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THE PROBLEM OF THE DETERMINACY OF INFINITE GAMES FROM AN INTUITIONISTIC POINT OF VIEW

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'We *must* have a bit of a fight, but I don't care about going on long,'
said Tweedledum. 'What's the time now?'
Tweedledee looked at his watch, and said, 'Half-past four.'
'Let's fight till six, and then have dinner,' said Tweedledum.

([2], page 190)

Abstract

We study finite and infinite games of perfect information for players I and II from an intuitionistic point of view, and are led to consider the following problem. Suppose that player I is allowed, each time he has to move, to ask player II a finite number of questions like: what is player II going to answer at this-or-that-position, in case we get there in the further course of the game? If player I, given this advantage with the certainty that player II will act according to his commitments, is able to win the game, must he also have then a winning strategy in the usual sense of the word? We show that he has one indeed, but only if player II has, for each one of his moves in the game, no more than a finite number of alternative possibilities. The proof is given in intuitionistic analysis, and uses the fan theorem, but no continuity principles.
We apply our result and find two intuitionistic theorems, one related to the continuum hypothesis, and one concerning Hausdorff's notion of a scattered subset of the set $\mathbb{Q}$ of rational numbers.

1 Intuitionistic determinacy: the problem, and the case of two-move-games

1.1 $\mathbb{N}$ is the set of natural numbers, $\mathcal{N}$ is the set of all infinite sequences of natural numbers.
We use $m, n, p, q, \ldots$ as variables over the set $\mathbb{N}$ and $\alpha, \beta, \ldots$ as variables over the set $\mathcal{N}$.
Let A be a subset of $\mathcal{N}$. We describe *the game for A*. There are two players, I and II, who together build an infinite sequence α in $\mathcal{N}$, as follows:

$$\begin{array}{ccccc} \text{Player I chooses } \alpha(0) & & & \text{Player I chooses } \alpha(2) & \\ & \searrow & \nearrow & & \searrow \\ & & \text{Player II chooses } \alpha(1) & & \ldots \end{array}$$

Player I is the winner if and only if the infinite sequence α belongs to the set A.
For which subsets A of $\mathcal{N}$ does player I have a sure method to win?
And for which subsets A of $\mathcal{N}$ does player II have a sure method to prevent player I from winning?
We study these questions from an intuitionistic point of view. In the first place, therefore, we follow the rules of intuitionistic logic. This is because we agree with Brouwer that every mathematical statement should be considered as a report on what we have been able to prove. It is natural then to interpret connectives and quantifiers and the corresponding set-theoretic operations constructively. In particular, a disjunctive statement $P \vee Q$ will be considered proven if and only if we either have a proof of P or a proof of Q.
In the second place, we are guided by the new axioms Brouwer proposed as a result of his reflection on the problem how to handle the concept of the continuum. We will mention the axioms at the places where we first need them.
Our arguments do not transcend the formal system for intuitionistic analysis developed by S.C. Kleene and R.E. Vesley in [8]. The reader also may consult [11].
We want to emphasize that our results make sense also from a classical point of view. Intuitionistic logic is only a restriction of classical logic. The axioms fall into two classes: those that are *not* classically acceptable, and those that are. Our main result, Theorem 4.4, does not depend on any axiom from the first class. Nevertheless, we do bring up these classically unacceptable axioms, the so-called *continuity principles*, as they will show us how to frame our definitions.

1.2 We slightly generalize the problem brought up in section 1.1.
$\mathbb{N}^*$ is the set of finite sequences of natural numbers.
We suppose that a bijective mapping $(a_0, a_1, \ldots, a_{n-1}) \mapsto \langle a_0, a_1, \ldots, a_{n-1}\rangle$ from $\mathbb{N}^*$ to $\mathbb{N}$ is given, a function *coding* the finite sequences of natural numbers by means of natural numbers, and we will identify a finite sequence of natural numbers with its code number. We assume that the code number of a finite sequence is never smaller than its length.
$*$ is the binary function on $\mathbb{N}$ which, via the coding, corresponds to the operation of concatenating finite sequences.
For each infinite sequence of natural numbers α, and each natural number n, we define $\overline{\alpha}(n)$ to be (the code number of) the finite sequence $\langle \alpha(0), \ldots, \alpha(n-1)\rangle$. If confusion seems unlikely, we write $\overline{\alpha}n$ rather than $\overline{\alpha}(n)$.
For each infinite sequence of natural numbers α, for each natural number s, we define: *α passes through s*, or: *s contains α*, or: *s is an initial part of α*, if and only if there exists n such that $\overline{\alpha}n = s$.
Let σ belong to $\mathcal{N}$. σ is called a *spread-law* if and only if $\sigma(\langle\ \rangle) = 0$ and, for each a, $\sigma(a) = 0$ if and only if, for some n, $|sigma(a * \langle n\rangle) = 0$. If $\sigma(a) = 0$, we will say that (the finite sequence coded by) a is *admitted by* σ, or that *a obeys σ*.
Let σ be a spread-law and let α belong to $\mathcal{N}$. We say: *α obeys the spread-law σ*, or: *σ admits α*, if and only if σ admits every initial part of α, that is, for each n, $\sigma(\overline{\alpha}n) = 0$.

The set of all infinite sequences of natural numbers α that obey the spread-law σ is called a *spread* and this set is also named σ.
Observe that a subset X of $\mathcal{N}$ coincides with a spread if and only if (i) X is *closed*, that is, every α such that every initial part of α contains an element of X belongs itself to X, and (ii) X is *located*, that is, there exists σ in $\mathcal{N}$ such that for every s, s contains an element of X if and only if $\sigma(s) = 0$.
Let σ be a spread and let A be a subset of σ. We describe *the game for A in σ*.
There are again two players, I and II, who join up to build an infinite sequence α in $\mathcal{N}$, but this time they take care that the infinite sequence α will belong to the spread σ:

Player I chooses $\alpha(0)$ such that $\sigma(\overline{\alpha}(1)) = 0$ $\qquad\qquad \cdots$

$\searrow \qquad\qquad\qquad\qquad \nearrow$

Player II chooses $\alpha(1)$ such that $\sigma(\overline{\alpha}(2)) = 0$

Player I is the winner if and only if the infinite sequence α belongs to the set A.
For which spreads σ and subsets A of σ does player I have a sure method to win the game for A in σ?
And for which spreads σ and subsets A of σ does player II have a sure method to prevent player I from winning the game for A in σ?

1.3 In order to give a precise meaning to the questions asked in sections 1.1 and 1.2, we introduce the concept of a strategy.
Let σ be a spread.
We let $\mathrm{Strat}_{\mathrm{I}}(\sigma)$, the set of *strategies for player I in σ*, be the set of all functions γ in $\mathcal{N}$ such that for each a, if σ admits a and length(a) is even, then σ admits $a * \langle\gamma(a)\rangle$, and, if σ does not admit a or length(a) is odd, then $\gamma(a) = 0$. Observe that $\mathrm{Strat}_{\mathrm{I}}(\sigma)$ itself is a spread.
Let α belong to σ and γ to $\mathrm{Strat}_{\mathrm{I}}(\sigma)$. We define: *$\alpha$ is played by player I according to the strategy γ*, or: *α I-obeys γ*, or *γ I-governs α*, if and only if, for each n, $\alpha(2n) = \gamma(\overline{\alpha}(2n))$.
Similarly, we let $\mathrm{Strat}_{\mathrm{II}}(\sigma)$, the set of *strategies for player II in σ*, be the set of all functions γ in $\mathcal{N}$ such that for each a, if σ admits a and length(a) is odd, then σ admits $a * \langle\gamma(a)\rangle$, and, if σ does not admit a or length(a) is even, then $\gamma(a) = 0$.
Let α belong to σ and γ to $\mathrm{Strat}_{\mathrm{II}}(\sigma)$. We define: *$\alpha$ is played by player II according to the strategy γ*, or: *α II-obeys γ*, or: *γ II-governs α*, if and only if, for each n, $\alpha(2n+1) = \gamma(\overline{\alpha}(2n+1))$.
Let A be a subset of σ and let γ be a strategy for player I in σ. We define: *the strategy γ wins the set A for player I* if and only if every α that I-obeys γ belongs to A. Let B be a subset of σ and let γ be a strategy for player II in σ. We define: *the strategy γ wins the set B for player II* if and only if every α that II-obeys γ belongs to B.
For every subset A of $\mathcal{N}$ we let $A^{\neg}$, the *complement of A* be the set of all α in $\mathcal{N}$ that do not belong to A.
Let A be a subset of σ. We define: A is *strongly determinate* if and only if *either* there is a strategy for player I in σ winning the set A for player I, *or* there is a strategy for player II in σ winning the set $A^{\neg}$ for player II.

We want to know, for each spread σ, which subsets of σ are strongly determinate in σ.

1.4 Alas, because of the disjunction occurring in its definition, the notion of strong determinacy introduced at the end of section 1.3 is, constructively, too strong to be useful. It is very easy to find non-determinate games. Even games in which player I, II make only finitely many moves need not be strongly determinate.
Consider for instance the game with no moves at all that is won by player I if and only if Riemann's hypothesis holds. Strongly determining this game is equivalent to deciding Riemann's hypothesis.
Obviously, we need a weaker notion.
Let σ be a spread and A a subset of σ. We define: A is *determinate in* σ if and only if: *if* every strategy for player II II-governs at least one element of A, *then* there is a strategy for player I in σ that wins the set A for player I.

We took the disjunctive formulation of strong determinacy, $P \vee Q$, changed it into $(\neg Q) \rightarrow P$, and then replaced the negative antecedent $\neg Q$ by a stronger, positive statement.
Of course the definition is biased as it considers the problem of the determinacy of A from the viewpoint of player I. It will follow from the examples that we will give that a set that is determinate in the sense we defined it, that is, from the viewpoint of player I, is not always determinate from the viewpoint of player II.

1.5 Before continuing the discussion of the notion of determinacy from Subsection 1.4, we want to mention one of the axioms of intuitionistic analysis.
Let σ be a spread and let ζ belong to $\mathcal{N}$. We define: ζ *codes a (continuous) function from* σ *to* $\mathcal{N}$ if and only if, for all n, for all α, there exists n such that $\alpha(\langle n\rangle * \overline{\alpha}(m)) \neq 0$.
Suppose that σ is a spread, and that ζ codes a function from σ to $\mathcal{N}$. For each $\alpha \in \sigma$ we define $\zeta|\alpha$ to be the sequence β such that, for all $n, p \in \mathbb{N}$, if p is the least m such that $\zeta(\langle n\rangle * \overline{\alpha}(m)) \neq 0$, then $\zeta(\langle n\rangle * \overline{\alpha}(p)) = \beta(n) + 1$.
We now are able to state the following axiom, see [14]:

Second Axiom of Continuous Choice:

> Let σ be a spread and let R be a subset of $\sigma \times \mathcal{N}$.
> If for all α in σ there exists β such that $\alpha R \beta$, then there exists ζ coding a function from σ to $\mathcal{N}$ such that, for all α in σ, $\alpha R(\zeta|\alpha)$.

(We write "$\alpha R \beta$" while intending "(α, β) belongs to R".)
This axiom is called *Brouwer's principle for functions* in [8], $\mathbf{GAC_{1,1}}$ in [5], and **C-C** in [11]. The axiom is unacceptable from a classical point of view. We do not go into the reasons one might have for adopting it. It seems to be the strongest possible formulation of a principle Brouwer is using in his intuitionistic papers.

1.6 We now continue the discussion of the notion of determinacy from Subsection 1.4.
Let σ be a spread, let A be a subset of σ and suppose that every strategy for player II in σ II-governs at least one element of A. Using the Second Axiom of Continuous Choice we find some ζ coding a continuous function from $\mathrm{Strat}_{\mathrm{II}}(\sigma)$ to $\mathcal{N}$ such that, for every γ in $\mathrm{Strat}_{\mathrm{II}}(\sigma)$, $\zeta|\gamma$ II-obeys γ and belongs to A.
When playing the game, player I may use ζ in the following way: each time he has to make a move he first questions his opponent, player II, on the strategy he intends to follow, asking: "What will be your reply if I make this move? And if I should continue so-and-so? Or if I make that move?" Only after having done so finitely many times, he will be able to take his decision with the help of ζ. He needs a finite piece of information on the strategy player II has in mind.
If player II, perhaps out of fear, answers and acts according to his answers, player I, provided he follows the advice of ζ, wins the game.
Our problem is: is it possible for player I to develop a *strategy* from the function ζ, i.e. a successful way of playing the game without asking unlawful questions and intimidating player II?

1.7 Let σ be a spread, and $\zeta \in \mathcal{N}$. We define: ζ is an *antistrategy for player I* in σ if and only if ζ codes a continuous function from $\mathrm{Strat}_{\mathrm{II}}(\sigma)$ to σ such that, for every γ in $\mathrm{Strat}_{\mathrm{II}}(\sigma)$, $\zeta|\gamma$ II-obeys γ.

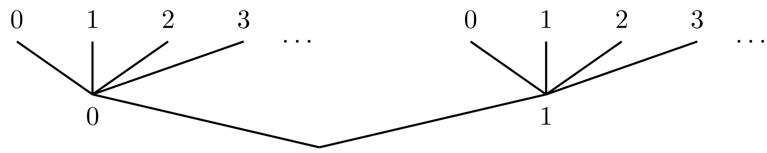
The notion of an antistrategy is close to the notion of a *delayed strategy*, as introduced in [1], but there is a difference, as the interested reader easily verifies: every delayed strategy is an antistrategy in our sense but not conversely.

Let σ be a spread, let A be a subset of σ and let ζ be an antistrategy for player I in σ.
We define: the antistrategy ζ *secures the set A for player I* if and only if, for each γ in $\mathrm{Strat}_{\mathrm{II}}(\sigma)$, $\zeta|\gamma$ belongs to A.
We also define: *A is predeterminate in σ* if and only if, *if* there is an antistrategy for player I in σ that secures the set A for player I, *then* there is a strategy for player I in σ that wins the set A for player I.

Observe that every subset of σ that is determinate in σ, is predeterminate in σ.
If one assumes the Second Axiom of Continuous Choice, then also every subset of σ that is predeterminate in σ, is determinate in σ.
We intend to prove, for many spreads σ and subsets A of σ, that A is predeterminate in σ.

1.8 Disappointingly, there exist two-move games already that thwart our expectations concerning the notion of determinacy introduced in Subsection 1.4, and the notion of predeterminacy introduced in Subsection 1.7.
Consider games of the following kind:

> Player I chooses either 0 or 1, player II chooses a natural number, and the game is over.



A strategy for player I in such a game consists of a single number, viz. his first and only move which is either 0 or 1.
A strategy for player II is a pair (p, q) of natural numbers, p being the answer player II will give to a first move 0, and q being the answer player II will give to a first move 1.
A subset A of $\{0,1\} \times \mathbb{N}$ is determinate in the sense of Subsection 1.4 if and only if: *if*, for all p, for all q, *either* $(0, p)$ belongs to A *or* $(1, q)$ belongs to A, *then: either*, for all p, $(0, p)$ belongs to A, *or*, for all q, $(1, q)$ belongs to A.
A subset A of $\{0,1\} \times \mathbb{N}$ is predeterminate in the sense of 1.7 if and only if: *if* there exists α such that for all p, for all q, either $\alpha(\langle p, q\rangle) = 0)$ and $(0, p)$ belongs to A, or $\alpha(\langle p, q\rangle) = 1)$ and $(1, q)$ belongs to A, *then: either* for all p, $(0, p)$ belongs to A, *or*, for all q, $(1, q)$ belongs to A. If we assume the following axiom, every predeterminate subset of $\{0,1\} \times \mathbb{N}$ is determinate.

First Axiom of Countable Choice:

For each subset R of $\mathbb{N} \times \mathbb{N}$, if for all m there exists n such that mRn, then there exists α such that, for all m, $mR(\alpha(m))$.

This axiom is called $^*2.2$ in [8] and $\mathbf{AC_{0,0}}$ in [5]. The First Axiom of Countable Choice is a weak consequence of the Second Axiom of Continuous Choice. This weaker axiom may be defended independently of the stronger one. It finds it justification in the fact that an infinite sequence $\alpha = \alpha(0), \alpha(1), \ldots$ may be constructed step by step, by successive free choices. One should notice that, in general, we can not *define* α by saying: let, for each n, $\alpha(n)$ be the least m such that (n, m) belongs to C. The reason is that C is not always a decidable subset of $\mathbb{N} \times \mathbb{N}$, and it may happen, for instance, that we know that $(0, 1)$ belongs to C but are uncertain if $(0, 0)$ belongs to C or not.
A subset C of $\mathbb{N}$ is a *decidable subset of* $\mathbb{N}$ if and only if, for each n, either n belongs to C or n does not belong to C. If we assume the First Axiom of Countable Choice, a subset C of $\mathbb{N}$ is a decidable subset of $\mathbb{N}$ if and only if there exists α such that, for every n, n belongs to C if and only if $\alpha(n) = 0$.
It will be clear what we understand by a decidable subset of $\{0,1\} \times \mathbb{N}$. Not every decidable subset A of $\{0,1\} \times \mathbb{N}$ is predeterminate. In order to see this, we construct a *counterexample in Brouwer's style*, as follows.
Let $p : \mathbb{N} \to \{0, 1, \ldots, 9\}$ be the decimal expansion of π. We let B be the set of all natural numbers k such that $k > 99$ and, for all $i < 99$, $p(k - i) = 9$. We let C be the set of all natural numbers k in B such that there is no m in B with the property $m < k$. Observe that both B and C are decidable subsets of $\mathbb{N}$ and that C has at most

one member. If C has a member, that member is the number of the first place in the decimal expansion of π at which an uninterrupted sequence of 99 9's is completed.
We are unable to decide whether C has a member or not, and we also do not know that every number that will turn up in C will be even, or that every such number will be odd.
We let A be the subset of $\{0,1\} \times \mathbb{N}$ consisting of all pairs (i,n) such that *either* $i = 0$ and every element of C smaller than n is odd, *or* $i = 1$ and every member of C smaller than n is even. A is a decidable subset of $\{0,1\} \times \mathbb{N}$ and for all p, for all q, either $(0,p)$ belongs to A or $(1,q)$ belongs to A. One easily defines a function α from $\mathbb{N}$ to $\{0,1\}$ such that for all p, for all q, either $\alpha(\langle p,q\rangle) = 0$ and $(0,p)$ belongs to A, or either $\alpha(\langle p,q\rangle) = 1$ and $(1,q)$ belongs to A.
Assuming that A is predeterminate we obtain the conclusion that *either* for all p, $(0,p)$ belongs to A, *or* for all q, $(1,q)$ belongs to A. In the first case, every member of C is an odd number, and in the second case, every member of C is an even number. The assumption that A is predeterminate thus leads to a conclusion for which we have no evidence.
We now want to show that the intuitionistic mathematician, should he accept the predeterminacy of decidable subsets of $\{0,1\} \times \mathbb{N}$ in general, is brought to a contradiction. The argument uses the following axiom.

Brouwer's Continuity Principle:

Let σ be a spread and R a subset of $\sigma \times \mathbb{N}$.
If for all α in σ there exists n such that $\alpha R n$, then for all α in σ there exist m, n such that for all β in σ, if β passes through $\overline{\alpha}m$, then $\beta R n$.

This axiom occurs as $^*27.15$ in [8] and is called **WC-N** in [11] and **CP** in [5].
Brouwer's Continuity Principle is a consequence of the stronger Second Axiom of Continuous Choice, and, like this axiom, it is classically unacceptable.
We now generalize the above construction. Let α belong to $\mathcal{N}$. We let C_α be the set of all natural numbers k such that $\alpha(k) \neq 0$ and, for each $m < k$, $\alpha(m) = 0$. We let A_α be the subset of $\{0,1\} \times \mathbb{N}$ consisting of all pairs (i,n) such that *either* $i = 0$ and every element of C_α smaller than n is odd, *or* $i = 1$ and every member of C_α smaller than n is even. Assuming that, for each α, A_α is predeterminate, we obtain the conclusion that, for each α, either every member of C_α is an odd number, or every member of C_α is an even number. Using Brouwer's Continuity Principle, we find m such that either for every α passing through $\underline{\overline{0}}m$, every member of C_α is odd, or for every α passing through $\underline{\overline{0}}m$, every member of C_α is even. This is false, as there exist α_0, α_1 in $\mathcal{N}$ such that $\overline{\alpha}_0(2m) = \overline{\alpha}_1(2m) = \underline{\overline{0}}(2m)$, and $\alpha_0(2m) = 1$, but $\alpha_1(2m) = 0$ and $\alpha_1(2m+1) = 1$.
Thus we see that the assumption of the predeterminacy of all decidable subsets of $\{0,1\} \times \mathbb{N}$ leads to a contradiction.
The technique that we applied here in order to obtain a contradiction from a suitable generalization of a weak counterexample is more or less standard in intuitionistic analysis. In [8], Section 7.10, for instance, one may see that Brouwer's Continuity Principle enables one to derive a contradiction from the assumption that for all α,

either $\alpha = \underline{0}$ or not $\alpha = \underline{0}$.

1.9 In Section 1.10 we intend to discuss another class of two-move-games. In the discussion we need the so-called *fan theorem*.
Unfortunately, in the literature, the name "fan theorem" is not used unequivocally. In Subsection 1.9.1 we introduce three precisely defined versions of the fan theorem. In Subsection 1.9.3 we prove a small combinatorial lemma that will be useful in 1.10.

1.9.1 Let σ be a spread-law. σ is called a *finitary spread-law* or a *fan-law* if and only if for each a, if σ admits a, then there are only finitely many numbers n such that σ admits $a * \langle n \rangle$.
The set of all infinite sequences obeying a fan-law is called a *fan*.
Let X be a subset of $\mathcal{N}$ and let B be a subset of $\mathbb{N}$. We say: *B is a bar in X* if and only if every infinite sequence in X has an initial part in B. We say: B is *bounded* if and only if there exists n such that, for each b in B, length$(b) \leq n$.

(Strict) Fan Theorem:

Let σ be a fan and let B be a *decidable* subset of $\mathbb{N}$ that is a bar in σ.
There exists a bounded subset B' of B that is a bar in σ.

(See $^*26.6a$ in [8] and FAN_D in [11].)
Removing the condition that B be a decidable subset of $\mathbb{N}$ we obtain a stronger statement.

(Generalized) Fan Theorem:

Let σ be a fan and let B be a subset of $\mathbb{N}$ that is a bar in σ.
There exists a bounded subset B' of B that is a bar in σ.

This statement occurs as $^*27.10$ [8] and as FAN in [11].
The next statement is much stronger and classically unacceptable.

Extended Fan Theorem:

Let σ be a fan and let R be a subset of $\sigma \times \mathbb{N}$ such that for all α in σ there exists m such that $\alpha R m$.
There exists n such that for all α in σ one may find $m \leq n$ with the property $\alpha R m$.

(The statements $^*27.8$ in [8] and FAN^* in [11] are slightly stronger.) From a classical point of view, the strict and the generalized fan theorem are equivalent reformulations of König's lemma. The usual formulation of König's lemma ("Every infinite finitely-branching tree has an infinite branch") is not valid intuitionistically.
The Second Axiom of Continuous Choice enables one to derive the Extended Fan Theorem from the Strict Fan Theorem. In fact a more simple axiom, that we want to mention now, suffices for this goal.
Let σ be a spread and let ζ belong to $\mathcal{N}$. We define: *ζ codes a (continuous) function from σ to* $\mathbb{N}$ if and only if for all α in σ there exists m such that $\zeta(\overline{\alpha}m) \neq 0$.

Suppose that σ is a spread and that ζ codes a continuous function from σ to $\mathbb{N}$ and that α belongs to σ. We let $\zeta(\alpha)$ be the natural number p such that, for some m, $\zeta(\overline{\alpha}m) = p+1$ and for each $i < m$, $\zeta(\overline{\alpha}i) = 0$.

First Axiom of Continuous Choice:

Let σ be a spread and let R be a subset of $\sigma \times \mathbb{N}$.
If for all α in σ there exists m such that $\alpha R m$, then there exists some ζ coding a function from σ to $\mathbb{N}$ such that, for all α in σ, $\alpha R(\zeta(\alpha))$.

This axiom is called *Brouwer's principle for numbers* in [8], $CONT_0$ in [11] and **CP** in [5].
The First Axiom of Continuous Choice is a statement slightly stronger than Brouwer's Continuity Principle.
The axiom also enables one to derive the Generalized Fan Theorem from the Strict Fan Theorem. We want to emphasize, however, that Brouwer's argument for the bar theorem, and its corollary, the fan theorem, *if* one accepts it, probably must be said to prove also the Generalized Fan Theorem, quite apart from the considerations that made Brouwer defend his continuity principles.

1.9.2 We will make use of the following easy consequence of the strict Fan Theorem.

Let σ be a fan and let ζ code a function from σ to $\mathbb{N}$.
The *range of* γ, that is, the set of all numbers $\zeta(\alpha)$, where α belongs to σ is a finite and therefore a decidable set of $\mathbb{N}$.

1.9.3 Let a be a natural number and let $m = \text{length}(a)$ be the length of the finite sequence coded by a. We consider this finite sequence as a function from the set $\{0, 1, \ldots, m-1\}$ to $\mathbb{N}$, and, for each n, if $n < m$, we define $a(n)$ to be the value of this function at n.
For all natural numbers m, p we let $S(p, m)$ be the set of all numbers a such that length(a)$= m$ and for each $i < m$, $a(i) < p$.

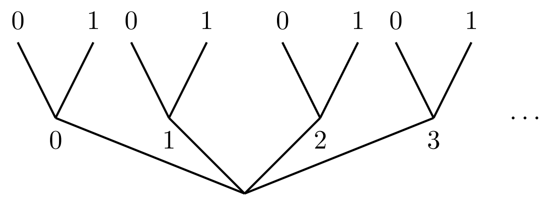
Lemma: For all m, p, for each subset A of $\mathbb{N} \times \mathbb{N}$, if for all a in $S(p, m)$ there exists $i < m$ such that $(i, a(i))$ belongs to A, then there exists $i < m$ such that, for all $q < p$, (i, q) belongs to A.

Proof. The proof uses induction to m.
The case $m = 1$ is obvious.
Suppose that m is a natural number and that the case m has been established. Let $p$be a natural number and let A be a subset of $\mathbb{N} \times \mathbb{N}$ such that for all a in $S(p, m+1)$ there exists $i < m+1$ such that $(i, a(i))$ belongs to A.
Let a belong to $S(p, m)$. Observe that for each $q < p$, either for some $i < m$, $(i, a(i))$ belongs to A, or (m, q) belongs to A. Therefore, either, for some $i < m$, $(i, a(i))$

belongs to A, or, for all $q < p$, (m, q) belongs to A. Let B be the set of all pairs (i, j) of natural numbers such that either (i, j) belongs to A, or, for all $q < p$, (m, q) belongs to A. Observe that for all a in $S(p, m)$ there exists i such that $(i, a(i))$ belongs to B. Applying the induction hypothesis we find $i < m$ such that for all $q < p$, (i, q) belongs to B, that is, either for all $q < p$, (i, q) belongs to A, or for all $q < p$, (m, q) belongs to A. ◇

1.10 We consider two-move-games of the following kind:

> Player I chooses a natural number, player II chooses 0 or 1, and the game is over.



A strategy for player I in a game like this consists of a single number, player I's first and only move. A strategy for player II, on the other hand, is a function from $\mathbb{N}$ to $\{0, 1\}$, assigning to each natural number p the answer player II will give if player I opens the game with p.
Observe that such a strategy is an infinitary object and that player I, if he is allowed to ask for finitely many values of the strategy player II intends to follow, cannot come to know this strategy completely.
The set of strategies for player II is the set of all functions from $\mathbb{N}$ to $\{0, 1\}$. This set is a finitary spread, called: the *binary fan*, or: *(intuitionistic) Cantor space* $\mathcal{C}$.
We now prove that every subset of $\mathbb{N} \times \{0, 1\}$ is determinate in the sense of Subsection 1.4:

> Let A be a subset of $\mathbb{N} \times \{0, 1\}$ such that for all α in $\mathcal{C}$ there exists n such that $(n, \alpha(n))$ belongs to A.
> Using the generalized Fan Theorem from 1.9.1, we find m such that for all α in $\mathcal{C}$ there exists $n \leq m$ such that $(n, \alpha(n))$ belongs to A. Therefore, for all a in $S(2, m+1)$ there exists $n < m + 1$ such that $(n, a(n))$ belongs to A.
> Using the combinatorial lemma from 1.9.3 we find n such that both $(n, 0)$ and $(n, 1)$ belong to A, and this number obviously is a winning strategy for player I.

In the special case that A is a decidable subset of $\mathbb{N} \times \{0,1\}$, the conclusion may be obtained without using the Fan Theorem, as follows.
Let A be a decidable subset of $\mathbb{N}\times\{0,1\}$. We use the First Axiom of Countable Choice and define α such that for every n, $\alpha(n) := 1$ if $(n,0)$ belongs to A and $\alpha(n) := 0$, if $(n,0)$ does not belong to A. We determine n such that $(n,\alpha(n))$ belongs to A and conclude that $\alpha(n) = 1$ and that both $(n,0)$ and $(n,1)$ belong to A.
The next case to consider is that A is not a decidable, but an *enumerable* subset of $\mathbb{N} \times \{0,1\}$, that is, there exists a function β in $\mathcal{N}$ such that, for each n, i, (n,i) belongs to A if and only if there exists p such that $\beta(\langle n,i,p\rangle) = 0$. The statement that every enumerable subset of $\mathbb{N} \times \{0,1\}$ is determinate in the above sense is an *equivalent* of the strict Fan Theorem, see [15], that is, in a weak formal system **BIM** for basic intuitionistic analysis introduced in [15] the strict Fan Theorem follows from this statement and, this statement follows from the strict Fan Theorem. The stronger statements we are to prove in this paper, Lemma 3.6, Corollary 3.8, Corollary 3.9, Lemma 3.11, Lemma 4.3 and Theorem 4.5 also are equivalents of the strict Fan Theorem.
The statement that every subset of $\mathbb{N} \times \{0,1\}$, without restriction, is determinate is equivalent to the generalized Fan Theorem. In the formal context of **BIM**, both the statement: "Every subset of $\mathbb{N} \times \{0,1\}$ is determinate" and the generalized Fan Theorem would have to be formulated as axiom schemes.

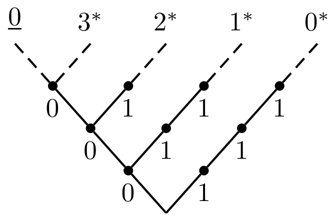
The result that every subset of $\mathbb{N} \times \{0,1\}$ is determinate in the sense of 1.4 occurs already in section 4 of [13]. Following a suggestion by J.R. Moschovakis in her review of [13], see **MR** 85g:03089, we gave here a slightly different proof.
The example given in Subsection 1.8 improves the one given in section 1 of [13].

1.11 We describe the contents of the remaining sections.
In Section 2 we consider infinite games of the simplest possible kind, games which result in a nondecreasing member of $\mathcal{C}$. We show, without using the fan theorem, that such games are predeterminate in the sense of Subsection 1.7.
In Section 3 we introduce II-finitary spreads, that is, spreads in which player II has only finitely many possibilities for each one of his moves. Using the fan theorem, we show that in such spreads, closed sets are predeterminate in the sense of Subsection 1.7.
In Section 4 we obtain the much stronger result that every subset of a II-finitary spread is predeterminate in the sense of Subsection 1.7. Our main tool is Lemma 4.2. The most important statement of this lemma might be paraphrased as follows. Let us call a subset A of a spread σ *secure* for player I if there is an antistrategy for player I securing A for player I, that is, the antistrategy associates to every strategy of player II a play according to that strategy that belongs to A. It turns out that in II-finitary spreads σ, if a countable union $A = \bigcup_{n\in\mathbb{N}} A_n$ is secure for player I, then player I is able to reach a position a in σ from which one of the sets $A_0, A_1, A_2, \ldots$ is secure for him.
In Section 5, we give two applications of our main result.

We will not repeat the classical story of the notion of determinacy. Great parts of it are splendidly told in [9] and [7].
A slightly different version of the main result of section 4 occurs already in chapter 16 of [12].

2 Playing in the monotone binary fan

An instructive example of a finitary spread is the *monotone binary fan*, that is, the set of all α such that, for each n, $\alpha(n) \leq \alpha(n+1) \leq 1$. We call this set σ_{2mon}.



We introduce an infinite sequence of elements of σ_{2mon}, $\underline{0}, 0^*, 1^*, 2^*, \ldots$, as follows. For each n, $\underline{0}(n) = 0$, and for each j, for each n, $j^*(n) = 1$ if and only if $n \geq j$.
Intuitionistically, σ_{2mon} does not coincide with its countable subset $\{\underline{0}, 0^*, 1^*, 2^*, \ldots\}$: an element α of σ_{2mon} in general is an unfinished object that is growing step-by-step: $\alpha(0)$, $\alpha(1)$, $\alpha(2), \ldots$ and one may be unable, at any finite stage of the development of α, to decide whether α will be equal to the zero-function $\underline{0}$ or not. More formally, Brouwer's Continuity Principle (see Subsection 1.8) implies that the range of any function from σ_{2mon} to $\mathbb{N}$ is a finite subset of $\mathbb{N}$, in the following way:

> Let f be a function from σ_{2mon} to $\mathbb{N}$. Using Brouwer's Continuity Principle, we calculate m in $\mathbb{N}$ such that for every α in σ_{2mon}, if α passes through $\underline{0}m$, then $f(\alpha) = f(\underline{0})$. Observe that for every α in σ_{2mon}, $f(\alpha)$ is one of the numbers $f(\underline{0}), f(0^*), f(1^*), \ldots, f(((m-1)^*)$.

Of course, if every function from σ_{2mon} to $\mathbb{N}$ has finite range, there can not be an effective enumeration of σ_{2mon}.

A game in σ_{2mon} is a war of nerves: the player who is the first to choose 1, for all practical purposes finishes the game; if neither player has the courage to do so, the game is endless.
We now prove that every subset of σ_{2mon} is predeterminate in the sense of section 1.7.
First, we associate to every γ in σ_{2mon} a strategy S_γ for player II in σ_{2mon} such that, for all n, $S_\gamma(\overline{\underline{0}}(2n+1)) = \gamma(n)$. Observe that, for every γ, α in σ_{2mon}, α II-obeys S_γ

if and only if, for all n, $\gamma(n) \leq \alpha(2n+1)$.
Suppose now that A is a subset of σ_{2mon} and that player I has an antistrategy for A in σ_{2mon}, that is, there exists a function ζ from σ_{2mon} to σ_{2mon} such that, for all γ in σ_{2mon}, for all n, $\gamma(n) \leq (\zeta|\gamma)(2n+1)$ and $\zeta|\gamma$ belongs to A.
We have to develop a strategy β for player I. We will ensure that for every α in σ_{2mon}, if α I-obeys β, then there exists γ in σ_{2mon}, such that $\alpha = \zeta|\gamma$ and thus α belongs to A. We first explain how to find $\beta(\langle\ \rangle)$, that is, we advise player I about his first move.

Consider the finite set consisting of all numbers $(\zeta|\gamma)(0)$ where γ belongs to σ_{2mon} and distinguish two cases:
Case (i): There exists γ in σ_{2mon} such that $(\zeta|\gamma)(0) = 1$. We define: $\beta(\langle\ \rangle) = 1$.
Case (ii): For all γ in σ_{2mon}, $(\zeta|\gamma)(0) = 0$. We define: $\beta(\langle\ \rangle) = 0$.
In case (i), player I makes a winning first move.The outcome of the game will be the sequence 0^* and there exists γ in σ_{2mon} such that $\zeta|\gamma = 0^*$.
In case (ii), we have to wait for the answer of player II. If player II answers 1, player I wins, as the outcome of the game will be the sequence 1^* and $1^* = \zeta|0^*$. If player II answers 0, we arrive in $\langle 0, 0\rangle = \overline{\underline{0}}2$. Observe that for all γ in σ_{2mon}, if γ passes through $\langle 0\rangle$, then $\zeta|\gamma$ passes through $\langle 0, 0\rangle$. We have to determine $\beta(\overline{\underline{0}}(2))$.
More generally, we assume that n is a natural number and that β has been defined at the positions $\langle\ \ \rangle, \overline{\underline{0}}(2), \ldots, \overline{\underline{0}}(2n)$ and that for all $j \leq n$, $\beta(\overline{\underline{0}}(2j)) = 0$. We also assume that for all γ in σ_{2mon}, if γ passes through $\overline{\underline{0}}(n+1)$, then $\zeta|\gamma$ passes through $\overline{\underline{0}}(2n+2)$. We have to determine $\beta(\overline{\underline{0}}(2n+2))$.
Consider the finite set consisting of all numbers $(\zeta|\gamma)(2n+2)$, where γ is an element of σ_{2mon} passing through $\overline{\underline{0}}(n+1)$ and distinguish two cases:
Case (i): There exists γ in σ_{2mon} passing through $\overline{\underline{0}}(n+1)$ such that $(\zeta|\gamma)(2n+2) = 1$. We define: $\beta(\overline{\underline{0}}(2n+2)) = 1$.
Case (ii): For all γ in σ_{2mon} passing through $\overline{\underline{0}}(n+1)$, $(\zeta|\gamma)(2n+2) = 0$. We define: $\beta(\overline{\underline{0}}(2n+2)) = 0$.
In case (i), player I makes a winning move: the outcome of the game will be $(2n+2)^*$ and there exists γ in σ_{2mon} passing through $\overline{\underline{0}}(n+1)$ such that $\zeta|\gamma = (2n+2)^*$.
In case (ii), we have to wait for the answer of player II. If player II answers 1, player I wins, as $\zeta|(n+1)^* = (2n+3)^*$ and the game results in $(2n+3)^*$. If player II answers 0, we arrive in $\overline{\underline{0}}(2n+4)$. Observe that for all γ in σ_{2mon}, if γ passes through $\overline{\underline{0}}(n+2)$, then $\zeta|\gamma$ passes through $\overline{\underline{0}}(2n+4)$. We have to determine $\beta(\overline{\underline{0}}(2n+4))$ and repeat our procedure.
This completes the description of the strategy β.
The set of all α in σ_{2mon} that I-obey the strategy β, that is, such that, for all n,$\alpha(2n) = \beta(\overline{\alpha}(2n))$, is a spread, that we want to call τ.
We now define δ coding a (continuous) function from τ to σ_{2mon} such that, for all α in τ, α coincides with $\zeta|(\delta|\alpha)$, in the following way:
. v

(i) For each α in τ, for each n, if $\overline{\alpha}(2n) = \overline{\underline{0}}(2n)$, then $\overline{(\delta|\alpha)}(n) = \overline{\underline{0}}(n)$.
(ii) For each α in τ, for each n, if $2n-2$ is the first j such that $\alpha(j) = 1$, then $\delta|\alpha = \gamma$, where γ is an element of σ_{2mon} passing through $\overline{\underline{0}}(n-1)$ such that $\zeta|\gamma = \alpha$. Observe that, if $2n-2$ is the first j such that $\alpha(j) = 1$, then player

I made the decisive move and there exists γ in σ_{2mon} passing through $\overline{\underline{0}}(n-1)$ such that $\zeta|\gamma = \alpha$.

(iii) For each α in τ, for each n, if $2n-1$ is the first j such that $\alpha(j) = 1$, then $\delta|\alpha = (n-1)^*$. Observe that, if $2n-1$ is the first j such that $\alpha(j) = 1$, then player II made the decisive move, and α coincides with $(2n-1)^*$ and with $\zeta|(n-1)^*$.

It is easy to see that δ fulfils our promises.
We conclude: for every α in σ_{2mon}, if α I-obeys β, then α belongs to A, that is, β is a winning strategy for player I in the game for A in σ_{2mon}.
Although the result of this section follows from Theorem 4.4, it is valuable in itself, as it does not depend on the fan theorem: the result is, for that reason, more elementary than Theorem 4.4.

3 The safe-move-lemma and the determinacy of closed sets and open sets in II-finitary spreads

3.1 Let σ be a spread and let a be a natural number admitted by σ, that is, such that $\sigma(a) = 0$. We define the spread-law $\sigma \downarrow a$ by: for all b in $\mathbb{N}$, $(\sigma \downarrow a)(b) = \sigma(a * b)$.

3.2 Let σ be a spread, and let γ be a strategy for player II in σ. Let a be a natural number such that $\sigma(a) = 0$ and length(a) is even, and let δ be a strategy for player II in $\sigma \downarrow a$. We define: γ *extends* δ, or: δ *extends to* γ, if and only if, for each b in $\mathbb{N}$, if $(\sigma \downarrow a)(b) = 0$ and length(b) is odd, then $\delta(b) = \gamma(a * b)$.

3.3 Recall that we defined, in Subsection 1.2, for each γ, for each a, γ *passes through* a, if and only if $\overline{\gamma}($ length$(a)) = a$, that is, the infinite sequence γ has the finite sequence coded by a as an initial part.

3.4 Let σ be a spread and let ζ be an antistrategy for player I in σ. Let a be a natural number such that $\sigma(a) = 0$ and length(a) is even. We define: a is ζ-*safe* if and only if every strategy δ for player II in the spread $\sigma \downarrow a$ extends to a strategy γ for player II in the spread σ such that $\zeta|\gamma$ passes through a.

3.5 Let σ be a spread. We define: σ is II-*finitary* if and only if, for each a, if σ admits a and length(a) is odd, then there exists n such that, for every m, if σ admits $a * \langle m \rangle$, then $m < n$.
Observe that, if σ is a II-finitary spread, then player II has only finitely many possibilities for each one of his moves. Therefore, for each strategy γ for player II in σ, for each a in $\mathbb{N}$, if $\sigma(a) \neq 0$ or length(a) is even, then $\gamma(a) = 0$, and if $\sigma(a) = 0$ and length(a) is odd, then there are finitely many possible values for $\gamma(a)$. This shows that, if σ is a II-finitary spread, then $\mathrm{Strat}_{\mathrm{II}}(\sigma)$ is a fan.

3.6 Lemma: (*the safe-move-lemma*).

Let σ be a II-finitary spread and let ζ be an antistrategy for player I in σ. Then:

(i) The set of all natural numbers a such that $\sigma(a) = 0$ and length(a) is even and a is ζ-safe is a decidable subset of $\mathbb{N}$.

(ii) For every natural number a, if $\sigma(a) = 0$, length(a) is even and a is ζ-safe, then there exists n such that $\sigma(a * \langle n \rangle) = 0$ and, for all m, if $\sigma(a * \langle n, m \rangle) = 0$, then $a * \langle n, m \rangle$ is ζ-safe.

Proof: Let σ, ζ fulfil the conditions of the lemma.
(i) Let a be a natural number such that $\sigma(a) = 0$ and length(a) is even. Using the strict Fan Theorem, we calculate a natural number N such that for all strategies γ, δ for player II in σ, if $\overline{\gamma}N = \overline{\delta}N$, then, for each $i <$ length(a), $(\zeta|\gamma)(i) = (\zeta|\delta)(i)$.
Consider the set B consisting of all natural numbers $\overline{\gamma}(N)$, where γ is a strategy for player II in σ, and observe that B is a finite set of natural numbers.
We assume that we coded the finite sequences of natural numbers in such a way that, for all a, b, $b \leq a * b$.
Let δ be a strategy for player II in the spread $\sigma \downarrow a$.
Considering the number $\overline{\delta}(N)$ and the set B we may decide whether δ extends to a strategy γ for player II in σ such that $\zeta|\gamma$ passes through a, or not. If δ does so indeed, we say that δ *fits* a. Observe that, for all strategies δ, ε for player II in the spread $\sigma \downarrow a$, if δ fits a and $\overline{\delta}N = \overline{\varepsilon}N$, then ε fits a.
Also observe that the set of all natural numbers $\overline{\delta}(N)$, where δ is a strategy for player II in the spread $\sigma \downarrow a$, is a finite set of natural numbers.
Therefore, we may decide if every strategy δ for player II in the spread $\sigma \downarrow a$ fits a, or not. If so, a is ζ-safe, if not, a is not ζ-safe.
(ii) Let a be a natural number such that $\sigma(a) = 0$, length(a) is even and a is ζ-safe. Using an easy corollary of the strict Fan Theorem (see 1.9.2) we calculate a natural number N such that for each strategy γ for player II in σ, $(\zeta|\gamma)(\text{length}(a)) < N$.
Observe that for each n, m, if $n \geq N$, then $a * \langle n, m \rangle$ is not ζ-safe.
We have to prove that there exists $n < N$ such that σ admits $a * \langle n \rangle$ and, for all m, if σ admits $a * \langle n, m \rangle$, then $a * \langle n, m \rangle$ is ζ-safe. Because of (i) we may reason by contradiction.
Let us assume that, for every n, if $n < N$ and σ admits $a*\langle n \rangle$, then there exists m such that $a * \langle n, m \rangle$ is not ζ-safe. Let $n_0, n_1, \ldots, n_{k-1}$ be an enumeration of the natural numbers n such that $n < N$ and σ admits $a * \langle n \rangle$. Determine $m_0, m_1, \ldots, m_{k-1}$ in $\mathbb{N}$ such that, for all $i < k$, $\sigma(a * \langle n_i, m_i \rangle) = 0$ and $a * \langle n_i, m_i \rangle$ is not ζ-safe.
Determine, for each $i < k$, a strategy δ_i for player II in $\sigma \downarrow (a * \langle n_i, m_i \rangle)$ such that δ_i does not fit $a * \langle n_i, m_i \rangle$.
Let γ be a strategy for player II in $\sigma \downarrow a$ be such that, for each $i < k$, γ extends δ_i and $\gamma(a * \langle n_i \rangle) = m_i$.
As a is ζ-safe, we may determine a strategy γ' for player II in σ, extending the strategy γ, and such that $\zeta|(\gamma')$ passes through a.
But then there exists $i < k$ such that $\zeta|(\gamma')$ passes through $a * \langle n_i, m_i \rangle$ and this contradicts the fact that γ' extends δ_i and δ_i does not fit $a * \langle n_i, m_i \rangle$.
We thus see that there exists $n < N$ such that σ admits $a * \langle n \rangle$ and, for all m, if σ

admits $a * \langle n, m\rangle$, then $a * \langle n, m\rangle$ is ζ-safe. $\diamond$

3.7 Let σ be a spread and let A a subset of σ.
We define: A is an *open* subset of σ if and only if there exists a decidable subset C of $\mathbb{N}$ such that for every α, α belongs to A if and only if, for some n, $\overline{\alpha}n$ belongs to C.
We define: A is a *closed* subset of σ if and only if there exists an open subset B of σ such that $A = B^{\neg}$, that is, for each α, α belongs to A if and only if α does not belong to B.
Observe that A is a closed subset of σ if and only if there exists a decidable subset C of $\mathbb{N}$ such that for all α, α belongs to A if and only if, for each n, $\overline{\alpha}n$ belongs to C.
If A itself is a spread, then A is a closed subset of σ, but not every closed subset of σ is a spread, see [12]. The reason is that, given a decidable subset C of $\mathbb{N}$, it is not always possible to decide if there exists α such that for every n, $\overline{\alpha}n$ belongs to C.

3.8 Corollary: *In II-finitary spreads, closed sets are predeterminate.*

Proof: Let σ be a II-finitary spread and let A be a closed subset of σ. Let C be a decidable subset of $\mathbb{N}$ such that for all α, α belongs to A if and only if, for all n, $\overline{\alpha}n$ belongs to C. Let ζ be an antistrategy for player I in σ such that for every strategy γ for player II in σ, $\zeta|\gamma$ belongs to A.
We will prove that there exists a strategy for player I in σ that wins the set A for him.
We apply the safe-move-lemma 3.6 and determine a strategy γ for player I in σ such that for every a, if σ admits a and length(a) is even and a is ζ-safe, then σ admits $a * \langle\gamma(a)\rangle$, and, for each m, if σ admits $a * \langle\gamma(a), m\rangle$, then $a * \langle\gamma(a), m\rangle$ is ζ-safe.
As the empty sequence $\langle\ \rangle$ is ζ-safe, every α that is played by player I according to γ will have the property that, for each n, $\overline{\alpha}(2n)$ is ζ-safe. Observe that, for each a, if length(a) is even and a is ζ-safe, then every initial part of a belongs to C. It follows that every α that I-obeys γ belongs to A. $\diamond$

3.9 Corollary: *In II-finitary spreads, open sets are predeterminate.*

Proof: Let σ be a II-finitary spread and let A be an open subset of σ.
Let C be a decidable subset of $\mathbb{N}$ such that, for all α in σ, α belongs to A if and only if, for some n, $\overline{\alpha}n$ belongs to C. Let ζ be an antistrategy for player I in σ such that for every strategy γ for player II in σ, $\zeta|\gamma$ belongs to A.
We will prove that there exists a strategy for player I in σ that wins the set A for him.
Using the strict Fan Theorem, we first calculate $N \in \mathbb{N}$ such that for every strategy γ for player II in σ there exists $n \leq N$ such that $\overline{(\zeta|\gamma)}n$ belongs to C. Let B be the set of all α in σ such that, for some $n \leq N$, $\overline{\alpha}n$ belongs to C. Observe that B is a closed subset of σ and that the antistrategy ζ secures the set B for player I. Using Corollary 3.8 we find a strategy γ for player I in σ that wins the set B for him. As B is a subset of A, γ also wins the set A for player I. $\diamond$

3.10 We want to prove a statement slightly stronger than corollary 3.9. This requires some new notations. As we agreed in Subsection 1.3, for each natural number a, length(a) is the length of the finite sequence coded by a. We consider such a finite sequence a as a function whose domain is the set $\{0, 1, \ldots, \text{length}(a) - 1\}$.
For each a, n such that $n < \text{length}(a)$, we let $a(n)$ be the value of the function a at n. For each a, n such that $n \leq \text{length}(a)$, we let $\overline{a}(n)$ be the code number of the finite sequence $(a(0), a(1), \ldots, a(n-1))$.
For each a, for each γ, we define: *a is played by player I according to the strategy γ*, or: *a I-obeys γ*, if and only if, for each n, if $2n < \text{length}(a)$, then $a(2n) = \gamma(\overline{a}(2n))$.
Similarly, for each a, for each γ, we define: *a is played by player II according to the strategy γ*, or: *a II-obeys γ*, if and only if, for each n, if $2n+1 < \text{length}(a)$, then $a(2n+1) = \gamma(\overline{a}(2n+1))$.
For each a, for each c, we define: *a is played by player I according to the partial strategy c*, or: *a I-obeys c*, if and only if, for each n, if $2n < \text{length}(a)$, then $\overline{a}(2n) < \text{length}(c)$ and $a(2n) = c(\overline{a}(2n))$.
Similarly, for each a, for each c, we define: *a is played by player II according to the partial strategy c*, or: *a II-obeys c*, if and only if, for each n, if $2n+1 < \text{length}(a)$, then $\overline{a}(2n+1) < \text{length}(c)$ and $a(2n+1) = c(\overline{a}(2n+1))$.

3.11 Corollary: *In II-finitary spreads, open sets are determinate.*

Proof: Let σ be a II-finitary spread and let A be an open subset of σ. Let C be a decidable subset of $\mathbb{N}$ such that, for every α in σ, α belongs to A if and only if, for some n, $\overline{\alpha}n$ belongs to C. Suppose that every strategy for player II in σ II-governs at least one element of A. Therefore, for every strategy γ for player II in σ there exists a in C such that a II-obeys γ. Therefore, for every strategy γ for player II in σ, there exist n, a such that a belongs to C and a II-obeys $\overline{\gamma}n$. Applying the strict Fan Theorem, we find N in $\mathbb{N}$ such that for every strategy γ for player II in σ, there exist n, a such that $n \leq N$ and a belongs to C and a II-obeys $\overline{\gamma}n$.
We now define an antistrategy ζ for player I in σ, as follows. Let γ be a strategy for player II in σ. Let b be the least a such that a II-obeys $\overline{\gamma}N$ and a belongs to C. Let $\zeta|\gamma$ be the sequence β passing through b such that β II-obeys γ and for each n, if $2n \geq \text{length}(b)$, then $\beta(2n)$ is the least p such that σ admits $\overline{\beta}(2n) * \langle p \rangle$.
It will be clear that, for each strategy γ for player II in σ, the sequence $\zeta|\gamma$ II-obeys γ and belongs to A. Applying Corollary 3.9 we conclude that there is a strategy for player I in σ that wins the set A for him. $\diamond$

4 The safe-conjecture-lemma and the intuitionistic determinacy theorem

4.1 We have seen, in Section 3, that player I, when playing in a II-finitary spread, is able to transform any given antistrategy ζ into a strategy γ which, in a way, keeps

close to the antistrategy ζ.
In this section, we will strengthen this result considerably: we show that, in any II-finitary spread σ, player I may form, given any antistrategy ζ, a strategy γ such that player I, while playing according to γ, and building, together with player II, an infinite sequence α, is able to conjecture a strategy δ which player II may be assumed to follow, and to which the resulting play α is the answer of player I according to ζ. It is not difficult to see that this result solves the determinacy problem for II-finitary spreads.

4.2 Let σ be a spread and let ζ be an antistrategy for player I in σ:
We want to refine the notion of a "ζ-safe position", introduced in Subsection 3.4. Let a be a natural number admitted by σ such that length(a) is even, and let c be a natural number. We define: *a is ζ-safe with conjecture c* if and only if each strategy for player II in the spread $\sigma \downarrow a$ extends to a strategy γ for player II in the spread σ *passing through* c such that $\zeta|\gamma$ passes through a.

4.3 Lemma: (*the safe-conjecture-lemma*).

Let σ be a II-finitary spread and let ζ be an anti-strategy for player I in σ.

(i) For each c, the set of all natural numbers a such that length(a) is even and $\sigma(a) = 0$ and a is ζ-safe with conjecture c is a decidable subset of $\mathbb{N}$.

(ii) For all natural numbers a, c, if $\sigma(a) = 0$, length(a) is even and a is ζ-safe with conjecture c, then there exists n such that $\sigma(a * \langle n \rangle) = 0$ and, for all m, if $\sigma(a * \langle n, m \rangle) = 0$, then $a * \langle n, m \rangle$ is ζ-safe with conjecture c.

(iii) For all natural numbers a, c, if $\sigma(a) = 0$, length(a) is even and a is ζ-safe with conjecture c, then, for every strategy δ for player II in the spread $\sigma \downarrow a$ there exists d, n such that length(d) is even and d II-obeys δ and $a * d$ is ζ-safe with conjecture $c * \langle n \rangle$.

Proof: Let σ, ζ fulfil the conditions of the lemma.
(i), (ii): We omit the proofs, as they are similar to the proofs of the corresponding statements in Lemma 3.7.
(iii) Let a, c be natural numbers such that $\sigma(a) = 0$ and length(a) is even and a is ζ-safe with conjecture c. Let δ be a strategy for player II in the spread $\sigma \downarrow a$. We determine a natural number n and a strategy γ for player II in the spread σ such that δ extends to γ and γ passes through $c * \langle n \rangle$ and $\zeta|\gamma$ passes through a. Using the continuity of the function coded by ζ we then find m such that for every strategy β for player II in the spread $\sigma \downarrow a$, if β passes through $\overline{\delta}m$, then there exists a strategy γ for player II in the spread σ such that γ extends β and γ passes through $c * \langle n \rangle$ and $\zeta|\gamma$ passes through a.
We define $k := 2m + \text{length}(a)$.
We let B be the set of all numbers $\overline{(\zeta|\gamma)}k$, where γ is a strategy for player II in the spread σ extending the strategy δ such that $\zeta|\gamma$ passes through a. As the set of all such strategies γ is a fan, B is a finite and thus a decidable subset of $\mathbb{N}$, (see Subsection 1.9.2).
Remark that for each b in B there exists d such that $b = a * d$ and length(d) is even

and d II-obeys δ.
Observe also that player II, if he arrives at some b belonging to B, will not be guided any more by $\overline{\delta}m$.
We claim that some member of B must be ζ-safe with conjecture $c * \langle n \rangle$. Because of (i) we may argue by contradiction.
Assume that no member of B is ζ-safe with conjecture $c * \langle n \rangle$. We then choose for each $b \in B$ a strategy δ_b for player II in the spread $\sigma \downarrow b$ such that δ_b does not extend to a strategy γ for player II in σ with the property that γ passes through $c * \langle n \rangle$ and $\zeta|\gamma$ passes through b. We then form a strategy β for player II in $\sigma \downarrow a$ passing through $\overline{\delta}m$ such that, for each b in B, β extends δ_b. We let γ be a strategy for player II in σ extending β and passing through $c * \langle n \rangle$ such that $\zeta|\gamma$ passes through a.
Consider $b := \overline{(\zeta|\gamma)}(k)$ and remark: b belongs to B and γ extends δ_b and γ passes through $c * \langle n \rangle$ and $\zeta|\gamma$ passes through b. Contradiction.
We conclude that some element of B must be ζ-safe with conjecture $c * \langle n \rangle$. Let b be such an element of B. Determine d such that $b = a * d$. Observe that length(d) is even and d II-obeys δ and that we have obtained the desired conclusion. $\diamond$

4.4 For each α, for each n, we let α^n be the element β of $\mathcal{N}$ such that, for all m, $\beta(m) = \alpha(\langle n, m \rangle$. In the proof of our main theorem, we use the following axiom:

Second Axiom of Countable Choice:

> For each subset R of $\mathbb{N} \times \mathcal{N}$, if for each n there exists α such that $nR\alpha$, then there exists α such that, for each n, $nR\alpha^n$.

This axiom occurs as *2.1 in [8] and as $\mathbf{AC_{01}}$ in [5] and as *AC-NF* in [11]. It is a consequence of the Second Axiom of Continuous Choice, that we mentioned in section 1.5.
Unlike the Second Axiom of Continuous Choice, the Second Axiom of Countable Choice is, from a classical point of view, a sensible assumption.

4.5 Theorem: *Let σ be a II-finitary spread. Every subset of σ is predeterminate.*

Proof: Let σ be a II-finitary spread. Let A be a subset of σ and let ζ be an antistrategy for player I in σ securing the set A for player I. We prove that there exist a strategy for player I in σ with the property that, for every α in σ, if α I-obeys γ, then there exists a strategy δ for player II in σ such that α coincides with $\zeta|\delta$. Obviously, the straegy γ then wins the set A for player I.
According to lemma 4.2 and corollary 3.11 we may determine, for each a, c such that $\sigma(a) = 0$, length(a) is even and a is ζ-safe with conjecture c, a natural number n and a strategy γ for player I in $\sigma \downarrow a$ with the property that for every α in σ I-obeying γ there exists p such that $a * \overline{\alpha}(2p)$ is ζ-safe with conjecture $c * \langle n \rangle$.
Let B be the set of all numbers $\langle a, c \rangle$ in $\mathbb{N}$ such that $\sigma(a) = 0, \text{length}(a)$ is even and a is ζ-safe with conjecture c. According to lemma 4.3, B is a decidable subset of $\mathbb{N}$.

Using the Second Axiom of Countable Choice we determine a function f from B to $\mathbb{N}$ and a function g from B to $\mathcal{N}$ with the property that, for each $\langle a, c\rangle$ in B, $g(\langle a, c\rangle)$ is a strategy for player I in $\sigma \downarrow a$ such that for every α in $\sigma \downarrow a$ I-obeying $g(\langle a, c\rangle)$ there exists p such that $a * \overline{\alpha}(2p)$ is ζ-safe with conjecture $c * \langle f(\langle a, c\rangle)\rangle$.

We now describe informally the strategy γ that player I should obey in σ.

> Observe that $\langle\ \rangle$ is ζ-safe with conjecture $\langle\ \rangle$. Define $\delta(0) = f(\langle\langle\ \rangle, \langle\ \rangle\rangle)$. Follow the strategy $g(\langle\langle\ \rangle, \langle\ \rangle\rangle)$, until, in cooperation with player II a position $\overline{\alpha}(2n_0)$ is reached that is ζ-safe with conjecture $\langle\delta(0)\rangle$, and such that $n_0 > 0$.
> Define $\delta(1) := f(\langle\overline{\alpha}(2n_0), \langle\delta(0)\rangle\rangle)$. Follow the strategy $g(\langle\overline{\alpha}(2n_0),\ \langle\delta(0)\rangle\rangle)$ until, in cooperation with player II, a position $\overline{\alpha}(2n_1)$ is reached that is ζ-safe with conjecture $\langle\delta(0), \delta(1)\rangle$, and such that $n_1 > n_0$.
> And so on.

Lemma 4.2(ii) ensures that it is indeed possible for player I to ensure that $n_1 > n_0$ and $n_2 > n_1$, and so on.

Suppose that α belongs to σ and is played by player I according to this strategy and that δ is the sequence of conjectures formed by player I during the play. Observe that, for all n, there exists a strategy β for player II in σ passing through $\overline{\delta}n$ such that $\zeta|\beta$ passes through $\overline{\alpha}(2n)$. It follows that δ is a strategy for player II in σ with the property : $\zeta|\delta = \alpha$. $\diamond$

4.6 In [1], and at many other places in the recent literature one finds a discussion of the following question that is related to the problem of the determinacy of infinite games:

> Let $\mathcal{C}$ be Cantor space.
> Is it true, for all subsets A of $\mathcal{C} \times \mathcal{C}$, that there exists a (continuous) function f from $\mathcal{C}$ to $\mathcal{C}$ such that *either*: for every α in $\mathcal{C}$, $(\alpha, f(\alpha))$ does not belong to A, *or*: for every α in $\mathcal{C}$, $(f(\alpha), \alpha)$ belongs to A.

What happens to this statement if we subject it to the treatment advocated in Section 1 of this paper?

We should consider the following question:

> Suppose that A is a subset of $\mathcal{C} \times \mathcal{C}$ and that G is an effective functional assigning to any (continuous) function f from $\mathcal{C}$ to $\mathcal{C}$ a member $G(f)$ of $\mathcal{C}$ such that $(G(f), f(G(f)))$ belongs to A.
> Is it possible to find a function f from $\mathcal{C}$ to $\mathcal{C}$ such that, for every α in $\mathcal{C}$, $(f(\alpha), \alpha)$ belongs to A?

The answer to this question is YES, and we may argue for it in two different ways:
Firstly, we may derive the result from Theorem 4.4. In order to see this we define for every α, infinite sequences α_I and α_{II} such that, for each n, $\alpha_I(n) = \alpha(2n)$ and $\alpha_{II}(n) = \alpha(2n+1)$. Let A and G be as above. Let B be the subset of $\mathcal{C}$ consisting of all infinite sequences α such that the pair (α_I, α_{II}) belongs to A, and consider the

game for B in $\mathcal{C}$.
Observe that every strategy γ for player II in $\mathcal{C}$ determines a (continuous) function f from $\mathcal{C}$ to $\mathcal{C}$ such that for every α in $\mathcal{C}$, $f(\alpha)$ II-obeys γ and $(f(\alpha))_I = \alpha$. Therefore, the effective functional G gives rise to an antistrategy G' for player I in $\mathcal{C}$ such that for every strategy γ for player II in $\mathcal{C}$, $G'(\gamma)$ II-obeys γ and belongs to B.
This antistrategy G', being effective, may be supposed to be continuous, and Theorem 4.4 applies: there exists a strategy γ for player I in $\mathcal{C}$ such that every α in $\mathcal{C}$ I-obeying γ belongs to B. It is easy to obtain from γ a continuous function f from $\mathcal{C}$ to $\mathcal{C}$ such that, for every α in $\mathcal{C}$, $(f(\alpha), \alpha)$ belongs to A.
Secondly, we may consider for every $\alpha \in \mathcal{C}$, the function α^* from $\mathcal{C}$ to $\mathcal{C}$ with the constant value α. We then define a function f from $\mathcal{C}$ to $\mathcal{C}$ such that, for every α in $\mathcal{C}$, $f(\alpha) = G(\alpha^*)$.
Observe that, for every α in $\mathcal{C}$, $(f(\alpha), \alpha)$ belongs to A. The function f, being effective, may be supposed to be continuous.
The second solution is, somewhat disappointingly, very simple.

5 Two applications

5.1 For each a, b in $\mathbb{N}$ we define: *the finite sequence (coded by) a is an initial part of the finite sequence (coded by) b* , notation: $a \sqsubseteq b$, if and only if there exists $n \leq \text{length}(b)$ such that $a = \overline{b}n$.
For each a, b in $\mathbb{N}$ we define: a, b *form a branching*, notation: $a \perp b$, if and only if a is not an initial part of b and b is not an initial part of a.

5.2 Let A be a subset of $\mathcal{N}$.
We consider the following game, sometimes called $G^*(A)$, that has been devised by Morton Davis in [3].

Player I chooses $\langle \ell_0, r_0 \rangle$ in $\mathbb{N} \times \mathbb{N}$ such that $\ell_0 \perp r_0$.

$\searrow$

Player II chooses i_0 in $\{0, 1\}$.
We define $a_0 := \ell_0$ if $i_0 = 0$, and
$a_0 := r_0$ if $i_0 = 1$.

$\swarrow$

Player I chooses $\langle \ell_1, r_1 \rangle$ in $\mathbb{N} \times \mathbb{N}$ such that $a_0 \sqsubseteq \ell_1, a_0 \sqsubseteq r_1$ and $\ell_1 \perp r_1$.

$\searrow$

Player II chooses i_1 in $\{0, 1\}$.
We define : $a_1 := \ell_1$ if $i_1 = 0$, and
$a_1 := r_1$ if $i_1 = 1$.

$\swarrow$

Player I chooses $\langle \ell_2, r_2 \rangle$ in $\mathbb{N} \times \mathbb{N}$ such that $a_1 \sqsubseteq \ell_2, a_1 \sqsubseteq r_2$ and $\ell_2 \perp r_2$.

$\searrow$

$$\begin{array}{ll} \text{Player II chooses } i_2 \text{ in } \{0,1\} & \\ \quad \text{We define} & a_2 := \ell_2 \text{ if } i_2 = 0, \text{and} \\ & a_2 := r_2 \text{ if } i_2 = 1. \end{array}$$

And so on.

In the end, we determine α in $\mathcal{N}$ such that, for all n, α passes through a_n.
Player I wins if and only if α belongs to A.

It will be clear that $G^*(A)$ may be described as a game in a II-finitary spread σ.
Observe that, if there exists a (continuous) strongly injective function from Cantor space $\mathcal{C}$ into A, then player I has a winning strategy in the game $G^*(A)$. Conversely, if player I has such a strategy, then this strategy is easily transformed into an embedding of $\mathcal{C}$ into A.
Observe also that, if the set A is enumerable, that is, if there exists a function f from $\mathbb{N}$ to $\mathcal{N}$ enumerating A, then player II may ensure that the result of a play in $G^*(A)$ will not belong to A by making his n-th move such that the result will differ from $f(n)$.
There is a classical argument showing, that for any strategy δ for player II in $G^*(A)$, the set of all α that cannot be the result of a play in $G^*(A)$ according to δ, is at most countable. Therefore, we may take the statement that player II has a strategy in $G^*(A)$ by which he may ensure that the result will not belong to A as one possible way of translating the classical statement: "A is countable" into intuitionistic language.

From Theorem 4.4 we know that player I may transform an antistrategy securing the game for him in $G^*(A)$ into a strategy winning the game for him.
This result then is another intuitionistic approximation to the continuum hypothesis, complementary to the one in section 2 of [5].

5.3 Let A be a subset of the set $\mathbb{Q}$ of rational numbers.
We consider the following game, that we call $\mathcal{H}(A)$, the letter $\mathcal{H}$ honouring F. Hausdorff.

Player I chooses q_0 in $\mathbb{Q}$.

$\searrow$

$$\begin{array}{ll} \text{Player II chooses } i_0 \text{ in } \{0,1\}. & \\ \quad \text{We define} & H_0 := \{q \in \mathbb{Q} \mid q < q_0\} \text{ if } i_0 = 0, \text{and} \\ & H_0 := \{q \in \mathbb{Q} \mid q > q_0\} \text{ if } i_0 = 1. \end{array}$$

$\swarrow$

Player I chooses q_1 in H_0.

$\searrow$

$$\begin{array}{ll} \text{Player II chooses } i_1 \text{ in } \{0,1\}. & \\ \quad \text{We define} & H_1 := H_0 \cap \{q \in \mathbb{Q} \mid q < q_1\} \text{ if } i_1 = 0, \text{and} \\ & H_1 := H_0 \cap \{q \in \mathbb{Q} \mid q > q_1\} \text{ if } i_1 = 1. \end{array}$$

$\swarrow$

Player I chooses q_2 in H_1.

$\searrow$

Player II chooses i_2 in $\{0, 1\}$.

We define $H_2 := H_1 \cap \{q \in \mathbb{Q} \mid q < q_2\}$ if $i_2 = 0$, and
$H_2 := H_1 \cap \{q \in \mathbb{Q} \mid q > q_2\}$ if $i_2 = 1$.

and so on.

In the end, player I wins if and only if, for each n, q_n belongs to A.

It is clear that the game $\mathcal{H}(A)$ may be described as a game in a II-finitary spread σ. Therefore, player I may transform any antistrategy of his securing this game for him into a strategy winning this game for him.
As to the meaning of the game $\mathcal{H}(A)$, observe that player I has a winning strategy in $\mathcal{H}(A)$ if and only if there exists an order-preserving embedding of $(\mathbb{Q}, <)$ into $(A, <)$. From a classical point of view, the game $\mathcal{H}(A)$ is determinate, as its winning set is a closed subset of σ. The class of all subsets A of $\mathbb{Q}$ such that player II has a winning strategy in the game $\mathcal{H}(A)$ therefore coincides classically with the class of all *scattered* subsets of $\mathbb{Q}$, that is, the class of all subsets A of $\mathbb{Q}$ such that it is impossible to embed $(\mathbb{Q}, <)$ into $(A, <)$.
Intuitionistically, the notion "player II has a method to ensure in the game $\mathcal{H}(A)$, for any resulting sequence $q_0, q_1, q_2, \ldots$, that not for every n, q_n belongs to A", is just one of the many possible intuitionistic approximations to the classical notion "A is a scattered subset of $\mathbb{Q}$".
Scattered sets are discussed in [10]. A famous characterization of scattered sets is given in [6].

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