π_1. (Also the same cyclic ordering as determined by their ends on π_2.) By the assumptions, every $\ell + 1$ consecutive curves of $\mathcal{S}_0 \cup \mathcal{S}_1$ must contain at least one curve from $\mathcal{S}_0$. So we find a, b such that $\sigma_{a(\ell+1)}, \sigma_{b(\ell+1)} \in \mathcal{S}_0$ divide the cyclic ordering of $\mathcal{S}_1$ into two parts of size at least $\frac{1}{2}(|\mathcal{S}_1| - \ell)$ each. Hence we may apply Claim 4.2 to $\theta = \sigma_{a(\ell+1)}, \theta' = \sigma_{b(\ell+1)}$ and to $\frac{1}{2}(|\mathcal{S}_1| - \ell)$ choices of disjoint pairs from $\mathcal{S}_1$, accounting for at least $|\mathcal{S}_1| - \ell$ crossings in $\mathcal{S}$.

More generally, with indices modulo p the pair $\sigma_{(a+i)(\ell+1)}, \sigma_{(b+i)(\ell+1)} \in \mathcal{S}_0$ divides the cyclic ordering of $\mathcal{S}_1$ into two parts of size at least $\frac{1}{2}(|\mathcal{S}_1| - (i+1)\ell)$ each, for $i = 0, 1, \ldots, \frac{2}{5}k - 2$. By applying the previous idea for each pair $\theta = \sigma_{(a+i)(\ell+1)}, \theta' = \sigma_{(b+i)(\ell+1)}$, we find at least this number of distinct crossings of curves in $\mathcal{S}$:

$$\sum_{i=0}^{2k/5-2} \bigl(|\mathcal{S}_1| - (i+1)\ell\bigr) \geqslant \sum_{i=0}^{2k/5-2} \left(\frac{2}{5}k\ell - (i+1)\ell\right) = \sum_{i=1}^{2k/5-1} i\ell = \binom{2k/5}{2}\ell$$
$$= \left(\frac{2}{25}k^2 - \frac{1}{5}k\right)\ell. \qquad \square$$

The following claim is an easy application of the pigeon-hole principle.

Lemma 4.3. *Let n, t be positive integers. Suppose that, for each $i = 1, 2, \ldots, n$, there is a set $\mathcal{R}_i$ of t closed curves. If the union $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \cdots \cup \mathcal{R}_n$ has fewer than t^2 intersecting pairs of curves, then there exist pairwise disjoint representatives $\varrho_i \in \mathcal{R}_i$, $i = 1, 2, \ldots, n$.*

For the next two lemmas, we define a set $\mathcal{X}(c, d)$ of pairwise disjoint cycles in a cyclic cubic grid $D = \mathcal{C}_{t,\ell}$ as follows: we shall use the notation from the definition of a cubic grid (Section 3, Fig. 2). Let C_j denote the cycle of the cubic grid D on vertices $v_{1,c+2j}, v_{2,c+2j}, w_{2,c+2j}, w_{3,c+2j}, v_{3,c+2j}, \ldots, w_{h-1,c+2j}, v_{h-1,c+2j}, v_{h,c+2j}, v_{h,c+2j+1}, v_{h-1,c+2j+1}, \ldots, v_{1,c+2j+1}$. (Such a cycle is also depicted as C in Fig. 6.) Then $\mathcal{X}(c, d) = \{C_j\colon\ 0 \leqslant j < \frac{1}{2}(d - c)\}$.

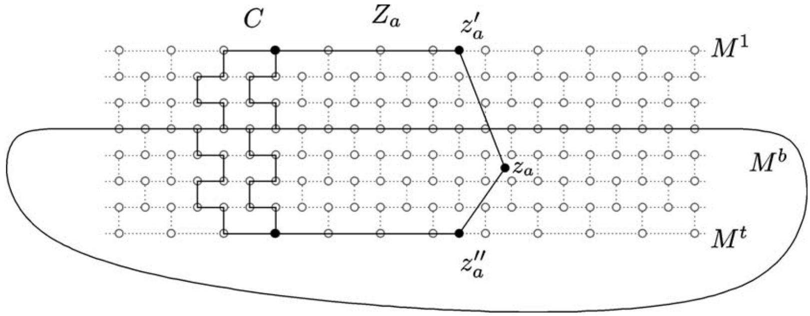

Fig. 6. An illustration to Claim 4.6.

Lemma 4.4. *Let k, ℓ, t be integers, and let $(p_1, p_2, \ldots, p_k)$ be an increasing sequence of integers such that $p_1 > 4kt$, $p_k < \ell$, and $p_{j+1} - p_i \geqslant 4kt$ for $j = 1, 2, \ldots, k$. Assume that the graph F is constructed from the cyclic cubic grid $D = \mathcal{C}_{t,\ell}$ by attaching a vertex z_j at position p_j for each $j = 1, 2, \ldots, k$. Then* $\mathrm{cr}(F) = k(t-2)$.

Proof. There is an obvious drawing of F with exactly $k(t-2)$ edge crossings—when the edges incident with each z_j cross all the main cycles of F except the outer ones. Conversely, we prove that every proper drawing of F must have at least $k(t-2)$ crossings. Let us fix $a \in \{0, 1, \ldots, k-1\}$. Assuming $p_0 = 0$, we denote by $\mathcal{X} = \mathcal{X}(p_a, p_{a+1})$ a collection of disjoint cycles in $D \subset F$. Since one edge crossing may involve edges of at most two of the cycles in $\mathcal{X}$, and since $\mathrm{cr}(F) < kt \leqslant \frac{1}{2}|\mathcal{X}|$, we conclude:

Claim 4.5. *For each $a \in \{0, 1, \ldots, k-1\}$, any optimal drawing of F must have a cycle $C \in \mathcal{X}(p_a, p_{a+1})$ with no crossed edge.*

Denote the ith main cycle in the cyclic cubic grid of F by M^i, for $i = 1, \ldots, t$. Denote the neighbours of z_a subdividing the outer main cycles M^1, M^t in F by z'_a, z''_a, respectively. We define a path Z_a in F consisting of the path $z'_a z_a z''_a$, and of the subpaths of M^1, M^t connecting z'_a, z''_a, respectively, to the cycle C from Claim 4.5, as depicted in Fig. 6. Let us further fix $b \in \{2, 3, \ldots, t-1\}$. By Claim 4.5 the cycle C is drawn as a simple closed curve with no edge crossing, and so a drawing of the main cycle M^b (which intersects C in two edges) separates the ends of Z_a on C. We conclude:

Claim 4.6. *The path Z_a must cross the main cycle M^b, for all pairs $a \in \{0, 1, \ldots, k-1\}$ and $b \in \{2, 3, \ldots, t-1\}$.*

Now observe that the main cycles $M^2, \ldots, M^{t-1}$ are pairwise disjoint, and that also the paths Z_a for $a = 0, 1, \ldots, k-1$ are chosen as pairwise disjoint subgraphs in F. Hence we account for at least $(t-2)k$ distinct edge crossings in F using Claim 4.6. □

Lemma 4.7. *Let q, t be integers. Let Π be a cylinder, and let ϱ_1, ϱ_2 be two disjoint curves on Π both connecting points on the opposite boundaries of Π. Assume that $\mathcal{D}$ is a drawing of the cyclic cubic grid $\mathcal{C}_{t,q}$ on Π. Moreover, assume the following*:

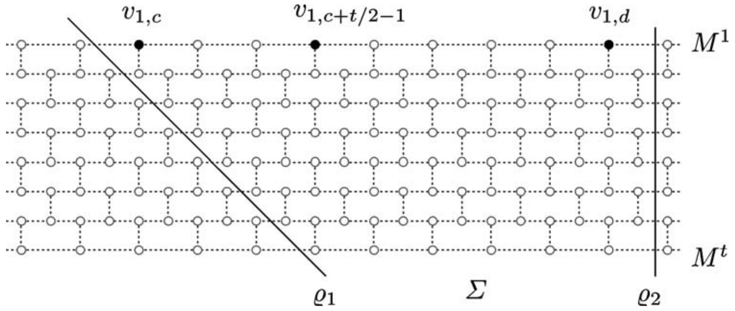

Fig. 7. An illustration to Lemma 4.7.

- *The drawing of $\mathcal{D}$ is such that each of the main cycles of $\mathcal{C}_{t,q}$ is drawn as a noncontractible closed curve on Π, intersecting each curve ϱ_1, ϱ_2 in exactly one point.*
- *No other edge of the drawing $\mathcal{D}$ is intersected by ϱ_1 or ϱ_2.*
- *There are indices c, d, $d > c + 2t$ such that the vertices $v_{1,c}$ and $v_{1,d}$ of the first main cycle $M^1 \subset D$ are drawn inside the same region Σ of $\Pi \setminus (\varrho_1 \cup \varrho_2)$.*

Then all the cycles in the set $\mathcal{X}(c+t, d-t)$ defined as above, are drawn inside the region Σ.

Proof. We may assume, without loss of generality, that ϱ_1 intersects the edge $v_{1,c-1}v_{1,c}$ of M^1. Denote by $P^i \subset M^i$ the subpath of the ith main cycle M^i of $\mathcal{D}$ on the vertices $v_{i,c-i}, \dots, v_{i,c+i-1}$. Recall that ϱ_1 intersects M^i exactly once. We establish the following claim by induction on $i \geqslant 1$:

Claim 4.8. *The curve ϱ_1 intersects M^i in an edge from P^i.*

The claim is true for $i = 1$ by our assumption. Suppose it is true for $i < t$, but false for $i + 1$. Then, up to symmetry, the vertices $w_{i+1,c+i}$ and $v_{i+1,c+i}$ of M^{i+1} would be drawn outside of Σ, while the vertices $w_{i,c+i}, v_{i,c+i}$ of M^i are drawn inside Σ (or vice versa). But, then the edge $w_{i,c+i}w_{i+1,c+i}$ or $v_{i,c+i}v_{i+1,c+i}$ (depending on parity of i) of $\mathcal{D}$ would have to be drawn intersecting $\varrho_1 \cup \varrho_2$, a contradiction to the assumptions.

(Actually, Claim 4.8 can be established in a stronger form, but we prefer this weak form with a straightforward proof. See Fig. 7.) A symmetric statement clearly holds for ϱ_2. Since the above paths P^i are disjoint from the cycles in $\mathcal{X}(c+t, d-t)$ by definition, the drawings of these cycles are not intersected by $\varrho_1 \cup \varrho_2$, and hence these cycles are all drawn inside Σ. □

5. Lower crossing bound

Recall the notation from Section 3, and assume that G is a graph on the vertex set $\{1, 2, \dots, n\}$. Let H_G denote the graph constructed along the description in Section 3. The following statement, together with Proposition 3.2, validates our reduction.

Proposition 5.1. *If an optimal linear arrangement of a graph G has weight A, then the crossing number of the graph H_G is at least*

$$\operatorname{cr}(H_G) \geqslant (s + rn)nt + 2(A + m)t - 8m.$$

(*See* (2) *and Proposition* 3.2 *for details on the notation.*)

Remark 5.2. The reader may notice here that the bound of this proposition is slightly different from the bound shown in Proposition 3.2. This small difference does not harm our reduction, while it makes some arguments significantly easier.

We proceed in the proof of Proposition 5.1 along the following sequence of claims. (Actually, all technical work has already been done in the previous section.) We choose some optimal drawing of the graph H_G and denote it by $\mathcal{H}_G$.

Lemma 5.3. *In the optimal drawing $\mathcal{H}_G$ of H_G, the boulders B_1, B_2 are drawn with no edge crossings.*

Proof. We assume, for a contradiction, that the boulder B_1 is drawn in $\mathcal{H}_G$ with some edge crossings. Notice that B_1 has (2) z pairwise disjoint main cycles by definition, and one edge crossing in $\mathcal{H}_G$ may involve at most two of them. Since the total number of crossings is fewer than $z/2$ by Corollary 3.3, we conclude that some main cycle N of B_1 is drawn with no crossings. Without loss of generality, we may suppose that the subgraph $\mathcal{H}_G - V(B_1)$ is drawn in the exterior face of N.

For each (free or ring) spoke S in H_G, we find a path P_S inside B_1 connecting an end of S with N, such that these paths are pairwise disjoint. We modify the drawing $\mathcal{H}_G$ into new $\mathcal{H}'_G$ in the following way. The subdrawing of B_1 in $\mathcal{H}_G$ is removed. The last edges e_S of all original spokes S are prolonged along the corresponding paths P_S to N. Finally, B_1 is drawn in the interior face of N such that the first main cycle N_1 of B_1 coincides with the original cycle N, and that ends of e_S (of all the prolonged spokes S) are in proper places.

In this way we introduce no new crossings to the drawing of $\mathcal{H}'_G$, and we eliminate previous crossings of $\mathcal{H}_G$ on edges of B_1, which contradicts optimality of the original drawing. □

Hence, in particular, the first main cycles N_j of the boulders B_j, $j = 1, 2$, are drawn with no crossings. Then there is a uniquely defined cylinder Π with the boundary curves N_1 and N_2 in the plane. Observe that the whole subgraph $\mathcal{H}_G - V(B_1) - V(B_2)$ is drawn on Π.

Lemma 5.4. *In the drawing $\mathcal{H}_G$, each main cycle M of every ring R_i, $i \in \{1, 2, \dots, n\}$, is drawn as a closed curve separating the subdrawing of the boulder B_1 from the subdrawing of B_2.*

Proof. Suppose, for a contradiction, that the claim is false for a main cycle M of R_h. Instead of the plane, let us consider the cylinder Π. Then our contradiction says that M is drawn as a contractible curve on Π.

We are going to apply Lemma 4.1 in this situation. Let $k = r$ and $\ell = m^3$ (see (2)). For $0 \leqslant i < m^3$ and $0 \leqslant j < r$, we denote by $\sigma_{i+1+j(m^3+1)}$ the drawing of the $(i + jm^3)$th free spoke—the edge κ_{i+jm^3} of $H_{m,n}$ (Section 3). Furthermore, for $0 \leqslant j < r$, we denote by $\sigma_{j(m^3+1)}$ the drawing of a path S_j associated with the jth ring spoke of the ring R_h: S_j consists of the edges $\mu^1_{h,j}, \mu^2_{h,j}$ and $\mu^4_{h,j}, \mu^5_{h,j}$, and of (one of) the shortest path connecting the ends of $\mu^2_{h,j}, \mu^4_{h,j}$ across the ring R_h.

One may easily verify that the collection $\mathcal{S} = \{\sigma_j\colon\ 0 \leqslant j < r(m^3 + 1)\}$ and $\tau = M$ satisfy the assumptions of Lemma 4.1. It follows from Lemma 5.3 that the ends of the curves in $\mathcal{S}$ are ordered on the boundaries of Π as required, and that τ is drawn in the interior of Π. Naturally,

τ intersects the drawings of each of the paths S_j since M shares a vertex with S_j. Moreover, a subcollection of disjoint paths $\mathcal{S}$ in a proper optimal drawing of a graph forms a nice set of curves by definition, and the same applies to the set $(\mathcal{S} \setminus \mathcal{S}_0) \cup \{\tau\}$ as in Lemma 4.1.

Since τ is contractible on Π, the possibility (ii) in Lemma 4.1 implies that the number of crossings in the drawing $\mathcal{H}_G$ is, using (3), at least

$$\left(\frac{2}{25}k^2 - \frac{1}{5}k\right)\ell = \left(\frac{2}{25}r^2 - \frac{1}{5}r\right)m^3 = \left(\frac{32}{25}m^4n^4 - \frac{4}{5}m^2n^2\right)m^3 = \frac{32}{25}m^7n^4 - \frac{4}{5}m^5n^2 > m^7n^4 > z,$$

which contradicts Corollary 3.3. Hence the possibility (i) of the lemma must be true, and there are at least

$$2 \cdot \frac{3}{5}k\ell = \frac{6}{5}rm^3 = \frac{6}{5}s = s + \frac{1}{5}rm^3 > s + rn + \frac{1}{8}rm^3 \tag{4}$$

crossings on the edges of M using (3). (Notice that we have not even considered crossings of M with the ring spokes of other rings than of R_h in this inequality.)

The above inequality (4) applies to every main cycle M in $\mathcal{H}_G$ which is drawn contractible on Π, while the noncontractible main cycles clearly have each at least $s + rn$ crossings with all the spokes in $\mathcal{H}_G$. Thus the total number of crossings in our $\mathcal{H}_G$ is at least

$$s + rn + \frac{1}{8}rm^3 + (nt-1)(s+rn) = \left(m^3+n\right)rnt + \frac{1}{8}rm^3 > \left(m^3+n\right)rnt + r = \frac{1}{2}z,$$

which again contradicts Corollary 3.3. Hence, indeed, every main cycle M must be drawn in $\mathcal{H}_G$ as a noncontractible closed curve on Π, and so M separates B_1 from B_2. □

Corollary 5.5. *There are at least $(s + rn)nt$ crossings in $\mathcal{H}_G$ between edges of the main cycles of the rings and edges of the free and ring spokes in H_G.*

Lemma 5.6. *There is a selection of main cycles $M_i \subset R_i$, $i = 1, 2, \ldots, n$, of the rings in H_G, such that the cycles $M_1, \ldots, M_n$ are drawn as pairwise disjoint closed curves in the drawing $\mathcal{H}_G$. Hence, there is a permutation π of $\{1, \ldots, n\}$ such that, for each $j = 1, \ldots, n$, the closed curve $M_{\pi(j)}$ separates the subdrawing of $B_1 \cup M_{\pi(1)} \cup \cdots \cup M_{\pi(j-1)}$ from the subdrawing of $B_2 \cup M_{\pi(j+1)} \cup \cdots \cup M_{\pi(n)}$.*

Proof. Combining Corollaries 3.3 and 5.5, we see that there are fewer than $r = t^2$ crossings between pairs of main cycles of the rings in $\mathcal{H}_G$. Let us, for $i = 1, \ldots, n$, form a collection $\mathcal{M}_i$ of closed curves—the drawings of the t main cycles of the ring R_i. Then we apply Lemma 4.3, and hence we find pairwise disjoint representatives $M_i \in \mathcal{M}_i$, as desired.

The second part then naturally follows from Lemma 5.4 and the Jordan Curve Theorem. □

Lemma 5.7. *For every $k = 0, 1, \ldots, 4n^2 - 1$, there is an index $c_k \in C_k = \{km^5 - 2m^4, \ldots, km^5 + 2m^4\}$ such that the edge of the c_kth free spoke κ_{c_k} is crossed exactly once by each of the main cycles of all the rings, and that κ_{c_k} has no more crossings than those in $\mathcal{H}_G$.*

Proof. By Lemma 5.4, each of the main cycles crosses each of the $s + rn$ spokes in $\mathcal{H}_G$. Suppose, for a contradiction, that for every $j \in C_k$ as above, $|C_k| = 4m^4 + 1$, the jth free spoke has at least two crossings with some main cycle in H_G. Then such a drawing $\mathcal{H}_G$ would have at least

$$(s+rn)nt+4m^4+1>(s+rn)nt+r=z/2$$

edge crossings, which is a contradiction to Corollary 3.3. □

Recall that the vertices of G are numbered as $\{1,2,\dots,n\}$, and that $X_{i,j}$ denotes the subgraph of the handle in the constructed graph H_G corresponding to an edge $ij \in E(G)$ (Section 3).

Lemma 5.8. *Let π be the permutation from Lemma 5.6, let Π be the cylinder defined after Lemma 5.3 for the drawing $\mathcal{H}_G$, and let $\{i,j\} \in E(G)$ be an edge. For $\ell = i+n(j-1)$, consider the indices $c_{4\ell-2}$ and $c_{4\ell+2}$ given by Lemma 5.7, and denote by Σ_ℓ the region on Π bounded by the drawings of the $c_{4\ell-2}$, $c_{4\ell+2}$th free spokes and containing the subdrawing of the handle $X_{i,j}$. Then Σ_ℓ contains at least*

$$t\big(\big|\pi^{-1}(i)-\pi^{-1}(j)\big|-1\big)$$

crossings of $\mathcal{H}_G$ between edges of the subgraph $X_{i,j} \cup R_i \cup R_j$ and edges of the main cycles of other rings R_k for $k \neq i,j$.

Proof. First, notice that Σ_ℓ is well defined since the drawings of the $c_{4\ell-2}$th and of the $c_{4\ell+2}$th free spokes are disjoint by Lemma 5.7, they do not cross $X_{i,j}$, and each of them connects two points on the opposite boundaries of Π. Moreover, by an analogous argument, the drawings of the $c_{4\ell-1}$th and $c_{4\ell+1}$th free spokes divide Σ_ℓ into three topological components Σ_ℓ^1, Σ_ℓ^2, Σ_ℓ^3 in this order, such that $X_{i,j}$ is drawn inside Σ_ℓ^2.

We denote by H'_G the subgraph of H_G obtained by deleting the two boulders and all the free and ring spokes, and by $\mathcal{H}'_G$ the corresponding subdrawing. Then, by Corollaries 3.3 and 5.5, $\mathcal{H}'_G$ contains fewer than r edge crossings. Consider now a ring R_k of H_G, for which $\pi^{-1}(i) < \pi^{-1}(k) < \pi^{-1}(j)$ (up to symmetry). By Lemma 5.6, there are main cycles M_i of R_i, M_j of R_j, and M_k of R_k, such that the drawing of M_k separates the drawings of M_i and M_j from each other on Π. Denote by $M_k^b \subset R_k$ the bth main cycle of the ring R_k.

Recall that $c_{4\ell-2} \leqslant (4\ell m^2-2m^2+2m)m^3$, and $c_{4\ell-1} \geqslant (4\ell m^2-m^2-2m)m^3$. So Lemma 5.7 also implies that the $(4\ell m^2-2m^2+3m-1)$th and $(4\ell m^2-m^2-3m+1)$th ring spokes of R_k are both drawn inside Σ_ℓ^1. Here we restrict the notation from the definition of a cyclic cubic grid just to the ring R_k. We set $c=(4\ell m^2-2m^2+3m)(m^3+n+1)$ and $d=(4\ell m^2-m^2-3m)(m^3+n+1)$. The previous argument implies that the vertices $v_{1,c}$, $v_{1,d}$ of the first main cycle M_k^1 of R_k are also drawn inside Σ_ℓ^1. Hence the situation corresponds with the setting of Lemma 4.7, and we conclude that all the cycles of R_k in the set $\mathcal{X}_1=\mathcal{X}(c+t,d-t)$ (defined in Section 4) are drawn inside Σ_ℓ^1.

Let us now estimate, using (2), (3):

$$\begin{aligned} d-c &= \big(m^2-6m\big)\big(m^3+n+1\big) > m^5-6\cdot 3m^4 > 100m^4-18m^4 > 16m^4+4m^4 \\ &> 4r+2t. \end{aligned}$$

So $|\mathcal{X}_1| = \frac{1}{2}(d-c-2t) > 2r$. Recall that the subdrawing $\mathcal{H}'_G$ has fewer than r edge crossings. Since one edge crossing may involve at most two of the cycles from $\mathcal{X}_1$, there exists a cycle

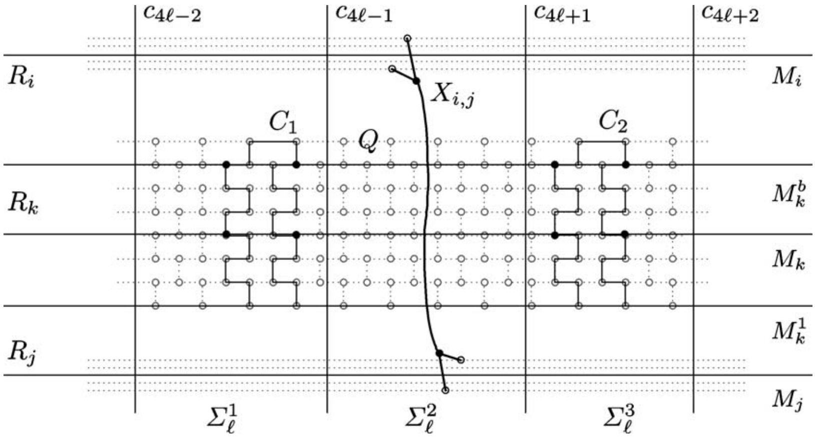

Fig. 8. An illustration to the proof of Lemma 5.8.

$C_1 \in \mathcal{X}_1$ which has no edge crossed in $\mathcal{H}'_G$. We analogously find a corresponding cycle C_2 drawn inside the region Σ_ℓ^3, and C_2 having no edge crossed in the subdrawing of H'_G. (See Fig. 8.)

Consider the drawing of the connected subgraph $R_i \cup R_j \cup X_{i,j}$ of H'_G which is disjoint from the drawings of C_1 and C_2. So it follows from our assumptions, and from $X_{i,j}$ being drawn inside Σ_ℓ^2, that the drawing of $R_i \cup R_j \cup X_{i,j}$ separates the drawings of C_1 and of C_2 from each other in Σ. We denote by $Q \subset M_k^b$ the shortest path connecting a vertex on C_1 to a vertex on C_2 and drawn in Σ. (Q is uniquely defined by our assumptions.) Then Q crosses the drawing of $R_i \cup R_j \cup X_{i,j}$ by the Jordan Curve Theorem. In this way we find distinct edge crossings for all choices of k such that $\pi^{-1}(i) < \pi^{-1}(k) < \pi^{-1}(j)$, and for all t choices of the main cycle M_k^b of R_k. The statement now follows. □

Now we are ready to finish the proof of crucial Proposition 5.1.

Proof of Proposition 5.1. We are going to count three collections of edge crossings in $\mathcal{H}_G$. These collections are pairwise disjoint since they involve different pairs of edges of H_G, as one may easily check. Firstly, there are (at least) $(s + rn)nt$ crossings described in Corollary 5.5.

Secondly, denote by d_i the degree of the vertex i in G. Let us consider the subgraph F_i of H_G formed by the ring R_i and by $2d_i$ pairs of incident edges from all handles which are attached to R_i in H_G. Then, by Lemma 4.4, the subgraph F_i itself has at least $2d_i(t-2)$ edge crossings in any drawing of H_G.

Thirdly, the permutation π from Lemma 5.6 defines a linear arrangement $\alpha = \pi^{-1}$ of the vertices of G. (An edge $\{i, j\} \in E(G)$ contributes with $|\alpha(i) - \alpha(j)|$ to the total weight of the arrangement α on G (1).) Recall the notation and conclusion of Lemma 5.8: an edge $\{i, j\}$ of G contributes (via its two handles in H_G) with at least $2t(|\alpha(i) - \alpha(j)| - 1)$ crossings in $\mathcal{H}_G$ which are contained in the regions Σ_ℓ and $\Sigma_{\ell'}$, where $\ell = i - 1 + n(j-1)$ and $\ell' = j - 1 + n(i-1)$. So in particular, the sets of crossings accounted here for distinct edges of G are pairwise disjoint, and also disjoint from the crossings contributed by the subgraphs F_i above.

Altogether, we have found at least this many distinct edge crossings in the optimal drawing $\mathcal{H}_G$ of our graph H_G:

$$\begin{aligned}
&(s+rn)nt + \sum_{i\in V(G)} 2d_i(t-2) + \sum_{\{i,j\}\in E(G)} 2t\big(\big|\alpha(i)-\alpha(j)\big|-1\big)\\
&\quad = (s+rn)nt + 2t \sum_{\{i,j\}\in E(G)} \big|\alpha(i)-\alpha(j)\big| - 2tm + 4tm - 8m\\
&\quad = (s+rn)nt + 2tA + 2tm - 8m. \qquad \square
\end{aligned}$$

6. Proof of the reduction

Finally, we conclude with the proof of our main result.

Proof of Theorem 2.1. Assume that G, a is an input instance of the OPTIMALLINEAR-ARRANGEMENT problem, and that G is sufficiently large (3). The graph H_G described in Section 3 is clearly cubic and 3-connected, it has polynomial size in $n = |V(G)|$, and H_G has been constructed efficiently. We now ask the problem CROSSINGNUMBER on the input $\langle H_G, (s+rn)nt + 2t(a+m)\rangle$, and give the same answer to OPTIMALLINEARARRANGEMENT on $\langle G, a\rangle$.

If there is a linear arrangement of G of weight at most a, then our correct answer is YES according to Proposition 3.2. Conversely, if the optimal linear arrangement of G has weight greater than a, then the crossing number of H_G is by Proposition 5.1

$$\mathrm{cr}(H_G) \geqslant (s+rn)nt + 2t(a+1+m) - 8m > (s+rn)nt + 2t(a+m),$$

and so the correct answer is NO. Since the OPTIMALLINEARARRANGEMENT problem is known to be NP-complete [3], the statement of Theorem 2.1 follows. $\square$

Remark 6.1. With a bit finer analysis, we can push forward the connectivity assumptions in Theorem 2.1: It is easy to see that a (not too small) cubic grid is a cyclically 5-connected graph. Joining a ring via at least five spokes with the boulders, maintains cyclical 5-connectivity of the "skeleton" graph $H_{m,n}$. Then, adding any edge handle to $H_{m,n}$ keeps the resulting graph cyclically 5-connected, and so the same applies also to the resulting graph H_G. Thus we may even assume the input graph in Theorem 2.1 to be cyclically 5-connected.

References

[1] D. Bokal, G. Fijavž, B. Mohar, The minor crossing number, manuscript, 2005.
[2] M.R. Fellows, The Robertson–Seymour theorems: A survey of applications, in: Contemp. Math., vol. 89, Amer. Math. Soc., Providence, RI, 1989, pp. 1–18.
[3] M.R. Garey, D.S. Johnson, Computers and Intractability, Freeman, New York, 1979.
[4] M.R. Garey, D.S. Johnson, Crossing number is NP-complete, SIAM J. Algebraic Discrete Methods 4 (1983) 312–316.
[5] I. Gitler, J. Leanos, G. Salazar, Crossing numbers of projective graphs, manuscript, 2005.
[6] L.Yu. Glebsky, G. Salazar, The crossing number of $C_m \times C_n$ is as conjectured for $n > m(m+1)$, J. Graph Theory 47 (2004) 53–72.
[7] M. Grohe, Computing crossing numbers in quadratic time, J. Comput. System Sci. 68 (2004) 285–302.
[8] R.K. Guy, The decline and fall of Zarankiewic's theorem, in: F. Harary (Ed.), Proc. Proof Techniques in Graph Theory, Academic Press, New York, 1969, pp. 63–69.
[9] F. Harary, P.C. Kainen, A.J. Schwenk, Toroidal graphs with arbitrarily high crossing numbers, Nanta Math. 6 (1973) 58–67.
[10] M. Klešč, R.B. Richter, I. Stobert, The crossing number of $C_5 \times C_n$, J. Graph Theory 22 (1996) 239–243.

[11] J. Pach, G. Tóth, Which crossing number is it, anyway? in: Proc. 39th Foundations of Computer Science, 1998, IEEE Press, 1999, pp. 617–626.
[12] M.J. Pelsmajer, M. Schaefer, D. Stefankovic, Odd crossing number is not crossing number, in: Graph Drawing 2005, in press.
[13] R.B. Richter, A problem submitted to: Problems in Topological Graph Theory (D. Archdeacon), http://www.emba.uvm.edu/~archdeac/problems/problems.html.
[14] N. Robertson, P.D. Seymour (Eds.), Open Problems, Graph Structure Theory, Proceedings of a AMS–IMS–SIAM Joint Summer Research Conference on Graph Minors, Seattle, 1991, Contemp. Math., vol. 147, Amer. Math. Soc., Providence, RI, 2003.