Chapter IX. The cohomology of line bundles.

In this chapter we study the cohomology of line bundles on abelian varieties. The main results are the Riemann-Roch Theorem (9.11) and the Vanishing Theorem for non-degenerate line bundles (9.14). The key step in deriving these results is the computation of the cohomology of the Poincaré bundle on $X \times X^t$.

(9.1) Theorem. *Let X be a g-dimensional abelian variety over a field k. Let $\mathscr{P}$ be the Poincaré bundle on $X \times X^t$ and write $p_2\colon X \times X^t \to X^t$ for the second projection. Then the sheaves $R^n p_{2,*}\mathscr{P}$ and the cohomology of $\mathscr{P}$ are given by*

$$R^n p_{2,*}\mathscr{P} = \begin{cases} 0 & \text{if } n \neq g; \\ i_0(k) & \text{if } n = g, \end{cases}$$

and

$$H^n(X \times X^t, \mathscr{P}) = \begin{cases} 0 & \text{if } n \neq g; \\ k & \text{if } n = g. \end{cases}$$

Here $i_0(k)$ denotes the skyscraper sheaf at $0 \in X^t$ with stalk k.

Proof. As the proof is a somewhat long we divide it into steps, (9.2)–(9.9).

(9.2) We look at the higher direct image sheaves $R^n p_{2,*}\mathscr{P}$ on X^t. If $y \in X^t \setminus \{0\}$ then the restriction of $\mathscr{P}$ to $X \times \{y\}$ is a non-trivial line bundle on X with class in Pic^0. As was proven in (7.19) such sheaves have zero cohomology. Applying (i) of (7.20), it follows that $R^n p_{2,*}\mathscr{P}$ has support only at $0 \in X^t$, for all n. As the closed point 0 is a zero-dimensional subscheme of X^t we have $H^i(X^t, R^n p_{2,*}\mathscr{P}) = 0$ for all $i \geqslant 1$. (Use HAG, III, Thm. 2.7 and Lemma 2.10.)

Applying the Leray spectral sequence

$$E_2^{p,q} = H^p(X^t, R^q p_{2,*}\mathscr{P}) \;\Rightarrow\; H^{p+q}(X \times X^t, \mathscr{P})$$

we find that

$$H^n(X \times X^t, \mathscr{P}) \cong H^0(X^t, R^n p_{2,*}\mathscr{P})\,. \tag{1}$$

As p_2 is projective of relative dimension g we have (HAG, III, Cor. 11.2) $R^n p_{2,*}\mathscr{P} = 0$ for all $n > g$. Hence also $H^n(X \times X^t, \mathscr{P}) = 0$ for $n > g$.

Next we apply Serre duality to the Poincaré bundle. We have $\mathscr{P}^{-1} \cong (-1,1)^*\mathscr{P} \cong (1,-1)^*\mathscr{P}$; see Exercise (7.4). In particular the cohomology of $\mathscr{P}^{-1}$ is the same as that of $\mathscr{P}$. As $X \times X^t$ is an abelian variety its dualizing sheaf is trivial, and Serre duality (in the form given by HAG, III, Cor. 7.7) gives

$$H^n(X \times X^t, \mathscr{P}) \cong H^{2g-n}(X \times X^t, \mathscr{P}^{-1})^\vee \cong H^{2g-n}(X \times X^t, \mathscr{P})^\vee\,.$$

Hence $H^n(X \times X^t, \mathscr{P}) = 0$ for all $n < g$ too. By (1) and the fact that the $R^n p_{2,*}\mathscr{P}$ are supported at 0 we also have $R^n p_{2,*}\mathscr{P} = 0$ for $n \neq g$.

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(9.3) Let $A := O_{X^t,0}$ be the local ring of X^t at 0. Let $\mathfrak{m} \subset A$ be the maximal ideal. It follows from (1) that $(R^g p_{2,*}\mathscr{P})_0$ is an A-module of finite length. By (??) the natural map

$$(R^g p_{2,*}\mathscr{P}) \otimes_{O_{X^t}} \kappa(0) \longrightarrow H^g\big(X \times \{0\}, \mathscr{P}_{|X\times\{0\}}\big) = H^g(X, O_X) \cong k$$

is an isomorphism. Using the Nakayama lemma, we find that $(R^g p_{2,*}\mathscr{P})_0 \cong A/\mathfrak{a}$ for some $\mathfrak{m}$-primary ideal $\mathfrak{a} \subset A$.

To complete the proof of (9.1) it remains to be shown that $\mathfrak{a} = \mathfrak{m}$. This is the hardest part of the proof. We need to exploit the fact that $\mathscr{P}$ is the universal line bundle on $X \times X^t$; thus far we have not made full use of this. In particular, we know that $\mathscr{P}$ is trivial over $X \times \{0\} = X \times \mathrm{Spec}(A/\mathfrak{m})$, but not over any "thickening" $X \times \mathrm{Spec}(A/J)$ for $J \subsetneq \mathfrak{m}$. The problem is how to translate this into information about $R^g p_{2,*}\mathscr{P}$.

We shall give two proofs of the fact that $\mathfrak{a} = \mathfrak{m}$. The first proof uses Grothendieck duality and is fairly short; the second relies on essentially the same ideas but is more elementary.

(9.4) Let Z be a scheme. Write $\mathrm{Mod}(Z)$ for the category of O_Z-modules and $D(Z)$ for its derived category. If F is a sheaf of O_Z-modules and $n \in \mathbb{Z}$, write $F[n]$ for the object of $D(Z)$ represented by the complex whose only non-zero term is the sheaf F, sitting in degree $-n$. The functor $\mathrm{Mod}(Z) \to D(Z)$ given by $F \mapsto F[0]$ realizes $\mathrm{Mod}(Z)$ as a full subcategory of $D(Z)$. If $C^\bullet$ is a complex of O_Z-modules with the property that $\mathscr{H}^i(C^\bullet) = 0$ for all $i \neq n$, for some integer n, then $C^\bullet \cong \mathscr{H}^n(C^\bullet)[-n]$ in $D(Z)$.

To simplify notation we write $Y := X \times X^t$. A corollary of Grothendieck duality, applied to the morphism $p_2\colon Y \to X^t$, is that for quasi-coherent O_{X^t}-modules G we have an isomorphism

$$\mathrm{Hom}_{O_Y}(\mathscr{P}, p_2^*G) \xrightarrow{\sim} \mathrm{Hom}_{O_{X^t}}(R^g p_{2,*}\mathscr{P}, G)\,, \tag{2}$$

which is functorial in G. Before we start exploiting this, let us indicate how this is obtained from the general machinery of Grothendieck duality.

We already know that p_2 is a smooth morphism of relative dimension g and that $\Omega^g_{Y/X^t} \cong O_Y$. Consider a bounded complex $F^\bullet$ of quasi-coherent O_Y-modules and a bounded complex $G^\bullet$ of quasi-coherent O_{X^t}-modules. Then a consequence of Grothendieck duality is that we have a canonical isomorphism

$$\mathrm{Hom}_{D(Y)}\big(F^\bullet, p_2^*G^\bullet[g]\big) \xrightarrow{\sim} \mathrm{Hom}_{D(X^t)}(Rp_{2,*}F^\bullet, G^\bullet)\,. \tag{3}$$

See Hartshorne [1], Chap. III, § 11, and use that the functor $p_2^!$ is given by $G^\bullet \mapsto p_2^*G^\bullet[g]$; see op. cit., Chap. III, § 2. We apply this with $F^\bullet = \mathscr{P}$. We already know that $Rp_{2,*}F^\bullet$ only has cohomology in degree g. As explained above, this implies that $Rp_{2,*}F^\bullet$ is isomorphic, in $D(X^t)$, to $R^g p_{2,*}\mathscr{P}[-g]$. If we now apply (3) with $G^\bullet = G[-g]$ for some quasi-coherent O_{X^t}-module G then we obtain (2).

(9.5) Let $J \subset A$ be a proper ideal (with $A = O_{X^t,0}$, as above). Write $Z(J) := \mathrm{Spec}(A/J)$, and let $i(J)\colon Z(J) \hookrightarrow X^t$ be the natural closed immersion. Write $Y(J) := X \times \mathrm{Spec}(A/J) = p_2^{-1}\big(Z(J)\big) \subset Y$. In particular, $Y(\mathfrak{m}) = X \times \{0\}$. If we write $\mathscr{P}(J)$ for the restriction of $\mathscr{P}$ to $Y(J)$ then $\mathrm{Hom}_{O_Y}(\mathscr{P}, O_{Y(J)}) = \mathrm{Hom}_{O_{Y(J)}}\big(\mathscr{P}(J), O_{Y(J)}\big)$.

Suppose J is an $\mathfrak{m}$-primary ideal. Via the natural map $O_{X^t} \to i(J)_*O_{Z(J)}$, the structure sheaf $O_{Z(J)}$ is then just the skyscraper sheaf $i_0(A/J)$ at $0 \in X^t$ with stalk A/J. Further $Y(J)$ is the closed subscheme of $Y = X \times X^t$ with underlying topological space $|X \times \{0\}|$ and structure sheaf $p_2^*O_{Z(J)} = O_X \otimes_k A/J$.

As explained in (9.3), we have $R^g p_{2,*}\mathscr{P} = i_0(A/\mathfrak{a})$ for some $\mathfrak{m}$-primary ideal $\mathfrak{a} \subset A$. Now consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{O_Y}(\mathscr{P}, O_{Y(\mathfrak{a})}) & \xrightarrow{\sim} & \mathrm{Hom}_{O_{X^t}}\big(i_0(A/\mathfrak{a}), i_0(A/\mathfrak{a})\big) & \cong & A/\mathfrak{a} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{O_Y}(\mathscr{P}, O_{Y(\mathfrak{m})}) & \xrightarrow{\sim} & \mathrm{Hom}_{O_{X^t}}\big(i_0(A/\mathfrak{a}), i_0(k)\big) & \cong & k \end{array},$$

where the horizontal arrows are given by (2) and the vertical arrows are induced by the quotient map $A/\mathfrak{a} \to A/\mathfrak{m} = k$.

We have a natural isomorphism h: $\mathscr{P}(\mathfrak{m}) = \mathscr{P}_{|X\times\{0\}} \xrightarrow{\sim} O_X$. This gives us an element

$$h \in \mathrm{Hom}_{O_Y}\big(\mathscr{P}, O_{Y(\mathfrak{m})}\big) = \mathrm{Hom}_{O_{X\times\{0\}}}\big(\mathscr{P}_{|X\times\{0\}}, O_{X\times\{0\}}\big)\,.$$

From the diagram we see that h can be lifted to an element

$$\tilde{h} \in \mathrm{Hom}_{O_Y}(\mathscr{P}, O_{Y(\mathfrak{a})}) = \mathrm{Hom}_{O_{Y(\mathfrak{a})}}\big(\mathscr{P}(\mathfrak{a}), O_{Y(\mathfrak{a})}\big)\,.$$

Then $\tilde{h}$: $\mathscr{P}(\mathfrak{a}) \to O_{Y(\mathfrak{a})}$ is a homomorphism of line bundles on $Y(\mathfrak{a})$ which is an isomorphism modulo $\mathfrak{m}$. It follows that $\tilde{h}$ is an isomorphism, too. This shows that the pull-back of $\mathscr{P}$ under $\mathrm{id}_X \times i(\mathfrak{a})$: $X \times \mathrm{Spec}(A/\mathfrak{a}) \hookrightarrow X \times X^t$ is trivial. By the universal property of $\mathscr{P}$ this implies that $i(\mathfrak{a})$: $\mathrm{Spec}(A/\mathfrak{a}) \hookrightarrow X^t$ factors through the closed point $\{0\} = \mathrm{Spec}(k) \subset X^t$. Hence $\mathfrak{a} = \mathfrak{m}$ and $R^g p_{2,*}\mathscr{P} = i_0(k)$. This finishes our (first) proof of Theorem (9.1). $\square$

(9.6) Our second proof that $(R^g p_{2,*}\mathscr{P})_0 \cong k$ is not very different from the first, but it replaces Grothendieck duality by more explicit arguments.

We use the notation introduced in (9.5). In particular, if $J \subset A$ is a proper ideal, the second projection p_2: $X \times X^t \to X^t$ restricts to a morphism p_2: $Y(J) \to Z(J)$. We shall systematically confuse $R^g p_{2,*}\mathscr{P}(J)$ with its A/J-module of global sections. Note that $Z\big((0)\big) = \mathrm{Spec}(A) \to X^t$ is a flat morphism; hence $R^g p_{2,*}\mathscr{P}\big((0)\big)$ is the same as the restriction of $R^g p_{2,*}\mathscr{P}$ to $Z\big((0)\big)$. (See HAG, Chap. III, Prop. 9.3.) Thus, our goal is to prove that $R^g p_{2,*}\mathscr{P}\big((0)\big) \cong k$.

We apply the results about cohomology and base-change explained in (??). This gives us a length g complex (with $g = \dim(X) = \dim(A)$) of finitely generated free A-modules

$$K^\bullet: \qquad 0 \longrightarrow K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} \cdots \longrightarrow K^{g-1} \xrightarrow{d^{g-1}} K^g \longrightarrow 0 \tag{4}$$

with the property that for all ideals $J \subset A$ and all n we have

$$R^n p_{2,*}\mathscr{P}(J) \cong \mathscr{H}^n(K^\bullet \otimes_A A/J)\,,$$

functorially in A/J. (In fact, a similar statement holds with A/J replaced by an arbitrary A-algebra, but we will not need this.) In particular $\mathscr{H}^n(K^\bullet) \cong R^n p_{2,*}\mathscr{P}$. But as shown in (9.2), $R^n p_{2,*}\mathscr{P} = 0$ for $n < g$; so $K^\bullet$ is a resolution of $\mathscr{H}^g := \mathscr{H}^g(K^\bullet)$. We want to show that $\mathscr{H}^g \cong A/\mathfrak{m} = k$.

Consider the "dual" complex

$$L^\bullet: \qquad 0 \longrightarrow L^0 \xrightarrow{\delta^0} L^1 \xrightarrow{\delta^1} \cdots \longrightarrow L^{g-1} \xrightarrow{\delta^{g-1}} L^g \longrightarrow 0$$

where $L^j := \mathrm{Hom}_A(K^{g-j}, A)$, and where δ^j is the map induced by d^{g-1-j}. Set

$$Q := \mathscr{H}^g(L^\bullet) = \mathrm{Coker}(\delta^{g-1}\colon L^{g-1} \to L^g)\,.$$

The next lemma (taken from MAV, p. 127) tells us that $L^\bullet$ is a free resolution of Q. (Note that all $\mathscr{H}^n(L^\bullet)$ are artinian A-modules, as easily follows from the corresponding fact for the complex $K^\bullet$.)

(9.7) Lemma. *Let A be a g-dimensional regular local ring. Let*

$$C^\bullet: \qquad 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^g \longrightarrow 0$$

be a complex of finitely generated free A-modules such that all cohomology groups $\mathscr{H}^j(C^\bullet)$ are artinian A-modules. Then $\mathscr{H}^j(C^\bullet) = 0$ for all $j < g$.

Proof. We use induction on g. For $g = 0$ there is nothing to prove, so we may assume that $g > 0$ and that the lemma holds in smaller dimensions. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, so that $A/(x)$ is regular of dimension $g-1$. Put $\overline{C}^\bullet := C^\bullet/(x)$, so that we have an exact sequence of complexes

$$0 \longrightarrow C^\bullet \xrightarrow{\cdot x} C^\bullet \longrightarrow \overline{C}^\bullet \longrightarrow 0\,.$$

In cohomology this gives the long exact sequence

$$\cdots \longrightarrow \mathscr{H}^i(C^\bullet) \xrightarrow{\cdot x} \mathscr{H}^i(C^\bullet) \longrightarrow \mathscr{H}^i(\overline{C}^\bullet) \longrightarrow \mathscr{H}^{i+1}(C^\bullet) \xrightarrow{\cdot x} \mathscr{H}^{i+1}(C^\bullet) \longrightarrow \cdots \quad .$$

We see from this that the $\mathscr{H}^i(\overline{C}^\bullet)$ are artinian modules, and by induction $\mathscr{H}^i(\overline{C}^\bullet) = 0$ for all $i < g-1$. Hence multiplication by x is injective on $\mathscr{H}^j(C^\bullet)$ for all $j < g$. But $\mathscr{H}^j(C^\bullet)$ is artinian, so it is killed by x^N for $N \gg 0$. This proves the induction step. $\square$

(9.8) From (7.27) we know the cohomology of the complex $K^\bullet \otimes_A k = [0 \to K^0/\mathfrak{m}K^0 \to K^1/\mathfrak{m}K^1 \to \cdots]$. In particular we have $\mathscr{H}^0(K^\bullet \otimes_A k) = H^0(X, O_X) = k$ and $\mathscr{H}^g(K^\bullet \otimes_A k) = H^g(X, O_X) = k$. This gives us that $\mathscr{H}^g/\mathfrak{m}\mathscr{H}^g \cong k$ and $Q/\mathfrak{m}Q \cong k$. By Nakayama's Lemma it follows that the A-modules $\mathscr{H}^g$ and Q are both generated by a single element, so there exist ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A with

$$\mathscr{H}^g \cong A/\mathfrak{a} \qquad \text{and} \qquad Q \cong A/\mathfrak{b}\,.$$

(For $\mathscr{H}^g$ this repeats what was explained in (9.3).)

Let $J \subset A$ be an ideal. Put

$$H^0_J := \mathrm{Ker}(K^0/JK^0 \to K^1/JK^1) = H^0\big(Y(J), \mathscr{P}(J)\big)\,.$$

Applying $\mathrm{Hom}_A(-, A/J)$ to the exact sequence $L^{g-1}/JL^{g-1} \to L^g/JL^g \to Q/JQ \to 0$ gives the exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(Q/JQ, A/J) \longrightarrow K^0/JK^0 \xrightarrow{\bar{d}^0} K^1/JK^1\,,$$

which shows that

$$H^0_J \cong \mathrm{Hom}_A(A/\mathfrak{b} + J, A/J)\,. \tag{5}$$

The isomorphism (5) is functorial in the ideal J, in the sense that for $J_1 \subseteq J_2$ the natural reduction map $H^0_{J_1} \to H^0_{J_2}$ corresponds to the natural map

$$\mathrm{Hom}_A(A/\mathfrak{b} + J_1, A/J_1) \to \mathrm{Hom}_A(A/\mathfrak{b} + J_1, A/J_2) = \mathrm{Hom}_A(A/\mathfrak{b} + J_2, A/J_2)\,.$$

We now use that, by definition of X^t, the closed point $0 \in X^t$ is the maximal closed subscheme over which $\mathscr{P}$ is trivial, in the sense of (2.4). Taking $J_1 = \mathfrak{b}$ and $J_2 = \mathfrak{m}$ in the above we find that the section $1 \in k = H^0(X, O_X) = H^0\big(Y(\mathfrak{m}), \mathscr{P}(\mathfrak{m})\big)$ lifts to a global section of $\mathscr{P}(\mathfrak{b})$. With the same arguments as in (9.5) it follows that $\mathscr{P}(\mathfrak{b}) \cong O_{Y(\mathfrak{b})}$, and by the universal property of $\mathscr{P}$ this is possible only if $\mathfrak{b} = \mathfrak{m}$.

(9.9) We have shown that $L^\bullet$ is a free resolution of the A-module $A/\mathfrak{m} = k$. Another way to obtain such a resolution is to use a Koszul complex. This works as follows. Choose a regular system of parameters $x_1, x_2, \ldots, x_g \in \mathfrak{m}$, i.e., a sequence of elements which generate $\mathfrak{m}$ and which give a k-basis for $\mathfrak{m}/\mathfrak{m}^2$. Consider the complex

$$F^\bullet: \qquad 0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots \longrightarrow F^g \longrightarrow 0$$

where $F^j = \wedge^j_A(A^g)$ and where, writing $\underline{x} = (x_1, \ldots, x_g) \in A^g$, the differential

$$d^j\colon \wedge^j_A(A^g) \longrightarrow \wedge^{j+1}_A(A^g)$$

is given by $v \mapsto \underline{x} \wedge v$. Then $F^\bullet$, the so-called *Koszul complex* associated to the sequence $\underline{x}$, is also a free resolution of k.

By (??) the complexes $L^\bullet$ and $F^\bullet$ are homotopy equivalent. "Dualizing back" we then find that the complex $K^\bullet$ is homotopy equivalent to the dual of the Koszul complex $F^\bullet$. The first terms of $F^\bullet$ are given by

$$0 \longrightarrow A \xrightarrow{d^0} A^g \longrightarrow \cdots \qquad \text{with } d^0\colon a \mapsto (x_1a, x_2a, \ldots, x_ga)\,.$$

The last non-zero terms of the dual complex are therefore given by

$$\cdots \longrightarrow A^g \xrightarrow{(d^0)^*} A \longrightarrow 0 \qquad \text{with } (d^0)^*\colon (a_1, a_2, \ldots, a_g) \mapsto x_1a_1 + x_2a_2 + \cdots + x_ga_g\,.$$

With this we can finally compute:

$$(R^g p_{2,*}\mathscr{P})_0 \cong \mathscr{H}^g(K^\bullet) \cong \mathscr{H}^g\big((F^\bullet)^*\big) = \mathrm{Coker}\big((d^0)^*\big) = A/\mathfrak{m} = k$$

and this finishes the (second) proof of Theorem (9.1). $\square$

(9.10) The following result we want to prove is the Riemann-Roch theorem for abelian varieties.

Let X be a proper scheme of finite type over a field k. If F is a quasi-coherent O_X-module then its *Euler characteristic* is defined to be the integer

$$\chi(L) := \sum_{i\geqslant 0} (-1)^i \cdot \dim_k H^i(X, F)\,.$$

Suppose X is projective and H is a very ample line bundle on X. Then $n \mapsto \chi(F \otimes H^n)$ is a polynomial function of n. More precisely, there is a polynomial with rational coefficients

$\Phi = \Phi_{F,H} \in \mathbb{Q}[t]$, called the *Hilbert polynomial of* F (with respect to H), such that $\Phi(n) = \chi(F \otimes H^n)$ for all $n \in \mathbb{Z}$. Note that there is a natural number n_0 such that $H^i(X, F \otimes H^n) = 0$ for all $i > 0$ and all $n \geqslant n_0$; hence $\Phi(n) = \dim_k H^0(X, F \otimes H^n)$ for all $n \geqslant n_0$.

This "polynomial behaviour" of the Euler characteristic with respect to its entries is a much more general phenomenon. For instance, suppose X is a smooth proper variety over k and $F_1, \ldots, F_r$ are vector bundles on X (or, more generally, coherent O_X-modules). Then the function $(n_1, \ldots, n_r) \mapsto \chi(F_1^{n_1} \otimes \cdots \otimes F_r^{n_r})$ is polynomial in the r-tuple of integers $(n_1, \ldots, n_r)$. This is a consequence of the Hirzebruch-Riemann-Roch theorem. When X is an abelian variety the Riemann-Roch formula takes a particularly simple form and the polynomial dependence of $\chi(F_1^{n_1} \otimes \cdots \otimes F_r^{n_r})$ on the exponents n_i becomes obvious; cf. (9.13).

(9.11) Riemann-Roch Theorem. *Let L be a line bundle on a g-dimensional abelian variety X. Then*

$$\chi(L) = c_1(L)^g/g! \quad \textit{and} \quad \chi(L)^2 = \deg(\varphi_L)\,.$$

Thus, if $L \cong O_X(D)$ for some divisor D then the first equation says that $\chi(L)$ equals $(D^g)/g!$, where (D^g) is g-fold self-intersection number of D. Notice that, by slight abuse of notation, we write $c_1(L)^g$ for $\deg\bigl(c_1(L)^g\bigr) = \int_X c_1(L)^g$.

We shall prove the theorem together with the following corollary.

(9.12) Corollary. *Let $f\colon Y \to X$ be an isogeny. If L is a line bundle on X then $\chi(f^*L) = \deg(f) \cdot \chi(L)$.*

Proof of (9.11) *and* (9.12). First we show that $\chi(L) = c_1(L)^g/g!$. For this we use the Hirzebruch-Riemann-Roch formula, which says that

$$\chi(L) = \int_X \mathrm{ch}(L) \cdot \mathrm{td}(T_X)\,. \tag{6}$$

Here $\mathrm{ch}(L)$, the Chern character of L, is the power series

$$\mathrm{ch}(L) = \exp\bigl(c_1(L)\bigr) = 1 + c_1(L) + c_1(L)^2/2 + \cdots$$

which should be thought of as a formal expression. Similiarly, $\mathrm{td}(T_X)$, the Todd class of the tangent bundle T_X, is a formal power series in the Chern classes of T_X. As T_X is trivial we have $c_i(T_X) = 0$ for all $i \geqslant 1$ and $\mathrm{td}(T_X) = 1$. This reduces (6) to the desired equality $\chi(L) = \int_X c_1(L)^g/g!$. Notice that, in particular,

$$\chi(L^m) = m^g \cdot \chi(L) \tag{7}$$

for all $m \in \mathbb{Z}$.

To prove (9.12) we may assume that $k = \overline{k}$. Let $f\colon Y \to X$ be an isogeny of degree d. Then $c_1(f^*L)^g = f^*\bigl(c_1(L)^g\bigr)$ in the Chow ring of Y. (Alternatively we may use any Weil cohomology, such as ℓ-adic cohomology for some $\ell \neq \mathrm{char}(k)$, or Betti cohomology in case the ground field is $\mathbb{C}$.) But $c_1(L)^g$ is represented by a 0-cycle (a formal sum of points), so all that remains to be shown is that $\int_Y f^*[P] = d$ for every point $P \in X$. This is clear if f is separable, for then $f^{-1}(P)$ consists of d distinct points, each with multiplicity 1. It is also clear if f is purely inseparable, because then $f^{-1}(P)$ consists of one single point, say Q, and $O_{Y,Q}$ is free of rank d over $O_{X,P}$. The general result follows by combining these two cases, using (5.8). This proves Cor. (9.12).

Next we show that $\chi(L)^2 = \deg(\varphi_L)$. We first do this for non-degenerate line bundles L. The idea is to compute $\chi\bigl(\Lambda(L)\bigr)$ in two different ways.

So, assume that L is non-degenerate. As usual we write $\Lambda(L)$ for the associated Mumford bundle on $X \times X$. We have a cartesian diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\mathrm{id}_X \times \varphi_L} & X \times X^t \\ {\scriptstyle p_2}\big\downarrow & & \big\downarrow{\scriptstyle p_2'} \\ X & \xrightarrow[\varphi_L]{} & X^t \end{array} .$$

Further we know that $\Lambda(L) = (\mathrm{id}_X \times \varphi_L)^* \mathscr{P}$, and φ_L is an isogeny with kernel $\{1\} \times K(L)$. By (9.1) and flat base change,

$$R^n p_{2,*}\Lambda(L) = \varphi_L^*\bigl(R^n p'_{2,*}\mathscr{P}\bigr) = \begin{cases} 0 & \text{if } n \neq g; \\ i_* O_{K(L)} & \text{if } n = g, \end{cases}$$

where $i\colon K(L) \hookrightarrow X$ is the inclusion. Using a Leray spectral sequence, as in (9.2), we find

$$h^n\bigl(X \times X, \Lambda(L)\bigr) = \begin{cases} 0 & \text{if } n \neq g; \\ \deg(\varphi_L) & \text{if } n = g. \end{cases} \tag{8}$$

Here, as usual, we write $h^n(-) := \dim H^n(-)$. In particular,

$$\chi\bigl(\Lambda(L)\bigr) = (-1)^g \cdot \deg(\varphi_L)\,. \tag{9}$$

(A quicker proof of (9) is to use (9.12), but we shall need (8) later.)

For the second computation of $\chi\bigl(\Lambda(L)\bigr)$, recall that $\Lambda(L) := m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$. The projection formula therefore gives

$$R^n p_{2,*}\Lambda(L) = R^n p_{2,*}\bigl(m^*L \otimes p_1^*L^{-1}\bigr) \otimes L^{-1}\,.$$

We know that $R^n p_{2,*}\Lambda(L)$ is supported on the finite subscheme $K(L) \subset X$. As L can be trivialized over $K(L)$ we find that

$$R^n p_{2,*}\bigl(m^*L \otimes p_1^*L^{-1}\bigr) \otimes L^{-1} = R^n p_{2,*}\bigl(m^*L \otimes p_1^*L^{-1}\bigr)\,.$$

Once again computing cohomology via a Leray spectral sequence we conclude that

$$H^n\bigl(X \times X, \Lambda(L)\bigr) \cong H^n(X \times X, m^*L \otimes p_1^*L^{-1}) \qquad \text{for all } n. \tag{10}$$

Now remark that $(m \times p_1)\colon X \times X \to X \times X$ is an isomorphism with $(m \times p_1)^*\bigl(p_1^*L \otimes p_2^*L^{-1}\bigr) = m^*L \otimes p_1^*L^{-1}$. By the Künneth formula it follows that

$$H^n(X \times X, m^*L \otimes p_1^*L^{-1}) \cong H^n(X \times X, p_1^*L \otimes p_2^*L^{-1}) \cong \bigoplus_{p+q=n} H^p(X,L) \otimes H^q(X,L^{-1})\,. \tag{11}$$

Combining (10) and (11) we find

$$\chi\bigl(\Lambda(L)\bigr) = \chi\bigl(p_1^*L \otimes p_2^*L^{-1}\bigr) = \chi(L) \cdot \chi(L^{-1}) = (-1)^g \cdot \chi(L)^2\,, \tag{12}$$

where the last equality follows from (7). Comparing the two answers (9) and (12) proves that $\chi(L)^2 = \deg(\varphi_L)$ for non-degenerate L.

Now suppose that L is degenerate. Then φ_L is not finite and, by convention, $\deg(\varphi_L) = 0$. We want to show that $\chi(L) = 0$ too. It is still true that $\Lambda(L) = (\mathrm{id}_X \times \varphi_L)^*\mathscr{P}$. We rewrite this as

$$m^*L \otimes p_2^*L^{-1} = (\mathrm{id}_X \times \varphi_L)^*\bigl(\mathscr{P} \otimes p_1^*L\bigr)\,.$$

The same argument as above gives that $\chi\bigl(m^*L \otimes p_2^*L^{-1}\bigr) = (-1)^g \cdot \chi(L)^2$. (Notice that this part of the above argument works without the assumption that L is non-degenerate.) If $H \subset K(L)$ is a subgroup scheme of order r then $\mathrm{id}_X \times \varphi_L$ factors through the projection $X \times X \to X \times X/H$, and by (9.12) it follows that $\chi\bigl(m^*L \otimes p_2^*L^{-1}\bigr)$ is divisible by r. But $K(L)$ contains subgroup schemes of arbitrarily large order (in fact, $K(L)^0_{\mathrm{red}}$ is an abelian subvariety of X of positive dimension), and we conclude that $\chi(L) = 0$. This finishes the proof of the theorem. $\square$

(9.13) Remark. If F is a coherent sheaf on a g-dimensional abelian variety X then Hirzebruch-Riemann-Roch gives $\chi(F) = \int_X \mathrm{ch}_g(F)$ where ch_g is a certain polynomial in the Chern classes of F. See Fulton [1], Example 3.2.3.

Looking at the proof of (9.11) we see that for non-degenerate bundles we can draw one further conclusion.

(9.14) Vanishing Theorem. *If L is a non-degenerate line bundle then there is a unique integer i (necessarily with $0 \leqslant i \leqslant g$) such that $H^i(X, L) \neq 0$.*

Proof. Combining (8), (10) and (11) we have shown that

$$\sum_{p+q=n} h^p(L) \cdot h^q(L^{-1}) = \begin{cases} 0 & \text{if } n \neq g; \\ \deg(\varphi_L) & \text{if } n = g. \end{cases}$$

As all $h^i(L)$ and $h^j(L^{-1})$ are in $\mathbb{Z}_{\geqslant 0}$ this is possible only if there are *unique* p and q (with $p + q = g$) such that $h^p(L) \neq 0$ and $h^q(L^{-1}) \neq 0$. $\square$

(9.15) Definition. If L is a non-degenerate line bundle then the unique index $i = i(L)$ such that $h^i(L) \neq 0$ is called the *index* of L.

Note that $i(L) = 0$ just means that L is effective.

(9.16) Example. Let D be a divisor of degree d on an elliptic curve E. Riemann-Roch for curves gives $\chi\bigl(O_E(D)\bigr) = d$. It follows that

$$\begin{aligned} D \text{ is degenerate} \quad &\Longleftrightarrow \quad d = 0 \\ D \text{ is non-degenerate of index } 0 \quad &\Longleftrightarrow \quad d > 0 \\ D \text{ is non-degenerate of index } 1 \quad &\Longleftrightarrow \quad d < 0 \end{aligned}$$

(9.17) Corollary. *Let X be an abelian variety over an algebraically closed field k. Let L be a non-degenerate line bundle on X with index $i = i(L)$. Then $H^i(X, L)$ is the unique irreducible weight 1 representation of the theta group $\mathscr{G}(L)$.*

Proof. That $H^i(X,L)$ is a $\mathscr{G}(L)$-representation of weight 1 is clear, for instance, using Čech cohomology. The corollary thus follows from (8.32) by a dimension count. Indeed, we have

$$\big(\dim H^i(X,L)\big)^2 = \chi(L)^2 = \deg(\varphi_L) = \mathrm{rank}\big(K(L)\big)\,,$$

as required. $\square$

If L is a non-degenerate line bundle with index i then $\chi(L) = (-1)^i \cdot h^i(L)$. In particular, $\chi(L)$ has sign equal to $(-1)^{i(L)}$. We shall later see how the index can be read off from the Hilbert polynomial of L. As a preparation for this we collect some properties of the index as a function on the set of non-degenerate bundles.

(9.18) Proposition. (i) *Let L be a non-degenerate line bundle on a g-dimensional abelian variety X. Then $i(L^{-1}) = g - i(L)$.*

(ii) *"The index is (locally) constant in algebraic families": If T is a locally noetherian k-scheme and M is a line bundle on $X \times T$ such that all $M_t := M_{|X\times\{t\}}$ are non-degenerate then the function $t \mapsto i(M_t)$ is locally constant on T. In particular, if L is as in* (i) *and L' is a line bundle on X with $[L'] \in \mathrm{Pic}^0_{X/k}$ then $i(L) = i(L\otimes L')$.*

(iii) *Let $f\colon X \to Y$ be an isogeny of degree prime to* $\mathrm{char}(k)$*. If M is a non-degenerate line bundle on Y then f^*M is non-degenerate too and $i(f^*M) = i(M)$.*

(iv) *If L is non-degenerate and $m \neq 0$ then L^m is non-degenerate too. Furthermore, if $m > 0$ and* $\mathrm{char}(k) \nmid m$ *then $i(L^m) = i(L)$.*

(v) *If L_1, L_2 and $L_1 \otimes L_2$ are all non-degenerate then $i(L_1\otimes L_2) \leqslant i(L_1) + i(L_2)$.*

(vi) *If H is ample and L and $L\otimes H$ are both non-degenerate then $i(L\otimes H) \leqslant i(L)$.*

Notes: In (9.23) below we shall sharpen (iv), showing that $i(L^m) = i(L)$ for all $m > 0$. In (9.26) we shall show that (iii) holds without the assumption that $\deg(f)$ is prime to $\mathrm{char}(k)$. If in (ii) the scheme T is geometrically connected then it suffices to require that M_t is non-degenerate for *some* $t \in T$ (as $K(M_t)$ does not jump in such families), and the conclusion is that $t \mapsto i(M_t)$ is constant on T. The requirement that T is locally noetherian is in fact superfluous, as we can reduce to the "universal" case $T = \mathrm{Pic}_{X/k}$.

Proof. Statement (i) was already found in the proof of (9.14). Alternatively, it follows from Serre duality.

The first statement of (ii) follows from the fact (HAG, III, Thm. 12.8) that for all j the function $t \mapsto \dim_{k(t)} H^j(X\otimes k(t), M_t)$ is upper semi-continuous. The second statement follows by applying this to the Poincaré bundle over $X \times \mathrm{Pic}_{X/k}$. Alternatively, passing to an algebraic closure of k the bundles $L\otimes L'$ with $[L'] \in \mathrm{Pic}^0_{X/k}$ are precisely the line bundles of the form t_x^*L. In cohomology the translation t_x induces an isomorphism between $H^j(X,L)$ and $H^j(X,t_x^*L)$.

(iii) As shown in (7.6), f^*M is again non-degenerate. We have $f_*(f^*M) = M \otimes_{O_Y} f_*O_X$. We claim that the sheaf O_Y is a direct summand of f_*O_X, hence M is a direct summand of f_*f^*M. Indeed, if $r = \deg(f)$ then f_*O_X is locally free of rank r over O_Y and by assumption r is invertible in O_Y. If trace: $f_*O_X \to O_Y$ is the trace map then $(1/r)\cdot\mathrm{trace}$ is a section of the natural map $O_Y \to f_*O_X$, so $f_*O_X = O_Y \oplus \mathrm{Ker}(\mathrm{trace})$.

Since f is finite, a Leray spectral sequence shows that $H^i(X,f^*M) \cong H^i(Y, f_*f^*M)$ for all i (see also HAG, III, Exercise 4.1), and we conclude that $H^i(Y,M)$ is isomorphic to a direct summand of $H^i(X,f^*M)$. This proves (iii).

(iv) We have $K(L^m) = m^{-1}\big(K(L)\big)$. Hence L^m is non-degenerate for $m \neq 0$. Now assume that $m > 0$ is relatively prime with $\mathrm{char}(k)$. We use the notation and the results of Exercise (7.8).

Consider the line bundle $L^{\boxtimes 4}$ on X^4 given by $L^{\boxtimes 4} = L^{\boxtimes \mathrm{id}_4} = \otimes_{i=1}^4 p_i^* L$. (Here id_4 denotes the identity matrix of size 4×4.) It is readily seen that $L^{\boxtimes 4}$ is again non-degenerate (in fact, $K\bigl(L^{\boxtimes 4}\bigr) = K(L)^4$), and by the Künneth formula we have $i\bigl(L^{\boxtimes 4}\bigr) = 4 \cdot i(L)$.

We write $m > 0$ as a sum of four squares, say $m = a^2 + b^2 + c^2 + d^2$. Consider the matrix

$$A = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

which should be thought of as representing the quaternion $a+bi+cj+dk$. We have $A^t \cdot A = m \cdot \mathrm{id}_4$. Now consider the homomorphism $\alpha = \alpha_A \colon X^4 \to X^4$ associated to A, and apply part (i) of Exercise (7.8). This gives that $\alpha^*\bigl(L^{\boxtimes 4}\bigr)$ and $(L^m)^{\boxtimes 4}$ differ by something in $\mathrm{Pic}^0_{X/k}$; hence by (ii) they have the same index. But by (iii) the index of $\alpha^*\bigl(L^{\boxtimes 4}\bigr)$ equals that of $L^{\boxtimes 4}$. Putting everything together we find that

$$i(L) = 1/4 \cdot i\bigl(L^{\boxtimes 4}\bigr) = 1/4 \cdot i\bigl((L^m)^{\boxtimes 4}\bigr) = i(L^m)\,,$$

as claimed.

(v) Let i_1, i_2 and ι be the indices of L_1, L_2 and $L_1 \otimes L_2$, respectively. Consider the line bundle $N := p_1^* L_1 \otimes p_2^* L_2$ on $X \times X$, and let $\nu \colon X \times X \to X$ be given by $\nu(x,y) = x - y$. The fibre of ν over 0 is the diagonal $X \cong \Delta(X) \subset X \times X$, over which N restricts to the bundle $L_1 \otimes L_2$. By (ii) it follows that all fibres of N have index ι, so that $R^j \nu_* N = 0$ for all $j < \iota$. By a Leray spectral sequence this implies that $H^j(X \times X, N) = 0$ for all $j < \iota$. But the Vanishing Theorem together with the Künneth decomposition show that $H^{i_1}(X, L_1) \otimes_k H^{i_2}(X, L_2) \cong H^{i_1+i_2}(X \times X, N)$.

Finally, (vi) follows from (v), as it follows from (iv) that ample bundles have index 0. $\square$

(9.19) Remark. The fact used in the proof of (iii) that O_Y is a direct summand of $f_* O_X$ is not necessarily true if the degree of f is divisible by $\mathrm{char}(k)$. For instance, suppose X is an abelian variety over a field k of characteristic $p > 0$, such that X is not ordinary, i.e., $f(X) < g$. Then the relative Frobenius map $F_{X/k} \colon X \to X^{(p)}$ is an isogeny of abelian varieties, but it can be shown that $O_{X^{(p)}}$ is in this case not a direct summand of $F_{X/k,*} O_X$. In the literature one finds this as the statement that a non-ordinary abelian variety is not Frobenius split; see Mehta-Srinivas [??].

For the proof of the following proposition we need a somewhat technical, but important lemma.

(9.20) Lemma. *Let Y be a d-dimensional projective scheme over a field. Let $L_1, \cdots, L_r$ be line bundles on Y. For $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$, set $|\underline{a}| := |a_1| + \cdots + |a_r|$ and $L^{\underline{a}} := L_1^{a_1} \otimes \cdots \otimes L_r^{a_r}$. Then there is a constant $C > 0$, only depending on Y and the bundles L_j, such that*

$$h^i(Y, L^{\underline{a}}) \leqslant C \cdot (1 + |\underline{a}|^r)$$

for all i and all $\underline{a} \in \mathbb{Z}^r$.

Proof. If all L_i are trivial then the assertion is clear; this covers the cases $d = 0$ and $r = 0$. Next we reduce to the case when all L_j are very ample. For this, choose a very ample bundle M such that each of the

$$M_j := L_j \otimes M \qquad (1 \leqslant j \leqslant r)$$

is very ample, too. Suppose we know the lemma for the line bundles $M_1, \ldots, M_r$, M_{r+1}. If C is the resulting constant then for all $\underline{a} \in \mathbb{Z}^r$, putting $\sigma(\underline{a}) := a_1 + \cdots + a_r$,

$$\begin{aligned} h^i(Y, L^{\underline{a}}) = h^i(Y, M_1^{a_1} \otimes \cdots \otimes M_r^{a_r} \otimes M_{r+1}^{-\sigma(\underline{a})}) &\leqslant C \cdot \bigl(1 + \{|\underline{a}| + |\sigma(\underline{a})|\}^r\bigr) \\ \leqslant C \cdot \bigl(1 + \{2|\underline{a}|\}^r\bigr) \leqslant (3^{r+1}C) \cdot (1 + |\underline{a}|^r)\,. \end{aligned}$$

From now on we may therefore assume all L_j to be very ample.

We proceed by induction on the integer $d + r$. The case $d + r = 0$ is already dealt with. Assume the lemma is true whenever $d + r \leqslant \nu$. As the lemma is true when $r = 0$, it suffices to do the case when we have $r + 1$ very ample bundles, say $L_1, \ldots, L_r$ and M, on a d-dimensional projective scheme Y, such that $d + r + 1 = \nu + 1$.

Let $Z \subset Y$ be a hyperplane section for the projective embedding given by M. For every $\underline{a} \in \mathbb{Z}^r$ and $b \in \mathbb{Z}$ we have an exact sequence

$$0 \longrightarrow L^{\underline{a}} \otimes M^{b-1} \longrightarrow L^{\underline{a}} \otimes M^b \longrightarrow (L^{\underline{a}} \otimes M^b)_{|Z} \longrightarrow 0\,.$$

In cohomology this gives an exact sequence

$$H^{i-1}(Z, L^{\underline{a}} \otimes M^b) \longrightarrow H^i(Y, L^{\underline{a}} \otimes M^{b-1}) \longrightarrow H^i(Y, L^{\underline{a}} \otimes M^b) \longrightarrow H^i(Z, L^{\underline{a}} \otimes M^b)$$

which gives

$$h^i(Y, L^{\underline{a}} \otimes M^b) \leqslant h^i(Y, L^{\underline{a}} \otimes M^{b-1}) + h^i(Z, L^{\underline{a}} \otimes M^b) \tag{13}$$

and

$$h^i(Y, L^{\underline{a}} \otimes M^{b-1}) \leqslant h^i(Y, L^{\underline{a}} \otimes M^b) + h^{i-1}(Z, L^{\underline{a}} \otimes M^b)\,. \tag{14}$$

By induction hypothesis we have estimates for $h^i(Y, L^{\underline{a}} \otimes M^b)$ when $b = 0$ and for the terms $h^i(Z, L^{\underline{a}} \otimes M^b)$. For $b > 0$ we get the desired estimates by iterated application of (13); for $b < 0$ we do the same using (14). $\square$

(9.21) To obtain further results on the index function, we investigate in more detail what happens in the situation of (vi) in (9.18). We fix a non-degenerate bundle L and an ample bundle H. As remarked above, ample bundles have index 0; in other words: they are effective.

Set $l = c_1(L)$ and $h = c_1(H)$. Consider the homogeneous polynomial of degree g

$$P(s, t) := (sl + th)^g \quad \in \mathbb{Z}[s, t]\,,$$

whose coefficients are intersection numbers. Notice that $P(m, n) = g! \cdot \chi(L^m \otimes H^n)$ for all integral m and n. Further note that P is homogeneous of degree g, so $P(m, n) = m^g P(1, n/m) = g! m^g \Phi_{L,H}(n/m)$ where $\Phi_{L,H}$ is the Hilbert polynomial of L with respect to H. In other words: P is "the Hilbert polynomial made homogeneous of degree g". If we want to indicate which bundles L and H we are working with then we use the notation $P_{L,H}$. For later use let us remark that

$$P_{L^m,H}(s, t) = P_{L,H}(ms, t) = m^g \cdot P_{L,H}(s, t/m) \tag{15}$$

for all integers $m \neq 0$.

(9.22) Proposition. *Suppose that both L and $L \otimes H$ are non-degenerate, and that $i(L) \neq i(L \otimes H)$. Then $P(1, t)$ has a root in the interval $[0, 1] \subset \mathbb{R}$.*

Proof. Let M be a square not divisible by $\mathrm{char}(k)$. By (iv) of (9.18) we have

$$i(L^M) = i(L) \neq i(L \otimes H) = i(L^M \otimes H^M)\,.$$

Assume that $P(1,t)$ does not vanish on $[0,1]$, so that there exists a constant $C > 0$ with $|P(1,t)| > C$ for all $t \in [0,1]$. As degenerate line bundles have zero Euler characteristic this implies that all line bundles $L^M \otimes H^n$ with $0 \leqslant n \leqslant M$ are non-degenerate. Let n be the smallest positive integer such that $i(L^M \otimes H^{n-1}) \neq i(L^M \otimes H^n)$. Set

$$\begin{aligned} i_1 &= i(L) = i(L^M) = \cdots = i(L^M \otimes H^{n-1}) \\ i_2 &= i(L^M \otimes H^n)\,, \end{aligned}$$

and observe that $i_2 < i_1$ by (vi) of (9.18).

Choose an effective divisor $D \in |H|$ and consider the short exact sequence

$$0 \longrightarrow L^M \otimes H^{n-1} \longrightarrow L^M \otimes H^n \longrightarrow (L^M \otimes H^n)_{|D} \longrightarrow 0\,.$$

Looking at the associated long exact cohomology sequence and using that $i_1 > i_2$ we find that

$$H^{i_2}(X, L^M \otimes H^n) \hookrightarrow H^{i_2}(D, L^M \otimes H^n)\,.$$

In particular, $h^{i_2}(D, L^M \otimes H^n) \geqslant M^g \cdot |P(1, n/M)|$, which by our choice of C is at least $M^g \cdot C$. Since this holds with arbitrarily large M, and since D has dimension $g-1$, we obtain a contradiction with (9.20). □

(9.23) Corollary. *If L is non-degenerate then $i(L^m) = i(L)$ for all $m > 0$.*

Proof. Write $L = H_1 \otimes H_2^{-1}$ as the difference of two ample bundles. Choose $M \geqslant 2$ big enough such that both polynomials $P_{L,H_1}(1,t)$ and $P_{L,H_2}(1,t)$ have no zeroes in the interval $[0, 1/M]$, which is possible since $P_{L,H_j}(1,0) = g! \cdot \chi(L) \neq 0$. By (15) it follows that for $m \geqslant M$ both $P_{L^m,H_1}(1,t)$ and $P_{L^m,H_2}(1,t)$ have no zeroes in the interval $[0,1]$. By the proposition this implies

$$i(L^{m+1}) = i(L^{m+1} \otimes H_2) = i(H_1^{m+1} \otimes H_2^{-m}) = i(L^m \otimes H_1) = i(L^m)\,.$$

Hence for large enough m the index of L^m is independent of m. Using properties (i) and (iv) in (9.18) the corollary follows. □

(9.24) Lemma. *Let L be non-degenerate, H ample, and let $P(s,t) := P_{L,H}(s,t)$ be the polynomial defined above. Suppose $P(1,t)$ has a unique root $\tau \in [0,1]$, of multiplicity μ and with $\tau \neq 1$. Then $i(L) \leqslant i(L \otimes H) + \mu$.*

Proof. As $P_{L^m,H^m}(s,t) = m^{2g} \cdot P_{L,H}(s,t)$ we may assume, using (9.23), that H is very ample. Also we may assume that $i(L) \neq i(L \otimes H)$, so that also $i(L^m) \neq i(L^m \otimes H^m)$ for all $m \neq 0$.

Let $m \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}$ with $n < m$. In the rest of the proof we shall only consider integers m which are coprime with all denominators of rational roots of $P(1,t)$. This ensures that $L^m \otimes H^n$ is non-degenerate; indeed, if $L^m \otimes H^n$ is degenerate then $P(1, n/m) = 0$.

With m and n as above, suppose that $i(L^m \otimes H^{n-1}) \neq i(L^m \otimes H^n)$. Note that

$$P_{L^m \otimes H^{n-1},H}(1,t) = m^g \cdot P_{L,H}(1, \frac{n-1+t}{m})\,,$$

so it follows from (9.22) that $P(1,t)$ has a root in the interval $[(n-1)/m, n/m]$. By the assumptions of the lemma we conclude that for given $m > 0$ there is a unique n with $1 < n \leqslant m$ (depending on m) such that

$$i(L^m) = \cdots = i(L^m \otimes H^{n-1}) \ > \ i(L^m \otimes H^n) = \cdots = i(L^m \otimes H^m)\,. \tag{16}$$

Let $X =: Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ be obtained by taking hyperplane sections for the projective embedding given by H. So, $Z_1 \subset X$ is a hyperplane section, Z_2 is a hyperplane section of Z_1, etc. We have exact sequences

$$0 \longrightarrow (L^m \otimes H^{q-1})_{|Z_r} \longrightarrow (L^m \otimes H^q)_{|Z_r} \longrightarrow (L^m \otimes H^q)_{|Z_{r+1}} \longrightarrow 0\,. \tag{17}$$

Fix $m > 0$ and let $n = n(m) < m$ be determined by (16). Set $i_1 := i(L) = i(L^m)$ and $i_2 := i(L \otimes H) = i(L^m \otimes H^m)$. Note that $i_1 > i_2$ and $i(L^m \otimes H^q) \geqslant i_1$ for all $q \leqslant n-1$. Similar to what we did in the proofs of (9.20) and (9.22), we shall use the exact sequences (17) to obtain dimension estimates for cohomology groups. As a first step, take $r = 0$ in (17). Since $i_2 < i_1$ we find that $H^{i_2}(X, L^m \otimes H^n)$ injects into $H^{i_2}(Z_1, L^m \otimes H^n)$ and that $H^j(Z_1, L^m \otimes H^q) = 0$ for all $j < i_1 - 1$ and $q \leqslant (n-1)$. Next we want to take $r = 1$, in which case we have the exact sequence

$$H^{i_2}(Z_1, L^m \otimes H^{n-1}) \longrightarrow H^{i_2}(Z_1, L^m \otimes H^n) \longrightarrow H^{i_2}(Z_2, L^m \otimes H^n)\,.$$

Applying the previous conclusions we see that the first term vanishes if $i_2 < i_1 - 1$. If this holds then $H^{i_2}(Z_1, L^m \otimes H^n)$ injects into $H^{i_2}(Z_2, L^m \otimes H^n)$; further we then find that $H^j(Z_2, L^m \otimes H^q) = 0$ for all $j < i_1 - 2$ and $q \leqslant (n-1)$.

Proceeding by induction we find that if $r < i_1 - i_2$ then

$$H^{i_2}(Z_{r-1}, L^m \otimes H^n) \hookrightarrow H^{i_2}(Z_r, L^m \otimes H^n)$$

and

$$H^j(Z_r, L^m \otimes H^q) = 0 \qquad \text{for all } j < i_1 - r \text{ and } q \leqslant (n-1).$$

(The induction breaks down for $r \geqslant i_1 - i_2$.) The conclusion of this (terminating) induction is that $H^{i_2}(X, L^m \otimes H^n)$ maps injectively to $H^{i_2}(Z_{i_1 - i_2}, L^m \otimes H^n)$. Comparing dimensions and using (9.20) we find that there exists a constant C such that

$$\left| m^g \cdot P(1, n/m) \right| \leqslant C \cdot \left| m^{g-(i_1-i_2)} \right| \tag{18}$$

for all sufficiently large m. Here $n = n(m) < m$ is a function of m.

Next we write $P(1,t) = (t-\tau)^\mu \cdot R(t)$ where $R(t)$ does not have roots in $[0,1]$. Choose a constant $C' > 0$ with $|R(t)| > C'$ for all $t \in [0,1]$. Combined with (18) this gives

$$\left| C' \cdot (n/m - \tau)^\mu \right| \leqslant \left| P\bigl(1, n/m\bigr) \right| \leqslant C \cdot \left| m^{-(i_1-i_2)} \right| \tag{19}$$

for all sufficiently large m.

To finish the argument we distinguish two cases. First assume that $\tau \in \mathbb{Q}$. Let f be its denominator. Recall that we only consider integers m that are coprime with f. For all such m and all $1 \leqslant n \leqslant m$ we have $|n/m - \tau| \geqslant 1/fm$. Using this in (19) and letting m get large we find the desired estimate $i_1 \leqslant i_2 + \mu$. Similarly, if τ is irrational then it suffices to show that there is an infinite sequence of values for m, say $m_1, m_2, \ldots$, and a constant C'' such that

$|n_j/m_j - \tau| \geqslant C''/m_j$ for all j. (Note that the n's are still a function of the m's, determined by the rule that τ lies in the interval $[(n-1)/m, n/m]$.) This is achieved by a theorem of Kronecker which says that the fractional parts of the numbers $m \cdot \tau$, for $m \in \mathbb{N}$, lie dense in the interval $]0,1[$; see Hardy and Wright [1], Chap. 23. $\square$

After all these preparations we are now ready for the main result about the relation between the index and the Hilbert polynomial of L.

(9.25) Theorem. (Kempf-Mumford-Ramanujam) *Let L be a non-degenerate line bundle on an abelian variety X. Let H be an ample line bundle on X and write $\Phi(t) \in \mathbb{Z}[t]$ for the Hilbert polynomial of L with respect to H. (So $\Phi(n) = \chi(L \otimes H^n)$ for all n.) Then all complex roots of Φ are real, and the index $i(L)$ equals the number of positive roots, counted with multiplicities.*

Proof. Writing $P(s,t) = (sl + th)^g$ for the 2-variable polynomial as introduced before (9.22), we have $\Phi(t) = P(1,t)/g!$. For the rest of the proof we may therefore work with $P(1,t)$. Notice that this is a polynomial of degree g.

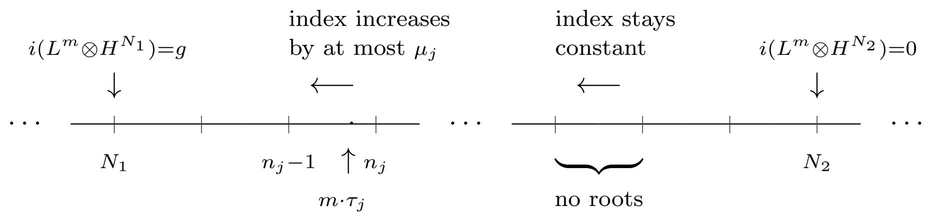
Let $\tau_1, \ldots, \tau_h$ be the real roots of $P(1,t)$, say with multiplicities $\mu_1, \ldots, \mu_h$, respectively. (It will be clear from the arguments below that $h > 0$.) Choose $m \in \mathbb{Z}_{>0}$ and $n_1, \ldots, n_h \in \mathbb{Z}$ such that τ_j lies in the interval $[(n_j - 1)/m, n_j/m]$. We can make these choices such that $P(1,t)$ has no roots of the form n/m, so that all bundles $L^m \otimes H^n$ are non-degenerate.

For $n >> 0$, say $n \geqslant N_2$, the bundle $L^m \otimes H^n$ is ample, so that $i(L^m \otimes H^n) = 0$. Similarly, for $n \leqslant N_1$ the bundle $L^m \otimes H^n$ is anti-ample, in which case $i(L^m \otimes H^n) = g$. (That $h > 0$ is now clear from (9.22).)

Applying Proposition (9.22) and Lemma (9.24) we find that for every $n \in \mathbb{Z}$,

either: $P(1,t)$ has no root in the interval $[(n-1)/m, n/m]$ and $i(L^m \otimes H^{n-1}) = i(L^m \otimes H^n)$,

or: $n = n_j$ (for some j), and $P(1,t)$ has a unique root in $[(n-1)/m, n/m]$, of multiplicity μ_j; in this case $i(L^m \otimes H^{n-1}) \leqslant i(L^m \otimes H^n) + \mu_j$.



Starting at $n = N_2$ and descending in steps of length 1 we find

$$g = i(N_1) - i(N_2) \leqslant \sum_j \mu_j \,.$$

On the other hand, as $P(1,t)$ has degree g we have $\sum_j \mu_j \leqslant g$. The conclusion is that we have equality everywhere: $P(1,t)$ has all its roots real and $i(L^m \otimes H^{n_j - 1}) = i(L^m \otimes H^{n_j}) + \mu_j$ for all j. This also gives that

$$i(L) = i(L^m) = i(L^m \otimes H^0) = \sum_{j; \tau_j > 0} \mu_j \,,$$

and the theorem is proven. $\square$

(9.26) Corollary. *Let $f\colon X \to Y$ be an isogeny. If L is a non-degenerate line bundle on Y then $i(L) = i(f^*L)$.*

Proof. Choose and ample line bundle H on Y. By (9.12), the Hilbert polynomial of f^*L with repsect to the ample bundle f^*H is just $\deg(f)$ times the Hilbert polynomial of L with respect to H. Now apply the theorem. $\square$

The reason that in (9.25) we restrict ourselves to non-degenerate bundles is that only for such bundles the index is well-defined. Without this restriction we still have a quantative result, though.

(9.27) Theorem. *Let L be a line bundle on an abelian variety X over a field k. Let H be an ample line bundle on X and write $\Phi(t) \in \mathbb{Z}[t]$ for the Hilbert polynomial of L with respect to H. Then the multiplicity of 0 as a root of Φ equals the dimension of $K(L)$.*

Proof. Write $Y := K(L)^0_{\mathrm{red}}$, which is an abelian subvariety of X. There exists an abelian subvariety $Z \subset X$ such that the homomorphism $\nu\colon Y \times Z \to X$ given by $(y, z) \mapsto y + z$ is an isogeny; see Exercise ?? or Theorem (12.2) below. Let $M := (\nu^*L)_{|\{0\}\times Z}$. Note that M is a non-degenerate bundle on Z. We claim that ν^*L differs from p_Z^*M by an element in $\mathrm{Pic}^0_{(Y\times Z)/k}$. Indeed, if we let $N := \nu^*L \otimes p_Z^*M^{-1}$ then $K(N)$ contains both $\{0\} \times Z$ (because $N_{|\{0\}\times Z}$ is trivial) and $Y \times \{0\}$ (because $N_{|Y\times\{0\}} = L_{|Y}$ and $Y \subset K(L)$); hence $K(N) = Y \times Z$, which by Cor. (7.22) means that the class of N lies in $\mathrm{Pic}^0_{(Y\times Z)/k}$. Writing $l = c_1(L)$ and $m = c_1(M)$ we therefore have $\nu^*l = p_Z^*m$. Let $g = \dim(X)$ and $s = \dim(Z)$, and write $h = c_1(H)$. Using Corollary (9.12) we find

$$\begin{aligned}\deg(\nu)\cdot\Phi(t) &= \deg(\nu)\cdot(l + t\cdot h)^g = \big(\nu^*(l + th)\big)^g = \big(p_Z^*m + t\cdot\nu^*h\big)^g\\ &= \sum_{j=0}^{s}\binom{g}{j}\big((p_Z^*m)^j\cdot(\nu^*h)^{g-j}\big)\cdot t^{g-j}\,,\end{aligned}$$

since $m^j = 0$ if $j > s = \dim(Z)$. Moreover, $m^s \neq 0$ because M is non-degenerate; and because ν^*h is an ample class then also $(p_Z^*m)^s\cdot(\nu^*h)^{g-s} \neq 0$. This shows that $\Phi(t)$ is exactly divisible by t^{g-s}. $\square$