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NILPOTENT SYMMETRIC JACOBIAN MATRICES AND THE JACOBIAN CONJECTURE

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Abstract

Let $H : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that the Jacobian $\mathsf{J}H$ of H is nilpotent and symmetric. The symmetric dependence problem, $SDP(n)$, asks whether the rows of the matrix $\mathsf{J}H$ are dependent over $\mathbb{C}$. We show that if $SDP(r)$ holds for all $r \le n$, then the Jacobian conjecture holds for all $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form $F = x + H$ with $\mathsf{J}H$ nilpotent and symmetric. As a consequence, we deduce the main result of [3].

Introduction

Write $\mathsf{J}F$ for the Jacobian of a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$. The Jacobian conjecture claims that F is an invertible polynomial map in case $\det \mathsf{J}F \in \mathbb{C}^*$. It was shown in [3] that in case $n \le 4$, the Jacobian conjecture holds for all polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form $F = x + H$, where H is homogeneous and $\mathsf{J}H$ is nilpotent and symmetric. Let $\mathsf{H}f$ be the matrix defined by

$$\mathsf{H}f = \begin{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_1} f & \frac{\partial^2}{\partial x_2 \partial x_1} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_1} f \\ \frac{\partial^2}{\partial x_1 \partial x_2} f & \frac{\partial^2}{\partial x_2 \partial x_2} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_2} f \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2}{\partial x_1 \partial x_n} f & \frac{\partial^2}{\partial x_2 \partial x_n} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_n} f \end{pmatrix}$$

Main ingredient in the proof is a result due to Gordan and Nöther in [4], which asserts the following: if $n \le 4$ and $h \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is a homogeneous polynomial such that $\det \mathsf{H}h = 0$, then h is degenerate, i.e. there exists a $T \in GL_n(\mathbb{C})$ such that $h(Tx) \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$.

In this paper we generalize the main result of [3] to the n-dimensional case. More precisely, we show that if F is of the form $F = x + H$ with $\mathsf{J}H$ nilpotent and symmetric (H does not need to be homogeneous), then F is invertible, provided a certain dependence problem (see $SDP(n)$ in section 1) has an affirmative answer. Since the Gordan-Nöther theorem implies that the homogeneous dependence problem (see $HSDP(n)$

is section 1) has an affirmative answer for $n \leq 4$ (corollary 1.3) and $SDP(n)$ has an affirmative answer for $n \leq 2$ (proposition 1.1), our main theorem (theorem 2.1) implies the main result of [3].

1 The symmetric dependence problem

Throughout this paper K denotes a field of characteristic 0 and $K[x] = K[x_1, x_2, \ldots, x_n]$ is the polynomial ring in n indeterminates over K. In search of the Jacobian conjecture the following problem arose naturally (see [6, cjct. 1, p. 80], [7, cjct. B, p. 135], [8, cjct. 11.3], [1] and [2, 7.1.7]).

Dependence problem $DP(n)$ Let $H = (H_1, H_2, \ldots, H_n) \in K^n$ such that $\mathsf{J}H$ is nilpotent. Are the rows of $\mathsf{J}H$ dependent over K?

It is not difficult to see that, in case $H_i(0) = 0$ for all i, the dependence of the rows of $\mathsf{J}H$ is equivalent to the linear dependence of the polynomials $H_1, H_2, \ldots, H_n$ over K.

Due to the embedding lemma (Lefschetz principle) (see [2, lm. 1.1.13]), we only need to examine the case $K = \mathbb{C}$ in the above and subsequent dependence problems.

In case $n \leq 2$, the dependence problem has an affirmative answer, however if $n \geq 3$ then there are counterexamples (see [2, th. 7.1.7]). On the other hand, if we additionally assume that that each H_i is either zero or homogeneous of a fixed degree $d \geq 1$, then the corresponding problem is still open for all $n \geq 3$:

Homogeneous dependence problem $HDP(n)$ Let $H = (H_1, H_2, \ldots, H_n) \in K^n$ be homogeneous of degree $d \geq 1$ such that $\mathsf{J}H$ is nilpotent. Are the rows of $\mathsf{J}H$ dependent over K?

In fact a highly non-trivial result obtained by Hubbers in [5] (see also [2, th. 7.1.2]) completely classifies all such maps H in case $n = 4$ and $d = 3$. From this result it follows that the homogeneous dependence problem has an affirmative answer in this case, see [2, col. 7.1.4].

In this section we discuss the dependence problem for symmetric nilpotent Jacobian matrices. So let $F = (F_1, F_2, \ldots, F_n) \in K[x]^n$ and assume that $\mathsf{J}F$ is a symmetric matrix. Then $F = (\mathsf{J}f)^{\mathsf{T}}$ for some $f \in K[x]$ (see for example [2, lm. 1.3.53]). Consequently, a symmetric Jacobian matrix is of the form

$$\mathsf{H}f = \begin{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_1} f & \frac{\partial^2}{\partial x_2 \partial x_1} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_1} f \\ \frac{\partial^2}{\partial x_1 \partial x_2} f & \frac{\partial^2}{\partial x_2 \partial x_2} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_2} f \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2}{\partial x_1 \partial x_n} f & \frac{\partial^2}{\partial x_2 \partial x_n} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_n} f \end{pmatrix}$$

The above matrix $\mathsf{H}f$ is called the Hessian of f. Observe that $\mathsf{H}f = \mathsf{H}\tilde{f}$, where $\tilde{f}$ is obtained from f by subtracting all monomials of degree ≤ 1. So $\tilde{f}$ has only monomials of degree ≥ 2. Such a polynomial is called *reduced*. Furthermore, we call $\tilde{f}$ the reduced part of f. Of course, the reduced part of a reduced polynomial is the polynomial itself.

Formulating the (homogeneous) dependence problem for symmetric Jacobian matrices then gives:

Symmetric dependence problem $SDP(n)$ Let $h \in K[x_1, x_2, \ldots, x_n]$ such that $\mathsf{H}h$ is nilpotent. Are the rows of $\mathsf{H}h$ dependent over K?

Homogeneous symmetric dependence problem $HSDP(n)$ Let $h \in K[x_1, x_2, \ldots, x_n]$ be homogeneous of degree $d \geq 2$ such that $\mathsf{H}h$ is nilpotent. Are the rows of $\mathsf{H}h$ dependent over K?

The following proposition is an immediate consequence of $DP(2)$, but a direct proof is easier and gives some ideas about nilpotent Hessians.

Proposition 1.1 *$SDP(2)$ has an affirmative answer.*

Proof: Let $M \in Mat_2(K)$ be symmetric and nilpotent (M does not need to be a Hessian matrix). Then $M^2 = 0$. So the first row v^{T} of M satisfies $v^{\mathsf{T}}M = 0$. Since v is the first row of M, $v^{\mathsf{T}} \cdot v = 0$. In other words, v is orthogonal to itself with respect to the common inner product $\langle a, b\rangle = \sum_{i=1}^n a_i b_i$ (here $n = 2$). Such a vector is called *isotropic*.

Since v is isotropic, $v_2^2 = -v_1^2$, i.e. $v_2 = \pm I v_1$, where $I = \sqrt{-1}$. Consequently, $(1, \pm I) \cdot M = 0$. In particular, $I \in K$ or $M = 0$. $\square$

One can easily see that for $j \geq r/2$ and symmetric $M \in Mat_n(K)$ with $M^r = 0$ and $M^{r-1} \neq 0$, all rows of M^j are orthogonal to each other. In particular, all rows of M^j are isotropic. Furthermore, all nonzero rows of M^{r-1} are eigenvectors of M. Isotropic vectors seem to play an important role in the theory of nilpotent Hessians.

Now we relate the $HSDP(n)$ to the Gordan-Nöther theorem mentioned in the introduction. Let $f \in K[x]$ and $T \in GL_n(K)$. Put $f \circ T = f(Tx)$. Then it is well-known that

$$\mathsf{H}(f \circ T) = T^{\mathsf{T}}(\mathsf{H}f)|_{Tx}T \qquad (1)$$

where $M|_{Tx}$ is the matrix with entries $M_{ij}(Tx)$.

We call f *degenerate* if there exists a $T \in GL_n(K)$ such that $f \circ T \in K[x_1, x_2, \ldots, x_{n-1}]$.

Proposition 1.2 *Let $f \in K[x_1, x_2, \ldots, x_n]$. Then the following statements are equivalent.*

i) The rows of $\mathsf{H}f$ are dependent over K.

ii) The columns of $\mathsf{H}f$ *are dependent over* K*.*

iii) There exists a nonzero $v \in K^n$ *such that* $\mathsf{H}f \cdot v = 0$*.*

iv) There exists a $T \in GL_n(K)$ *such that the last column of* $\mathsf{H}(f \circ T)$ *equals zero.*

v) The reduced part $\tilde{f}$ *of* f *is degenerate, i.e. there exists a* $T \in GL_n(K)$ *such that* $\tilde{f} \circ T \in K[x_1, x_2, \ldots, x_{n-1}]$*.*

Moreover, the $T \in GL_n(K)$ *for which iv) holds match those for which v) holds.*

Proof: iii) is a reformulation of ii). Further, i) and ii) are equivalent, since $\mathsf{H}f$ is symmetric. So it suffices to show iii) $\Rightarrow$ iv), iv) $\Rightarrow$ v), and v) $\Rightarrow$ iii).

First assume iii). Extend v to a $T \in GL_n(K)$ such that v is the last column of T. Then the last column of $\mathsf{H}f \cdot T$ equals $\mathsf{H}f \cdot v = 0$. From (1), iv) now follows.

Next assume iv). Since $\mathsf{H}(f \circ T)$ is symmetric, the last row of $\mathsf{H}(f \circ T)$ is zero. So $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_n}(f \circ T) = 0$ for all i. Since $\mathrm{char} K = 0$, it follows that $\frac{\partial}{\partial x_n}(f \circ T) \in K$. Notice that the reduced part of $f \circ T$ is just $\tilde{f} \circ T$. Consequently, $\frac{\partial}{\partial x_n}(\tilde{f} \circ T) = 0$ and v) follows.

Finally assume v). Since $\tilde{f} \circ T \in K[x_1, x_2, \ldots, x_{n-1}]$, the last column of $\mathsf{H}(f \circ T) = \mathsf{H}(\tilde{f} \circ T)$ equals zero. So the n-th standard basis vector e_n satisfies $\mathsf{H}(f \circ T) \cdot e_n = 0$. From (1), it follows that $\mathsf{H}f \cdot Te_n = 0$ and iii) follows. $\square$

Corollary 1.3 *$HSDP(n)$ has an affirmative answer for all $n \leq 4$.*

Proof: Suppose that $\mathsf{H}h$ is nilpotent. Then $\det \mathsf{H}h = 0$ in particular. By the theorem of Gordan and Nöther mentioned in the introduction, h is degenerate. If $\deg h \leq 1$, then the reduced part of h equals zero. If $\deg h \geq 2$, then h is reduced, for h is homogeneous. In either case, the reduced part of h is degenerate. Now apply proposition 1.2, v) $\Rightarrow$ i). $\square$

In the remainder of this section, we assume that K is algebraically closed (in fact it is sufficient that K is closed under taking square roots).

Suppose that $\mathsf{H}h$ is nilpotent and T is orthogonal, i.e. $T^{\mathsf{T}}T = 1$. ¿From (1), it follows that $\mathsf{H}(h \circ T)$ is nilpotent as well. So an interesting question is whether T can always be chosen orthogonal in the definition of degenerate, in which case we call h *orthogonally degenerate*. The answer is no. Take $h = (x_1 + Ix_2)^2$ and suppose that $h \circ T \in K[x_1]$. Then $\mathsf{H}(h \circ T)$ is of the form

$$\mathsf{H}(h \circ T) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

So $\mathsf{H}(h \circ T)$ cannot both be nilpotent and have rank 1.

We call f *isotropically degenerate* if there is an orthogonal $T \in GL_n(K)$ such that $f \circ T \in K[x_1, x_2, \ldots, x_{n-2}, x_{n-1} + Ix_n]$. Clearly, the above h with $n = 2$ is isotropically degenerate (take $T = 1$).

The following lemma gives a class of $f \in K[x]$ that are orthogonally degenerate.

Lemma 1.4 *Let $f \in K[x_1, x_2, \ldots, x_n]$ be reduced such that $\mathsf{H}f \cdot v = 0$ for some non-isotropic v. Then f is orthogonally degenerate.*

Proof: Replacing v by $v/\langle v, v\rangle$, we may assume that $\langle v, v\rangle = 1$. Then, using the Gram-Schmidt process, we can find an orthogonal $T \in GL_n(K)$ such that v is the last column of T, i.e.

$$v = T \cdot e_n \tag{2}$$

So $\mathsf{H}f \cdot T \cdot e_n = \mathsf{H}f \cdot v = 0$. ¿From (1), it follows that

$$\mathsf{H}(f \circ T) \cdot e_n = 0$$

So the last column of $\mathsf{H}(f \circ T)$ equals zero. Now apply proposition 1.2, iv) $\Rightarrow$ v). □

The following lemma, which gives a class of $f \in K[x]$ that is isotropically degenerate, is harder to prove than the above lemma. A problem is that iv) $\Rightarrow$ v) of proposition 1.2 cannot be applied directly.

Lemma 1.5 *Let $f \in K[x_1, x_2, \ldots, x_n]$ be reduced such that $\mathsf{H}f \cdot v = 0$ for some isotropic $v \neq 0$. Then f is isotropically degenerate.*

Proof: Since permutation matrices are orthogonal, we may assume that $v_1 \neq 0$. Replacing v by v/v_1, we may assume that $v_1 = 1$. The first standard basis vector e_1 and the vector $\tilde{v} = -I(0, v_2, v_3, \ldots, v_n)$ satisfy $\langle e_1, \tilde{v}\rangle = 0$, $\langle e_1, e_1\rangle = 1$, and $\langle \tilde{v}, \tilde{v}\rangle = \langle v_1, v_1\rangle - \langle v, v\rangle = 1$. By the Gram-Schmidt process there exists an orthogonal $T \in GL_n(K)$ such that the last two columns of T are e_1 and $\tilde{v}$, in this order. So

$$\begin{aligned} T \cdot (e_{n-1} + Ie_n) &= (Te_{n-1} + ITe_n) & (3)\\ &= (e_1 + I\tilde{v}) & (4)\\ &= v & (5) \end{aligned}$$

and therefore, $\mathsf{H}f \cdot T \cdot (e_{n-1} + Ie_n) = \mathsf{H}f \cdot v = 0$.

In order to prove this lemma, we define $S \in GL_n(K)$ by

$$S(x) = (x_1, x_2, \ldots, x_{n-2}, x_{n-1} + Ix_n, -Ix_n)$$

Then $S^{-1}(x) = (x_1, x_2, \ldots, x_{n-2}, x_{n-1} + x_n, Ix_n)$. In particular, $S^{-1} \cdot e_n = e_{n-1} + Ie_n$. Therefore, it follows from (3) that

$$T \cdot S^{-1} \cdot e_n = T \cdot (e_{n-1} + Ie_n) = v \tag{6}$$

Consequently, $\mathsf{H}f \cdot T \cdot S^{-1} \cdot e_n = \mathsf{H}f \cdot v = 0$. ¿From (1), it follows that $g = f \circ T \circ S^{-1}$ satisfies $\mathsf{H}g \cdot e_n = 0$ and therefore

$$g \in K[x_1, x_2, \ldots, x_{n-1}]$$

Consequently, $f \circ T = g \circ S$ is a polynomial in the first $n - 1$ coordinates of Sx, i.e. $f \circ T \in K[x_1, x_2, \ldots, x_{n-2}, x_{n-1} + Ix_n]$. □

The following proposition claims that a degenerate f is either orthogonally or isotropically degenerate.

Proposition 1.6 *Let $f \in K[x_1, x_2, \ldots, x_n]$ be degenerate (not necessarily reduced). Say that $f \circ T \in K[x_1, x_2, \ldots, x_{n-1}]$ with $T \in GL_n(K)$. Let $v = Te_n$ be the last column of T.*

1. *If v is not isotropic, then f is orthogonally degenerate.*

2. *If v is isotropic, then f is isotropically degenerate.*

Proof: Write $\tilde{f}$ for the reduced part of f. Suppose first that v is not isotropic. Assume without loss of generality that $\langle v, v\rangle = 1$. From lemma 1.4, it follows that there is an orthogonal $\tilde{T} \in GL_n(K)$ such that $\tilde{f} \circ \tilde{T} \in K[x_1, x_2, \ldots, x_{n-1}]$. Without loss of generality, we may assume that $f(0) = 0$. Since $f - \tilde{f}$ is linear, it follows that

$$\begin{aligned}(f \circ \tilde{T}) &= ((f - \tilde{f}) \circ \tilde{T}) + (\tilde{f} \circ \tilde{T}) \\ &= ((f - \tilde{f}) \circ (\tilde{T} - T)) + ((f - \tilde{f}) \circ T) + (\tilde{f} \circ \tilde{T}) \\ &= ((f - \tilde{f}) \circ (\tilde{T} - T)) + (f \circ T) - (\tilde{f} \circ T) + (\tilde{f} \circ \tilde{T}) \end{aligned} \tag{7}$$

According to (2), we may assume that the last column of $\tilde{T} - T$ equals zero. So $\tilde{T} - T \in K[x_1, x_2, \ldots, x_{n-1}]^n$. Since $f \circ T \in K[x_1, x_2, \ldots, x_{n-1}]$ by assumption and $\tilde{f} \circ T$ is the reduced part of $f \circ T$, it follows from (7) that $f \circ \tilde{T} \in K[x_1, x_2, \ldots, x_{n-1}]$.

Suppose next that v is isotropic. Assume without loss of generality that $v_1 = 1$. From lemma 1.5, it follows that there is an orthogonal $\tilde{T} \in GL_n(K)$ such that $\tilde{f} \circ \tilde{T} \in K[x_1, x_2, \ldots, x_{n-1} + Ix_n]$. Take S as in the proof of lemma 1.5. Similar to the case that v is orthogonal, it follows from (6) that $g = f \circ \tilde{T} \circ S^{-1} \in K[x_1, x_2, \ldots, x_{n-1}]$ and therefore $f \circ \tilde{T} = g \circ S \in K[x_1, x_2, \ldots, x_{n-1} + Ix_n]$. □

To describe the next results, it is convenient to introduce some new notation. Let $x = x_1, x_2, \ldots, x_n$, $M \in \mathrm{Mat}_n(K[x])$, and $f \in K[x]$. Then write x_* for $x_1, x_2, \ldots, x_{n-1}$, M_* for $(M_{ij})_{1 \le i,j \le n-1} \in \mathrm{Mat}_{n-1}(K[x])$, and f_* for f, viewed as polynomial in x_* over $K[x_n]$. Similarly, we define $x_{**} = x_1, x_2, \ldots, x_{n-2}$, $M_{**} = (M_{ij})_{1 \le i,j \le n-2}$, etc. Also, we define J_* J_{**}, H_*, and H_{**} in a similar way. So we have for example $\mathsf{H}_* f_* = (\mathsf{H} f)_*$.

Suppose that $f \in K[x]$ such that $\mathsf{H} f \cdot v = 0$. Take T as in proposition 1.6 and write $g = f \circ T$. If v is not isotropic, then $g \in K[x_*]$ and therefore

$$\mathsf{H} g = \begin{pmatrix} & & & 0 \\ & \mathsf{H}_* g_* & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

Consequently

$$\mathsf{H} g \text{ is nilpotent, if and only if } \mathsf{H}_* g_* \text{ is nilpotent} \tag{8}$$

in case v is not isotropic.

If v is isotropic, then $g \in K[x_{**}, x_{n-1} + Ix_n]$ and therefore

$$\mathsf{H} g = \begin{pmatrix} \mathsf{H}_{**} g_{**} & w & Iw \\ w^{\mathsf{T}} & a & Ia \\ Iw^{\mathsf{T}} & Ia & -a \end{pmatrix}$$

for some $w \in K[x]^{n-2}$ and $a \in K[x]$. In order to obtain the 'isotropic analogon'

$$\mathsf{H}g \text{ is nilpotent, if and only if } \mathsf{H}_{**}g_{**} \text{ is nilpotent} \tag{9}$$

of (8), we need the following lemma.

Lemma 1.7 *Let R be a commutative ring with $I = \sqrt{-1}$ and $M \in \mathrm{Mat}_n(R)$ symmetric and of the form*

$$M = \begin{pmatrix} M_{**} & w & Iw \\ w^{\mathsf{T}} & a & Ia \\ Iw^{\mathsf{T}} & Ia & -a \end{pmatrix}$$

with $w \in R^{n-2}$ and $a \in R$. Then M is nilpotent, if and only if M_{**} is nilpotent.

Proof: Suppose by induction that M^r is of the form

$$M^r = \begin{pmatrix} (M_{**})^r & u & Iu \\ u^{\mathsf{T}} & b & Ib \\ Iu^{\mathsf{T}} & Ib & -b \end{pmatrix} \tag{10}$$

Then,

$$M^{r+1} = M \cdot M^r = \begin{pmatrix} (M_{**})^{r+1} & M_{**}u & IM_{**}u \\ (M_{**}u)^{\mathsf{T}} & \langle w, u\rangle & I\langle w, u\rangle \\ (IM_{**}u)^{\mathsf{T}} & I\langle w, u\rangle & -\langle w, u\rangle \end{pmatrix}$$

So M^r is of the form (10) for all r.

If M is nilpotent, then M_{**} is nilpotent as well, since $(M_{**})^r$ is a submatrix of M^r for all r. Hence assume that M_{**} is nilpotent. Take r such that $(M_{**})^r = 0$. Then

$$M^r = \begin{pmatrix} \emptyset & u & Iu \\ u^{\mathsf{T}} & b & Ib \\ Iu^{\mathsf{T}} & Ib & -b \end{pmatrix}$$

and one can easily verify that $M^{2r} = (M^r)^2 = 0$. So M is nilpotent. $\square$

2 The main result

The following result is the main theorem of this paper. Again, we assume that K is a field of characteristic zero.

Theorem 2.1 *Let $n \geq 1$ and suppose that $H \in K[x]^n$ such that $\mathsf{J}H$ is nilpotent and symmetric.*

1. *If $SDP(p)$ holds for all $p \leq n$, then $x + H$ is invertible.*
2. *If H is homogeneous, $SDP(p)$ holds for all $p \leq n-2$, and $HSDP(n-1)$ and $HSDP(n)$ hold, then $x + H$ is invertible.*

Proof: In case $n = 1$, $\mathsf{J}H = 0$ and therefore H is constant and $x - H$ is the inverse of $x + H$. So assume that $n \geq 2$. Since $\mathsf{J}H$ is symmetric, we have $\mathsf{J}H = \mathsf{H}h$ for some reduced h. If $H(0) = 0$, then $(\mathsf{J}h)^{\mathsf{T}} = H$ as well. Since translations are invertible, we assume that $H(0) = 0$.

We shall show the following assertions.

i) If h is degenerate and $(H)SDP(n-1)$, then h is isotropically degenerate.

ii) If $h \in K[x_{**}, x_{n-1} + Ix_n]$, then $x + H$ is invertible over K, if and only if $x_{**} + H_{**}$ is invertible over $K(x_{n-1} + Ix_n)$.

Suppose first that these assertions hold. Since $(H)SDP(n)$ is assumed, it follows from i) and proposition 1.2 that h is isotropically degenerate. Suppose that the conclusion of this theorem does not hold. Take n minimal such that H satisfies the conditions of this theorem, but $x + H$ is not invertible. Since h is isotropically degenerate, there is an orthogonal $T \in GL_n(K)$ such that $h \circ T \in K[x_{**}, x_{n-1} + Ix_n]$.

If $n \geq 3$, then it follows from (9) and ii) that there is a $G \in K(y)[x_{**}]^n$ such that $\mathsf{J}G$ is nilpotent and symmetric, but $x_{**} + G$ is not invertible over $K(y)$. Since G satisfies the conditions of this theorem as well as H, we have a contradiction, so $n \leq 2$. The case $n = 1$ is trivial, so assume that $n = 2$. Take $G = (H_1, H_2, 0)$. Then $x + H$ is invertible, if and only if $(x, x_3) + G$ is invertible. Furthermore, the invertibility of G reduces to the case $n = 1$ of this theorem, which is trivially satisfied.

First, we show assertion i). Suppose that h is not isotropically degenerate. Since h is assumed to be degenerate, h is orthogonally degenerate according to proposition 1.6. Take T orthogonal such that $h \circ T \in K[x_*]$. Since $g = h \circ T$ is reduced, it follows from (8) and $(H)SDP(n-1)$ that g is degenerate, so g is either isotropically or orthogonally degenerate according to proposition 1.6. If g is isotropically degenerate, then h is isotropically degenerate as well. If, on the other hand, g is orthogonally degenerate, then there is an orthogonal $S \in GL_n(K)$ such that $g \circ S \in K[x_{**}]$. Therefore $h \circ (T \circ S) \in K[x_{**}] \subseteq K[x_{**}, x_{n-1} + Ix_n]$ and h is isotropically degenerate.

Next, we show assertion ii). Suppose that $h \in K[x_{**}, x_{n-1} + Ix_n]$, say that $h = g(x_{**}, x_{n-1} + Ix_n)$ with $g \in K[x_*]$. Put $H_{**} = (J_{**}h)^{\mathsf{T}}$. Then

$$
\begin{aligned}
P_1 &= x + H \\
&= \left(x_{**} + H_{**}, x_{n-1} + \left(\frac{\partial}{\partial x_{n-1}} g\right)(x_{**}, x_{n-1} + Ix_n),\right. \\
&\qquad \left. x_n + I\left(\frac{\partial}{\partial x_{n-1}} g\right)(x_{**}, x_{n-1} + Ix_n)\right)
\end{aligned}
$$

is invertible, if and only if

$$
\begin{aligned}
P_2 &= (x_{**}, x_{n-1} + Ix_n, x_n) \circ P_1 \\
&= \left(x_{**} + H_{**}, x_{n-1} + Ix_n, x_n + I\left(\frac{\partial}{\partial x_{n-1}} g\right)(x_{**}, x_{n-1} + Ix_n)\right)
\end{aligned}
$$

is invertible. Put $G_{**} = \mathsf{J}_{**}g$, then $G_{**} = H_{**}(x_{**}, x_{n-1} - Ix_n, x_n)$. So P_2 is invertible, if and only if

$$
P_3 = P_2 \circ (x_{**}, x_{n-1} - Ix_n, x_n)
$$

$$= \left(x_{**} + G_{**}, x_{n-1}, x_n + I\left(\frac{\partial}{\partial x_{n-1}} g\right)(x_{**}, x_{n-1})\right)$$

is invertible. P_3 is invertible, if and only

$$P_4 = \left(x_*, x_n - I\left(\frac{\partial}{\partial x_{n-1}} g\right)(x_*)\right) \circ P_3 = (x_{**} + G_{**}, x_{n-1}, x_n)$$

is invertible. Since $G_{**} \in K[x_*]^{n-2}$, it follows that P_4 is invertible, if and only if $x_{**} + G_{**}$ is invertible over $K[x_{n-1}]$, i.e. if and only if $x_{**} + H_{**}$ is invertible over $K[x_{n-1} + Ix_n]$. By [2, lm. 1.1.8], this last statement is equivalent to the assertion that $x_{**} + H_{**}$ is invertible over $K(x_{n-1} + Ix_n)$. This gives assertion ii). □

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