Relations Between Crossing Numbers of Complete and Complete Bipartite Graphs

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1. INTRODUCTION. In his "A Note of Welcome" in the first issue of the *Journal of Graph Theory* [5], Paul Turán wrote of his experience in a labor camp during the Second World War.

> There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled trucks to the storage yards. ... the work was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time... the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings?... This problem has become a notoriously difficult unsolved problem; the present state of it and the ensuing general problems one can see in the interesting paper of Guy [2].

We can abstract from Turán's story the following general question. Given m "kilns" to be joined by "tracks to n "storage areas", what is the minimum number of crossings of tracks possible? If we call the kilns and storage areas "vertices" and the tracks "edges", we are asking what is the minimum number of pairwise crossings of edges in a planar drawing of the *complete bipartite graph* $K_{m,n}$, which has two sets of vertices, one with m vertices and the other with n, such that each vertex in one set is joined to every vertex in the other set. See Figure 1 for two drawings of $K_{3,4}$ in the plane.

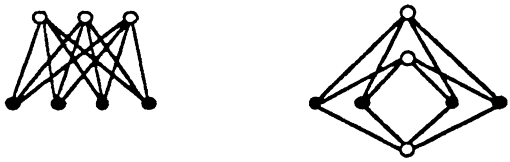

Figure 1

In general, for a graph G, the minimum number of pairwise crossings of edges among all drawings of G in the plane is the *crossing number* of G and is denoted by $cr(G)$. Thus, $cr(K_{3,4}) = 2$. We remark that Figure 1 shows that $cr(K_{3,4}) \leq 2$. It is an interesting exercise for the reader to prove that $cr(K_{3,4}) = 2$. At present, there is no known efficient algorithm to calculate the crossing number of an arbitrary graph. In fact, the problem of calculating the crossing number of a graph is NP-complete [1], so it is unlikely that such an efficient algorithm exists. Yet one might hope that the crossing number of a graph with special structure can be calculated.

The *complete graph* on n vertices is the graph K_n having n vertices such that every pair is joined by an edge. Figure 2 shows a drawing of K_6 with only 3

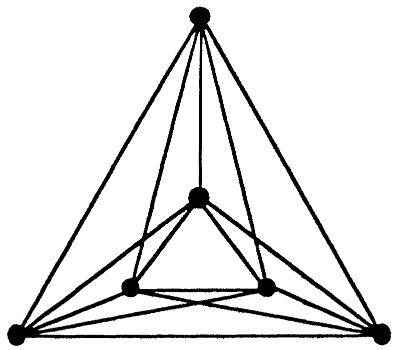

Figure 2

crossings, which turns out to be optimal. Since the complete graphs have a very special structure indeed, we can hope to calculate their crossing numbers.

There are conjectures for the crossing numbers of both the complete and complete bipartite graphs [3]:

$$cr(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

and

$$cr(K_{m,n}) = \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$$

However, these remain open. Some partial results are known: the former has been verified for $n \leq 10$, while the latter holds for $m \leq 6$ and all n [4] and for $m = 7$ and $n \leq 10$ [7].

The best known drawings of $K_{m,n}$ and K_n achieve these values. The description of such a drawing for $K_{m,n}$ is quite simple. Divide both the m-set and the n-set into two as-equal-as-possible parts. Place the m along the y-axis, with half above the x-axis and half below. Similarly, place the n along the x-axis, with half to the left of the y-axis and half to the right. Now join the m to the n using straight lines. The second drawing in Figure 1 is such a drawing of $K_{3,4}$.

Turan's story suggests a variant of the crossing number problem for complete bipartite graphs: find the smallest number of crossings in a *cylindrical* drawing of $K_{n,n}$, that is a drawing of $K_{n,n}$ on a cylinder such that each class of n vertices is on one of the two boundaries of the cylinder.

One way to get a drawing of K_{2n} in the plane is start with a cylindrical drawing of $K_{n,n}$ and then use the top and bottom of the cylinder to complete the drawing of K_{2n}. See Figure 3 for the case $n = 4$.

Obviously, this drawing of K_{2n} has $2\binom{n}{4}$ more crossings than the cylindrical drawing of $K_{n,n}$; this type of drawing of K_{2n} is described in [6]. With an appropriate choice of cylindrical drawing of $K_{n,n}$, the conjectured crossing number of K_{2n} is obtained this way.

One might hope that some better cylindrical drawing of $K_{n,n}$ exists and, therefore, a better drawing of K_{2n} would result. In Section 2, we associate a quadratic form with such drawings. Minimizing the quadratic form, we find the best cylindrical drawing of $K_{n,n}$, and so get the best drawing of K_{2n} *of this type.*

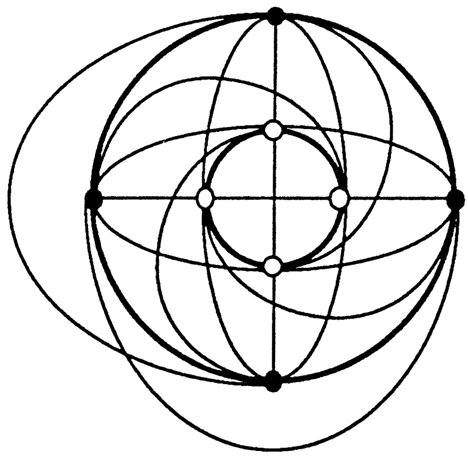

Figure 3

In Section 3 we shall discuss asymptotic values of the crossing numbers of K_n and $K_{n,n}$. It is easy to see (and will be discussed in Section 3) that the sequences $cr(K_n)/\binom{n}{4}$ and $cr(K_{n,n})/\binom{n}{2}^2$ are monotonically increasing and each term is less than 1. Therefore, the limits

$$\lim_{n\to\infty} \frac{cr(K_n)}{\binom{n}{4}} \quad \text{and} \quad \lim_{n\to\infty} \frac{cr(K_{n,n})}{\binom{n}{2}^2}$$

both exist and are at most 1. The conjectures on the values of the crossing numbers $cr(K_n)$ and $cr(K_{n,n})$ imply that the limits are $3/8$ and $1/4$, respectively. We prove in Section 3 that the latter implies the former.

2. CYLINDRICAL DRAWINGS OF $K_{n,n}$. We want to determine a lower bound on the number of crossings in any cylindrical drawing of $K_{n,n}$. We need to discover just what forces a crossing in the drawing. Consider, first, a single vertex v of $K_{n,n}$. All the edges incident with v are drawn across the cylinder to vertices on the other boundary. No two of these edges cross in an optimal drawing; see Figure 4.

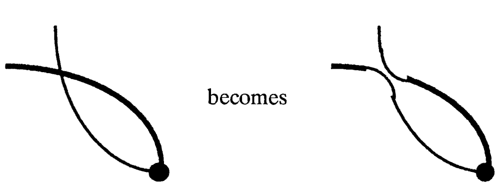

Figure 4

Now consider two vertices v and w on the same boundary. There are several possibilities for how the edges incident with these vertices are drawn, but we can see (Figure 5) that no two edges cross more than once in an optimal drawing. So how can two edges be forced to cross?

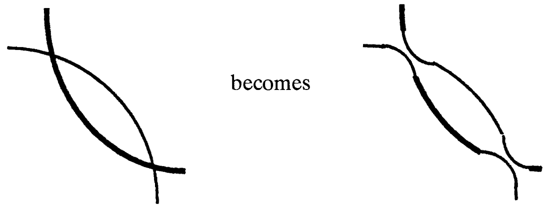

Figure 5

A little thought yields a simple observation.

For each vertex i on the inside boundary, there is a vertex $x_i \in \{1,2,\ldots,n\}$ on the outside boundary such that the simple closed curve consisting of the edges from i to each of x_i and $x_i + 1$ (the arithmetic being taken modulo n), together with the little segment of the outer boundary of the cylinder joining x_i and $x_i + 1$ bounds a disc containing the inner boundary of the cylinder.

As examples, in Figure 6, $x_1 = 5$ and $x_2 = 7$.

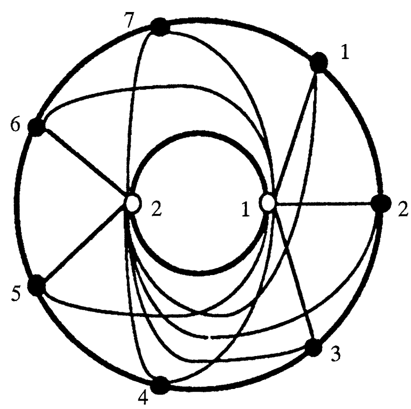

Figure 6

Now it is a simple matter to get a lower bound on the number of crossings given that the values of $x_1, x_2, \ldots, x_n$ are known. We need only deal with these in pairs, i.e., it suffices to calculate the number of crossings among edges incident with the vertices i and j on the inside boundary. If we pick two vertices r and s between $x_i + 1$ and x_j, say, then, among the four edges with ends i or j and r or s, there must be at least one crossing (Figure 7a). Similarly, if r and s are both between $x_j + 1$ and x_i. But if one is between $x_i + 1$ and x_j and the other is between $x_j + 1$ and x_i, then there need not be a crossing (Figure 7b).

Assuming that $1 \leq x_i \leq x_j \leq n$, it follows that there are at least

$$\binom{x_j - x_i}{2} + \binom{n + x_i - x_j}{2}$$

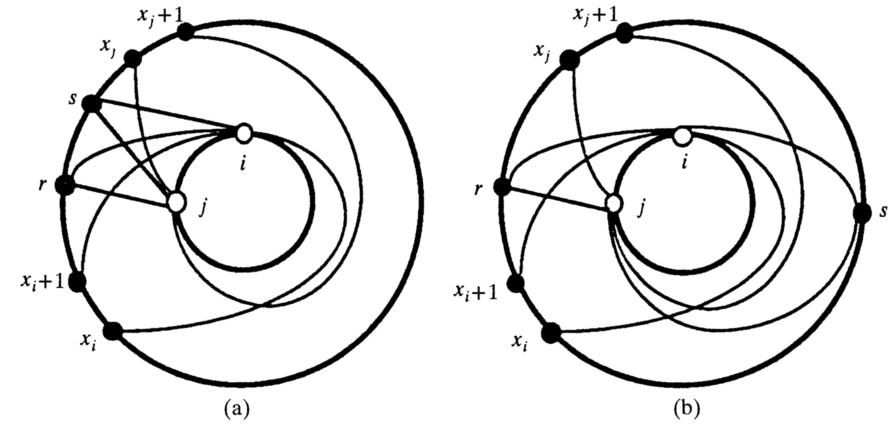

Figure 7

crossings in the drawing among edges incident with i and j. Therefore, a lower bound for the total number of crossings in the drawing is

$$\sum_{1\le i<j\le n} \binom{|x_j - x_i|}{2} + \binom{n - |x_j - x_i|}{2}.$$

Using the relation $\binom{y}{2} = y(y-1)/2$, we see that the lower bound is the function

$$f(x_1, x_2, \dots, x_n) = \binom{n}{2}^2 + \left(\sum_{1\le i<j\le n} |x_i - x_j|^2\right) - n\left(\sum_{1\le i<j\le n} |x_i - x_j|\right).$$

Ordering the variables so $1 \le x_1 \le x_2 \le \cdots \le x_n \le n$, we see that the lower bound is given by the quadratic function

$$F(x_1, x_2, \dots, x_n) = \binom{n}{2}^2 + \left(\sum_{1\le i<j\le n} (x_j - x_i)^2\right) - n\left(\sum_{1\le i<j\le n} (x_j - x_i)\right).$$

Clearly F has a minimum, which we shall determine.

The function F is differentiable and

$$\frac{\partial F}{\partial x_i} = 2\sum_{j\ne i}(x_i - x_j) + n(n - 2i + 1) = 2nx_i - 2\sum_{j=1}^{n} x_j + n(n - 2i + 1).$$

Setting $S = \sum_{j=1}^{n} x_j$ and $\nabla F = 0$, we find that

$$x_i = \frac{2S - n(n - 2i + 1)}{2n}$$

It is an easy calculation to see that $x_{i+1} - x_i = 1$ and, therefore, setting $x_i = i$ yields a solution to these equations. Moreover, every other solution is obtained from this one by adding the same quantity t to each x_i.

This means that there is an integral minimum for F, namely $x_i = i$, $i = 1, 2, \dots, n$. Thus, a lower bound for the number of crossings in a cylindrical drawing of $K_{n,n}$ is

$$F(1,2,\dots,n) = \sum_{1\le i<j\le n} \binom{j-i}{2} + \sum_{1\le i<j\le n} \binom{n-j+i}{2}$$
$$= \sum_{k=1}^{n} \binom{k}{2}(n-k) + \sum_{k=1}^{n-1} \binom{k}{2} k$$
$$= n\sum_{k=1}^{n-1} \binom{k}{2} = n\binom{n}{3}.$$

This is attainable; see Figure 3 for the case $n = 4$. The drawing of K_{2n} obtained from this optimal cylindrical drawing of $K_{n,n}$ has the same number of crossings as the conjectured crossing number of K_{2n}.

3. ASYMPTOTICS. The following classical counting argument estimates the crossing number of K_{n+1} in terms of the crossing number of K_n. Deleting in turn each vertex from a drawing of K_{n+1} yields $n + 1$ different drawings of K_n. Each of these must have at least $cr(K_n)$ crossings, so we estimate the crossing number of K_{n+1} by $(n + 1)cr(K_n)$.

How many times do we count a given crossing? A given crossing from K_{n+1} occurs in one of the drawings of K_n if the four vertices that are the ends of the edges involved in the crossing are all in the K_n we pick. Given that we must have these four vertices, there are $n - 4$ vertices left to be picked from the remaining $n - 3$ vertices of the K_{n+1}. Thus, the four vertices (and so the particular crossing) are in $n - 3$ of the K_n. Thus, each crossing is counted $n - 3$ times and we have the estimate

$$cr(K_{n+1}) \ge \frac{n+1}{n-3} cr(K_n).$$

This estimate is equivalent to

$$\frac{cr(K_{n+1})}{\binom{n+1}{4}} \ge \frac{cr(K_n)}{\binom{n}{4}}.$$

Therefore, the sequence $cr(K_n)/\binom{n}{4}$ is nondecreasing. Since it is bounded above by 1, it has a limit, say LC (for Limit of Complete graphs).

An entirely analogous argument shows that $cr(K_{n,n})/\binom{n}{2}^2$ has a limit LB (for Limit of complete Bipartite graphs). The drawings of $K_{n,n}$ such as the second drawing in Figure 1 show that $LB \le 1/4$.

It is easy to see that the conjectures as to the crossing numbers for K_n and $K_{m,n}$ imply that $LC = 3/8$ and $LB = 1/4$. We now show there is a relation between these limits.

Theorem. *$LC \ge (3/2)LB$. If $LB = 1/4$, then $LC = 3/8$.*

Proof: Let K_{2n} be drawn with $cr(K_{2n})$ crossings. Within this drawing, there are many different drawings of $K_{n,n}$. We need to estimate how many drawings of $K_{n,n}$ there are and how many of these contain a given crossing.

We shall count *ordered* $K_{n,n}$'s, i.e., those where we first pick one set of n and then the other set of n. There are, evidently, $\binom{2n}{n}$ such graphs.

Now consider a given crossing involving the edges ab and cd of K_{2n}. One of a and b must be in the first set of n chosen, and similarly for c and d. Thus, there are 4 ways to distribute a, b, c, d into the first set of n chosen, if this crossing is to occur in the resulting $K_{n,n}$. There are $2n - 4$ vertices left, of which $n - 2$ are to be put into the first set of n. Therefore, there are $4\binom{2n-4}{n-2}$ different $K_{n,n}$'s that contain the given crossing, and hence

$$cr(K_{2n}) \geq \frac{\binom{2n}{n}}{4\binom{2n-4}{n-2}} cr(K_{n,n}).$$

Divide both sides of this inequality by $\binom{2n}{4}$ and do some easy arithmetic to get

$$\frac{cr(K_{2n})}{\binom{2n}{4}} \geq \frac{3}{2} \frac{cr(K_{n,n})}{\binom{n}{2}^2}.$$

Now taking the limit as n tends to infinity, we have the relation

$$LC \geq (3/2)LB.$$

It follows that if $LB = 1/4$, then $LC \geq 3/8$. Since we have previously noted $LC \leq 3/8$, it follows that if $LB = 1/4$, then $LC = 3/8$. ■

This theorem shows that the conjecture for $cr(K_{n,n})$ implies the conjecture for $cr(K_{2n})$, at least asymptotically. Does the converse hold?

Probably this cannot be derived by counting. The reason why the proof of the theorem works (as the proof shows!) is that *any* (almost) optimal drawing of K_{2n} contains a drawing of $K_{n,n}$ that is economical in the sense that it has (almost) as few crossings as the conjectured value for $cr(K_{n,n})$.

For the converse, however, we do not know of a natural way to extend (almost) optimal drawings of $K_{n,n}$ to economical drawings of K_{2n}. The optimal cylindrical drawings of $K_{n,n}$ have many more than $cr(K_{n,n})$ crossings.

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