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THE LINEARISATION CONJECTURE AND OTHER PROBLEMS OVER NONREDUCED RINGS

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Abstract

Theorems in the theory of polynomial mappings which are true over fields are considered for nonreduced rings: counterexamples are given, and some generalisations are made. Equivalence of the cancellation problem over a ring with the cancellation problem over the ring modulo its nilradical is proved. For polynomial maps $F \in R[X_1, \dots, X_n]^n$ satisfying $F^s = X$ it is shown that there is equivalence between 1) F is linearisable by conjugation; 2) $\bar{F}$ is linearisable by conjugation, where $\bar{F}$ is F modulo the nilradical of R.

1 Introduction

If one likes to prove statements for general rings, sometimes it is possible to prove it first for fields, then for domains, from that prove it for reduced rings and finally prove it for (nonreduced) general rings. Also, as a rule, "many" statements are true for fields, "some" for domains, "less" for reduced rings and "just a few" for nonreduced rings. So, in some sense, nonreduced rings are "dirty".

This paper is dedicated to finding out if certain properties over a ring are unchanged if one calculates modulo the nilradical, and vice versa. Section 3 will focus on Rentschler's theorem(see [13]), and show that it can only be generalised for derivations having a slice. This has the consequence that there is equivalence of the cancellation problem over a ring with the cancellation problem over the ring modulo its nilradical. Also the (not) finitely generatedness of kernels of derivations over unreduced rings is studied.

Section 4 deals with the problem that if $F \in R[X_1, \dots, X_n]^n$ and $F^s = (X_1, \dots, X_n)$ for some integer $s > 0$, whether F should be linearisable by a conjugation. This is still an unsolved question if $R = \mathbb{C}$ and $n \geq 3$. In this section it is shown that, for this problem, one may restrict to reduced rings, by proving that if $F^s = (X_1, \dots, X_n)$ then F is linearisable if and only if $\bar{F}$ is linearisable.

Section 2 defines used notations and discusses some prerequisites.

2 Notations

First let us define a whole list of notations for this paper, even though some of them will be introduced later on.

Definition 2.1.

- k is a field of characteristic zero. R is a commutative ring. R^* is the subset of R consisting of units. η is the nilradical of the ring. A ring whose nilradical is (0) is called *reduced.* Otherwise it is called nonreduced. We denote R/η by $\bar{R}$.
- $R_m := \mathbb{C}[T]/(T^m)$. We will denote $\bar{T}$ by ϵ.
- $A := R[X_1, \ldots, X_n]$ (except in this section, where it can be a commutative R-algebra). We denote $\bar{R}[X_1, \ldots, X_n]$ by $\bar{A}$.
- $X = (X_1, \ldots, X_n)$, the identity map.
- If $F \in A^n$ then $F = (F_1, \ldots, F_n)$ where $F_i \in A$; hence F_i is defined as the i-th coordinate of F.
- $\partial_{X_i} = \partial_i$ is the map on A taking derivative with respect to X_i.

In the rest of this section we define notations and several objects of interest.

Let R be some commutative ring. A polynomial mapping is an element $F \in R[X_1, \ldots, X_n]^n$. A polynomial automorphism is a polynomial map which has a polynomial inverse G, i.e. $G \circ F = F \circ G = X$. The collection of these polynomial automorphisms is denoted by $Aut(R[X_1, \ldots, X_n])$. Any element $F \in Aut(R[X_1, \ldots, X_n])$ gives an automorphism $R[X_1, \ldots, X_n] \longrightarrow R[X_1, \ldots, X_n]$ by $P \longrightarrow P(F)$.

Definition 2.2. Let R be a ring, let A be an R-algebra. Then an R-derivation on A is an R-linear map $D : A \longrightarrow A$ (which means $D(ra) = rD(a)$ and $D(a+b) = D(a) + D(b)$ for all $a, b \in A, r \in R$) such that $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$.

We denote the set of all R-derivations on A by $Der_R(A)$. We denote the set of *all* derivations on A by $Der(A)$. Both are additive groups.

Definition 2.3. Let $D \in Der_R(A)$. A *slice* of D is an element $s \in A$ such that $D(s) = 1$.

Definition 2.4. Let $D \in Der(A)$. We say D is *locally nilpotent* if for all $x \in A$ there exists $n \in \mathbb{N}$ such that $D^n(x) = 0$.

We recall some facts about derivations and polynomial maps:

- $\varphi : A \longrightarrow A$ is an R-automorphism if and only if $\bar{\varphi} : \bar{A} \longrightarrow \bar{A}$ is an $\bar{R}$-automorphism.
- $D \in Der(A)$, $\varphi : A \longrightarrow A$ an automorphism. Then $\varphi^{-1} D \varphi$ again is a derivation. Furthermore, if D is locally nilpotent, so is $\varphi^{-1} D \varphi$.
- D has a slice $\Longleftrightarrow$ $\bar{D}$ has a slice. (See [3])
- $D \in Der_R(R[X_1, \ldots, X_n] \Longleftrightarrow \exists a_1, \ldots, a_n \in R[X_1, \ldots, X_n]$ such that $D = a_1\partial_1 + \ldots + a_n\partial_n$.

- If D is locally nilpotent and has a slice, then D is surjective. More in detail: for every $a \in A$ there exists a "primitive" $p \in A$ (just take $p = \sum_{i=0}^{\infty} D^i(a)(-s)^{i+1}/(i+1)!$, which in fact is a finite sum).
- If D is a locally nilpotent derivation on A having a slice s, then $A = A^D[s]$, where $A^D := ker(D)$ (see [6]).

3 Generalisations of known theorems.

This section concentrates on generalising known theorems about fields to nonreduced rings (especially to R_m), or showing that they cannot be generalised. Now and then we quote a known theorem about fields. For proofs of these theorems we refer to [6].

We know the following theorem(see [13]):

Theorem 3.1. *(Rentschler) Let $0 \neq D$ be a locally nilpotent derivation on $k[X,Y]$. Then there exists $\varphi \in Aut_k k[X,Y]$ and $f(Y) \in k[Y]$ such that $\varphi^{-1}D\varphi = f(Y)\partial_X$.*

This theorem cannot be completely generalized, as the following example shows:

Example 3.2. Let $D = \epsilon XY\partial_X$ on $R_2[X,Y]$. This derivation is clearly locally nilpotent (ϵ is nilpotent). Suppose we have $f(Y) \in R_2[Y]$, $\varphi \in Aut_{R_2}R_2[X,Y]$ such that $\varphi^{-1}D\varphi = f(Y)\partial_X$. Write $\bar{\varphi}$ for the part "without ϵ". Then

$$\begin{aligned} f(Y,\epsilon)\partial_X = &\quad \varphi^{-1}D\varphi \\ = &\quad \varphi^{-1}\epsilon XY\partial_X\varphi \\ = &\quad \epsilon(\bar{\varphi}^{-1}XY\partial_X\bar{\varphi}) \end{aligned}$$

and hence

$$\bar{\varphi}^{-1}XY\partial_X\bar{\varphi} = \tilde{f}(Y)\partial_X$$

for some $\tilde{f}(Y) \in k[X,Y]$ but this would mean that $XY\partial_X$ can be made locally nilpotent by some automorphism φ and that is impossible.

However, we have the following general form for derivations having a slice:

Proposition 3.3. *Let R be a $\mathbb{Q}$-algebra. Let D be a locally nilpotent R-derivation having a slice on $A := R[X_1, \ldots, X_n]$. Then the following are equivalent.*

1. *There exists $\varphi \in Aut_R A$, $f \in R^*$ such that $\varphi^{-1}D\varphi = f\partial_{X_1}$;*
2. *There exists $\varphi_0 \in Aut_{\bar{R}}\bar{A}$, $\bar{f} \in \bar{R}^*$ such that $\varphi_0^{-1}\bar{D}\varphi_0 = \bar{f}\partial_{X_1}$.*

Proof. (1⇒2) is trivial: just calculate modulo the nilradical and take $\varphi_0 := \bar{\varphi}$. (2⇒1) Let $\bar{\varphi}$ be a map such that $\bar{\varphi}^{-1}\bar{D}\bar{\varphi} = \bar{f}\partial_{X_1}$ for some $f \in R$. Then $\widetilde{D} := \varphi^{-1}D\varphi = f\partial_{X_1} + \partial$ where $\partial \in \eta Der_R A$. D has a slice and hence $\widetilde{D}$ must have one too. Now write $\partial = g_1\partial_{X_1} + \ldots + g_n\partial_{X_n}$ where each g_i must be nilpotent. Now since $\widetilde{D}$ is locally nilpotent and has a slice there exist elements G_i such that $\widetilde{D}(G_i) = g_i$.

(See the remarks about derivations towards the end of section 2.) Notice that these G_i are nilpotent too. Hence the map $\varphi := (X_1 - G_1, \ldots, X_n - G_n)$ defines an R-automorphism of A sending $P \in A$ to $P \circ \varphi$. Thus

$$\begin{aligned}(\varphi^{-1}\widetilde{D}\varphi)(X_i) = &\ \varphi^{-1}\widetilde{D}(X_i - G_i)\\ = &\ \varphi^{-1}(\widetilde{D}(X_i) - \widetilde{D}(G_i))\\ = &\ \varphi^{-1}(\delta_{1i}f + g_i - g_i)\\ = &\ \varphi^{-1}(\delta_{1i}f)\\ = &\ \delta_{1i}f\end{aligned}$$

where δ_{1i} equals 1 if $i = 1$ and zero if $i \geq 2$. Hence $\varphi^{-1}\widetilde{D}\varphi = f\partial_{X_1}$. Finally, since $\varphi^{-1}D\varphi$ has a slice, it follows that $f \in R^*$. □

Another interesting conjecture is the **Cancellation Problem.** Let V be a non-singular affine algebraic variety of dimension $d \geq 1$ over the complex numbers, and suppose that $V \times \mathbb{C} \cong \mathbb{C}^{d+1}$ as algebraic varieties for some $d \in \mathbb{N}$. Conjectured is that $V \cong \mathbb{C}^d$ as algebraic varieties. Only for the cases $d = 1$ and $d = 2$ it is known that the answer is affirmative. (For a proof of the $d = 2$ case see [9].) The similar implication

$$V \times \mathbb{C} \cong W \times \mathbb{C} \overset{?}{\Longrightarrow} V \cong W$$

(the biregular cancellation problem) is generally not true (see [1]). Also, if V is smooth and diffeomorphic to $\mathbb{C}^d$, then V need not be isomorphic to $\mathbb{C}^d$ as algebraic varieties. (See [11], [5].)

One of the algebraic reformulisations of the Cancellation Problem is the

Cancellation Problem (Algebraic version). Let D be a locally nilpotent derivation on $\mathbb{C}[X_1, \ldots, X_n]$ having a slice. Then there exists $\varphi \in Aut_{\mathbb{C}}\mathbb{C}[X_1, \ldots, X_n]$ such that $\varphi^{-1}D\varphi = \partial_1$.

The above question can be asked for general rings, replacing $\mathbb{C}$ by R. (So if D is a locally nilpotent derivation on $R[X_1, \ldots, X_n]$ having a slice, does there exist some $\varphi \in Aut_R R[X_1, \ldots, X_n]$ such that $\varphi^{-1}D\varphi = \partial_1$.) This question can in general not be affirmatively answered (see [7]). On the other hand as a consequence of 3.3 we get

Corollary 3.4. *Let R be a $\mathbb{Q}$-algebra. Then the cancellation problem over $R[X_1, \ldots, X_n]$ is equivalent to the cancellation problem over $\bar{R}[X_1, \ldots, X_n]$.*

Proof. Using the algebraic reformulisation and proposition 3.3 we are done. □

Lemma 3.3 is a useful tool when studying derivations having a slice on rings having nilpotent elements. If the assumption that a derivation needs to have a slice is dropped most things don't work anymore.

Theorem 3.5. *(Nagata, Nowicki [12]) Let k be a field of characteristic zero. If $0 \neq D$ is a k-derivation on $k[X_1, \ldots, X_n]$, then $k[X_1, \ldots, X_n]^D$ is a finitely generated k-algebra if $n \leq 3$.*

Notice that for $n \geq 5$ there do exist locally nilpotent derivations on $k[X_1, \dots, X_n]$ for which $k[X_1, \dots, X_n]^D$ is not finitely generated (see [8],[4]). For unreduced rings things go wrong even in dimension one:

Example 3.6. Let $D := \epsilon\partial_X$ on $R_2[X]$. Then $R_2[X]^D = R_2 \oplus \epsilon R_2[X] = R_2[\epsilon X, \epsilon X^2, \dots]$ and no generator can be omitted. Hence $R_2[X]^D$ is not finitely generated over R_2.

One also might consider the question under what conditions $R[X_1, \dots, X_n]^D$ is finitely generated as R-algebra if and only if $\bar{R}[X_1, \dots, X_n]^{\bar{D}}$ is finitely generated as $\bar{R}$-algebra. It is not enough to require D locally nilpotent and $\bar{D} \neq 0$:

Example 3.7. Let $D := \epsilon\partial_X + Z\partial_Y$. Then $R_2[X,Y,Z]^D = R_2[Z] \oplus \epsilon R_2[X] = R_2[Z, \epsilon X, \epsilon X^2, \dots]$ not finitely generated but $\bar{R}_2[X,Y,Z]^{\bar{D}} = k[X,Y,Z]^{Z\partial_Y} = k[X,Z] = \bar{R}_2[X,Z]$.

However, if D is locally nilpotent and has a slice then it is true:

Lemma 3.8. *Let R be a ring. Let D be a locally nilpotent R-derivation on $A := R[X_1, \dots, X_n]$ having a slice. Then the following are equivalent.*

1. $\bar{A}^{\bar{D}} = \bar{R}[\bar{f}_1, \dots, \bar{f}_m]$ *for some* $f_i \in A \backslash R$;
2. $A^D = R[f_1, \dots, f_m]$ *for some* $f_i \in A \backslash R$.

Proof. Let $s \in A$ be a slice. Suppose $f_i \in A$ is such that $\bar{D}(\bar{f}_i) = 0$. Then $D(f_i) \in \eta A$. Let $g_i = D(f_i)$. Define $G_i := \sum_{j=0}^{\infty} (-s)^{j+1}/(j+1)! D^j(g_i)$. Then $G_i \in \eta A$ and $D(G_i) = g_i$. So let us replace f_i by $f_i - G_i$, if necessary, to obtain $D(f_i) = 0$ as well as $\bar{D}(\bar{f}_i) = 0$ (so we can assume that).

(2)$\longrightarrow$(1): $A = A^D[s] = R[f_1, \dots, f_n, s]$, so $\bar{A} = \bar{R}[\bar{f}_1, \dots, \bar{f}_n, \bar{s}] \subseteq \bar{A}^{\bar{D}}[\bar{s}] \subseteq \bar{A}$ hence $\bar{A} = \bar{A}^{\bar{D}}[\bar{s}]$ thus $\bar{A}^{\bar{D}} = \bar{R}[\bar{f}_1, \dots, \bar{f}_n]$. (1)$\longrightarrow$(2): $\bar{A} = \bar{R}[\bar{f}_1, \dots, \bar{f}_n, \bar{s}]$ if and only if $A = R[f_1, \dots, f_n, s]$. Now $A^D[s] = A = R[f_1, \dots, f_n, s]$ hence $A^D = R[f_{,1}, \dots, f_n, s]^D = R[f_1, \dots, f_n]$. □

Readers who are interested in generalisations of several theorems in this section to UFD's, should check [2]

4 The linearisation conjecture on nonreduced rings

We introduce some notations. Let R be a $\mathbb{Q}$-algebra. s will be some positive integer. If we write $F = L + H$ we mean that L is **linear** and H **contains no linear monomials** (all monomials are of degree at least 2). Furthermore k denotes a field of characteristic zero.

Definition 4.1. We say that F is *linearisable over R* if there exists an R-automorphism φ of A such that $\varphi^{-1} F \varphi = L$ where L is a linear map.

Linearisation conjecture (over a field) (see [10]) Let $s \geq 1$ and $F \in Aut_k k[x_1, \ldots, x_n]$ with $F^s = (x_1, \ldots, x_n)$, T hen there exists $\varphi \in Aut_k k[x_1, \ldots, x_n]$ with $\varphi^{-1} F \varphi = L$, a linear map.

If n=2 this conjecture is true, and is an immediate consequence of the Jung-van der Kulk theorem. The case $n \geq 3$ is still open.

The more general conjecture, where k is replaced by an arbitrary commutative $\mathbb{Q}$-algebra is open for all $n \geq 2$. However, the next result, which is the main result of this section, shows that we may assume that R is a reduced ring.

Theorem 4.2. *Let $F^s = X$. Then there is equivalence between:*

1. *F is linearisable over R.*
2. *$\bar{F}$ is linearisable over $\bar{R}$.*

$1 \longrightarrow 2$ is clear. The following lemmas are dedicated to the proof of $2 \longrightarrow 1$.

Lemma 4.3. *Suppose theorem 4.2 has been proved for maps F satisfying*

1. *$F^s = X$,*
2. *$F = L + H$ where $H \in (IA)^n$ and I is an ideal in R satisfying $I^2 = (0)$.*

Then theorem 4.2 is true in general.

Proof. Suppose $\bar{F}$ is linearisable. That is, one may assume F to be of a form such that $F^s = X$ and $F = L + H$ and $H \in (\eta A)^n$. Now we have to prove that F is linearisable. First we show that we may assume R to be noetherian. Write $F = (F_1, \ldots, F_n)$, $F_i = \sum c_{\alpha,i} X^\alpha$. Define $R' := \mathbb{Q}[c_{\alpha,i}] \subseteq R$ and notice $F \in R'[X]^n$. This ring is finitely generated over $\mathbb{Q}$ hence noetherian. We are going to show that there exists an automorphism $\varphi \in R'[X_1, \ldots, X_n]^n$ (which hence is an automorphism $\in R[X_1, \ldots, X_n]^n$ such that $\varphi^{-1} F \varphi$ is linear. So replacing R by R' we may assume R to be noetherian). Now $H \in (\eta A)^n$ and $\eta^N = 0$ for some integer $N \geq 1$ (since R is noetherian). Calculating modulo η^2 we have $\bar{F} = \bar{L} + \bar{H}$ where $\bar{H} \in (\eta A / \eta^2 A)^n$, $\bar{\eta}^2 = \bar{0}$. Hence there exists (by assumption) a polynomial map $\bar{\varphi} \in A/\eta^2 A$ such that $\bar{\varphi}^{-1} F \bar{\varphi}$ is linear. So there exists $\varphi \in A$ such that $\tilde{F} := \varphi^{-1} F \varphi = \tilde{L} + \tilde{H}$ where $\tilde{H} \in (\eta^2 A)^n$. Now calculating modulo $(\eta^2)^2$ we find in the same way some $\tilde{\varphi}$ such that $\tilde{\varphi}^{-1} \tilde{F} \tilde{\varphi} = L' + H'$ where $H' \in (\eta^4 A)^n$, and after a finite number of conjugations we get $F'' = L'' + H''$ where $H'' \in (\eta^N A)^n = (0A)^n = 0$. ☐

Hence this lemma says that we only need to prove theorem 4.2 for maps $F^s = X$, $F = L + H$ where $H \in (IA)^n$ and I is an ideal in R satisfying $I^2 = (0)$.

Lemma 4.4. *Let $F = L + H$ where L linear and $H \in (IA)^n$ such that $I^2 = (0)$. Then there is equivalence between:*

1. *$F^s = X$ for some integer s*

2. $L^s = X$ *and* $\sum_{i=0}^{s-1} L^{s-1-i}HL^i = 0$

Proof. By induction we will prove $X = F^s = L^s + \sum_{i=0}^{s-1} L^{s-1-i}HL^i = 0$. Since $deg(L^s) = 1$ and $L^{s-1-i}HL^i$ only contains monomials of degree 2 and up, the theorem follows. So suppose $F^{k-1} = L^{k-1} + \sum_{i=0}^{k-2} L^{k-2-i}HL^i$ Then

$$\begin{aligned} F^k = &\quad F \circ F^{k-1} = (L+H)(L^{k-1} + \textstyle\sum_{i=0}^{k-2} L^{k-2-i}HL^i) \\ = &\quad (L^k + \textstyle\sum_{i=0}^{k-2} L^{k-1-i}HL^i + H(L^{k-1} + \textstyle\sum_{i=0}^{k-2} L^{k-2-i}HL^i)) \\ (*) = &\quad (L^k + \textstyle\sum_{i=0}^{k-2} L^{k-1-i}HL^i + H(L^{k-1})) \\ = &\quad (L^k + \textstyle\sum_{i=0}^{k-1} L^{k-1-i}HL^i) \end{aligned}$$

where $(*)$ holds since $H(L^{k-1} + \sum_{i=0}^{k-2} L^{k-2}HL^i) = H(L^{k-1})$ because $H \in (IA)^n$ where $I^2 = (0)$. ☐

Definition 4.5. Let $F \in A^n$. Define $\sigma_F : A^n \longrightarrow A^n$ by $\sigma_F(G) = [F,G] = FG - GF$. Define $\tau_F : A^n \longrightarrow A^n$ by $\tau_F(G) = \frac{1}{s}\sum_{i=0}^{s-1} iF^iGF^{s-1-i}$.

Lemma 4.6. *Let $I^2 = (0)$ and let $F \in A^n, F = L + H$ with L linear , $H \in (IA)^n$, and $G, G_1, G_2 \in (IA)^n$. Then*

1. $\sigma_F(G) = \sigma_L(G)$
2. $\sigma_F(G_1 + G_2) = \sigma_F(G_1) + \sigma_F(G_2)$
3. $\tau_F(G) = \tau_L(G)$
4. $\tau_F(G_1 + G_2) = \tau_F(G_1) + \tau_F(G_2)$
5. $L^i\tau_L(G)L^j = \tau_L(L^iGL^j)$
6. $\sigma_F\tau_F\sigma_F(G) = \sigma_F(G)$

Proof.

1. Notice that if $E_1, E_2 \in (IA)^n$ then $(F+E_1)(E_2) = FE_2$ and $E_2(F+E_1) = E_2F$ for any $F \in A^n$, since $I^2 = (0)$. Hence $\sigma_F(G) = FG - GF = (L+H)(G) - G(L+H) = LG - GL = \sigma_L(G)$.

2. Notice that if L linear then $L(F_1 + F_2) = LF_1 + LF_2$ for any $F_i \in A^n$ and $(F_1+F_2)(F_3) = F_1F_3 + F_2F_3$ for any $F_i \in A^n$ (by definition of the map $F_1 + F_2$). Hence

$$\begin{aligned} \sigma_F(G_1+G_2) =_{(1)} &\quad \sigma_L(G_1+G_2) \\ = &\quad L(G_1+G_2) - (G_1+G_2)L \\ = &\quad LG_1 + LG_2 - G_1L - G_2L \\ = &\quad \sigma_L G_1 + \sigma_L G_2 \\ =_{(1)} &\quad \sigma_F G_1 + \sigma_F(G_2). \end{aligned}$$

3. Similar to 1).
4. Similar to 2).
5. Easy.
6.

$$\begin{aligned}
\sigma_F\tau_F\sigma_F(G) &=_{(1),(3)} \sigma_L\tau_L\sigma_L(G) \\
&= \sigma_L\tau_L(LG-GL) \\
&=_{(4)} \sigma_L(\tau_L(LG)-\tau_L(GL)) \\
&=_{(2)} \sigma_L(\tau_L(LG))-\sigma_L\tau_L(GL) \\
&= L\tau_L(LG)-\tau_L(LG)L-L\tau_L(GL)+\tau_L(GL)L \\
&=_{(5)} \tau_L(L^2G)-2\tau_L(LGL)+\tau_L(GL^2) \\
&= \tfrac{1}{s}\big(\textstyle\sum_{i=0}^{s-1} iL^{i+2}GL^{s-i-1}-2\sum_{i=0}^{s-1} iL^{i+1}GL^{s-i}+ \\
&\qquad \textstyle\sum_{i=0}^{s-1} iL^{i}GL^{s-i+1}\big) \\
&= \tfrac{1}{s}\big(\textstyle\sum_{i=1}^{s}(i-1)L^{i+1}GL^{s-i}-2\sum_{i=0}^{s-1} iL^{i+1}GL^{s-i}+ \\
&\qquad \textstyle\sum_{i=-1}^{s-2}(i+1)L^{i+1}GL^{s-i}\big) \\
&= \tfrac{1}{s}\big(\textstyle\sum_{i=1}^{s}(i-1)L^{i+1}GL^{s-i}-2\sum_{i=1}^{s-1} iL^{i+1}GL^{s-i}+ \\
&\qquad \textstyle\sum_{i=0}^{s-2}(i+1)L^{i+1}GL^{s-i}\big) \\
&= \tfrac{1}{s}\big((s-1)L^{s+1}GL^{s-s}+\textstyle\sum_{i=1}^{s-1}(i-1)L^{i+1}GL^{s-i} \\
&\qquad -2\textstyle\sum_{i=1}^{s-1} iL^{i+1}GL^{s-i}+\sum_{i=1}^{s-2}(i+1)L^{i+1}GL^{s-i} \\
&\qquad +LGL^{s-0}-(s-1+1)L^{s-1+1}GL^{s-(s-1)}\big) \\
&= \tfrac{1}{s}\big((s-1)LG+LG-sGL\big) \\
&= LG-GL=\sigma_L(G) \\
&=_{(1)} \sigma_F(G).
\end{aligned}$$

□

Now we can finish the

Proof. (of theorem 4.2, $2 \longrightarrow 1$): By lemma 4.3 we may assume that we have a map F satisfying $F^s = X$ and $F = L + H$ where L is linear and $H \in (IA)^n$ and $I^2 = (0)$.

Write $\tau := \tau_L$, $\sigma := \sigma_L$. Let $\varphi := X + \tau(H)$. Then

$$\begin{aligned}(X - \tau(H))(X + \tau(H)) = & \quad X + \tau(H) - \tau(H)(X + \tau(H)) \\ = & \quad X - \tau(H) - \tau(H(X)) \\ = & \quad X.\end{aligned}$$

So $\varphi^{-1} = (X - \tau(H))$. Define $L + \tilde{H} := \tilde{F} := \varphi^{-1} F \varphi$. "$=_{(i)}$" means that part i of the previous lemma is used.

$$\begin{aligned}\tilde{F} = & \quad L + \tilde{H} \\ = & \quad \varphi^{-1} F \varphi \\ = & \quad (X - \tau(H))(L + H)(X + \tau(H)) \\ = & \quad (X - \tau(H))(L + L\tau(H) + H(X + \tau(H))) \\ =_{(5)} & \quad (X - \tau(H))(L + \tau(LH) + H(X)) \\ = & \quad L + \tau(LH) + H - \tau(H)(L + \tau(LH) + H) \\ = & \quad L + \tau(LH) + H - \tau(HL) \\ =_{(4)} & \quad L + \tau(LH - HL) + H \\ = & \quad L - \tau\sigma(H) + H.\end{aligned}$$

Now $\sigma(H - \tau\sigma(H)) =_{(2)} \sigma(H) - \sigma\tau\sigma(H) =_{(6)} \sigma(H) - \sigma(H) = 0$. So $\tilde{H} = H - \tau\sigma(H) \in ker(\sigma)$. So $0 = \sigma(\tilde{H}) = L\tilde{H} - \tilde{H}L$ hence $\tilde{H}L = L\tilde{H}$. By lemma 4.4 we have $0 = \sum_{i=0}^{s-1} L^{s-1-i}\tilde{H}L^i$ hence $0 = \sum_{i=0}^{s-1} \tilde{H}L^{s-1} = s\tilde{H}L^{s-1}$. So $s\tilde{H}L^s = 0$ i.e. $\tilde{H} = 0$ since $L^s = X$. Hence $\tilde{F} = L$ linear. So F is linearisable. $\square$

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