Cyclic-Order Graphs and Zarankiewicz's Crossing-Number Conjecture

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ABSTRACT

Zarankiewicz's conjecture, that the crossing number of the complete-bipartite graph $K_{m,n}$ is $\lfloor \frac{1}{2} m \rfloor \lfloor \frac{1}{2}(m-1) \rfloor \lfloor \frac{1}{2} n \rfloor \lfloor \frac{1}{2}(n-1) \rfloor$, was proved by Kleitman when $\min(m, n) \leqslant 6$, but was unsettled in all other cases. The cyclic-order graph CO_n arises naturally in the study of this conjecture; it is a vertex-transitive harmonic diametrical (even) graph. In this paper the properties of cyclic-order graphs are investigated and used as the basis for computer programs that have verified Zarankiewicz's conjecture for $K_{7,7}$ and $K_{7,9}$; thus the smallest unsettled cases are now $K_{7,11}$ and $K_{9,9}$. © 1993 John Wiley & Sons, Inc.

1. ZARANKIEWICZ'S CONJECTURE

We shall discuss Zarankiewicz's conjecture in Section 1, cyclic-order graphs in Section 2 (and the Appendix), and the connection between them in Section 3. Section 4 describes the principles underlying the computer programs, and Section 5 describes the results of the individual programs.

The *crossing number* cr(G) of a graph G is the smallest crossing number of any drawing of G in the plane, where the *crossing number* cr(D) of a drawing D is the number of pairs of nonadjacent edges that intersect in the drawing. It is impiicit that the edges in a drawing are Jordan arcs (hence, nonselfintersecting), and it is easy to see that a drawing with minimum crossing number must be a *good* drawing; that is, each two edges have at most one point in common, which is either a common end vertex or a crossing.

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The problem of determining the crossing number of the complete-bipartite graph $K_{m,n}$ is sometimes known as Turán's brick-factory problem: for the history of the problem, see [4]. The "obvious" way of drawing $K_{m,n}$ is to put the two sets of vertices along two parallel straight lines, and to join them by mn straight-line segments; this drawing evidently has $\binom{m}{2}\binom{n}{2}$ crossings. A more economical drawing, pointed out by Zarankiewicz [8], is to arrange the m and n vertices along the x axis and y axis respectively; in each case with half of them either side of the origin (as nearly as possible, with no vertex at the origin itself), again joined by mn straight-line segments; this drawing has $Z(m)Z(n)$ crossings, where

$$Z(n) := \binom{\lfloor \frac{1}{2}n \rfloor}{2} + \binom{\lceil \frac{1}{2}n \rceil}{2} = \left\lfloor \tfrac{1}{2}n \right\rfloor \left\lfloor \tfrac{1}{2}(n-1) \right\rfloor.$$

This drawing shows that

$$\operatorname{cr}(K_{m,n}) \leqslant Z(m)Z(n). \tag{1}$$

The conjecture that equality holds in (1) has been called *Zarankiewicz's conjecture.* This conjecture was proved by Kleitman [5] for the case $\min(m,n) \leqslant 6$. Kleitman used an ingenious argument to show that if the conjecture holds for $K_{5,5}$ and $K_{5,7}$, then it holds for $K_{5,n}$ for all n. Unfortunately, this argument does not seem to work for $K_{7,7}$, $K_{7,9}$, and $K_{7,n}$. Kleitman also used the following theorem and corollary.

Theorem 1. A drawing of $K_{m,n}$ with crossing number k contains a drawing of $K_{m-1,n}$ with crossing number at most $k(m-2)/m$.

Proof. There are m drawings of $K_{m-1,n}$ in the drawing of $K_{m,n}$, and each crossing occurs in $m-2$ of them. ∎

Corollary 1.1. If m is even, then Zarankiewicz's conjecture holds for $K_{m,n}$ if it holds for $K_{m-1,n}$ (since then $Z(m)/Z(m-1) = m/(m-2)$). ∎

Thus it suffices to consider the case when m and n are both odd. Another basic result, simple cases of which were used by Kleitman and Zarankiewicz, is the following.

Theorem 2. Let m and n be odd integers and $m' < m$ an even integer such that Zarankiewicz's conjecture holds for $K_{m'+1,n}$ and $K_{m-m',n}$. Then there are at least $Z(m)Z(n)$ crossings in any drawing of $K_{m,n}$ that contains a drawing of $K_{m',n}$ with $Z(m')Z(n)$ or fewer crossings.

Proof. Let the partite sets of $K_{m,n}$ be M and N, those of $K_{m',n}$ be M' and N, and let the drawing of $K_{m',n}$ have $k \leqslant Z(m')Z(n)$ crossings.

Each crossing in the drawing of $K_{m,n}$ involves exactly one pair of vertices u, v in M. The number of crossings corresponding to vertices u, v in M' is k, the number corresponding to u and v both in $M \setminus M'$ is at least $\mathrm{cr}(K_{m-m',n})$, and the number corresponding to u in M' and v in $M \setminus M'$ is at least $(m - m')[\mathrm{cr}(K_{m'+1,n}) - k]$. These three numbers add up to at least $Z(m)Z(n)$ in view of the hypotheses of the theorem, the upper bound on k, and the easily verifiable equation

$$Z(m') + Z(m - m') + (m - m')[Z(m' + 1) - Z(m')] = Z(m).$$

This completes the proof. ∎

2. THE CYCLIC-ORDER GRAPH CO_n

This graph is defined as follows. Its vertices are the $(n - 1)!$ different cyclic orderings of a set V_n of n elements, and two such orderings are adjacent in the graph if one can be obtained from the other by transposing two adjacent elements. (The significance of *cyclic* ordering is of course that 012 is the same as 120 or 201, so that 0 and 2 are adjacent elements of the cyclic ordering 012.) Thus CO_n is vertex-transitive and n-regular, and each edge ab can be labeled with one of the $\binom{n}{2}$ 2-subsets xy of V_n, indicating that x and y are transposed in getting from a to b. The graph CO_4 is shown in Figure 1. For a related graph defined on the same set of vertices, see [2].

If $a \in V(CO_n)$, then $\overline{a}$ denotes the reverse ordering of a; for example, in CO_7, if $a = 0354162$, then $\overline{a} = 0261453$. The mapping $a \mapsto \overline{a}$ determines an automorphism of CO_n that preserves edge-labelings. It seems intuitively

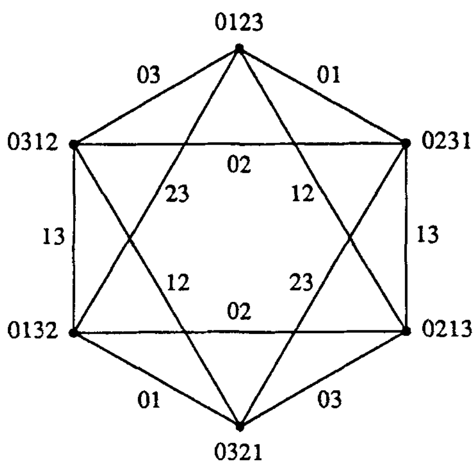

FIGURE 1. The cyclic-order graph CO_4.

obvious that $\overline{a}$ is the unique vertex of CO_n that is at maximum distance from a. We do not use this fact, but for completeness we shall prove it in the Appendix. However, not every vertex b of CO_n is necessarily on a shortest $a\overline{a}$-path; for example, if $n = 5$, $a = 01234$, $\overline{a} = 04321$, and $b = 02413$, then the distance $d(a,\overline{a}) = 4$ but $d(a,b) = d(\overline{a},b) = 3$.

A graph G is called *diametrical* [6], *even* [3], or a *self-centered unique-eccentric-point graph* [7] if, for each vertex a, there is a unique vertex $\overline{a}$ such that $d(a,\overline{a})$ is equal to the diameter of G. Such a graph is a *harmonic* [3] if $\overline{a}\,\overline{b} \in E(G)$ whenever $ab \in E(G)$. It is *symmetric* [3] or *antipodal* [1] if, for each two vertices a, b, $d(a,b) + d(b,\overline{a}) = d(a,\overline{a})$. Clearly CO_n is diametrical and harmonic; but, as we have just seen, it is not antipodal if $n \geqslant 5$. Nevertheless, we shall refer to $\overline{a}$ as the *antipode* of a rather than as its *buddy* [3].

Theorem 3. (a) If n is odd, then CO_n is bipartite.
(b) A shortest path between two vertices in CO_n cannot contain two edges with the same label.
(c) If $a \in V(CO_n)$, then $d(a,\overline{a}) = Z(n)$.

Proof. (a) The $n!$ *linear* orderings of V_n divide into two classes (even and odd permutations) such that every transposition causes a move from one class to the other. If n is odd, then any n orderings that we are going to regard as identical, because they all represent the same cyclic ordering, all belong to the same class (since an n-cycle is an even permutation if n is odd). Thus the vertices of CO_n also divide into two classes such that any transposition (hence, *a fortiori,* any transposition of adjacent elements) causes a move from one class to the other; that is, CO_n is bipartite.

(b) Recall that each edge of the path denotes the transposition of two adjacent elements of the cyclic ordering. If the label xy occurs on more than one edge in the path, delete the first and second edges of the path with this label, and for each edge on the segment between them, replace x by y and y by x wherever they occur—that is, replace every transposition of the form (x,z) by (y,z), and vice versa. We obtain a new path (sequence of transpositions) connecting the same two vertices and with two edges fewer than before.

(c) To get from a to $\overline{a}$ in $Z(n)$ steps, divide the cyclic ordering a into two segments of lengths $\lfloor \frac{1}{2}n \rfloor$ and $\lceil \frac{1}{2}n \rceil$ respectively, and reverse each segment separately by means of $\binom{\lfloor \frac{1}{2}n \rfloor}{2} + \binom{\lceil \frac{1}{2}n \rceil}{2}$ adjacent interchanges. To see that fewer steps will not work, use induction on n. The result is obvious if $n = 1$ or 2, since then $a = \overline{a}$; so suppose $n \geqslant 3$. Let P be a shortest $a\overline{a}$-path and let xy be a label that does not appear in P (which clearly exists, since otherwise P has length $\binom{n}{2} > Z(n)$). Since every triple xyz reverses its orientation between a and $\overline{a}$, exactly one of xz and yz must occur in P. Thus exactly $n - 2$ edges of P involve x or y. By the induction hypothesis applied to the vertices of CO_{n-2} obtained by deleting x and y from a and

$\overline{a}$, the number of edges of P that do not involve x or y is at least $Z(n-2)$. Since $Z(n) = Z(n-2) + n - 2$, the result follows. ∎

The *antidistance* $\overline{d}(a,b)$ between two vertices a and b is the distance $d(a,\overline{b})$ between a and $\overline{b}$ (or between $\overline{a}$ and b). An ab-*antipath* is an $a\overline{b}$-path or an $\overline{a}b$-path. If M is a multiset whose elements are all in $V(CO_n)$, and $|M| = m$, then M is called an (m, n)-*set*, and the sum of all $\binom{m}{2}$ antidistances between pairs of elements of M is the *antisum* $\overline{A}(M)$ of M. ($|M|$ denotes the number of elements of M *counted according to multiplicity*.)

Theorem 4. (a) Every $(3,n)$-set has antisum at least $Z(3)Z(n) = Z(n)$.

(b) If m and n are both odd, then the antisum of any (m,n)-set is even unless both m and $n \equiv 3 \pmod 4$, when the antisum is odd.

Proof. (a) If $\{a,b,c\}$ is a $(3,n)$-set, then

$$\begin{aligned}\overline{d}(a,b) + \overline{d}(b,c) + \overline{d}(c,a) &= d(a,\overline{b}) + d(\overline{b},c) + d(c,\overline{a})\\ &\geqslant d(a,\overline{a}) = Z(n).\end{aligned}$$

(b) If n is odd, then $Z(n)$ and $\binom{n}{2}$ have the same parity, being even if $n \equiv 1 \pmod 4$ and odd if $n \equiv 3 \pmod 4$. Since CO_n is bipartite, $d(x,y) + d(y,z) + d(z,x)$ is even for every triple of vertices x, y, z, and so

$$d(a,b) + d(a,\overline{b}) + d(b,\overline{b}) = d(a,b) + \overline{d}(a,b) + Z(n)$$

is even for every pair of vertices a, b. Thus $d(a,b)$ and $\overline{d}(a,b)$ have the same parity or different parity according as $Z(n)$ is even or odd. Since the sum of the distances between any odd number of vertices in a bipartite graph is necessarily even, the sum of the antidistances of any (m,n)-set has the same parity as $\binom{m}{2}Z(n)$, which is of the required form. ∎

3. CYCLIC-ORDER GRAPHS AND CROSSING NUMBERS

The following result is stated without (much) proof by Kleitman [5].

Theorem 5. In a drawing D of $K_{2,n}$ on two sets, $\{A,B\}$ and V_n, let the clockwise orders in which the edges leave A and B to go to V_n be the elements a and b of $V(CO_n)$. Then $\mathrm{cr}(D) \geqslant \overline{d}(a,b)$, and if n is odd then $\mathrm{cr}(D) \equiv \overline{d}(a,b) \pmod 2$.

Proof. If two of the arcs AxB and AyB in the drawing cross each other more than once, then we can "open out" two of the crossings to obtain a drawing with two fewer crossings, without changing the corresponding orders a and b or their antidistance $\overline{d}(a,b)$; thus we may suppose that each two arcs cross at most once. Assuming there is at least one crossing in the drawing, this implies the existence of a crossing-point P such that neither of the two segments of arc connecting A to P contains any crossings. (One way of seeing that P exists is to imagine the drawing in the sphere rather than the plane, to consider only those crossing-points P such that *at least one* of the two segments of arc connecting A to P is free of crossings, and among such points P to choose one that minimizes the area of the region bounded by the two arcs AP and not containing B.)

Now, suppose there are r crossings in the drawing. In view of the definition of $\overline{d}(a,b)$ and the fact that CO_n is bipartite if n is odd by Theorem 3(a), it suffices to construct an $a\overline{b}$-path in CO_n of length r. If $r = 0$ there is nothing to do, since $a = \overline{b}$; so suppose $r > 0$. Let P be as in the previous paragraph and let the arcs that cross at P be AxB and AyB. If this crossing is opened out so that the arc AP that was formerly part of AxB is now part of AyB and vice versa, we obtain a drawing with $r - 1$ crossings in which the order a is replaced by a new order a_1 that is obtained from a by transposing the adjacent elements x and y. Iteration of this procedure yields the required path $a, a_1, \ldots, a_r = \overline{b}$. ∎

We shall say that a drawing of $K_{m,n}$ *corresponds to* a given (m,n)-set M if the clockwise orders in which the edges leave the m vertices to go to the n vertices are precisely the elements of M (with the right multiplicities). By an abuse of terminology we shall use the same letter to denote an element a of M and the vertex in the drawing that it corresponds to. Let D be a good drawing corresponding to M, and let N be the (n,m)-set to which D also corresponds, interchanging the rôles of the two partite sets of vertices. If $a, b \in M$ and $x, y \in N$, then the ab-antipath constructed in Theorem 5 contains an edge labeled xy if and only if the arcs axb and ayb cross; that is, if and only if there is a crossing between ax and by or between ay and bx (we cannot tell which); that is, if and only if the xy-antipath constructed in Theorem 5 (with M and N interchanged) contains an edge labeled ab. (It is not difficult to see that, since D is a good drawing, the arcs axb and ayb cannot cross more than once.)

We shall call an (m,n)-set M *tight* if there exist

(a) an (n,m)-set N,
(b) for each a, b in M, a shortest ab-antipath P_{ab}, and
(c) for each x, y in N, an xy-antipath Q_{xy} (not necessarily shortest), such that, for each a, b in M and x, y in N, xy occurs as a label in P_{ab} if and only if ab occurs as a label in Q_{xy}.

We shall call M *pseudo-tight* if there exist

(a) an (n,m)-set N,
(b) for each x, y in N, an xy-antipath Q_{xy} (not necessarily shortest), such that, for each a, b in M and x, y in N, ab occurs as a label in Q_{xy}

if xy occurs as a label in *every* shortest ab-antipath and *only if* xy occurs as a label in *at least one* shortest ab-antipath.

It is easy to see that every subset of a tight set is tight, every subset of a pseudo-tight set is pseudo-tight, and every tight set is pseudo-tight; there may be some nontight sets that are also pseudo-tight, although I do not know of any.

By a *lobspacron function* I mean a function that assigns, to each (m, n)-set M, an integer that is a *LO*wer *B*ound for, and has the *S*ame *P*arity *A*s, the *CRO*ssing *N*umber of any good drawing corresponding to M.

Theorem 6. If n is odd, then each of the following defines a lobspacron function f.

(a) $f(M)$ is the smallest number of crossings in any drawing of $K_{m,n}$ that corresponds to M.
(b) $f(M) = \overline{A}(M)$, the antisum of M.
(c) $f(M) = \overline{A}(M) + 2\lceil 2t/(m-2)(m-3) \rceil$, where t is the number of $(4, n)$-subsets of the (m, n)-set M that are not pseudo-tight.

Proof. (a) and (b) follow from Theorem 5 and the succeeding discussion, since the number of crossings in a drawing of $K_{m,n}$ is equal to the sum of the numbers of crossings in all the drawings of $K_{2,n}$ contained in it. To prove (c), note that if some $(4, n)$-subset of M is not pseudo-tight, hence necessarily not tight, then there must be at least one $(2, n)$-subset $\{a, b\}$ of it such that the corresponding $K_{2,n}$ in the drawing has at least $\overline{d}(a, b) + 2$ crossings. Each such $(2, n)$-subset occurs in exactly $\frac{1}{2}(m-2)(m-3)$ $(4, n)$-subsets of M, and so the term $\lceil 2t/(m-2)(m-3) \rceil$ in (c) gives a lower bound for the number of different such $(2, n)$-subsets in M. ∎

The reason for the precise formulation of Theorem 6(c) is that it is easier to check by computer whether $(4, n)$-subsets of M are pseudo-tight than whether M itself is tight. In fact, I know of no counterexample to the conjecture that an (m, n)-set is tight if and only if every $(4, n)$-subset of it is pseudo-tight. A proper understanding of tight sets might well provide the clue to proving Zarankiewicz's conjecture.

4. THE COMPUTER PROGRAMS—GENERAL PRINCIPLES

We illustrate the principles by reference to the program for $K_{7,7}$. Let f be a lobspacron function. (Usually f was taken to be the antisum; in one program the augmented antisum of Theorem 6(c) was used instead.) Suppose there is a good drawing of $K_{7,7}$ with crossing number less than 81. Then using Theorem 1, the parity considerations of Theorem 4(b) and the definition of a lobspacron function, we see that there is a (7, 7)-set $\{a, b, c, d, e, f, g\}$ (or, by an abuse of terminology, simply $abcdefg$), such that

$$\left.\begin{array}{ccc} f(abcdefg) \leqslant 79, & f(abcdef) \leqslant 56, & f(abcde) \leqslant 36, \\ f(abcd) \leqslant 21, & f(abc) \leqslant 9, & f(ab) \leqslant 3\,. \end{array}\right\} \quad (2)$$

The 720 elements of $V(CO_7)$ were written as 7-digit integers starting with 0, and so acquired a natural order 0123456, 0123465, ..., 0654321. This order induces a lexicographic ordering of all possible ordered (7, 7)-sets. To avoid generating a given (7, 7)-set in more than one order, or generating more than one representative of a given isomorphism class of (7, 7)-sets, each (7, 7)-set was reduced to a canonical form by first permuting its elements so as lexicographically to minimize the sequence

$$\begin{array}{ccc} f(abcdefg), & f(abcdef), & f(abcde), \\ f(abcd), & f(abc), & f(ab)\,; \end{array} \quad (3)$$

then, among all permutations and isomorphs of the set that give the same minimum sequence, the one was chosen that lexicographically minimizes the ordered set $abcdefg$.

The set $abcdefg$ was generated an element at a time. We illustrate the process with a few examples. The above considerations imply that a must be 0123456. The choice $b = 0123456$ was not considered because then (with $f = \overline{A}$) $f(ab) = 9$, violating the inequality in (2). The choice $b = 0154326$ was not considered because the mapping

$$\begin{array}{cccc} 0 \longrightarrow 1, & 1 \longrightarrow 2, & 2 \longrightarrow 6, & 3 \longrightarrow 5, \\ 4 \longrightarrow 4, & 5 \longrightarrow 3, & 6 \longrightarrow 0 & \end{array}$$

maps a into $a' = 0126543$ and b into $b' = 0123456$, and so $b'a'$ is an ordered set isomorphic to ab but lexicographically smaller. For the choice $b = 0126543$, which *was* considered, $c = 0123456$ was not considered because then $f(abc) = 15$, violating (2); $c = 0345216$ was not considered because $f(bc) = 2 < f(ab) = 3$ and so, for any choice of d, e, f, and g, the ordering $bcadefg$ will give a lexicographically smaller sequence (3) than $abcdefg$; $c = 0465231$ was not considered because the mapping

$$\begin{array}{cccc} 0 \longrightarrow 2, & 1 \longrightarrow 1, & 2 \longrightarrow 0, & 3 \longrightarrow 6, \\ 4 \longrightarrow 5, & 5 \longrightarrow 4, & 6 \longrightarrow 3 & \end{array}$$

maps a into $a' = 0654321$, b into $b' = 0345621$ and c into $c' = 0612534$, and $\overline{a}'\,\overline{b}'\,\overline{c}'$ is an ordered set isomorphic to abc but lexicographically smaller; and so on.

The aim of this process was not just to eliminate (7, 7)-sets from consideration, but to eliminate them *as early as possible.* It is a waste of time to consider possible choices for e (say), if there is some way of proving that the current choices for a, b, c, and d cannot possibly give rise to a configuration with antisum less than 81. This process of early elimination was helped by Theorems 7 and 8 (below).

5. THE COMPUTER PROGRAMS—RESULTS

Five programs produced interesting results. The first three ran quite quickly and used just the ideas of the previous section.

The first program showed that every (5, 5)-set has antisum at least $16 = Z(5)Z(5)$, and there are exactly 32 nonisomorphic (5, 5)-sets with antisum exactly 16 ("nonisomorphic" being used in the rather loose sense illustrated in the previous section). This verifies the (known) result of Zarankiewicz's conjecture for $K_{5,5}$.

The second program showed that every (5, 7)-set has antisum at least $36 = Z(5)Z(7)$, and there are exactly 96 821 nonisomorphic (5, 7)-sets with antisum exactly 36. This verifies Zarankiewicz's conjecture for $K_{5,7}$ (which, again, was already known [5]).

The third program showed that every (7, 5)-set has antisum at least 36 with exactly one exception (up to isomorphism), which has antisum 34; this set was found by Kleitman [5], and is necessarily not tight, since every (5, 7)-set has antisum at least 36.

In addition to the ideas of the previous section, the fourth program used the following theorem.

Theorem 7. Let m and n be odd integers and $m' < n$ an even integer such that every $(m' + 1, n)$-set has antisum at least $Z(m' + 1)Z(n)$ and every $(m - m', n)$-set has antisum at least $Z(m - m')Z(n)$. Then any (m, n)-set M that contains an (m', n)-subset M' with antisum $Z(m')Z(n)$ or less has antisum at least $Z(m)Z(n)$.

Proof. This follows by the argument of Theorem 2 with "number of crossings" replaced by "sum of antidistances." ∎

The hypotheses of this theorem are satisfied for $(m', m, n) = (2, 7, 7)$ and $(4, 7, 7)$, in view of Theorem 4(a) and the result of the second computer program. Thus a (7, 7)-set that has antisum less than $81 = Z(7)Z(7)$ cannot contain a (2, 7)-subset with antisum $0 = Z(2)Z(7)$ (that is, it cannot contain both a and $\overline{a}$), nor a (4, 7)-subset with antisum $18 = Z(4)Z(7)$ or less. These facts were built into the program so as to speed the early elimination of impossible sets. The program then ran for about 4 hours on a Sun 4/110. It showed that there are no (7, 7)-sets with antisum less than 79, but there are exactly 13 448 nonisomorphic such sets with antisum exactly 79. However, each of these contains at least 10 (4, 7)-subsets that are not pseudo-tight, and so none of them can correspond to a counterexample to Zarankiewicz's conjecture. The conjecture is thus verified for $K_{7,7}$.

It was clearly not going to be practicable to generate all (9, 7)-sets with antisum less than $144 = Z(9)Z(7)$, and so in the fifth and final program the augmented antisum of Theorem 6(c) was used in order to eliminate as early as possible any set that, despite having a sufficiently small antisum, could not correspond to a counterexample to the conjecture for $K_{7,9}$. The following theorem was also used.

Theorem 8. Let M be a (9, 7)-set that corresponds to a drawing D that is a counterexample to Zarankiewicz's conjecture for $K_{7,9}$. Then M cannot contain any of the following:

(a) a (2, 7)-subset M' with antisum 0,
(b) a (4, 7)-subset M' with antisum $18 = Z(4)Z(7)$ or less,
(c) a (6, 7)-subset M' with antisum $52 = Z(6)Z(7) - 2$ or less,
(d) a (6, 7)-subset M' with antisum $54 = Z(6)Z(7)$ or less, all of whose (4, 7)-subsets are pseudo-tight.

Proof. Since every (3, 7)-set has antisum at least $Z(3)Z(7)$ by Theorem 4(a), and Zarankiewicz's conjecture holds for $K_{7,7}$ by the fourth computer program, (a) follows by the same argument as Theorems 2 and 7. (For, $\text{cr}(D)$ is at least as large as the crossing number of the copy of $K_{7,7}$ determined by $M \setminus M'$ within D, plus the sum of all the other antidistances counted in Theorem 7.) Since every (5, 7)-set has antisum at least $Z(5)Z(7)$ by the second computer program, (b) follows directly from Theorem 7. Since, by the fourth computer program, every (7, 7)-set has antisum at least $Z(7)Z(7) - 2$, and every (7, 7)-set with this antisum has a (4, 7)-subset that is not pseudo-tight, it follows by the argument of Theorem 7 that if M has a (6, 7)-subset with antisum $Z(6)Z(7) - 2$ or less then its own antisum $\overline{A}(M) \geqslant Z(9)Z(7) - 2$, and equality implies that M has a (4, 7)-subset that is not pseudo-tight so that M does not correspond to a counterexample to Zarankiewicz's conjecture. This proves (c).

Finally, we prove (d). If $a, b \in M$, let $\text{cr}(a, b)$ denote the crossing number of the drawing of $K_{2,n}$ corresponding to a and b. If M' exists as in (d), then the argument of Theorem 7 gives $\overline{A}(M) \geqslant Z(9)Z(7) - 6$. Moreover, if, for each of the three elements b of $M \setminus M'$, *either* $\overline{A}(M' \cup \{b\}) \geqslant 81 = Z(7)Z(7)$ *or* there is at least one a in M' such that $\text{cr}(a, b) > \overline{d}(a, b)$ (hence, $\text{cr}(a, b) \geqslant \overline{d}(a, b) + 2$, by Theorem 5), then $\text{cr}(D) \geqslant Z(9)Z(7)$. So suppose that, for some b in $M \setminus M'$, $\overline{A}(M' \cup \{b\}) = 79$ and $\text{cr}(a, b) = \overline{d}(a, b)$ for each a in M'. By the fourth computer program, $M' \cup \{b\}$ contains at least ten (4, 7)-subsets that are not pseudo-tight, each of which must contain b (since by hypothesis M' has no such subsets), and each of which must contain a pair $\{a, a'\} \subseteq M'$ such that

$$\text{cr}(a, a') > \overline{d}(a, a') . \tag{4}$$

Since a pair of elements of M' is contained in only four (4, 7)-subsets of $M' \cup \{b\}$ that contain b, there must be at least three different pairs $\{a, a'\}$ in

M' satisfying (4), whence $\mathrm{cr}(D) \geqslant \overline{A}(M) + 6$ and the required conclusion follows. ∎

After the results of this theorem had been incorporated into the program, it ran to completion in about 6 hours, showing that there are no (9, 7)-sets that can correspond to counterexamples to Zarankiewicz's conjecture. The conjecture is thus verified for $K_{7,9}$.

APPENDIX

This appendix is devoted to a proof that $\overline{a}$ is the antipode of a in CO_n; that is, that $d(a,b) < d(a,\overline{a}) = Z(n)$ if $a, b \in V(\mathrm{CO}_n)$ and $b \neq \overline{a}$. The value of n is assumed fixed throughout.

We start with some definitions and a technical lemma, the proof of which is left to the reader. (Part (c) is most easily "proved" by tabulating small values!) If $0 \leqslant r \leqslant \frac{1}{2}n$ and $0 \leqslant s \leqslant n$, let

$$f(r) := \begin{cases} 0, & \text{if } r = 0, \\ 2r - 1, & \text{if } 0 < r < \frac{1}{2}n, \\ 2r - 2, & \text{if } r = \frac{1}{2}n, \end{cases}$$

and

$$g(s) := \begin{cases} \binom{s}{2}, & \text{if } s \leqslant \frac{1}{2}(n+1), \\ \binom{\frac{1}{2}(n+1)}{2} + \binom{s - \frac{1}{2}(n+1)}{2}, & \text{if } s \geqslant \frac{1}{2}n + 1 \text{ and } n \text{ is odd}, \\ \binom{\frac{1}{2}n + 1}{2} + \binom{s - \frac{1}{2}n - 1}{2} - 1, & \text{if } s \geqslant \frac{1}{2}n + 1 \text{ and } n \text{ is even}. \end{cases}$$

(Note that $\binom{0}{2} = \binom{1}{2} = 0$.)

Lemma 1. (a) $g(n) = Z(n)$.

(b) If $1 \leqslant r \leqslant \frac{1}{2}n$ then $g(r+1) = g(r-1) + f(r)$.

(c) If $s > r + 1$ then $g(r+1) + g(s-r-1) \leqslant g(s)$, with equality if and only if $r + 1$ or $s - r - 1$ equals $\lfloor \frac{1}{2}n \rfloor + 1$ or $r + 1 = s - r - 1 = \frac{1}{2}n$. ∎

Without loss of generality, let $a = 012\ldots(n-1)$, and imagine the entries of a spaced regularly round a circle. We shall give a prescription for converting b into a by fewer than $Z(n)$ adjacent interchanges (assuming

$b \neq \bar{a}$), and so we describe the position of an entry j in a as the *final position* of j.

Imagine b superimposed on a. This can be done in n different rotational positions, which we label arbitrarily as orientations $1, \ldots, n$. Let $\alpha(i,j)$ denote the number in a that coincides with j in b when b is in orientation i; $j - \alpha(i,j)$ is equal (mod n) to the distance that j has to move, in the positive direction, from its position in b to its final position, when b is in orientation i. Let $\beta(i,j)$ denote the integer of minimum absolute value that is congruent (mod n) to $j - \alpha(i,j)$ (taking $\beta(i,j) := +\frac{1}{2}n$ in the only case where there is a choice), so that $|\beta(i,j)|$ is the shortest distance that j must move. Let $\gamma(i,j) := \frac{1}{2}f(|\beta(i,j)|)$, $\delta_i(a,b) := \sum_{j=0}^{n-1} \gamma(i,j)$, and $\delta(a,b) := \min_i \delta_i(a,b)$.

Lemma 2. (a) $\delta(a,b) \leqslant Z(n)$, with equality only if $\delta_i(a,b) = Z(n)$ for each i.

(b) For each i, $\sum_{j=0}^{n-1} \beta(i,j) \equiv 0 \pmod{n}$.

(c) There is at least one i such that, for every j, $\beta(i,j) \neq \frac{1}{2}n$.

Proof. As i runs through the values $1, \ldots, n$, b runs through all n different orientations with respect to a, and $\beta(i,j)$ runs through the values $0, \pm 1, \pm 2, \ldots, \pm\frac{1}{2}(n-1)$ (n odd) or $0, \pm 1, \pm 2, \ldots, \pm\frac{1}{2}(n-2)$, $\frac{1}{2}n$ (n even). Thus, for each j,

$$\sum_{i=1}^{n} \gamma(i,j) =$$

$$\begin{cases} 1 + 3 + 5 + \cdots + (n-2) = \frac{1}{2}(n-1)\frac{1}{2}(n-1) = Z(n), & (n \text{ odd}), \\ 1 + 3 + 5 + \cdots + (n-3) + \frac{1}{2}(n-2) = \frac{1}{2}(n-2)\frac{1}{2}n = Z(n), & (n \text{ even}). \end{cases}$$

Hence

$$\delta(a,b) \leqslant \frac{1}{n}\sum_{i=1}^{n} \delta_i(a,b) = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=1}^{n} \gamma(i,j) = Z(n).$$

This proves (a).

(b) holds trivially if $b = a$, and an adjacent interchange (the transposition of two adjacent elements of b) does not alter $\sum_j \beta(i,j)$ (mod n), so (b) always holds.

If (c) fails, then n is even, and for each i, $\beta(i,j) = \frac{1}{2}n$ for exactly one value of j (since for each j there is always exactly one i for which $\beta(i,j) = \frac{1}{2}n$). Now, if $\beta(i,j) = \beta(i,j')$ for some j and j', then rotating b relative to a would make $\beta(i',j) = \beta(i',j') = \frac{1}{2}n$ for some i', which we have just seen to be impossible. It follows that, for each i, the n numbers $\beta(i,j)$ are all different, and so are equal to $0, \pm 1, \pm 2, \ldots, \pm\frac{1}{2}(n-2)$, $\frac{1}{2}n$. But this violates (b). ∎

We shall prove, by induction on $\delta(a,b)$, that b can be converted into a by at most $\delta(a,b)$ adjacent interchanges. This is obvious if $\delta(a,b) = 0$, when evidently $b = a$. So suppose $\delta(a,b) \neq 0$, choose i so that $\delta_i(a,b) = \delta(a,b)$, and imagine b superimposed on a in position i. For some r $(1 \leqslant r \leqslant \frac{1}{2}n)$ b must contain a segment $x_0x_1 \ldots x_r$ such that $x_1, \ldots, x_{r-1}$ are in final position (this list being empty if $r = 1$) and x_0 needs to move $r + p$ places "to the right" and x_r needs to move $r + q$ places "to the left" to reach their final positions by the shortest route $(p, q \geqslant 0)$. Interchange x_0 and x_r in b to form a new cyclic ordering b_1. This can be achieved by $f(r)$ adjacent interchanges, since if $r = \frac{1}{2}n$ then x_0 and x_r can both move in the same direction so that they do not cross each other. Also, $f(r + p) - f(p) \geqslant f(r)$, with equality if and only if $p = 0$ or $\frac{1}{2}n - r$. Thus, by the definition of δ_i,

$$\delta(a,b_1) \leqslant \delta_i(a,b_1) \leqslant \delta_i(a,b) - f(r) = \delta(a,b) - f(r) \qquad (5)$$

with equality only if $\{p,q\} \subseteq \{0, \frac{1}{2}n - r\}$. By the induction hypothesis, b_1 can be converted into a by at most $\delta(a,b_1)$ adjacent interchanges, and it follows that b can be converted into a by at most $\delta(a,b)$ adjacent interchanges; that is, $d(a,b) \leqslant \delta(a,b)$.

It follows from this and Lemma 2(a) that $d(a,b) \leqslant Z(n)$. We must now prove that, assuming $b \neq \overline{a}$, equality cannot occur. It would seem natural to do this by proving that $\delta(a,b) < Z(n)$, but this seems difficult, especially when n is a multiple of 4. So we adopt a different approach. By Lemma 2(a) and (c), if $\delta(a,b) = Z(n)$ we can choose i so that $\delta_i(a,b) = \delta(a,b)$ and, for every j, $\beta(i,j) \neq \frac{1}{2}n$. Then for equality to occur in (5), $p = q = 0$, that is, x_0 and x_r must both be in their final positions after their interchange—we refer to such an interchange as a *final-position interchange*—and also $\delta_i(a,b_1) = \delta(a,b_1) = \min_h \delta_h(a,b_1)$. So if $d(a,b) = Z(n)$, it must be possible to convert b into a by means of a sequence of final-position interchanges, without rotating b relative to a between successive interchanges. It will therefore follow from Lemma 3 that $d(a,b) < Z(n)$ when $b \neq \overline{a}$, since $g(n) = Z(n)$ by Lemma 1(a), and if a is of the form $y_1 \ldots y_n$ and b of the form $y_ty_{t-1} \ldots y_1y_ny_{n-1} \ldots y_{t+1}$ then $b = \overline{a}$.

Lemma 3. Let $b = b_0, b_1, b_2, \ldots, b_k = a$ be a sequence of cyclic orderings such that, for each $j \geqslant 1$, b_j is obtained from b_{j-1} by a final-position interchange (without rotating it relative to a). Suppose that some b_j contains a segment $y_0y_1 \ldots y_sy_{s+1}$ such that $y_1, \ldots, y_s$ are in final position and *either* $s = n$ (so that $y_0 = y_s$ and $y_1 = y_{s+1}$) *or* y_0 and y_{s+1} are not in final position. Then, in converting b_0 into b_j, the number of *adjacent* interchanges carried out that involve $y_1, \ldots, y_s$ is at most $g(s)$, with equality if and only if $s \leqslant \lfloor \frac{1}{2}n \rfloor + 1$ and b contains the segment $y_sy_{s-1} \ldots y_1$, or $s > \lfloor \frac{1}{2}n \rfloor + 1$ and

b contains the segment $y_t y_{t-1} \dots y_1 y_s y_{s-1} \dots y_{t+1}$ where either t or $s - t$ is equal to $\lfloor \frac{1}{2}n \rfloor + 1$

Proof. We prove the result by induction on s. It is trivial if $y_1, \dots, y_s$ were all already in final position in b, which covers the case $s = 1$. So suppose that at least one final-position interchange has been needed to bring them all into final position. Let the last such interchange be between x_0 and x_r in the segment $x_0 x_1 \dots x_r$, where $r \leqslant \lfloor \frac{1}{2}n \rfloor$ by the definition of final-position interchange. Note that all such final-position interchanges involve only the elements $y_1, \dots, y_s$, and so $s \geqslant r + 1$ and the segment $x_0 x_1 \dots x_r$ is contained in $y_1 \dots y_s$. There are three cases to consider.

Case 1. $s = r + 1$. Then $s \leqslant \lfloor \frac{1}{2}n \rfloor + 1$ and $x_0 \dots x_r$ is the same as $y_1 \dots y_s$. Applying the induction hypothesis to $y_2 \dots y_{s-1}$ ($= x_1 \dots x_{r-1}$), we see that the number of adjacent interchanges used that involve $y_1, \dots, y_s$ is at most $g(r - 1) + f(r) = g(r + 1) = g(s)$ by Lemma 1(b), with equality if and only if b contains the segment $y_{s-1} y_{s-2} \dots y_2$, when it must also contain $y_s y_{s-1} \dots y_1$.

Case 2. $s > r + 1$ and $x_0 \dots x_r$ occurs at one end of $y_1 \dots y_s$: w.l.o.g. $x_0 = y_1$. Applying the induction hypothesis to $y_2 \dots y_r$ and to $y_{r+2} \dots y_s$ we see that the number of adjacent interchanges used is at most

$$g(r - 1) + g(s - r - 1) + f(r) = g(r + 1) + g(s - r - 1) \leqslant g(s)$$

by Lemma 1(c), with equality if and only if b contains the segment $y_t y_{t-1} \dots y_1 y_s y_{s-1} \dots y_{t+1}$ where either t or $s - t$ is equal to $\lfloor \frac{1}{2}n \rfloor + 1$. (Here $t = r + 1$ if $r + 1$ or $s - r - 1$ equals $\lfloor \frac{1}{2}n \rfloor + 1$ in Lemma 1(c), and $t = r + 2$ if $r + 1 = s - r - 1 = \frac{1}{2}n$ in Lemma 1(c), when $s = n$ and $b_j = \bar{b}$.)

Case 3. $s > r + 1$ and $x_0 \dots x_r$ is not at one end of $y_1 \dots y_s$: say $x_0 = y_t$ for some t, $2 \leqslant t \leqslant s - r - 1$. If $s = n$, so that $y_1 \dots y_s$ is the whole of b_j, relabel b_j so that $x_0 \dots x_r = y_1 \dots y_{r+1}$ and go back to Case 2. Otherwise, applying the induction hypothesis to $y_1 \dots y_{t-1}$, to $y_{t+1} \dots y_{t+r-1}$ and to $y_{t+r+1} \dots y_s$, we see that the number of adjacent interchanges is at most

$$g(t - 1) + g(r - 1) + g(s - t - r) + f(r) = g(t - 1) + g(r + 1) + g(s - t - r) < g(s),$$

since the conditions for equality in Lemma 1(c) cannot be satisfied in *both* of these additions. This completes the proof of Lemma 3 and of the result $d(a, b) < Z(n)$ when $b \neq \bar{a}$. ∎

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