Chapter VI. The Picard scheme of an abelian variety.

§ 1. *Relative Picard functors.*

To place the notion of a dual abelian variety in its context, we start with a short discussion of relative Picard functors. Our goal is to sketch some general facts, without much discussion of proofs.

Given a scheme X we write

$$\mathrm{Pic}(X) = H^1(X, O_X^*) = \{\text{isomorphism classes of line bundles on } X\}\,,$$

which has a natural group structure. (If τ is either the Zariski, or the étale, or the fppf topology on $\mathsf{Sch}_{/X}$ then we can also write $\mathrm{Pic}(X) = H^1_\tau(X, \mathbb{G}_m)$, viewing the group scheme $\mathbb{G}_m = \mathbb{G}_{m,X}$ as a τ-sheaf on $\mathsf{Sch}_{/X}$; see Exercise ??.)

If C is a complete non-singular curve over an algebraically closed field k then its Jacobian $\mathrm{Jac}(C)$ is an abelian variety parametrizing the degree zero divisor classes on C or, what is the same, the degree zero line bundles on C. (We refer to Chapter 14 for further discussion of Jacobians.) Thus, for every $k \subset K$ the degree map gives a homomorphism $\mathrm{Pic}(C_K) \to \mathbb{Z}$, and we have an exact sequence

$$0 \longrightarrow \mathrm{Jac}(C)(K) \longrightarrow \mathrm{Pic}(C_K) \longrightarrow \mathbb{Z} \longrightarrow 0\,.$$

In view of the importance of the Jacobian in the theory of curves one may ask if, more generally, the line bundles on a variety X are parametrized by a scheme which is an extension of a discrete part by a connected group variety.

If we want to study this in the general setting of a scheme $f\colon X \to S$ over some basis S, we are led to consider the contravariant functor $P_{X/S}\colon (\mathsf{Sch}_{/S})^0 \to \mathsf{Ab}$ given by

$$P_{X/S}\colon T \mapsto \mathrm{Pic}(X_T) = H^1(X \times_S T, \mathbb{G}_m)\,.$$

However, one easily finds that this functor is not representable (unless $X = \emptyset$.). The reason for this is the following. Suppose $\{U_\alpha\}_{\alpha\in A}$ is a Zariski covering of S and L is a line bundle on X such that the restrictions $L_{|X\times_S U_\alpha}$ are trivial. Then it is not necessarily the case that L is trivial. This means that $P_{X/S}$ is not a sheaf for the Zariski topology on $\mathsf{Sch}_{/S}$, hence not representable. (See also Exercise (6.1).)

The previous arguments suggest that in order to arrive at a functor that could be representable we should first sheafify (or "localize") $P_{X/S}$ with respect to some topology.

(6.1) Definition. The *relative Picard functor* $\mathrm{Pic}_{X/S}\colon (\mathsf{Sch}_{/S})^0 \to \mathsf{Ab}$ is defined to be the fppf sheaf (on $(S)_{\mathrm{FPPF}}$) associated to the presheaf $P_{X/S}$. An S-scheme representing $\mathrm{Pic}_{X/S}$ (if such a scheme exists) is called the *relative Picard scheme* of X over S.

Concretely, if T is an S-scheme then we can describe an element of $\mathrm{Pic}_{X/S}(T)$ by giving an fppf covering $T' \to T$ and a line bundle L on $X_T \times_T T'$ such that the two pull-backs of L to

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$X_T \times_T (T' \times_T T')$ are isomorphic. Now suppose we have a second datum of this type, say an fppf covering $U' \to T$ and a line bundle M on $X_T \times_T U'$ whose two pull-backs to $X_T \times_T (U' \times_T U')$ are isomorphic. Then $(T' \to T, L)$ and $(U' \to T, M)$ define the same element of $\mathrm{Pic}_{X/S}(T)$ if there is a common refinement of the coverings T' and U' over which the bundles L and M become isomorphic.

As usual, if $\mathrm{Pic}_{X/S}$ is representable then the representing scheme is unique up to S-isomorphism; this justifies calling it *the* Picard scheme.

(6.2) Let us study $\mathrm{Pic}_{X/S}$ in some more detail in the situation that

$$(*) \qquad \begin{cases} \text{the structure morphism } f\colon X \to S \text{ is quasi-compact and quasi-separated,} \\ f_*(O_{X \times_S T}) = O_T \text{ for all } S\text{-schemes } T, \\ f \text{ has a section } \varepsilon\colon S \to X. \end{cases}$$

For instance, this holds if S is the spectrum of a field k and X is a complete k-variety with $X(k) \neq \emptyset$ (see also Exercise ??); this is the case we shall mostly be interested in.

Rather than sheafifying $P_{X/S}$ we may also rigidify the objects we are trying to classify. This is done as follows. If L is a line bundle on X_T for some S-scheme T then, writing $\varepsilon_T\colon T \to X_T$ for the section induced by ε, by a *rigidification of L along ε_T* we mean an isomorphism $\alpha\colon O_T \xrightarrow{\sim} \varepsilon_T^* L$. (In the sequel we shall usually simply write ε for ε_T.)

Let (L_1, α_1) and (L_2, α_2) be line bundles on X_T with rigidification along ε. By a homomorphism $h\colon (L_1, \alpha_1) \to (L_2, \alpha_2)$ we mean a homomorphism of line bundles $h\colon L_1 \to L_2$ with the property that $(\varepsilon^* h) \circ \alpha_1 = \alpha_2$. In particular, an endomorphism of (L, α) is given by an element $h \in \Gamma(X_T, O_{X_T}) = \Gamma\big(T, f_*(O_{X_T})\big)$ with $\varepsilon^*(h) = 1$. By the assumption that $f_*(O_{X_T}) = O_T$ we therefore find that rigidified line bundles on X_T have no nontrivial automorphisms.

Now define the functor $P_{X/S,\varepsilon}\colon (\mathsf{Sch}_{/S})^0 \to \mathsf{Ab}$ by

$$P_{X/S,\varepsilon}\colon T \mapsto \left\{ \begin{matrix} \text{isomorphism classes of rigidified} \\ \text{line bundles } (L, \alpha) \text{ on } X \times_S T \end{matrix} \right\},$$

with group structure given by

$$\begin{aligned} &(L, \alpha) \cdot (M, \beta) = (L \otimes M, \gamma)\,, \\ &\gamma = \alpha \otimes \beta\colon O_T = O_T \underset{O_T}{\otimes} O_T \to \varepsilon^* L \underset{O_T}{\otimes} \varepsilon^* M = \varepsilon^*(L \otimes M)\,. \end{aligned}$$

If $h\colon T' \to T$ is a morphism of S-schemes and (L, α) is a rigidified line bundle on $X \times_S T$ then $P_{X/S,\varepsilon}(h)\colon P_{X/S,\varepsilon}(T) \to P_{X/S,\varepsilon}(T')$ sends (L, α) to (L', α'), where $L' = (\mathrm{id}_X \times h)^* L$ and where $\alpha'\colon O_{T'} \xrightarrow{\sim} \varepsilon_{T'}^* L' = h^*\big(\varepsilon_T^* L\big)$ is the pull-back of α under h.

Suppose $P_{X/S,\varepsilon}$ is representable by an S-scheme. On $X \times_S P_{X/S,\varepsilon}$ we then have a universal rigidified line bundle $(\mathscr{P}, \nu)$; it is called the *Poincaré bundle*. The universal property of $(\mathscr{P}, \nu)$ is the following: if (L, α) is a line bundle on $X \times_S T$ with rigidification along the section ε then there exists a unique morphism $g\colon T \to P_{X/S,\varepsilon}$ such that $(L, \alpha) \cong (\mathrm{id}_X \times g)^*(\mathscr{P}, \nu)$ as rigidified bundles on X_T.

Under the assumptions $(*)$ on f it is not so difficult to prove the following facts. (See for example BLR, § 8.1 for details.)

(i) For every S-scheme T there is a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(T) \xrightarrow{\mathrm{pr}_T^*} \mathrm{Pic}(X_T) \longrightarrow \mathrm{Pic}_{X/S}(T)\,. \tag{1}$$

This can be viewed as a short exact sequence obtained from a Leray spectral sequence. The existence of a section is not needed for this.

(ii) For every S-scheme T, we have an isomorphism

$$\mathrm{Pic}(X_T)/\mathrm{pr}_T^*\mathrm{Pic}(T) \xrightarrow{\sim} P_{X/S,\varepsilon}(T)$$

obtained by sending the class of a line bundle L on X_T to the bundle $L \otimes f^*\varepsilon^*L^{-1}$ with its canonical rigidification.

(iii) The functor $P_{X/S,\varepsilon}$ is an fppf sheaf. (Descent theory for line bundles.)

Combining these facts we find that $P_{X/S,\varepsilon} \cong \mathrm{Pic}_{X/S}$ and that these functors are given by

$$T \mapsto \frac{\mathrm{Pic}(X_T)}{\mathrm{pr}_T^*\mathrm{Pic}(T)} = \frac{\{\text{line bundles on } X_T\}}{\{\text{line bundles of the form } f^*L, \text{ with } L \text{ a line bundle on } T\}}\,.$$

In particular, the exact sequence (1) extends to an exact sequence

$$0 \longrightarrow \mathrm{Pic}(T) \longrightarrow \mathrm{Pic}(X_T) \longrightarrow \mathrm{Pic}_{X/S}(T) \longrightarrow 0\,. \tag{2}$$

It also follows that $\mathrm{Pic}_{X/S}$ equals the Zariski sheaf associated to $P_{X/S}$.

(6.3) Returning to the general case (i.e., no longer assuming that f satisfies the conditions $(*)$ in (6.2)), one finds that $\mathrm{Pic}_{X/S}$ cannot be expected to be representable unless we impose further conditions on X/S. (See Exercise ?? for an example.) The most important general results about representability all work under the assumption that $f\colon X \to S$ is proper, flat and of finite presentation. We quote some results:

(i) If f is flat and projective with geometrically integral fibres then $\mathrm{Pic}_{X/S}$ is representable by a scheme, locally of finite presentation and separated over S. (Grothendieck, FGA, Exp. 232.)

(ii) If f is flat and projective with geometrically reduced fibres, such that all irreducible components of the fibres of f are geometrically irreducible then $\mathrm{Pic}_{X/S}$ is representable by a scheme, locally of finite presentation (but not necessarily separated) over S. (Mumford, unpublished.)

(iii) If $S = \mathrm{Spec}(k)$ is the spectrum of a field and f is proper then $\mathrm{Pic}_{X/S}$ is representable by a scheme that is separated and locally of finite type over k. (Murre [1], using a theorem of Oort [1] to reduce to the case that X is reduced.)

If we further weaken the assumptions on f, e.g., if in (ii) we omit the condition that the irreducible components of the fibres are geometrically irreducible, then we may in general only hope for $\mathrm{Pic}_{X/S}$ to be representable by an algebraic space over S. Also if we only assume X/S to be proper, not necessarily projective, then in general $\mathrm{Pic}_{X/S}$ will be an algebraic space rather than a scheme. For instance, in Grothendieck's FGA, Exp. 236 we find the following criterion.

(iv) If $f\colon X \to S$ is proper and locally of finite presentation with geometrically integral fibres then $\mathrm{Pic}_{X/S}$ is a separated algebraic space over S.

We refer to ??, ?? for further discussion.

(6.4) Remark. Let X be a complete variety over a field k, let Y be a k-scheme and let L be a line bundle on $X \times Y$. The existence of maximal closed subscheme $Y_0 \hookrightarrow Y$ over which L is trivial, as claimed in (2.4), is an immediate consequence of the existence of $\mathrm{Pic}_{X/k}$. Namely, the line bundle L gives a morphism $Y \to \mathrm{Pic}_{X/k}$ and Y_0 is simply the fibre over the zero section of $\mathrm{Pic}_{X/k}$ under this morphism. (We use the exact sequence (1); as remarked earlier this does not require the existence of a rational point on X.)

Let us now turn to some basic properties of $\mathrm{Pic}_{X/S}$ in case it is representable. Note that $\mathrm{Pic}_{X/S}$ comes with the structure of an S-group scheme, so that the results and definitions of Chapter 3 apply.

(6.5) Proposition. *Assume that $f\colon X \to S$ is proper, flat and of finite presentation, with geometrically integral fibres. As discussed above, $\mathrm{Pic}_{X/S}$ is a separated algebraic space over S. (Those who wish to avoid algebraic spaces might add the hypothesis that f is projective, as in that case $\mathrm{Pic}_{X/S}$ is a scheme.)*

(i) Write $\mathscr{T}$ for the relative tangent sheaf of $\mathrm{Pic}_{X/S}$ over S. Then the sheaf $e^*\mathscr{T}$ ("the tangent space of $\mathrm{Pic}_{X/S}$ along the zero section") is canonically isomorphic to $R^1f_*O_X$.

(ii) Assume moreover that f is smooth. Then every closed subscheme $Z \hookrightarrow \mathrm{Pic}_{X/S}$ which is of finite type over S is proper over S.

For a proof of this result we refer to BLR, Chap. 8.

(6.6) Corollary. *Let X be a proper variety over a field k.*

(i) *The tangent space of $\mathrm{Pic}_{X/S}$ at the identity element is isomorphic to $H^1(X,O_X)$. Further, $\mathrm{Pic}^0_{X/S}$ is smooth over k if and only if $\dim \mathrm{Pic}^0_{X/S} = \dim H^1(X,O_X)$, and this always holds if $\mathrm{char}(k) = 0$.*

(ii) *If X is smooth over k then all connected components of $\mathrm{Pic}_{X/k}$ are complete.*

Proof. This is immediate from (6.5) and the results discussed in Chapter 3 (notably (3.17) and (3.20)). As we did not prove (6.5), let us here give a direct explanation of why the tangent space of $\mathrm{Pic}_{X/S}$ at the identity element is isomorphic to $H^1(X,O_X)$, and why the components of $\mathrm{Pic}_{X/k}$ are complete.

Let $S = \mathrm{Spec}\,\bigl(k[\varepsilon]\bigr)$, where $k[\varepsilon]$ is the ring of dual numbers over k. Note that X and X_S have the same underlying topological space. On this space we have a short exact sequence of sheaves

$$0 \longrightarrow O_X \xrightarrow{h} O_{X_S}^* \xrightarrow{\mathrm{res}} O_X^* \longrightarrow 1$$

where h is given on sections by $f \mapsto \exp(\varepsilon f) = 1 + \varepsilon f$ and where res is the natural restriction map. On cohomology in degree zero this gives the exact sequence

$$0 \longrightarrow k \longrightarrow k[\varepsilon]^* \longrightarrow k^* \longrightarrow 1$$

where the maps are given by $f \mapsto 1+\varepsilon f$ and $a + \varepsilon b \mapsto a$. On cohomology in degree 1 we then find an exact sequence

$$0 \longrightarrow H^1(X,O_X) \xrightarrow{h} \mathrm{Pic}(X_S) \xrightarrow{\mathrm{res}} \mathrm{Pic}(X)\,. \tag{3}$$

Concretely, if $\gamma \in H^1(X,O_X)$ is represented, on some open covering $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$, by a Čech 1-cocyle $\{f_{\alpha\beta} \in O_X(U_\alpha \cap U_\beta)\}$ then $h(\gamma)$ is the class of the line bundle on X_S which is trivial on each U_α (now to be viewed as an open subset of X_S) and with transition functions $1+\varepsilon f_{\alpha\beta}$.

Write T for the tangent space of $\mathrm{Pic}_{X/k}$ at the identity element. We can descibe T as the kernel of the restriction map $\mathrm{Pic}_{X/k}(S) \to \mathrm{Pic}_{X/k}(k)$; see Exercise 1.3. If $\gamma \in H^1(X,O_X)$ then $h(\gamma)$ restricts to the trivial class on X. Hence γ defines an element of T, and this gives a linear map $\xi\colon H^1(X,O_X) \to T$. As $\mathrm{Pic}(S) = \{1\}$ it follows from the exact sequences (1) and (3) that ξ is injective.

So far we have not used anything about X. To prove that ξ is also surjective it suffices to show that $\dim\bigl(H^1(X,O_X)\bigr) = \dim(T)$. Both numbers do not change if we extend the ground field. Without loss of generality we may therefore assume that $X(k)$ is non-empty, so that assumptions $(*)$ in (6.2) are satisfied. Then the surjectivity of the map ξ follows from the exact sequence (2). This proves that $H^1(X, O_X) \xrightarrow{\sim} T$.

Next let us explain why the components of $\mathrm{Pic}_{X/S}$ are complete. We already know that $\mathrm{Pic}_{X/S}$ is a group scheme, locally of finite type over k. By Propositions (3.12) and (3.17), all connected components are separated and of finite type over k. To show that they are complete, we may extend the ground field; hence we can again assume that the assumptions $(*)$ in (6.2) are satisfied. In this situation we apply the valuative criterion for properness. Let R be a k-algebra which is a dvr. Let K be its fraction field, and suppose we have a K-valued point of $\mathrm{Pic}_{X/k}$, say represented by a line bundle L on X_K. We want to show that L extends to a line bundle on X_R. Since X/k is smooth, L is represented by a Weil divisor. But if $P \subset X_K$ is any prime divisor then the closure of P inside X_R is a prime divisor of X_R. It follows that L extends to a line bundle on X_R. $\square$

(6.7) Remark. If $\mathrm{char}(k) = p > 0$ then $\mathrm{Pic}_{X/k}$ is in general not reduced, even if X is smooth and proper over k. An example illustrating this will be given in (7.31) below.

(6.8) Let C be a complete curve over a field k. Then $\mathrm{Pic}_{C/k}$ is a group scheme, locally of finite type over k; see (6.3). We claim that $\mathrm{Pic}_{C/k}$ is smooth over k. To see this we may extend the ground field and assume that $C(k) \neq \emptyset$, so that the assumptions $(*)$ in (6.2) are satisfied. Because $\mathrm{Pic}_{C/k}$ is locally of finite type over k, it suffices to show that any point of $\mathrm{Pic}_{C/k}$ with values in $R_0 := k[t]/(t^n)$ can be lifted to a point with values in $R := k[t]/(t^{n+1})$. But if we have a line bundle L_0 on $C \otimes_k R_0$ then the obstruction for extending L_0 to a line bundle on $C \otimes_k R$ lies in $H^2(C, O_C)$, which is zero because C is a curve.

In particular, we find that the identity component $\mathrm{Pic}^0_{C/k}$ is a group variety over k. If in addition we assume that C is smooth then by Cor. (6.6) $\mathrm{Pic}^0_{C/k}$ is complete, and is therefore an abelian variety. In this case we usually write $\mathrm{Jac}(C)$ for $\mathrm{Pic}^0_{C/k}$; it is called the *Jacobian* of C. Jacobians will be further discussed in Chapter 14. We remark that the term "Jacobian of C", for a complete and smooth curve C/k, usually refers to the abelian variety $\mathrm{Jac}(C) := \mathrm{Pic}^0_{C/k}$ together with its natural principal polarisation.

(6.9) Remark. Suppose X is a smooth proper variety over an algebraically closed field k. Recall that two divisors D_1 and D_2 are said to be algebraically equivalent (notation $D_1 \sim_{\mathrm{alg}} D_2$) if there exist (i) a smooth k-variety T, (ii) codimension 1 subvarieties $Z_1, \ldots, Z_n$ of $X \times_k T$ which are flat over T, and (iii) points $t_1, t_2 \in T(k)$, such that $D_1 - D_2 = \sum_{i=1}^n (Z_i)_{t_1} - (Z_i)_{t_2}$ as divisors on X; here $(Z_i)_t := Z_i \cap (X \times \{t\})$, viewed as a divisor on X. Translating this to line bundles we find that $D_1 \sim_{\mathrm{alg}} D_2$ precisely if the classes of $L_1 = O_X(D_1)$ and $L_2 = O_X(D_2)$ lie in the same connected component of $\mathrm{Pic}_{X/k}$. (Note that the components of the reduced scheme underlying $\mathrm{Pic}_{X/k}$ are smooth k-varieties.) The discrete group $\pi_0(\mathrm{Pic}_{X/k}) = \mathrm{Pic}_{X/k}/\mathrm{Pic}^0_{X/k}$ is therefore naturally isomorphic to the *Néron-Severi group* $\mathrm{NS}(X) := \mathrm{Div}(X)/\sim_{\mathrm{alg}}$. For a more precise treatment, see section (7.24).

§ 2. *Digression on graded bialgebras.*

In our study of duality, we shall make use of a structure result for certain graded bialgebras.

Before we can state this result we need to set up some definitions.

Let k be a field. (Most of what follows can be done over more general ground rings; for our purposes the case of a field suffices.) Consider a graded k-module $H^\bullet = \oplus_{n\geqslant 0} H^n$. An element $x \in H^\bullet$ is said to be homogeneous if it lies in H^n for some n, in which case we write $\deg(x) = n$. By a graded k-algebra we shall mean a graded k-module $H^\bullet$ together with a unit element $1 \in H^0$ and an algebra structure map (multiplication) $\gamma\colon H^\bullet \otimes_k H^\bullet \to H^\bullet$ such that

(i) the element 1 is a left and right unit for the multiplication;

(ii) the multiplication γ is associative, i.e., $\gamma(x, \gamma(y, z)) = \gamma(\gamma(x, y), z)$ for all x, y and z;

(iii) the map γ is a morphism of graded k-modules, i.e., it is k-linear and for all homogeneous elements x and y we have that $\gamma(x, y)$ is homogeneous of degree $\deg(x) + \deg(y)$.

If no confusion arises we shall simply write xy for $\gamma(x, y)$.

A homomorphism between graded k-algebras $H_1^\bullet$ and $H_2^\bullet$ is a k-linear map $f\colon H_1^\bullet \to H_2^\bullet$ which preserves the gradings, with $f(1) = 1$ and such that $f(xy) = f(x)f(y)$ for all x and y in $H_1^\bullet$.

We say that the graded algebra $H^\bullet$ is graded-commutative if

$$xy = (-1)^{\deg(x)\deg(y)} yx$$

for all homogeneous $x, y \in H^\bullet$. (In some literature this is called anti-commutativity, or sometimes even commutativity.) The algebra $H^\bullet$ is said to be connected if $H^0 = k \cdot 1$; it is said to be of finite type over k if $\dim_k(H^n) < \infty$ for all n (which is weaker than saying that $H^\bullet$ is finite-dimensional).

If $H_1^\bullet$ and $H_2^\bullet$ are graded k-algebras then the graded k-module $H_1^\bullet \otimes_k H_2^\bullet$ inherits the structure of a graded k-algebra: for homogeneous elements $x, \xi \in H_1^\bullet$ and $y, \eta \in H_2^\bullet$ one sets $(x \otimes y) \cdot (\xi \otimes \eta) = (-1)^{\deg(y)\deg(\xi)} \cdot (x\xi \otimes y\eta)$. As an exercise the reader may check that $H^\bullet$ is graded-commutative if and only if the map $\gamma\colon H^\bullet \otimes H^\bullet \to H^\bullet$ is a homomorphism of graded k-algebras. The field k itself shall be viewed as a graded k-algebras with all elements of degree zero.

(6.10) Definition. A *graded bialgebra over* k is a graded k-algebra $H^\bullet$ together with two homomorphisms of k-algebras

$$\begin{aligned} \mu\colon H^\bullet &\to H^\bullet \otimes_k H^\bullet &&\text{called co-multiplication,}\\ \varepsilon\colon H^\bullet &\to k &&\text{the identity section,} \end{aligned}$$

such that

$$(\mu \otimes \mathrm{id}) \circ \mu = (\mathrm{id} \otimes \mu) \circ \mu\colon H^\bullet \to H^\bullet \otimes_k H^\bullet \otimes_k H^\bullet$$

and

$$(\varepsilon \otimes \mathrm{id}) \circ \mu = (\mathrm{id} \otimes \varepsilon) \circ \mu\colon H^\bullet \to H^\bullet$$

(using the natural identifications $H^\bullet \otimes_k k = H^\bullet = k \otimes_k H^\bullet$).

(6.11) Examples. (i) If all elements of $H^\bullet$ have degree zero, i.e., $H^\bullet = H^0$, then we can ignore the grading and we "almost" find back the definition of a Hopf algebra as in (3.9). The main distinction between Hopf algebras and bialgebras is that for the latter we do not require an antipode.

(ii) If V is a vector space over k then we can form the exterior algebra $\wedge^\bullet V = \oplus_{n\geqslant 0} \wedge^n V$. The multiplication is given by the "exterior product", i.e.,

$$(x_1 \wedge \cdots \wedge x_r) \cdot (y_1 \wedge \cdots \wedge y_s) = x_1 \wedge \cdots \wedge x_r \wedge y_1 \wedge \cdots \wedge y_s\,.$$

By definition we have $\wedge^0 V = k$.

A k-linear map $V_1 \to V_2$ induces a homomorphism of graded algebras $\wedge^\bullet V_1 \to \wedge^\bullet V_2$. Furthermore, there is a natural isomorphism $\wedge^\bullet(V \oplus V) \xrightarrow{\sim} (\wedge^\bullet V) \otimes (\wedge^\bullet V)$. Therefore, the diagonal map $V \to V \oplus V$ induces a homomorphism $\mu\colon \wedge^\bullet V \to \wedge^\bullet V \otimes \wedge^\bullet V$. Taking this as co-multiplication, and defining $\varepsilon\colon \wedge^\bullet V \to k$ to be the projection onto the degree zero component we obtain the structure of a graded bialgebra on $\wedge^\bullet V$.

(iii) If $H_1^\bullet$ and $H_2^\bullet$ are two graded bialgebras over k then $H_1^\bullet \otimes_k H_2^\bullet$ naturally inherits the structure of a graded bialgebra; if $a \in H_1^\bullet$ with $\mu_1(a) = \sum x_i \otimes \xi_i$ and $b \in H_2^\bullet$ with $\mu_2(b) = \sum y_j \otimes \eta_j$ then the co-multiplication $\mu = \mu_1 \otimes \mu_2$ is described by

$$\mu(a \otimes b) = \sum_{i,j} (-1)^{\deg(y_j)\deg(\xi_i)} (x_i \otimes y_j) \otimes (\xi_i \otimes \eta_j)\,.$$

(iv) Let $x_1, x_2, \ldots$ be indeterminates. We give each of them a degree $d_i = \deg(x_i) \geqslant 1$ and we choose $s_i \in \mathbb{Z}_{\geqslant 2} \cup \{\infty\}$. Then we can define a graded-commutative k-algebra $H^\bullet = k\langle x_1, x_2, \ldots\rangle$ generated by the x_i, subject to the conditions $x_i^{s_i} = 0$. Namely, we take the monomials

$$m = x_1^{r_1} x_2^{r_2} \cdots \qquad (r_i \neq 0 \text{ for finitely many } i)$$

as a k-basis, with $\deg(m) = r_1 d_1 + r_2 d_2 + \cdots$, and where we set $x_i^{s_i} = 0$. Then there is a unique graded-commutative multiplication law such that $\gamma(x_i, x_j) = x_i x_j$ for $i \leqslant j$, and with this multiplication $k\langle x_1, x_2, \ldots\rangle$ becomes a graded k-algebra. Note that $k\langle x_1, x_2, \ldots x_N\rangle$ is naturally isomorphic to $k\langle x_1\rangle \otimes_k \cdots \otimes k\langle x_N\rangle$.

It is an interesting question whether $k\langle x_1, x_2, \ldots\rangle$ can have the structure of a bialgebra. It turns out that the existence of such a structure imposes conditions on the numbers d_i and s_i. Let us first do the case of one generator; the case of finitely many generators will follow from this together with Borel's theorem to be discussed next. So, we consider a graded k-algebra $H^\bullet = k\langle x \mid x^s = 0\rangle$ with $\deg(x) = d > 0$. Suppose that $H^\bullet$ has the structure of a bialgebra. Then:

	conditions on s:
$\mathrm{char}(k) = 0$, d odd	$s = 2$
$\mathrm{char}(k) = 0$, d even	$s = \infty$
$\mathrm{char}(k) = 2$	either $s = \infty$ or $s = 2^n$ for some n
$\mathrm{char}(k) = p > 2$, d odd	$s = 2$
$\mathrm{char}(k) = p > 2$, d even	either $s = \infty$ or $s = p^n$ for some n

For a proof of this result (in fact a more general version of it) we refer to Milnor and Moore [1], § 7. Note that the example given in (ii) is of the form $k\langle x_1, x_2, \ldots\rangle$ where all x_i have $d_i = 1$ and $s_i = 2$.

(6.12) Theorem. (Borel-Hopf structure theorem) *Let $H^\bullet$ be a connected, graded-commutative bialgebra of finite type over a perfect field k. Then there exist graded bialgebras $H_i^\bullet$ $(i = 1, \ldots, r)$ and an isomorphism of bialgebras*

$$H^\bullet \cong H_1^\bullet \otimes_k \cdots \otimes_k H_r^\bullet$$

such that the algebra underlying $H_i^\bullet$ is generated by one element, i.e., the algebras $H_i^\bullet$ are of the form $k\langle x_i \mid x_i^{s_i} = 0\rangle$, with $\deg(x_i) = d_i > 0$.

For a proof of this result, which is due to A. Borel, we refer to Borel [1] or Milnor and Moore [1].

(6.13) Corollary. *Let $H^\bullet$ be as in (6.12). Assume there is an integer g such that $H^n = (0)$ for all $n > g$. Then $\dim_k(H^1) \leqslant g$. If $\dim_k(H^1) = g$ then $H^\bullet \cong \wedge^\bullet H^1$ as graded bialgebras.*

Proof. Decompose $H^\bullet = H_1^\bullet \otimes_k \cdots \otimes_k H_r^\bullet$ as in (6.12). Note that $\dim_k(H^1)$ equals the number of generators x_i such that $d_i = 1$. Now $x_1 \cdots x_r$ ($:= x_1 \otimes \cdots \otimes x_r$) is a nonzero element of $H^\bullet$ of degree $d_1 + \cdots + d_r$. Therefore $d_1 + \cdots + d_r \leqslant g$, which implies $\dim_k(H^1) \leqslant g$. Next suppose $\dim_k(H^1) = g$, so that all generators x_i have degree 1. If $x_i^2 \neq 0$ for some i then $x_1 \cdots x_{i-1} x_i^2 x_{i+1} \cdots x_g$ is a nonzero element of degree $g+1$, contradicting our assumptions. Hence $x_i^2 = 0$ for all i which means that $H^\bullet \cong \wedge^\bullet H^1$. □

(6.14) Let us now turn to the application of the above results to our study of abelian varieties. Given a g-dimensional variety X over a field k, consider the graded k-module

$$H^\bullet = H^\bullet(X, O_X) := \bigoplus_{n=0}^{g} H^n(X, O_X)\,.$$

Cup-product makes $H^\bullet$ into a graded-commutative k-algebra, which is connected since X is connected.

In case X is a group variety the group law induces on $H^\bullet$ the structure of a graded bialgebra. Namely, via the Künneth formula $H^\bullet(X \times_k X, O_{X\times X}) \cong H^\bullet(X, O_X) \otimes_k H^\bullet(X, O_X)$ (which is an isomorphism of graded k-algebras), the group law $m\colon X \times_k X \to X$ induces a co-multiplication

$$\mu\colon H^\bullet \to H^\bullet \otimes_k H^\bullet\,.$$

For the identity section $\varepsilon\colon H^\bullet \to k$ we take the projection onto the degree zero component, which can also be described as the map induced on cohomology by the unit section $e\colon \mathrm{Spec}(k) \to X$. Now the statement that these μ and e make $H^\bullet$ into a graded bialgebra over k becomes a simple translation of the axioms in (1.2) satisfied by m and e.

As a first application of the above we thus find the estimate $\dim_k(H^1(X, O_X)) \leqslant g$ for a g-dimensional group variety X over a field k. (Note that $\dim_k(H^1(X, O_X))$ does not change if we pass from k to an algebraic closure; we therefore need not assume k to be perfect.) For abelian varieties we shall prove in (6.18) below that we in fact have equality.

We summarize what we have found.

(6.15) Proposition. *Let X be a group variety over a field k. Then $H^\bullet(X, O_X)$ has a natural structure of a graded k-bialgebra. We have $\dim_k(H^1(X, O_X)) \leqslant \dim(X)$.*

To conclude this digression on bialgebras, let us introduce one further notion that will be useful later.

(6.16) Definition. Let $H^\bullet$ be a graded bialgebra with comultiplication $\mu\colon H^\bullet \to H^\bullet \otimes_k H^\bullet$. Then an element $h \in H^\bullet$ is called a *primitive* element if $\mu(h) = h \otimes 1 + 1 \otimes h$.

(6.17) Lemma. *Let V be a finite dimensional k vector space, and consider the exterior algebra $\wedge^\bullet V$ as in (6.11). Then $V = \wedge^1 V \subset \wedge^\bullet V$ is the set of primitive elements in $\wedge^\bullet V$.*

Proof. We follow Serre [1]. Since the co-multiplication μ is degree-preserving, an element of a graded bialgebra $H^\bullet$ is primitive if and only if all its homogeneous components are primitive. Thus we may restrict our attention to homogeneous elements of $\wedge^\bullet V$.

It is clear that the non-zero elements of $\wedge^0 V = k$ are not primitive. Further we see directly from the definitions that the elements of $\wedge^1 V = V$ are primitive. Let now $y \in \wedge^n V$ with $n \geqslant 2$. Write

$$[(\wedge^\bullet V) \otimes (\wedge^\bullet V)]^n = \bigoplus_{p+q=n} \wedge^p V \otimes \wedge^q V\,,$$

and write $\mu(y) = \sum \mu(y)^{p,q}$ with $\mu(y)^{p,q} \in \wedge^p V \otimes \wedge^q V$. For instance, one easily finds that $\mu(y)^{n,0} = y = \mu(y)^{0,n}$ via the natural identifications $\wedge^n V \otimes k = \wedge^n V = k \otimes \wedge^n V$. Similarly, we find that the map $y \mapsto \mu(y)^{1,n-1}$ is given (on decomposable tensors) by

$$v_1 \wedge \cdots \wedge v_n \mapsto \sum_{i=1}^{n} (-1)^{i+1} v_i \otimes (v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_n)\,.$$

It follows that for $\lambda \in V^*$ the composition $\wedge^n V \to V \otimes \wedge^{n-1} V \to \wedge^{n-1} V$ given by $y \mapsto (\lambda \otimes \mathrm{id})(\mu(y)^{1,n-1})$ is just the interior contraction $y \mapsto y \lrcorner \lambda$. The assumption that y is primitive and $n \geqslant 2$ implies that $\mu(y)^{1,n-1} = 0$ so we find $y \lrcorner \lambda = 0$ for all $\lambda \in V^*$. This only holds for $y = 0$. $\square$

§ 3. *The dual of an abelian variety.*

From now on, let $\pi\colon X \to S = \mathrm{Spec}(k)$ be an abelian variety over a field k. We shall admit from the general theory that $\mathrm{Pic}_{X/k}$ is a group scheme over k with projective connected components. One of the main results of this section is that $\mathrm{Pic}^0_{X/k}$ is reduced, and is therefore again an abelian variety.

Note that $\mathrm{Pic}_{X/k}$ also represents the functor $P_{X/k,0}$ of line bundles with rigidification along the zero section. As above, the identification between the two functors is given by sending the class of a line bundle L on $X \times_k T$ to the class of $L \otimes \mathrm{pr}_T^* e^* L^{-1}$ with its canonical rigidification along $\{0\} \times T$. (In order to avoid the notation 0^*L we write e for the zero section of X_T.) In particular, we have a Poincaré bundle $\mathscr{P}$ on $X \times_k \mathrm{Pic}_{X/k}$ together with a rigidification $\alpha\colon O_{\mathrm{Pic}_{X/k}} \xrightarrow{\sim} \mathscr{P}_{|\{0\}\times \mathrm{Pic}_{X/k}}$.

If L is a line bundle on X we have the associated Mumford bundle $\Lambda(L)$ on $X \times X$. In order to distinguish the two factors X, write $X^{(1)} = X \times \{0\}$ and $X^{(2)} = \{0\} \times X$. Viewing $\Lambda(L)$ as a family of line bundles on $X^{(1)}$ parametrised by $X^{(2)}$ we obtain a morphism

$$\varphi_L\colon X = X^{(2)} \longrightarrow \mathrm{Pic}_{X/k}$$

which is the unique morphism with the property that $(\mathrm{id}_X \times \varphi_L)^* \mathscr{P} \cong \Lambda(L)$. On points, the morphism φ_L is of course given by $x \mapsto [t_x^* L \otimes L^{-1}]$, just as in (2.10). We have seen in (2.10), as a consequence of the Theorem of the Square, that φ_L is a homomorphism. Further we note that φ_L factors through $\mathrm{Pic}^0_{X/k}$, as X is connected and $\varphi_L(0) = 0$.

(6.18) Theorem. *Let X be an abelian variety over a field k. Then $\mathrm{Pic}^0_{X/k}$ is reduced, hence it is an abelian variety. For every ample line bundle L the homomorphism $\varphi_L\colon X \to \mathrm{Pic}^0_{X/k}$ is an isogeny with kernel $K(L)$. We have $\dim(\mathrm{Pic}^0_{X/k}) = \dim(X) = \dim_k H^1(X, O_X)$.*

Proof. Let L be an ample line bundle on X. By Lemma (2.17), φ_L has kernel $K(L)$. Since $K(L)$ is a finite group scheme it follows that $\dim(\mathrm{Pic}^0_{X/k}) \geqslant \dim(X)$. Combining this with (6.6) and (6.15) we find that $\dim(\mathrm{Pic}^0_{X/k}) = \dim(X) = \dim_k H^1(X, O_X)$ and that $\mathrm{Pic}^0_{X/k}$ is reduced. □

(6.19) Definition and Notation. The abelian variety $X^t := \mathrm{Pic}^0_{X/k}$ is called the *dual* of X. We write $\mathscr{P}$, or $\mathscr{P}_X$, for the Poincaré bundle on $X \times X^t$ (i.e., the restriction of the Poincaré bundle on $X \times \mathrm{Pic}_{X/k}$ to $X \times X^t$). If $f\colon X \to Y$ is a homomorphism of abelian varieties over k then we write $f^t\colon Y^t \to X^t$ for the induced homomorphism, called the *dual* of f or the *transpose* of f. Thus, f^t is the unique homomorphism such that

$$(\mathrm{id} \times f^t)^* \mathscr{P}_X \cong (f \times \mathrm{id})^* \mathscr{P}_Y$$

as line bundles on $X \times Y^t$ with rigidification along $\{0\} \times Y^t$.

(6.20) Remark. We do not yet know whether $f \mapsto f^t$ is additive; in other words: if we have two homomorphisms f, $g\colon X \to Y$, is then $(f+g)^t$ equal to $f^t + g^t$? Similarly, is $(n_X)^t$ equal to n_{X^t} ? We shall later prove that the answer to both questions is "yes"; see (7.17). Note however that such relations certainly do not hold on all of $\mathrm{Pic}_{X/k}$; for instance, we know that if L is a line bundle with $(-1)^*L \cong L$ then $n^*L \cong L^{n^2}$ which is in general not isomorphic to L^n.

Exercises.

(6.1) Show that the functor $P_{X/S}$ defined in §1 is *never* representable, at least if we assume X to be a non-empty scheme.

(6.2) Let X and Y be two abelian varieties over a field k.

(i) Write $i_X\colon X \to X \times Y$ and $i_Y\colon Y \to X \times Y$ for the maps given by $x \mapsto (x, 0)$ and $y \mapsto (0, y)$, respectively. Show that the map $(i_X^t, i_Y^t)\colon (X \times Y)^t \to X^t \times Y^t$ that sends a class $[L] \in \mathrm{Pic}^0_{(X\times Y)/k}$ to $\big([L_{|X\times\{0\}}], [L_{|\{0\}\times Y}]\big)$, is an isomorphism. [*Note:* in general it is certainly not true that the full Picard scheme $\mathrm{Pic}_{X\times Y/k}$ is isomorphic to $\mathrm{Pic}_{X/k} \times \mathrm{Pic}_{Y/k}$.]

(ii) Write

$$p\colon X \times Y \times X^t \times Y^t \longrightarrow X \times X^t \quad \text{and} \quad q\colon X \times Y \times X^t \times Y^t \longrightarrow Y \times Y^t$$

for the projection maps. Show that the Poincaré bundle of $X \times Y$ is isomorphic to $p^*\mathscr{P}_X \otimes q^*\mathscr{P}_Y$.

(6.3) Let L be a line bundle on an abelian variety X. Consider the homomorphism $(1, \varphi_L)\colon X \to X \times X^t$. Show that $(1, \varphi_L)^*\mathscr{P}_X \cong L \otimes (-1)^*L$.

(6.4) The goal of this exercise is to prove the restrictions listed in (iv) of (6.11). We consider a graded bialgebra $H^\bullet$ over a field k, with co-multiplication μ. We define the height of an element $x \in H^\bullet$ to be the smallest positive integer n such that $x^n = 0$, if such an n exists, and to be ∞ if x is not nilpotent.

(i) If $y \in H^\bullet$ is an element of odd degree, and $\mathrm{char}(k) \neq 2$, show that $y^2 = 0$.

(ii) If $x \in H^\bullet$ is primitive, show that $\mu(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i}$. Conclude that if x has height $n < \infty$ then $\mathrm{char}(k) = p > 0$ and n is a power of p.

(iii) If $H^\bullet = k\langle x \mid x^s = 0\rangle$ with $\deg(x) = d$, show that x is a primitive element. Deduce the restrictions on the height of x listed in (iv) of (6.11).