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The Toroidal Crossing Number of $K_{m,n}$

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ABSTRACT

It is shown that the toroidal crossing number of the complete bipartite graph, $K_{m,n}$, lies between

$$\frac{1}{15}\binom{m}{2}\binom{n}{2} \quad \text{and} \quad \frac{1}{6}\binom{m-1}{2}\binom{n-1}{2},$$

the lower bound holding for sufficiently large m and n.

1. DEFINITIONS

The (planar) *crossing number*, $cr(G)$, of a graph G has been defined [5] as the minimum number of crossings in any drawing of the graph on a plane (or sphere). A *drawing* is a mapping of the vertices of the graph into distinct points (*nodes*) of a 2-manifold, and of the edges into Jordan arcs of that manifold, having the two appropriate nodes for end-points, and no other node as interior point. A *crossing* is a common point of two arcs other than a node. We assume that three arcs do not have points in common other than nodes. In a drawing exhibiting the minimum number of crossings, two arcs have at most one point in common. If the 2-manifold is a torus (genus 1) we have the corresponding definition of the *toroidal crossing number*, $cr_1(G)$. We investigate $cr_1(K_{m,n})$, the toroidal crossing number of the complete bipartite (2-colored) graph, $K_{m,n}$, i.e., the graph on $m + n$ vertices, whose edges are exactly those which join one of the m vertices to one of the n.

2. PROLOG

In 1952, Zarankiewicz [6] gave an attempted solution of the problem to find $cr(K_{m,n})$, and in fact showed

$$cr(K_{m,n}) \leqslant [\tfrac{1}{2}m][\tfrac{1}{2}(m-1)][\tfrac{1}{2}n][\tfrac{1}{2}(n-1)], \tag{1}$$

where brackets denote "integer part," and that equality holds when $m = 3$. In a footnote he states that the problem was also solved by K. Urbaník, though now the validity of this solution may never be known. In 1965–6, P. Kainen and G. Ringel independently observed a hiatus [6, p. 141, or 1, p. 149] in Zarankiewicz's proof of the reverse inequality to (1). He assumes without proper justification that, if $n = 2s$ or $2s + 1$, then in a minimal drawing of $K_{m,n}$, from the n m-claws ($K_{m,1}$ graphs) of which it is composed, it is possible to select s pairs, each of which forms a crossing-free subdrawing of $K_{m,2}$.

Since (1) is true for $m = 3$, it can be deduced, as in the proof of (52) in Section 6 below, that

$$cr(K_{m,n}) \geqslant \tfrac{1}{6}m(m-1)[\tfrac{1}{2}n][\tfrac{1}{2}(n-1)]. \tag{2}$$

This implies equality in (1) for $m = 4$. In fact the truth of (1) for m odd implies its truth for $m + 1$. An analogous "one-legged induction" occurs [2] in the corresponding problem for the complete graph, K_n. The problems [1–6] to find $cr(K_n)$, $cr_1(K_n)$, $cr(K_{m,n})$, $cr_1(K_{m,n})$ remain open. In some of them there is evidence to conjecture that the best known upper bounds are the correct results. Various methods have been proposed, including those of Section 6 of this paper, which may be used to produce improvements in the lower bounds, but in each problem a clear gap remains.

3. Preliminary Considerations

We note that, in a crossing-free drawing of a (connected) subgraph of $K_{m,n}$, every circuit has an even number of nodes, and in particular every region into which the arcs divide the surface is bounded by an even circuit. So if t_j is the number of regions with j bounding arcs, F the number of regions, E the number of arcs, and V the number of nodes, then $t_j = 0$ for j odd, and

$$F = t_4 + t_6 + t_8 + \cdots, \tag{3}$$

$$2E = 4t_4 + 6t_6 + 8t_8 + \cdots, \tag{4}$$

and by Euler's theorem for the torus,

$$V \geqslant E - F, \tag{5}$$

so

$$V \geqslant t_4 + 2t_6 + 3t_8 + \cdots \geqslant F. \tag{6}$$

Suppose we have a minimal drawing of $K_{m,n}$ on the torus, i.e., one with $cr_1(K_{m,n})$ crossings, and that by removing e arcs, a crossing-free drawing is produced. Then (5) and (6) give

$$E - V = (mn - e) - (m + n) \leqslant F \leqslant V = m + n,$$

so

$$cr_1(K_{m,n}) \geqslant e \geqslant mn - 2(m + n). \tag{7}$$

In particular,

$$cr_1(K_{3,n}) \geqslant n - 6, \tag{8}$$

$$cr_1(K_{4,n}) \geqslant 2n - 8. \tag{9}$$

Figures 1 and 2, which show the reverse inequalities in the respective ranges $6 \leqslant n \leqslant 12$, $4 \leqslant n \leqslant 8$, confirm equality in (8) and (9) in these cases. It is also clear that

$$cr_1(K_{3,n}) = 0, \qquad n \leqslant 6, \tag{10}$$

$$cr_1(K_{4,n}) = 0, \qquad n \leqslant 4. \tag{11}$$

Similarly, Figures 3 and 4 with formula (7) show that

$$cr_1(K_{5,n}) = 3n - 10, \qquad 4 \leqslant n \leqslant 6, \tag{12}$$

$$cr_1(K_{6,6}) = 12. \tag{13}$$

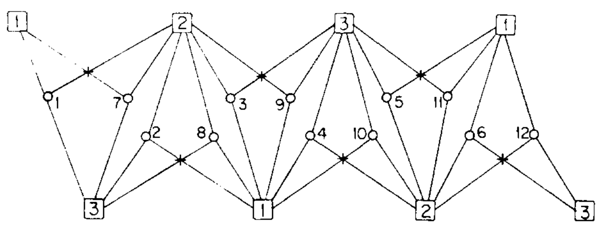

FIGURE 1

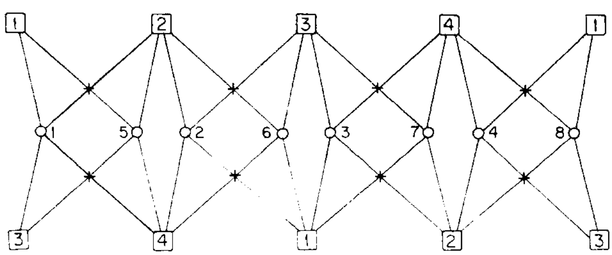

FIGURE 2

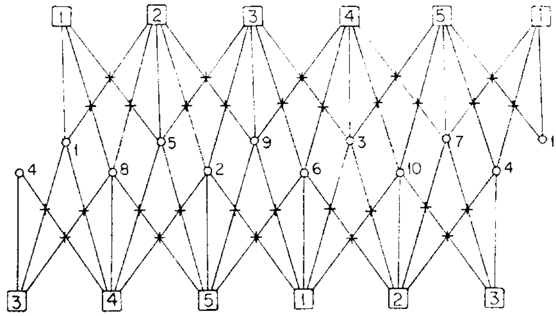

FIGURE 3

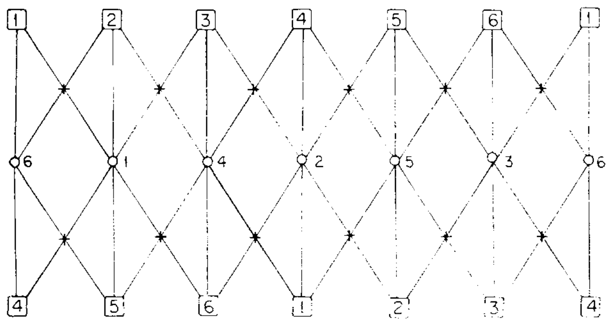

FIGURE 4

4. ANALOG OF ZARANKIEWICZ'S LEMMA

We prove a theorem for the case $m = 3$.

THEOREM 1.

$$cr_1(K_{3,n}) = [(n-3)^2/12]. \tag{14}$$

PROOF: The theorem is already proved for $n \leqslant 12$. Let $n = 6q + r$, $0 \leqslant r \leqslant 5$. In Figure 5, the $m = 3$ vertices are represented by squares and the $n = 6q + r$ by six groups of $q + \epsilon_i$ circles, $1 \leqslant i \leqslant 6$, $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_r = 1$, $\epsilon_{r+1} = \cdots = \epsilon_6 = 0$. The number of crossings in Figure 5 is

$$\begin{aligned}\sum_{i=1}^{6} \sum_{j=1}^{q+\epsilon_i} (j-1) &= \tfrac{1}{2} rq(q+1) + \tfrac{1}{2}(6-r)\, q(q-1) \\ &= \tfrac{1}{12}[36q^2 + 12q(r-3)] \\ &= [(6q+r-3)^2/12] = [(n-3)^2/12],\end{aligned}$$

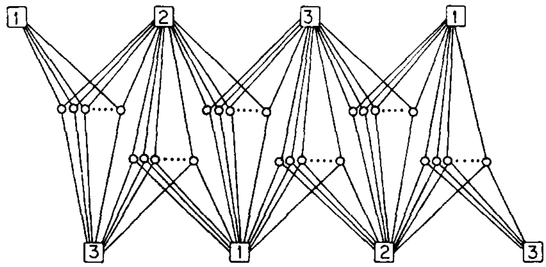

FIGURE 5

so $cr_1(K_{3,n})$ does not exceed this value. We show that the value is at least this by induction on n. Suppose that the result is true for $n = 6q + r$. Then $K_{3,n+1}$ contains $n + 1$ subgraphs $K_{3,n}$, each of which contains at least $3q^2 + qr - 3q$ crossings by the induction hypothesis. A crossing arises from two of the $n + 1$ nodes, so a crossing will have been counted $n - 1$ times. Hence

$$cr_1(K_{3,n+1}) \geqslant \frac{n+1}{n-1}(3q^2 + qr - 3q),$$

$$cr_1(K_{3,n+1}) \geqslant 3q^2 + q(r-2) + \frac{q(r-5)}{6q+r-1},$$

$$cr_1(K_{3,n+1}) \geqslant 3q^2 + q(r+1-3) = [(n+1-3)^2/12],$$

since

$$-1 < \frac{q(r-5)}{6q+r-1} \leqslant 0$$

for $q \geqslant 1$ and $0 \leqslant r \leqslant 5$, except in the case $q = 1, r = 0$, for which the theorem has already been proved. This completes the proof of Theorem 1.

5. UPPER BOUNDS

Figure 5 is a generalization of Figure 1, and the corresponding generalization of Figure 2 is Figure 6, which contains $m = 4$ nodes represented by squares and $n = 4q + r, 0 \leqslant r \leqslant 3$, nodes represented by circles, in four groups of $q + \epsilon_i$, $1 \leqslant i \leqslant 4$, $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_r = 1$, $\epsilon_{r+1} = \cdots = \epsilon_4 = 0$. From this it follows that

$$cr_1(K_{4,n}) \leqslant rq(q+1) + (4-r)\,q(q-1),$$

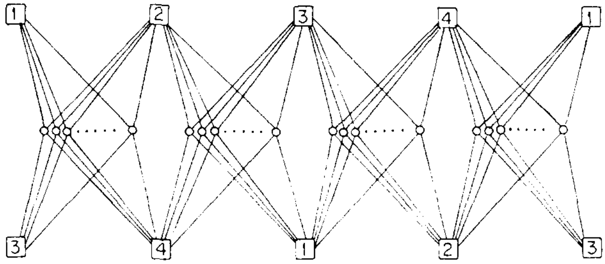

FIGURE 6

or

$$cr_1(K_{4,n}) \leqslant \tfrac{1}{4}n(n-4), \qquad (n, 4) = 4, \tag{15}$$

$$cr_1(K_{4,n}) \leqslant \tfrac{1}{4}(n-1)(n-3), \qquad (n, 4) = 1, \tag{16}$$

$$cr_1(K_{4,n}) \leqslant \tfrac{1}{4}(n-2)^2, \qquad (n, 4) = 2, \tag{17}$$

where (a, b) is the highest common factor of a and b. An attempt to prove the reverse inequalities to (15) to (17), using the method of Theorem 1, fails, as the induction does not make the step from $n = 4q$ to $4q + 1$. For example, we know that $cr_1(K_{4,8}) = 8$, so that

$$cr_1(K_{4,9}) \geqslant \tfrac{9}{7} \times 8, \qquad cr_1(K_{4,9}) \geqslant 11,$$

whereas (16) gives $cr_1(K_{4,9}) \leqslant 12$.

Figure 3 may be generalized in a similar way by taking $n = 10q + r$, $0 \leqslant r \leqslant 9$, and replacing nodes 1 to r by sets of $q + 1$ nodes and nodes $r + 1$ to 10 by sets of q nodes. The thirty crossings in Figure 3 will be replaced by thirty sets of crossings, each set containing q^2, $q(q + 1)$, or $(q + 1)^2$ crossings. In addition, there will be

$$\left[\binom{2}{2} + \binom{3}{2}\right]\binom{q+1}{2} \quad \text{or} \quad \left[\binom{2}{2} + \binom{3}{2}\right]\binom{q}{2}$$

additional crossings arising from each of the sets of $q + 1$ or q nodes which replace the first r or the last $10 - r$ nodes of Figure 3. On counting the crossings in the various cases $r = 0$, $r = 1, 3, 7$ or 9, $r = 2, 4, 6$ or 8, and $r = 5$, we obtain the formulas

$$cr_1(K_{5,n}) \leqslant \tfrac{1}{2}n(n-4), \qquad (10, n) = 10, \tag{18}$$

$$\leqslant \tfrac{1}{2}(n-1)(n-3), \qquad (10, n) = 1, \tag{19}$$

$$\leqslant \tfrac{1}{2}(n-2)^2, \qquad (10, n) = 2, \qquad (20)$$

$$\leqslant \tfrac{1}{2}(n^2 - 4n + 5), \qquad (10, n) = 5. \qquad (21)$$

Similarly we can generalize Figure 4, yielding

$$cr_1(K_{6,n}) \leqslant \tfrac{1}{6}n(5n - 18), \qquad (6, n) = 6, \qquad (22)$$

$$\leqslant \tfrac{1}{6}(n-1)(5n-13), \qquad (6, n) = 1, \qquad (23)$$

$$\leqslant \tfrac{1}{6}(n-2)(5n-8), \qquad (6, n) = 2, \qquad (24)$$

$$\leqslant \tfrac{1}{6}(n-3)(5n-3), \qquad (6, n) = 3, \qquad (25)$$

and we may continue with drawings having $n = 2mq + r, 0 \leqslant r \leqslant 2m - 1$, nodes in $2m$ sets when m is odd, and $n = mq + r, 0 \leqslant r \leqslant m - 1$, nodes in m sets when m is even, and obtain the following upper bounds:

$$cr_1(K_{7,n}) \leqslant \tfrac{1}{4}n(5n - 18), \qquad (14, n) = 14, \qquad (26)$$

$$\leqslant \tfrac{1}{4}(n-1)(5n-13), \qquad (14, n) = 1, \qquad (27)$$

$$\leqslant \tfrac{1}{4}(n-2)(5n-8), \qquad (14, n) = 2, \qquad (28)$$

$$\leqslant \tfrac{1}{4}(5n^2 - 18n + 21), \qquad (14, n) = 7. \qquad (29)$$

$$cr_1(K_{8,n}) \leqslant \tfrac{1}{4}n(7n - 24), \qquad (8, n) = 8, \qquad (30)$$

$$\leqslant \tfrac{1}{4}(n-1)(7n-17), \qquad (8, n) = 1, \qquad (31)$$

$$\leqslant \tfrac{1}{4}(n-2)(7n-10), \qquad (8, n) = 2, \qquad (32)$$

$$\leqslant \tfrac{1}{4}(7n^2 - 24n + 24), \qquad (8, n) = 4. \qquad (33)$$

$$cr_1(K_{9,n}) \leqslant \tfrac{1}{3}n(7n - 24), \qquad (18, n) = 18, \qquad (34)$$

$$\leqslant \tfrac{1}{3}(n-1)(7n-17), \qquad (18, n) = 1, \qquad (35)$$

$$\leqslant \tfrac{1}{3}(n-2)(7n-10), \qquad (18, n) = 2, \qquad (36)$$

$$\leqslant \tfrac{1}{3}(7n^2 - 24n + 18), \qquad (9, n) = 3, \qquad (37)$$

$$\leqslant \tfrac{1}{3}(7n^2 - 24n + 27), \qquad (18, n) = 9. \qquad (38)$$

$$cr_1(K_{10,n}) \leqslant n(3n - 10), \qquad (10, n) = 10, \qquad (39)$$

$$\leqslant (n-1)(3n-7), \qquad (10, n) = 1, \qquad (40)$$

$$\leqslant (n-2)(3n-4), \qquad (10, n) = 2, \qquad (41)$$

$$\leqslant 3n^2 - 10n + 5, \qquad (10, n) = 5. \qquad (42)$$

$$cr_1(K_{11,n}) \leqslant \tfrac{5}{4}n(3n-10), \qquad (22,n)=22, \tag{43}$$

$$\leqslant \tfrac{5}{4}(n-1)(3n-7), \qquad (22,n)=1, \tag{44}$$

$$\leqslant \tfrac{5}{4}(n-2)(3n-4), \qquad (22,n)=2, \tag{45}$$

$$\leqslant \tfrac{5}{4}(3n^2-10n+11), \qquad (22,n)=11. \tag{46}$$

In general

$$cr_1(K_{m,n}) \leqslant \tfrac{1}{24}(m-2)[n^2(m-1)-3mn],$$
$$(m,2)=2, \qquad (m,n)=m, \tag{47}$$

$$\leqslant \tfrac{1}{24}(m-1)[n^2(m-2)-3n(m-1)],$$
$$(m,2)=1, \qquad (m,n)=2m, \tag{48}$$

$$\leqslant \tfrac{1}{24}(m-1)[n^2(m-2)-3n(m-1)+3m],$$
$$(m,2)=1, \qquad (m,n)=m. \tag{49}$$

With only slight loss of sharpness, formulas (47) to (49) may be combined:

$$cr_1(K_{m,n}) \leqslant \frac{1}{6}\binom{m-1}{2}\binom{n-1}{2}, \tag{50}$$

and (50) may be verified for all m and n. The right member of (49) reduces to $\binom{m}{4}$ when $n = m$.

6. Lower Bounds

If $p \leqslant m, q \leqslant n$, then $K_{m,n}$ contains $\binom{m}{p}\binom{n}{q}$ subgraphs $K_{p,q}$. If we count the minimum number of crossings in these, noting that each crossing arises from just two nodes among the m and just two among the n, so that it is counted $\binom{m-2}{p-2}\binom{n-2}{q-2}$ times,

$$cr_1(K_{m,n}) \geqslant \binom{m}{p}\binom{n}{q} cr_1(K_{p,q}) \Big/ \binom{m-2}{p-2}\binom{n-2}{q-2},$$

$$cr_1(K_{m,n}) \geqslant \frac{mn(m-1)(n-1)}{pq(p-1)(q-1)} cr_1(K_{p,q}). \tag{51}$$

Put $p = 3, q = n$, in (51):

$$cr_1(K_{m,n}) \geqslant \tfrac{1}{6}m(m-1)[(n-3)^2/12]. \tag{52}$$

Alternatively, put $p = m$, $q = n - 1$, or $p = m - 1$, $q = n$:

$$cr_1(K_{m,n}) \geqslant \left\{\frac{n}{n-2}\, cr_1(K_{m,n-1})\right\}, \tag{53}$$

$$cr_1(K_{m,n}) \geqslant \left\{\frac{m}{m-2}\, cr_1(K_{m-1,n})\right\}, \tag{54}$$

where braces denote the least integer not less than their contents. These two formulas combine to give

$$cr_1(K_{m,n}) \geqslant \max\left(\left\{\frac{m}{m-2}\, cr_1(K_{m-1,n})\right\}, \left\{\frac{n}{n-2}\, cr_1(K_{m,n-1})\right\}\right), \tag{55}$$

which has been used to compile Table 1, of known lower bounds for $cr_1(K_{m,n})$.

THEOREM 2. *If $n \geqslant 5$, then*

$$[(n-1)(n-2)/5] \leqslant cr_1(K_{4,n}) \leqslant [(n-2)^2/4]\,. \tag{56}$$

PROOF: The second inequality follows from (15) to (17). We prove the first by induction on n. It is true for $5 \leqslant n \leqslant 7$ from Table 1. Assume it true for $n - 1$; then, from (53) with $m = 4$,

$$cr_1(K_{4,n}) \geqslant \left\{\frac{n}{n-2}\, cr_1(K_{4,n-1})\right\} \geqslant \left\{\frac{n}{n-2}\left[\frac{(n-2)(n-3)}{5}\right]\right\}$$

$$\geqslant \left\{\frac{n}{n-2}\left[\frac{(n-2)(n-3)-2}{5}\right]\right\},$$

since $(n-2)(n-3) \equiv 0, 1$, or 2, modulo 5. Hence

$$cr_1(K_{4,n}) \geqslant \left\{\frac{(n-1)(n-2)}{5} - \frac{4n-4}{5(n-2)}\right\} = \left[\frac{(n-1)(n-2)}{5}\right],$$

since

$$-1 < -\frac{4n-4}{5n-10} < -\frac{2}{5}$$

for $n \geqslant 7$ and $(n-1)(n-2) \equiv 0, 1$ or 2, modulo 5. This proves the theorem. On putting $p = 4$, $q = n$ in (51):

TABLE 1

LOWER BOUNDS FOR $cr_1(K_{m,n})$

m \ n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
4	0	2	4	6	8	11	14	18	22	26	31	36	42	48	54	61	68	76	84	92	101	110	120
5		5	8	12	16	21	27	33	40	48	56	65	75	85	96	108	120	133	147	161	176	192	208
6			12	18	24	32	41	51	62	74	87	101	116	132	149	167	186	206	227	249	272	296	321
7				26	35	45	58	72	87	104	122	142	163	185	209	234	261	289	318	349	381	415	450
8					47	61	78	96	116	139	163	190	218	248	279	312	348	386	425	466	509	554	601
9						79	101	124	150	179	210	245	281	319	359	402	448	497	547	600	655	713	773
10							127	156	188	224	263	307	352	399	449	503	560	622	685	751	820	892	967
11								191	230	274	322	376	431	489	551	616	685	761	838	918	1003	1091	1182
12									276	329	387	452	518	588	662	740	823	914	1006	1102	1204	1310	1420
13										389	458	535	613	695	783	876	974	1081	1190	1304	1423	1549	1679
14											535	625	716	812	914	1022	1137	1262	1389	1522	1661	1808	1959
15												722	827	938	1056	1181	1313	1457	1603	1757	1917	2087	2261
16													946	1073	1208	1351	1502	1666	1833	2008	2191	2386	2585
17														1217	1370	1532	1703	1889	2078	2276	2484	2705	2931
18															1542	1724	1916	2126	2339	2562	2795	3044	3298
19																1927	2142	2377	2615	2865	3126	3403	3687
20																	2380	2642	2907	3184	3474	3782	4098

m \ n	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
4	130	140	151	162	174	186	198	211	224	238	252	266	281	296	312	328	344	361	378	396
5	225	243	261	280	300	320	341	363	385	408	432	456	481	507	533	560	588	616	645	675
6	347	374	402	431	461	492	524	557	591	626	662	699	737	776	816	857	899	942	986	1031
7	486	524	563	604	646	690	735	781	829	878	929	981	1035	1090	1146	1204	1263	1324	1386	1449
8	650	700	752	806	862	920	980	1042	1106	1172	1239	1308	1380	1454	1529	1606	1685	1766	1849	1934
9	836	901	968	1038	1110	1184	1261	1340	1422	1507	1594	1683	1775	1870	1966	2065	2167	2271	2378	2487
10	1045	1127	1211	1298	1388	1481	1577	1676	1778	1884	1993	2104	2219	2338	2458	2582	2709	2839	2973	3109
11	1278	1378	1481	1587	1697	1811	1928	2049	2174	2303	2436	2572	2713	2858	3005	3156	3311	3470	3634	3800
12	1534	1654	1778	1905	2037	2174	2315	2460	2610	2764	2924	3087	3256	3430	3606	3788	3974	4164	4361	4560
13	1814	1955	2102	2253	2409	2570	2736	2908	3085	3267	3456	3649	3848	4054	4262	4477	4697	4922	5154	5390
14	2117	2281	2453	2629	2811	2999	3193	3393	3600	3812	4032	4258	4490	4730	4973	5224	5480	5743	6013	6289
15	2443	2632	2831	3034	3244	3461	3685	3916	4154	4399	4653	4914	5181	5458	5738	6028	6324	6627	6939	7257
16	2792	3008	3236	3468	3708	3956	4212	4476	4748	5028	5318	5616	5922	6238	6558	6890	7228	7574	7931	8294
17	3166	3410	3668	3931	4203	4484	4774	5073	5382	5699	6028	6365	6712	7070	7433	7809	8192	8584	8989	9400
18	3562	3837	4127	4423	4729	5045	5371	5708	6055	6412	6782	7161	7551	7954	8363	8786	9216	9657		
19	3982	4289	4613	4944	5286	5639	6003	6380	6768	7167	7580	8004	8440	8890	9347	9820				
20	4426	4767	5126	5494	5874	6266	6671	7089	7520	7964	8423	8894	9378	9878						

COROLLARY.

$$cr_1(K_{m,n}) \geqslant \frac{1}{6}\binom{m}{2}\left[\frac{(n-1)(n-2)}{5}\right] \quad \text{for} \quad m \geqslant 4, n \geqslant 5. \tag{57}$$

THEOREM 3. *If $n \geqslant 5$, then*

$$\left[\frac{(n-1)^2}{3}\right] \leqslant cr_1(K_{5,n}) \leqslant \frac{1}{2}(n^2 - 4n + 5). \tag{58}$$

PROOF: The second inequality follows from (18) to (21). The first follows by induction on n, since it is true for $n = 5$ from Table 1, and, as in the proof of Theorem 2,

$$\begin{aligned} cr_1(K_{5,n}) &\geqslant \left\{\frac{n}{n-2}\, cr_1(K_{5,n-1})\right\} \geqslant \left\{\frac{n}{n-2}\left[\frac{(n-2)^2}{3}\right]\right\} \\ &\geqslant \left\{\frac{n}{n-2}\left[\frac{(n-2)^2-1}{3}\right]\right\} \\ &= \left\{\frac{(n-1)^2}{2} - \frac{2n-2}{3n-6}\right\} = \left[\frac{(n-1)^2}{3}\right], \end{aligned}$$

since

$$-1 < -\frac{2n-2}{3n-6} < -\frac{1}{3}$$

for $n \geqslant 5$, and $(n-1)^2 \equiv 0$ or 1, modulo 3.

COROLLARY 1.

$$cr_1(K_{m,n}) \geqslant \frac{1}{10}\binom{m}{2}\left[\frac{(n-1)^2}{3}\right], \qquad m, n \geqslant 5. \tag{59}$$

COROLLARY 2.

$$cr_1(K_{5,7}) = 12. \tag{60}$$

THEOREM 4. *If $n \geqslant 8$, then*

$$\binom{n}{2} - 4 \leqslant cr_1(K_{6,n}) \leqslant \frac{1}{6}(n-2)(5n-8). \tag{61}$$

PROOF: Similar to those of the previous two theorems.

COROLLARY.

$$cr_1(K_{m,n}) \geqslant \frac{1}{15}\binom{m}{2}\left[\binom{n}{2}-4\right], \qquad m \geqslant 6, n \geqslant 8. \tag{62}$$

THEOREM 5. *If* $n \geqslant 15$, *then*

$$9(n-2)(3n+1)/38 < cr_1(K_{7,n}) \leqslant \tfrac{1}{4}(5n^2-18n+21). \tag{63}$$

PROOF: The second inequality follows from (26) to (29). We prove the first by induction on n in the sharper form

$$cr_1(K_{7,n}) \geqslant (27n^2-45n+a_n)/38, \tag{64}$$

where a_n is given in Table 2. It depends only on c, where $n = 19k + c$, $|c| \leqslant 9$. Note that $27n^2 - 45n + a_n$ is a multiple of 38 for all values of n. The theorem may be verified for $15 \leqslant n \leqslant 20$ from Table 1. By (53) and the inductive hypothesis,

$$\begin{aligned} cr_1(K_{7,n}) &\geqslant \left\{\frac{n}{n-2}\, cr_1(K_{7,n-1})\right\} \\ &\geqslant \left\{\frac{n}{n-2}\left[\frac{27(n-1)^2-45(n-1)+a_{n-1}}{38}\right]\right\} \\ &= \left\{\left[27n^2-45n+\left(1+\frac{2}{n-2}\right)(a_{n-1}-18)\right]\Big/38\right\} \\ &= (27n^2-45n+a_n)/38, \end{aligned}$$

since the last expression is an integer, and, for $n > 20$, it may be verified from Table 2 that

$$a_n - 38 < \left(1+\frac{2}{n-2}\right)(a_{n-1}-18) \leqslant a_n \tag{65}$$

in each of the 19 cases. The most critical case for the first inequality is $c = 1$, $a_n - 38 = -20$, $a_{n-1} - 18 = -18$; the inequality is not strict when $n = 20$. Equality occurs between the second and last members of (65) when $c = 8$.

COROLLARY.

$$cr_1(K_{m,n}) \geqslant \frac{1}{21}\binom{m}{2}\left[\frac{9(3n^2-5n+3)}{38}\right], \qquad m \geqslant 7, n \geqslant 15. \tag{66}$$

TABLE 2

c		−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	−9	−8	−7	−6	−5	−4
n		15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
$27n^2 - 45n$		5400	6192	7038	7938	8892	9900	10962	12078	13248	14472	15750	17082	18468	19908	21402	22950	24552	26208	27918	29682
a_n		−4	2	−8	4	0	18	20	6	14	6	20	18	0	4	−8	2	−4	12	12	−4
a_{n-1}	−18	−6	−22	−16	−26	−14	−18	0	2	−12	−4	−12	2	0	−18	−14	−26	−16	−22	−6	−6
$18n^2 - 30n$		3600	4128	4692	5292	5928	6600	7308	8052	8832	9648	10500	11388	12312	13272	14268	15300	16368	17472	18612	19788
b_n		10	14	20	9	0	12	26	23	22	23	26	31	38	28	20	14	10	8	8	10
b_{n-1}	−12	−4	−2	2	8	−3	−12	0	14	11	10	11	14	19	26	16	8	2	−2	−4	−4

THEOREM 6. *If $n \geqslant 15$, then*

$$\frac{6n(3n-5)}{19} \leqslant cr_1(K_{8,n}) \leqslant \frac{7n^2-24n+24}{4}. \tag{67}$$

PROOF: This follows closely that of Theorem 5, but using (30) to (33) and establishing the first inequality in the form

$$cr_1(K_{8,n}) \geqslant (18n^2 - 30n + b_n)/19, \tag{68}$$

where b_n is the analog of a_n and is also given in Table 2. Then

$$\begin{aligned} cr_1(K_{8,n}) &\geqslant \left\{\frac{n}{n-2}\left[\frac{18(n-1)^2 - 30(n-1) + b_{n-1}}{19}\right]\right\} \\ &= \left\{\left[18n^2 - 30n + \left(1 + \frac{2}{n-2}\right)(b_{n-1} - 12)\right]\Big/19\right\} \\ &= \frac{18n^2 - 30n + b_n}{19}, \end{aligned}$$

since this is an integer, and for $n \geqslant 15$ we have

$$b_n - 19 < \left(1 + \frac{2}{n-2}\right)(b_{n-1} - 12) \leqslant b_n, \tag{69}$$

except in the cases $c = 1$ and $c = 2$. Note that the equality in (69) occurs for $n = 18$, and that for $c = 8$ the first inequality is strict by an amount $38/(n-2)$.

To cover the exceptional cases we use (54) and (64), giving

$$cr_1(K_{8,n}) \geqslant \left\{\frac{8}{6}\, cr_1(K_{7,n})\right\} \geqslant \left\{\frac{4}{3}\,\frac{27n^2 - 45n + a_n}{38}\right\},$$

where $a_n = 18$ and 20 in the cases $c = 1$ and 2. So

$$cr_1(K_{8,n}) \geqslant \left\{\frac{18n^2 - 30n + 2a_n/3}{19}\right\} = \frac{18n^2 - 30n + b_n}{19},$$

since $b_n - 19 < \frac{2}{3}a_n \leqslant b_n$ in these two cases.

COROLLARY.

$$cr_1(K_{m,n}) \geqslant \frac{3}{14 \cdot 19}\binom{m}{2} n(3n-5), \qquad m \geqslant 8, m \geqslant 15. \tag{70}$$

7. Conclusion

THEOREM 7.

$$\frac{1}{15}\binom{m}{2}\binom{n}{2} < cr_1(K_{m,n}) \leqslant \frac{1}{6}\binom{m-1}{2}\binom{n-1}{2},$$

provided, in the lower bound, m and n are at least equal to one of the (unordered) pairs (7,45), (8,44), (10,43), (14,42) *or* (19,41).

PROOF: The theorem follows from (50) and (70), since, for sufficiently large n,

$$\frac{1}{15}\binom{n}{2} < \frac{3n(3n-5)}{266},$$

and the values for which the theorem is stated may be checked from the continuation of Table 1.

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