## Chapter XIII. The Fourier transform and the Chow ring.

In this chapter we study the Chow ring of an abelian variety. For a nonsingular variety over a field the classes of cycles modulo rational equivalence form a ring with respect to the intersection product of cycles. For an abelian variety the Chow ring carries a second product, called the Pontryagin or convolution product. Here the product cycle is obtained, loosely speaking, by adding the points on the two cycles. These two aspects of the Chow ring are related by duality. The transition is provided by the Fourier transform, a transformation from the Chow ring of an abelian variety to the Chow ring of the dual abelian variety, under which the intersection product on the abelian variety corresponds to the convolution product on the dual. This Fourier transform is a wonderful tool for investigating the structure of the Chow ring of an abelian variety $X$. Using the Fourier transform one can decompose the diagonal correspondence in $X \times X$ as a sum of orthogonal idempotents. In the motivic language this gives a decomposition of the Chow motive of an abelian variety as $R(X) = \oplus_{i=0}^{2g} R^i(X)$, analogous to the decomposition $H^*(X) = \oplus_{i=0}^{2g} H^i(X)$ in cohomology. We close the chapter with a theorem of Künnemann which says that $R(X) \cong \wedge^* R^1(X)$.

Along the way we need some properties of the Chern classes of the Hodge bundle. These properties, like the so-called Key Formula and the vanishing of the top Chern class are of independent interest and are proved in section 2.

### § 1. *The Chow ring.*

We review some properties of the Chow ring and correspondences. An excellent reference book is Fulton [1]. Note that we are mainly interested in intersection theory on non-singular varieties, hence we do not need the theory developed in Fulton's book in its full strength.

**(13.1)** Let $X$ be a variety over a field $k$. The group $Z_r(X)$ of $r$-cycles on $X$ is defined as the free abelian group on the $r$-dimensional closed subvarieties of $X$. We usually write $[V] \in Z_r(X)$ for the element corresponding to a subvariety $V \subset X$. Thus, an $r$-cycle on $X$ is a finite formal sum $\sum n_i \cdot [V_i]$ where the $V_i \subset X$ are closed subvarieties of dimension $r$ and $n_i \in \mathbb{Z}$. For $r = \dim(X) - 1$ an $r$-cycle is the same as a Weil divisor.

In general, $Z_r(X)$ is a very big group. We arrive at a much more manageable group by taking the quotient modulo rational equivalence. This is done as follows. (Further details and proofs of some properties can be found in Fulton [1], Chap. 1.)

Let $W$ be an $(r+1)$-dimensional subvariety of $X$. Let $V \subset W$ be a subvariety of codimension 1. The local ring $O_{W,V}$ of $W$ along $V$ is a 1-dimensional local domain with fraction field $k(W)$, the field of rational functions on $W$. (Note: $V$ corresponds to a single point $x \in |W|$, and $O_{W,V}$ is just the stalk $O_{W,x}$ of $O_W$ at $x$.) For $0 \neq a \in O_{W,V}$, define the order of vanishing of $a$ along $V$ to be the integer

$$\mathrm{ord}_V(a) := \mathrm{length}_{O_{W,V}}\Big(O_{W,V}/(a)\Big)\,.$$

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We can extend this to a homomorphism $\mathrm{ord}_V\colon k(W)^* \to \mathbb{Z}$ by writing $f \in k(W)^*$ as $f = a/b$ with $a, b \in O_{W,V}$; then let $\mathrm{ord}_V(f) := \mathrm{ord}_V(a) - \mathrm{ord}_V(b)$. Note that if $V$ is not contained in the singular locus of $W$ then $O_{W,V}$ is a discrete valuation ring, and $\mathrm{ord}_V$ is just the valuation homomorphism.

Given $f \in k(W)^*$, there are only finitely many codimension 1 subvarieties $V \subset W$ such that $\mathrm{ord}_V(f) \neq 0$. This allows us to define an $r$-cycle on $X$, called the divisor of $f$ on $W \subset X$, by

$$\mathrm{div}(f) := \sum_V \mathrm{ord}_V(f) \cdot [V]\,,$$

where the sum runs over the subvarieties $V \subset W$ of codimension 1.

An $r$-cycle $\alpha \in Z_r(X)$ is said to be rationally equivalent to zero, notation $\alpha \sim 0$ or $\alpha \sim_{\mathrm{rat}} 0$, if there exist $(r+1)$-dimensional subvarieties $W_1, \ldots, W_n$ of $X$ and rational functions $f_i \in k(W_i)^*$ such that $\alpha = \sum_{i=1}^n \mathrm{div}(f_i)$. The cycles rationally equivalent to zero form a subgroup $\mathrm{Rat}_r(X)$ of $Z_r(X)$ and one defines the Chow group of $r$-cycles to be the factor group

$$\mathrm{CH}_r(X) := Z_r(X)/\mathrm{Rat}_r(X)\,.$$

We set $\mathrm{CH}^r(X) := \mathrm{CH}_{\dim(X)-r}(X)$; this is called the Chow group of codimension $r$ cycles. Let

$$\mathrm{CH}^*(X) := \oplus_r \mathrm{CH}^r(X)\,, \qquad \text{and} \qquad \mathrm{CH}^*_{\mathbb{Q}}(X) := \mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

It is a fundamental fact that for $X$ a non-singular variety, there exists an intersection pairing

$$\mathrm{CH}^r(X) \times \mathrm{CH}^s(X) \to \mathrm{CH}^{r+s}(X)\,, \qquad (\alpha, \beta) \mapsto \alpha \cdot \beta$$

which makes $\mathrm{CH}^*(X)$ into a commutative graded ring with identity. This ring is called the *Chow ring* of $X$. The identity element is $1_X = [X] \in \mathrm{CH}^0(X)$. (If $X$ is singular, there is still a good intersection theory, but this may not give a ring structure on $\mathrm{CH}^*(X)$. See Fulton [1].)

**(13.2)** Let $f\colon X \to Y$ be a morphism of $k$-varieties. Then we have a pull-back homomorphism $f^*\colon \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$. If $f$ is flat then $f^*$ is given by $f^*[V] = [f^{-1}(V)]$. The definition in the general case requires a little more care; we refer to Fulton [1], Chap. 8 for details. If $X$ and $Y$ are non-singular then $f^*$ is a homomorphism of graded rings.

Now assume that $f$ is proper. Let $V$ be a closed subvariety of $X$. Then $W = f(V)$ is a closed subvariety of $Y$. If $\dim(W) = \dim(V)$, let $\deg(V/W)$ be the degree of the function field extension $\bigl[k(V) : k(W)\bigr]$ defined by $f$; if $\dim(W) < \dim(V)$ let $\deg(V/W) := 0$. We set $f_*[V] = \deg(V/W)\cdot[W]$. By extending this linearly, we get a homomorphism $f_*\colon Z_r(X) \to Z_r(Y)$ which induces a homomorphism $f_*\colon \mathrm{CH}_r(X) \to \mathrm{CH}_r(Y)$.

For a proper morphism $f\colon X \to Y$ we have the *projection formula*

$$f_*(f^*\eta \cdot \xi) = \eta \cdot f_*\xi \qquad \text{for all } \xi \in \mathrm{CH}^*(X) \text{ and } \eta \in \mathrm{CH}^*(Y).$$

Furthermore, if

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{h} & Y \end{array}$$

is a Cartesian square with $h$ flat and $f$ proper ("flat base change of a proper morphism"), then $g$ is flat and $f'$ is proper, and for all $\alpha \in \mathrm{CH}^*(X)$ we have

$$f'_* g^* \alpha = h^* f_* \alpha. \tag{1}$$

**(13.3)** Let $X$ be a variety. Let $K^0(X)$ be the Grothendieck group of vector bundles on $X$. Then $K^0(X)$ has a natural structure of a commutative ring, with product $[E_1] \cdot [E_2] = [E_1 \otimes E_2]$. Let $K_0(X)$ be the Grothendieck group of coherent sheaves on $X$. Then $K_0(X)$ has a natural structure of a $K^0(X)$-module, by $[E] \cdot [F] = [E \otimes_{O_X} F]$. If $f\colon X \to Y$ is a morphism of varieties then we have a natural ring homomorphism $f^*\colon K^0(Y) \to K^0(X)$. If $f$ is proper then we have a homomorphism $f_*\colon K_0(X) \to K_0(Y)$ given by $f_*[F] = \sum_{i \geqslant 0} (-1)^i \big[R^i f_* F\big]$.

Now assume $X$ is non-singular. The natural homomorphism $K^0(X) \to K_0(X)$, sending a vector bundle to the corresponding $O_X$-module, is in this case an isomorphism. If there is no risk of confusion we simply write $K(X)$ for $K^0(X)$. Just as for the Chow ring, we have pull-backs $f^*$ for arbitrary morphisms $f$ between non-singular varieties, and push-forwards $f_*$ for proper morphisms. We write $K_{\mathbb{Q}}(X) := K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

There is a ring homomorphism

$$\mathrm{ch}\colon K(X) \to \mathrm{CH}^*_{\mathbb{Q}}(X),$$

called the *Chern character*. For a line bundle $L$ with associated divisor class $\ell = c_1(L) \in \mathrm{CH}^1_{\mathbb{Q}}(X)$, it is given by

$$[L] \mapsto e^{\ell} := 1 + \ell + \frac{1}{2}\ell^2 + \frac{1}{3!}\ell^3 + \cdots .$$

(Note that $e^{\ell}$ only involves a finite sum, as $\mathrm{CH}^i(X) = 0$ for $i > \dim(X)$.) For further details about the definition of the Chern character, see Fulton [1], sections 3.2 and 15.1.

Still assuming that $X$ is non-singular, the homomorphism $K_{\mathbb{Q}}(X) \to \mathrm{CH}^*_{\mathbb{Q}}(X)$ induced by the Chern character is an isomorphism. See Fulton [1], Example 15.2.16.

If $f\colon X \to Y$ is a morphism between non-singular varieties then the Chern character commutes with $f^*$, in the sense that $f^*\big(\mathrm{ch}(\alpha)\big) = \mathrm{ch}\big(f^*(\alpha)\big)$ for all $\alpha \in K(Y)$. But if $f$ is proper then "ch" does *not*, in general, commute with $f_*$. The difference between $f_* \circ \mathrm{ch}$ and $\mathrm{ch} \circ f_*$ is made precise by the Grothendieck-Riemann-Roch theorem; see Fulton [1], Thm. 15.2.

**(13.4)** Let $X$ and $Y$ be non-singular varieties. Elements in $\mathrm{CH}^*_{\mathbb{Q}}(X \times Y)$ are called *correspondences* from $X$ to $Y$. For a correspondence $\xi \in \mathrm{CH}^*_{\mathbb{Q}}(X \times Y)$ the transpose correspondence ${}^{\mathrm{t}}\xi$ from $Y$ to $X$ is defined as ${}^{\mathrm{t}}\xi := s_*(\xi)$, where $s\colon X \times Y \to Y \times X$ is the morphism reversing the factors.

Assume $Y$ is complete. If $Z$ is a third non-singular variety then we can compose correspondences: Given $\varphi \in \mathrm{CH}^*_{\mathbb{Q}}(X \times Y)$ and $\psi \in \mathrm{CH}^*_{\mathbb{Q}}(Y \times Z)$ we define their composition, which is a correspondence from $X$ to $Z$, by

$$\psi \circ \varphi = p_{XZ,*}\big(p^*_{XY}(\varphi) \cdot p^*_{YZ}(\psi)\big) \quad \in \mathrm{CH}^*_{\mathbb{Q}}(X \times Z)\,.$$

Here $p_{XZ}$ denotes the projection $X \times Y \times Z \to X \times Z$, and similarly for the other projections. We have ${}^{\mathrm{t}}(\psi \circ \varphi) = {}^{\mathrm{t}}\varphi \circ {}^{\mathrm{t}}\psi$.

If $f\colon X \to Y$ is a morphism with graph map $\gamma_f\colon X \to X \times Y$, then the correspondence $\Gamma_f = \big[\gamma_f(X)\big]$ in $\mathrm{CH}^*_{\mathbb{Q}}(X \times Y)$ is called the graph correspondence of $f$. Note that $\Gamma_f = \gamma_{f,*}[X]$. If $f\colon X \to Y$ and $g\colon Y \to Z$ then $\Gamma_g \circ \Gamma_f = \Gamma_{gf}$.

Assume $X$ is complete. A correspondence $\Gamma$ from $X$ to $Y$ gives rise to a homomorphism of groups $\gamma\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$ by

$$\gamma(\alpha) = p_{Y,*}\big(p_X^*(\alpha)\cdot\Gamma\big)\,, \tag{2}$$

where $p_X$ and $p_Y$ are the projections from $X \times Y$ to $X$ and $Y$, respectively. If $\Gamma = \Gamma_f$ for some morphism $f$ then $\gamma = f_*$. (Note that $f$ is automatically proper, as we have assumed that $X$ is complete.) If $\Gamma = {}^{\mathrm{t}}\Gamma_f$ then $\gamma = f^*$. If $\Gamma = \Gamma'\circ\Gamma''$ then for the associated homomorphisms we have $\gamma = \gamma'\circ\gamma''$.

We have a similar construction with Chow rings replaced by $K$-groups. So, if $X$ is complete then an element $\Gamma \in K(X\times Y)$ gives rise to a homomorphism $\gamma_K\colon K(X) \to K(Y)$ by the same formula as in (2). Further we write $\gamma_{\mathrm{CH}}\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$ for the homomorphism associated to the correspondence $\mathrm{ch}(\Gamma)$ from $X$ to $Y$.

**(13.5)** We shall need a variant of the above relative to a given base variety. For this, let $k$ be a field and let $S$ be a smooth quasi-projective $k$-scheme. Consider the category $\mathscr{V}(S)$ of smooth projective $S$-schemes. Note that if $X \to S$ and $Y \to S$ are in $\mathscr{V}(S)$ then so is the fibre product $X\times_S Y \to S$. Note further that if $X \to S$ is in $\mathscr{V}(S)$ then $X$ itself is again a smooth quasi-projective $k$-scheme. In particular this implies that $X$ is geometrically regular. If $X = \amalg X_i$ is the decomposition of $X$ as a union of its connected components then the $X_i$ are $k$-varieties in the sense of Fulton [1] (but not in our sense, as they may not be geometrically irreducible) and we set $\mathrm{CH}^*(X) := \oplus \mathrm{CH}^*(X_i)$.

Let $X$ and $Y$ be two smooth projective $S$-schemes. Elements in $\mathrm{CH}^*_{\mathbb{Q}}(X\times_S Y)$ are called *relative correspondences* between $X$ and $Y$. As before we can compose correspondences.

We shall make repeated use of the following lemma.

**(13.6) Lemma.** *Suppose given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathscr{V}(S)$ and classes $\alpha \in \mathrm{CH}^*_{\mathbb{Q}}(X\times_S Y)$ and $\beta \in \mathrm{CH}^*_{\mathbb{Q}}(Y\times_S Z)$. Then we have the identities of correspondences*

$$[\Gamma_g]\circ\alpha = (\mathrm{id}_X\times g)_*(\alpha)\,, \qquad \textit{and} \qquad \beta\circ[\Gamma_f] = (f\times \mathrm{id}_Z)^*(\beta)\,.$$

*Similarly, if $f'\colon Y \to X$ and $g'\colon Z \to Y$ are also morphisms in $\mathscr{V}(S)$ then*

$$[{}^{\mathrm{t}}\Gamma_{g'}]\circ\alpha = (\mathrm{id}_X\times g')^*(\alpha)\,, \qquad \textit{and} \qquad \beta\circ[{}^{\mathrm{t}}\Gamma_{f'}] = (f'\times \mathrm{id}_Z)_*(\beta)\,.$$

*Proof.* The first identities are proven as in Fulton [1], Prop. 16.1.1(c); the last two follow by transposition. $\square$

The Grothendieck-Riemann-Roch theorem has a variant for correspondences. As usual we write $\mathrm{Td}(E)$ for the Todd class of a vector bundle $E$; see Fulton [1], Example 3.2.4.

**(13.7) Proposition.** (GRR) *Let $X$ and $Y$ be in $\mathscr{V}(S)$, with $X \to S$ proper. For $\Gamma \in K(X\times_S Y)$ with associated homomorphisms $\gamma_K$ and $\gamma_{\mathrm{CH}}$, we have*

$$\mathrm{ch}\big(\gamma_K(\alpha)\big) = p_{Y,*}\Big[p_X^*\big(\mathrm{ch}(\alpha)\big)\cdot\mathrm{ch}(\Gamma)\cdot\mathrm{Td}\big(p_X^*T_{X/S}\big)\Big]\,.$$

*Proof.* This follows from the usual GRR theorem (see Fulton [1], 15.2.8) applied to the morphism $p_Y\colon X \times Y \to Y$ and the element $p_X^*(\alpha)\cdot\Gamma$ of $K(X \times_S Y)$. One gets

$$\mathrm{ch}\big[p_{Y,*}\big(p_X^*(\alpha)\cdot\Gamma\big)\big] = p_{Y,*}\big[\mathrm{ch}\big(p_X^*(\alpha)\cdot\Gamma\big)\cdot \mathrm{Td}(T_{X\times Y/Y})\big]\,.$$

Now use the definition of $\gamma_K(\alpha)$, the fact that ch is a ring homomorphism, and that $T_{X\times Y/Y} = p_X^* T_{X/S}$. $\square$

For an abelian scheme $\pi : X \to S$ the cotangent bundle $\Omega^1_X/S$ is a pull-back from a bundle $\mathbb{E}$ of rank $g$ from the base $S$ as it is trivial on the fibres. This is the Hodge bundle which we will consider more thoroughly in the next section.

**(13.8) Corollary.** *With $S$ as in (13.5), let $\xi\colon X \to S$ and $\eta\colon Y \to S$ be abelian schemes over $S$. Let $\mathbb{E}$ be the Hodge bundle of $X/S$. Let $\Gamma$ be an element of $K(X\times_S Y)$ with associated homomorphisms $\gamma_K$ and $\gamma_{\mathrm{CH}}$. Then for $\alpha \in \mathrm{CH}^*_{\mathbb{Q}}(X)$ we have*

$$\mathrm{ch}\big(\gamma_K(\alpha)\big) = \gamma_{\mathrm{CH}}\big(\mathrm{ch}(\alpha)\big)\cdot \eta^*\mathrm{Td}(\mathbb{E}^\vee)\,.$$

*In particular, if $S = \mathrm{Spec}(k)$ then the diagram*

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathrm{ch}} & \mathrm{CH}^*_{\mathbb{Q}}(X) \\ {\scriptstyle\gamma_K}\big\downarrow & & \big\downarrow{\scriptstyle\gamma_{\mathrm{CH}}} \\ K(Y) & \xrightarrow{\mathrm{ch}} & \mathrm{CH}^*_{\mathbb{Q}}(Y) \end{array}$$

*is commutative.*

*Proof.* We have $T_{X/S} = \xi^*\mathbb{E}^\vee$, so $p_X^* T_{X/S} = p_Y^*\eta^*\mathbb{E}^\vee$. The corollary now follows from (13.7) using the projection formula. $\square$

§ 2. *The Hodge bundle.*

In this section we consider the Hodge bundle of an abelian scheme and prove several basic properties of its Chern classes.

**(13.9) Definition.** Let $S$ be a quasi-projective non-singular variety over a field $k$. Let $\pi\colon X \to S$ be an abelian scheme over $S$ of relative dimension $g$ and with zero section $s$. The *Hodge bundle* $\mathbb{E} = \mathbb{E}_X$ of $X$ is the vector bundle (locally free sheaf) $\pi_*(\Omega^1_{X/S})$ of rank $g$ on $S$. By $\mathbb{E}^t$ we mean the Hodge bundle of the dual abelian scheme $X^t$. For $i = 1, \ldots, g$ we denote by $\lambda_i \in \mathrm{CH}^i(S)$ the $i$-th Chern class of $\mathbb{E}$ and by $\lambda_i^t$ the $i$-th Chern class of $\mathbb{E}^t$.

Alternatively, the Hodge bundle $\mathbb{E}$ may be defined as $\mathbb{E} = s^*\omega_{X/S}$ and we can view it as the cotangent bundle to the zero section $s$. It satisfies $\pi^*(\mathbb{E}) \cong \Omega^1_{X/S}$. Note that we have

$$(\mathbb{E}^t)^\vee = \mathrm{Lie}(X^t) \cong R^1\pi_* O_X.$$

**(13.9) Lemma.** *We have* $\det(\mathbb{E}) \cong \det(\mathbb{E}^t)$, *i.e.* $\lambda_1^t = \lambda_1$.

*Proof.* Note that $R^g\pi_* O_X \cong \wedge^g R^1\pi_* O_X$, and by Grothendieck duality (see [1], Thm. ?? or ??) we have $R^g\pi_* O_X \cong R^0\pi_*(\Omega^g_{X/S})^\vee$, i.e., we get $\det(\mathbb{E}) \cong \det(\mathbb{E}^t)$. $\square$

If $X$ carries a separable polarization then the corresponding map $X \to X^t$ induces an isomorphism $\mathbb{E} \cong \mathbb{E}^t$. (Is there always an isomorphism??)

The Grothendieck-Riemann-Roch theorem allows us to obtain relations in the Chow ring of the base space. We apply this to an ample line bundle on an abelian variety.

**(13.10) Theorem.** *Let $S$ be a smooth quasi-projective scheme over $k$ and $\pi : X \to S$ be an abelian scheme over $S$ with zero section $s$. Furthermore, let $L$ be a symmetric line bundle on $X/S$ such that $s^*L$ is trivial and giving a polarization on each fibre. If $\Theta$ is the divisor class in $\mathrm{CH}^1_{\mathbb{Q}}(X)$ representing $L$ then we have the identity*

$$\pi_*(\sum_{k=0}^{\infty} \frac{\Theta^{g+k}}{(g+k)!}) = d \cdot 1 \qquad \text{in } CH^*_{\mathbb{Q}}(S),$$

*where $d = \deg(\Theta^g/g!)$.*

*Proof.* The idea is to apply the Grothendieck-Riemann-Roch theorem to $L$. Actually, before doing that we first replace $X$ by $Y = X^4$, $g$ by $g' = 4g$ and $L$ by $M = L^{\otimes 4}$ (in shorthand using the exterior tensor product, i.e., $M = p_1^*L \otimes p_2^*L \otimes p_3^*L \otimes p_4^*L$). Then by the Zarhin Trick there exists for any $n \in \mathbb{Z}_{\geqslant 1}$ an isogeny of $\alpha$: $Y \to Y$ over $S$ such that $\alpha^*(M) \cong M^{\otimes n}$. Moreover, if $H$ is the kernel of $\alpha$ (a finite flat group scheme of rank $n^{4g}$ over $S$) then we claim that $\det(O_H)$ is a trivial $O_S$-module. To see this, note that $\alpha$ is given by an integral $4 \times 4$ matrix corresponding to a quaternion $z = a + bi + cj + dk$. Since $z$ lies in a quadratic subfield of the quaternions, the kernel of $\alpha$ is a direct sum of an even number of copies of group schemes $X[m]$ for divisors $m$ of $n$. Now $X[m]$ is self-dual, hence the square of the determinant of $O_{X[m]}$ is trivial. This implies that $\det(O_H)$ is trivial, cf. [Faltings-Chai, p. 25?]

Now we take an integer $n$ prime to the degree of $L$. Then we have a direct sum decomposition $K(M^{\otimes n}) \cong K(M) \oplus Y[n]$ and a similar decomposition $G(M^{\otimes n}) \cong G(M) \oplus Y[n]$. Theorem (8.14) tells us that we can lift $H$ to a level subgroup (again denoted by $H$) of $G(M^{\otimes n})$ (the theory works over base schemes as well). Let $H^c$ be the commutator of $H$ in $G(M^{\otimes n})$ so that $H^c/H$ is isomorphic to $G(M)$ by (8.16). By the representation theory of the theta group we find

$$\pi_*(M^{\otimes n}) \cong \mathrm{Ind}_{H^c}^{G(M^{\otimes n})} \pi_*(M).$$

Restrict the representation to the inverse image of $Y[n]$ in $G(M^{\otimes n})$. Then it decomposes as $\pi_*(M)$ tensor a representation of $G(M^{\otimes n})$ induced from a rank 1 representation of $H^c$ with the property that its $n$-th power extends to a representation of $G(M^{\otimes n})$. If we ignore elements of finite order (and we do because we work in $\mathrm{CH}^*_{\mathbb{Q}}(S)$) then we may conclude that the determinant of this representation is equal to $\det(O_H)$, hence trivial. We thus get

$$\mathrm{ch}(\pi_*(M^{\otimes n})) = n^{g'} \, \mathrm{ch}(\pi_*(M)) \qquad \text{in} \quad \mathrm{CH}^*_{\mathbb{Q}}(S). \tag{1}$$

The Grothendieck-Riemann-Roch theorem applied to $\pi : Y \to S$ and $M$ says

$$\begin{aligned} \mathrm{ch}(\pi_! M^{\otimes n}) &= \pi_*(\mathrm{ch}(M^{\otimes n}) \cdot \mathrm{Td}(\Omega^\vee_{Y/S}) \\ &= \pi_*(\mathrm{ch}(M^{\otimes n}) \cdot \mathrm{Td}(\pi^*(\mathbb{E}^\vee_Y))) \\ &= \pi_*(\mathrm{ch}(M^{\otimes n})) \cdot \mathrm{Td}(\mathbb{E}^\vee_Y) \end{aligned}$$

by the projection formula. Here $\mathbb{E}_Y$ is the Hodge bundle of $Y/S$. Since $R^i\pi_*(M) = 0$ for $i > 0$ it follows that $\pi_!(M) = \pi_*(M)$ is a vector bundle.

The relation (1) now gives writing $e^{n\Theta'}$ for $\mathrm{ch}(M^{\otimes n})$:

$$\pi_* \left( \sum_{k=0}^{\infty} \frac{n^{g'+k} {\Theta'}^{g'+k}}{(g'+k)!} \right) \mathrm{Td}(\mathbb{E}_Y^\vee) = n^{g'} \pi_* \left( \sum_{k=0}^{\infty} \frac{{\Theta'}^{g'+k}}{(g'+k)!} \right) \mathrm{Td}(\mathbb{E}_Y^\vee).$$

Comparing coefficients of $n^m$ and using that $\mathrm{Td}(\mathbb{E}_Y) = 1 + \ldots$ gives the result immediately for $Y$, $M$ and $\Theta'$. It is easy to derive it then for $X$, $L$ and $\Theta$. $\square$.

**(13.10) Corollary.** *With $L$ as in the Theorem we have $ch(\pi_!(L)) = d\,\mathrm{Td}(\mathbb{E}^\vee)$.*

By comparing codimension 1 classes in the Grothendieck-Riemann-Roch formula applied to $\pi$ and $L$ as in 13.10

$$\mathrm{ch}(\pi_! L) = \pi_*(e^{\Theta})\mathrm{Td}(\mathbb{E}^\vee = d\,\mathrm{Td}(\mathbb{E}^\vee) \tag{2}$$

and using $\mathrm{Td}_1(\mathbb{E}^\vee) = -\lambda_1/2$ we find the following Corollary.

**(13.11) Corollary.** (Key Formula) *For $L$ as in the theorem we have the formula in* $\mathrm{CH}^1_{\mathbb{Q}}(S)$

$$c_1(\pi_! L) = -\mathrm{rank}(\pi_*(L))\,\lambda_1/2.$$

By Zarhin's trick we know that for any abelian variety $X/S$ the abelian variety $Y = (X \times_S X^t)^4$ carries a principal polarization $L$. This implies that $\pi_! L$ lives in degree 0 and is given by a line bundle $\pi_*(L)$, so

$$\mathrm{ch}(\pi_*(L)) = e^{-\lambda_1(\mathbb{E}_Y)/2}.$$

On the other hand, equation (2) implies $\mathrm{ch}(\pi_*(L)) = \mathrm{Td}(\mathbb{E}_Y^\vee)$. By comparing these we get the following corollary.

**(13.12) Corollary.** *Let $X/S$ be an abelian scheme over a smooth quasi-projective basis $S$. Then if $\lambda_i = c_i(\mathbb{E})$ we have in $\mathrm{CH}^*_{\mathbb{Q}}(S)$ the relation*

$$\mathrm{Td}(\mathbb{E}^\vee)\mathrm{Td}((\mathbb{E}^t)^\vee) = e^{-\lambda_1}.$$

*If $X$ carries a separable polarization then we have $\mathrm{Td}(\mathbb{E}^\vee) = e^{-\lambda_1/2}$.*

*Proof.* Note that $\mathrm{Td}(\mathbb{E}_Y^\vee) = \mathrm{Td}(\mathbb{E}^\vee)^4 \mathrm{Td}((\mathbb{E}^t)^\vee)^4$ and $\lambda_1(\mathbb{E}_Y) = 4\lambda_1 + 4\lambda_1^t = 8\lambda_1$ by (10.9). If $X/S$ carries a separable polarization then we get a separable isogeny $X \to X^t$ inducing an isomorphism between $\mathbb{E}$ and $\mathbb{E}^t$. $\square$

As a consequence of the basic relation deduced in 13.10 we get the following fundamental relation for the Chern classes of the Hodge bundle.

**(13.13) Theorem.** *If $X/S$ carries a separable polarization then we have in* $\mathrm{CH}^*_{\mathbb{Q}}(S)$ *the relation*

$$(1 + \lambda_1 + \lambda_2 + \ldots + \lambda_g)(1 - \lambda_1 + \ldots + (-1)^g \lambda_g) = 1. \tag{3}$$

*Proof.* The relation $\mathrm{Td}(\mathbb{E}^\vee) = e^{-\lambda_1/2}$ implies that $\mathrm{Td}(\mathbb{E} \oplus \mathbb{E}^\vee) = 1$. This again implies that if $\alpha_1, \ldots, \alpha_g$ are the Chern roots of $\mathbb{E}$ then

$$\prod_{i=1}^{g} \frac{\alpha_i}{e^{\alpha_i} - 1} = \prod_{i=1}^{g} e^{-\alpha_i/2},$$

equivalently, that

$$\prod_{i=1}^{g}(e^{\alpha_i/2}-e^{-\alpha_i/2})=\prod_{i=1}^{g}\alpha_i$$

and this implies that the even degree power sums of the $\alpha_i$ vanish. This is easily see to be equivalent to $\mathrm{ch}(\mathbb{E}\oplus\mathbb{E}^\vee)=2g$ or to the relation (3). $\square$

Another important result on the Hodge bundle deals with the top Chern class $\lambda_g$ of $\mathbb{E}$ in the rational Chow group $\mathrm{CH}^g_{\mathbb{Q}}(S)$.

**(13.14) Theorem.** *Let $\pi\colon X\to S$ be an abelian scheme of relative dimension $g$ over the smooth quasi-projective scheme $S$. Then the top Chern class $\lambda_g\in\mathrm{CH}^g_{\mathbb{Q}}(S)$ of the Hodge bundle $\mathbb{E}$ vanishes.*

*Proof.* We apply the Grothendieck-Riemann-Roch theorem to the structure sheaf $O_X$ and the morphism $\pi: X\to S$. It says

$$\mathrm{ch}(\pi_!(O_X))=\pi_*(\mathrm{ch}(O_X)\mathrm{Td}((\Omega^1_{X/S})^\vee))=\pi_*(1)\,\mathrm{Td}(\mathbb{E}^\vee),$$

since $\Omega^1_{X/S}=\pi^*(\mathbb{E})$. We know the cohomology of $O_X$:

$$\pi_!(O_X)=1-\mathbb{E}^\vee+\wedge^2\mathbb{E}^\vee-\ldots+(-1)^g\wedge^g\mathbb{E}^\vee.$$

We thus get the identity

$$\mathrm{ch}(1-\mathbb{E}^\vee+\wedge^2\mathbb{E}^\vee-\ldots+(-1)^g\wedge^g\mathbb{E}^\vee)=\pi_*(1)\mathrm{Td}(\mathbb{E}^\vee)=0.$$

A general relation, due to Borel and Serre 1, p. 128 says that for a vector bundle $B$ of rank $r$ one has

$$\sum_{j=0}^{r}(-1)^j\mathrm{ch}(\wedge^jB^\vee)=c_r(B)\,\mathrm{Td}(B)^{-1}.$$

So we see $\lambda_g\,\mathrm{Td}(\mathbb{E}^\vee)=0$. Since Td is invertible the result follows. $\square$

§ 3. *The Fourier transform of an abelian variety.*

**(13.15) Definition.** Let $S$ be a quasi-projective non-singular variety over a field $k$. Let $X$ be an abelian scheme over $S$ with multiplication map $m\colon X\times_S X\to X$. The *Pontryagin product*, or *convolution product*

$$*\colon \mathrm{CH}^*(X)\times\mathrm{CH}^*(X)\longrightarrow\mathrm{CH}^*(X)$$

(relative to $S$) is the map defined by

$$\alpha*\beta=m_*(p_1^*\alpha\cdot p_2^*\beta)\,.$$

Intuitively, the product $\alpha*\beta$ is obtained by adding the points on cycles representing $\alpha$ and $\beta$. Note that the Pontryagin product depends on the base variety $S$, though this is not indicated in the notation.

**(13.16) Lemma.** *Let $g=\dim(X/S)$. The Pontryagin product makes $\mathrm{CH}^*(X)=\oplus_i\mathrm{CH}^i(X)$ into a commutative ring for which the cycle $\big[e(S)\big]\in\mathrm{CH}^g(X)$ given by the identity section $e(S)\subset X$ is the identity element.*

The proof of this fact is straightforward and is left to the reader.

**(13.17) Lemma.** *Let $f\colon X \to Y$ be a homomorphism of abelian schemes over $S$. Then we have $f_*(\alpha * \beta) = f_*(\alpha) * f_*(\beta)$ for all $\alpha$, $\beta \in \mathrm{CH}^*(X)$.*

*Proof.* Denote the projections of $X \times_S X$ (resp. $Y \times_S Y$, resp. $Y \times_S X$) on the two factors by $p_i$ (resp. $q_i$, resp. $r_i$), $i = 1, 2$. Since $f \circ m_X = m_Y \circ (f \times f) = m_Y \circ (\mathrm{id}_Y \times f) \circ (f \times \mathrm{id}_X)$, we have

$$\begin{aligned} f_*(\alpha * \beta) &= f_* m_{X,*}(p_1^*\alpha \cdot p_2^*\beta) \\ &= m_{Y,*}(\mathrm{id}_Y \times f)_*(f \times \mathrm{id}_X)_*(p_1^*\alpha \cdot p_2^*\beta) \\ &= m_{Y,*}(\mathrm{id}_Y \times f)_*(f \times \mathrm{id}_X)_*\big(p_1^*\alpha \cdot (f \times \mathrm{id}_X)^* r_2^*\beta\big) \\ &= m_{Y,*}(\mathrm{id}_Y \times f)_*\big((f \times \mathrm{id}_X)_* p_1^*\alpha \cdot r_2^*\beta\big), \end{aligned}$$

where in the last step we use the projection formula. Applying (1) to the Cartesian diagram

$$\begin{tikzcd} X \times_S X \arrow[r, "f \times \mathrm{id}_X"] \arrow[d, "p_1"'] & Y \times_S X \arrow[d, "r_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

gives that

$$(f \times \mathrm{id}_X)_* p_1^*\alpha = r_1^* f_*\alpha = \big(q_1 \circ (\mathrm{id}_Y \times f)\big)^* f_*\alpha = (\mathrm{id}_Y \times f)^* q_1^* f_*\alpha\,.$$

Again using the projection formula, this gives $f_*(\alpha * \beta) = m_{Y,*}\big(q_1^* f_*\alpha \cdot (\mathrm{id}_X \times f)_* r_2^*\beta\big)$. Finally we apply (1) to the Cartesian diagram

$$\begin{tikzcd} Y \times_S X \arrow[r, "\mathrm{id}_Y \times f"] \arrow[d, "r_2"'] & Y \times_S X \arrow[d, "q_2"] \\ X \arrow[r, "f"] & Y\,. \end{tikzcd}$$

This gives the desired conclusion that $f_*(\alpha * \beta) = m_{Y,*}(q_1^* f_*\alpha \cdot q_2^* f_*\beta) = f_*\alpha * f_*\beta$. □

We now come to the main notion of this chapter.

**(13.18) Definition.** Situation as in (13.15). Let $\ell = c_1(\mathscr{P}_X) \in \mathrm{CH}^1(X \times_S X^t)$ be the class of the Poincaré bundle of $X$. We define the *Fourier transform* $T$ of $X$ as the correspondence from $X$ to $X^t$ given by

$$T = \mathrm{ch}(\mathscr{P}) = \exp(\ell) = 1 + \ell + \frac{1}{2!}\ell^2 + \cdots \quad \in \mathrm{CH}^*_{\mathbb{Q}}(X \times_S X^t)\,.$$

We write

$$\tau_K\colon K(X) \longrightarrow K(X^t) \qquad \text{and} \qquad \tau = \tau_{\mathrm{CH}}\colon \mathrm{CH}^*_{\mathbb{Q}}(X) \longrightarrow \mathrm{CH}^*_{\mathbb{Q}}(X^t)$$

for the homomorphisms associated to the element $[\mathscr{P}] \in K(X \times_S X^t)$, as explained in (13.4). Concretely,

$$\begin{aligned} \tau_K(x) &= p_{X^t,*}(\mathscr{P} \cdot p_X^* x) && \text{for } x \in K(X); \\ \tau_{\mathrm{CH}}(x) &= p_{X^t,*}(e^\ell \cdot p_X^* x) = p_{X^t,*}(T \cdot p_X^* x) && \text{for } x \in \mathrm{CH}^*_{\mathbb{Q}}(X). \end{aligned}$$

**(13.19) Proposition.** *Let $X/S$ be an abelian scheme of relative dimension $g$. Let $\xi^t\colon X^t \to S$ with zero section $e^t\colon S \to X^t$ be the dual abelian scheme. Then we have*

$$\tau_{\mathrm{CH}}(1_X) = (-1)^g \cdot e^t_*(1_S)$$

*in* $\mathrm{CH}^*_{\mathbb{Q}}(X^t)$.

*Proof.* Let $\mathbb{E}$ and $\mathbb{E}^t$ be the Hodge bundles of $X/S$ and $X^t/S$, respectively. By (13.8) we have

$$\mathrm{ch}\big(\tau_K[O_X]\big) = \tau_{\mathrm{CH}}(1_X) \cdot \xi^{t,*}\mathrm{Td}(\mathbb{E}^\vee)\,. \tag{3}$$

On the other hand we can calculate $\tau_K(1_X) = \tau_K[O_X]$ directly. Namely,

$$\begin{aligned}\tau_K[O_X] = p_{X^t,*}\big(\mathscr{P} \cdot p_X^*[O_X]\big) = p_{X^t,*}(\mathscr{P}) &= \sum_{i=0}^{g}(-1)^i \cdot [R^i p_{X^t,*}\mathscr{P}] \\ &= (-1)^g \cdot e^t_*\big(\det(\mathbb{E}^t)^{-1}\big) \\ &= (-1)^g \cdot e^t_*(O_S) \cdot \xi^{t,*}\det(\mathbb{E}^t)^{-1}\,,\end{aligned}$$

according to our calculation of the cohomology of the Poincaré bundle $\mathscr{P}$. Now we apply GRR to the morphism $e\colon S \to X^t$. This gives

$$\mathrm{ch}\big(e^t_*(O_S)\big) \cdot \mathrm{Td}(T_{X^t}) = e^t_*\big(\mathrm{Td}(T_S)\big)\,,$$

hence

$$\mathrm{ch}\big(\tau_K[O_X]\big) = (-1)^g \cdot e^t_*\big(\mathrm{Td}(T_S)\big) \cdot \mathrm{Td}(T_{X^t})^{-1} \cdot \xi^{t,*}\mathrm{ch}\big(\det(\mathbb{E}^t)\big)\,.$$

We have an exact sequence $0 \to \xi^{t,*}\mathbb{E}^{t,\vee} \to T_{X^t} \to \xi^{t,*}T_S \to 0$. This gives the relation $\mathrm{Td}(T_{X^t}) = \xi^{t,*}\mathrm{Td}(\mathbb{E}^{t,\vee}) \cdot \xi^{t,*}\mathrm{Td}(T_S)$. Since $e^{t,*} \circ \xi^{t,*} = \mathrm{id}$ we get, using the projection formula,

$$e^t_*\big(\mathrm{Td}(T_S)\big) \cdot \mathrm{Td}(T_{X^t})^{-1} = e^t_*\big(\mathrm{Td}(\mathbb{E}^{t,\vee})^{-1}\big) = e^t_*(1_S) \cdot \xi^{t,*}\mathrm{Td}(\mathbb{E}^{t,\vee})^{-1}\,.$$

In total this gives

$$\mathrm{ch}\big(\tau_K[O_X]\big) = (-1)^g \cdot e^t_*(1_S) \cdot \xi^{t,*}\Big[\mathrm{Td}(\mathbb{E}^{t,\vee})^{-1} \cdot \mathrm{ch}\big(\det(\mathbb{E}^t)\big)^{-1}\Big]\,. \tag{4}$$

Let $\lambda_1 = c_1(\mathbb{E})$ and $\lambda_1^t = c_1(\mathbb{E}^t)$. As shown in 13.12 we have $\mathrm{Td}(\mathbb{E}^\vee)\mathrm{Td}((\mathbb{E}^t)^\vee) = \exp(-\lambda_1/2 - \lambda_1^t/2)$ and as we remarked in the beginning of section 2 we have $\lambda_1 = \lambda_1^t$. Comparison of the two expressions (3) and (4) gives the desired identity. □

Let $T^t$ be the Fourier transform of $X^t$. It is associated to the Poincaré bundle on $X^t \times X^{tt}$. If we apply the isomorphism $\kappa_X\colon X \xrightarrow{\sim} X^{tt}$ then $T^t$ can be identified with the transpose of the correspondence $T$.

**(13.20) Proposition.** *Let $f\colon X \to Y$ be a homomorphism of abelian schemes over $S$. Then $T_Y \circ [\Gamma_f] = [{}^{\mathrm{t}}\Gamma_{f^t}] \circ T_X$ in $\mathrm{CH}^*_{\mathbb{Q}}(X \times_S Y^t)$. If $f$ is an isogeny then we further have the relation $T_X \circ [{}^{\mathrm{t}}\Gamma_f] = [\Gamma_{f^t}] \circ T_Y$ in $\mathrm{CH}^*_{\mathbb{Q}}(Y \times_S X^t)$.*

*Proof.* Lemma (13.6) gives $T_Y \circ [\Gamma_f] = (f \times \mathrm{id}_{Y^t})^*\mathrm{ch}(\mathscr{P}_Y)$ and $[{}^{\mathrm{t}}\Gamma_{f^t}] \circ T_X = (\mathrm{id}_X \times f^t)^*\mathrm{ch}(\mathscr{P}_X)$. So for the first assertion we have to show that

$$(f \times \mathrm{id}_{Y^t})^*\mathrm{ch}(\mathscr{P}_Y) = (\mathrm{id}_X \times f^t)^*\mathrm{ch}(\mathscr{P}_X)\,. \tag{5}$$

But the dual $f^t$ of $f$ is defined by the identity $(\mathrm{id}_X \times f^t)^*(\mathscr{P}_X) = (f \times \mathrm{id}_{Y^t})^*(\mathscr{P}_Y)$. Applying the Chern character we get (5).

In a similar way, again using (13.6), the second assertion is equivalent to

$$(f \times \mathrm{id}_{X^t})_*\mathrm{ch}(\mathscr{P}_X) = (\mathrm{id}_Y \times f^t)_*\mathrm{ch}(\mathscr{P}_Y)\,. \tag{6}$$

We use the Cartesian diagram

$$\begin{array}{ccc} X \times_S Y^t & \xrightarrow{\mathrm{id}_X \times f^t} & X \times_S X^t \\ {\scriptstyle f\times \mathrm{id}_{Y^t}}\downarrow & & \downarrow{\scriptstyle f\times \mathrm{id}_{X^t}} \\ Y \times_S Y^t & \xrightarrow{\mathrm{id}_Y \times f^t} & Y \times_S X^t \end{array} \quad .$$

This gives the identity

$$\begin{aligned}(\mathrm{id}_Y \times f^t)^*(f \times \mathrm{id}_{X^t})_*\mathrm{ch}(\mathscr{P}_X) &= (f \times \mathrm{id}_{Y^t})_*(\mathrm{id}_X \times f^t)^*\mathrm{ch}(\mathscr{P}_X) \\ &= (f \times \mathrm{id}_{Y^t})_*(f \times \mathrm{id}_{Y^t})^*\mathrm{ch}(\mathscr{P}_Y) \qquad \text{by (5)} \\ &= \deg(f)\mathrm{ch}(\mathscr{P}_Y)\,.\end{aligned}$$

Applying $(\mathrm{id}_Y \times f^t)_*$ to both sides gives $\deg(f^t)(f \times \mathrm{id}_{X^t})_*\mathrm{ch}(\mathscr{P}_X) = \deg(f)(\mathrm{id}_Y \times f^t)_*\mathrm{ch}(\mathscr{P}_Y)$, and if $f$ is an isogeny then (6) follows because $\deg(f^t) = \deg(f) \neq 0$. □

**(13.21) Theorem.** *Let $m$: $X \times_S X \to X$ and $m^t$: $X^t \times_S X^t \to X^t$ be the group laws of $X$ and $X^t$, respectively, let $\Delta$: $X \to X \times_S X$ and $\Delta^t$: $X^t \to X^t \times_S X^t$ be the diagonal morphisms, and let $T\otimes T$ denote the Fourier transform of $X \times_S X$. Then we have identities of correspondences*

$$\begin{aligned} T^t \circ T &= (-1)^g \cdot [{}^{\mathrm{t}}\Gamma_{-\mathrm{id}_X}] && \textit{in} \quad \mathrm{CH}^*_{\mathbb{Q}}(X \times_S X)\,; \\ T \circ [\Gamma_m] &= [{}^{\mathrm{t}}\Gamma_{\Delta^t}] \circ (T \otimes T) && \textit{in} \quad \mathrm{CH}^*_{\mathbb{Q}}(X \times_S X \times_S X^t)\,; \\ T \circ [{}^{\mathrm{t}}\Gamma_{\Delta}] &= (-1)^g \cdot [\Gamma_{m^t}] \circ (T \otimes T) && \textit{in} \quad \mathrm{CH}^*_{\mathbb{Q}}(X \times_S X \times_S X^t)\,. \end{aligned}$$

*Proof.* For the second identity one applies the previous proposition to the homomorphism $m$. (Use Exercise 7.1.)

Next remark that, by definition, the correspondence $T^t \circ T$ on $X \times_S X$ is

$$p_{13,*}(p_{12}^* e^{\ell} \cdot p_{23}^* e^{\ell^t}) = p_{13,*}\big(\exp(p_{12}^*\ell + p_{23}^*\ell^t)\big)\,.$$

Let $\mu$: $X \times_S X^t \times_S X \to X \times_S X^t$ be the homomorphism given on points by $(a,b,c) \mapsto (a+c, b)$ and let $s$: $X \times_S X^t \to X^t \times_S X$ be the map reversing the factors. Let $\mathscr{P}$ be the Poincaré bundle on $X \times_S X^t$. In $\mathrm{Pic}_{(X\times_S X^t \times_S X)/S}$ we have the identity

$$p_{12}^*(\mathscr{P}) + p_{23}^* s^*(\mathscr{P}) = \mu^*(\mathscr{P})\,, \tag{7}$$

as follows from the Theorem of the Cube by checking that the two sides have the same restrictions to $X \times_S X^t \times e(S)$, to $X \times e(S) \times X$ and to $e(S) \times X^t \times X$. So we find that $T^t \circ T = p_{13,*}(e^{\mu^*(\ell)}) = p_{13,*}(\mu^* e^{\ell})$. From the Cartesian diagram

$$\begin{array}{ccc} X \times_S X^t \times_S X & \xrightarrow{\mu} & X \times_S X^t \\ {\scriptstyle p_{13}}\downarrow & & \downarrow{\scriptstyle p_1} \\ X \times_S X & \xrightarrow{m} & X \end{array}$$

we get $T^t \circ T = m^* p_{1,*}(e^\ell) = m^* \tau^t_{\mathrm{CH}}(1_{X^t})$. Application of Prop. (13.19) then gives $T^t \circ T = (-1)^g m^* e_*(1_S)$. But by the Cartesian square

$$\begin{array}{ccc} X & \xrightarrow{(\mathrm{id}_X, -\mathrm{id}_X)} & X \times X \\ \downarrow & & \downarrow{\scriptstyle m} \\ \mathrm{Spec}(k) & \xrightarrow{\quad e \quad} & X \end{array}$$

we get $m^* e_*(1_S) = \Gamma_{-\mathrm{id}_X}$. This proves the first identity.

For the third identity, start from the relation

$$T^t \circ [\Gamma_{m^t}] = [{}^{\mathrm{t}}\Gamma_\Delta] \circ (T^t \otimes T^t)\,,$$

which is the second identity for $X^t$. Multiply by $T$ from the left, by $(T \otimes T)$ from the right, and use the first identity (both for $X^t$ and for $X \times_S X$). This gives

$$\begin{aligned}(-1)^g \cdot [{}^{\mathrm{t}}\Gamma_{-\mathrm{id}_{X^t}}] \circ [\Gamma_{m^t}] \circ (T \otimes T) &= T \circ [{}^{\mathrm{t}}\Gamma_\Delta] \circ [{}^{\mathrm{t}}\Gamma_{-\mathrm{id}_{X \times X}}] \\ &= T \circ [{}^{\mathrm{t}}\Gamma_{-\mathrm{id}_X}] \circ [{}^{\mathrm{t}}\Gamma_\Delta]\,.\end{aligned}$$

Now observe that $T \circ [{}^{\mathrm{t}}\Gamma_{-\mathrm{id}_X}] = [{}^{\mathrm{t}}\Gamma_{-\mathrm{id}_{X^t}}] \circ T$, because both equal $\exp(-\ell)$. Since $[{}^{\mathrm{t}}\Gamma_{-\mathrm{id}_{X^t}}] = [\Gamma_{-\mathrm{id}_{X^t}}]$ is a unit in the ring of correspondences from $X^t$ to itself, this proves the third identity. □

**(13.22) Corollary.** *Situation as in* (13.15). *Let* $g = \dim(X/S)$.

(i) *We have* $\tau^t_{\mathrm{CH}} \circ \tau_{\mathrm{CH}} = (-1)^g(-\mathrm{id}_X)^*$. *For all* $x$, $y \in \mathrm{CH}^*_{\mathbb{Q}}(X)$ *we have the relations* $\tau_{\mathrm{CH}}(x * y) = \tau_{\mathrm{CH}}(x) \cdot \tau_{\mathrm{CH}}(y)$ *and* $\tau_{\mathrm{CH}}(x \cdot y) = (-1)^g \tau_{\mathrm{CH}}(x) * \tau_{\mathrm{CH}}(y)$.

(ii) *For a homomorphism* $f\colon X \to Y$ *we have* $\tau_Y \circ f_* = f^{t,*} \circ \tau_X$. *If* $f$ *is an isogeny then also* $\tau_X \circ f^* = f^t_* \circ \tau_Y$.

*Proof.* These relations follow directly from Prop. (13.20) and Thm. (13.21). For example, for ii) note that $T \circ [\Gamma_m]$ induces a map $\mathrm{CH}^*_{\mathbb{Q}}(X \times_S X) \to \mathrm{CH}^*_{\mathbb{Q}}(X^t)$ with $p_1^*\alpha \cdot p_2^*\beta \mapsto \tau m_*(p_1^*\alpha \cdot p_2^*\beta) = \tau(\alpha * \beta)$. On the other hand, since $P_{X \times_S X} = p_1^* P_X \otimes p_2^* P_X$ we have

$$\begin{aligned}\tau_{X \times_S X}(p_1^*\alpha \cdot p_2^*\beta) &= p_{X^t \times_S X^t}(p_1^*(\alpha \cdot P_X) p_2^*(\beta \cdot P_X)) \\ &= p_1'^*(\tau(\alpha) \cdot p_2'^*(\tau(\beta))\end{aligned}$$

with $p_i'$ the projections of $X^t \times_S X^t$ onto its factors. Now $[{}^t\Gamma_{\Delta^t}]$ induces $(\Delta^t)^*$ so that $[{}^t\Gamma_{\Delta^t}] \circ T \otimes T$ induces a map sending $p_1^*(\alpha) \cdot p_2^*(\beta)$ to $\tau(\alpha) \cdot \tau(\beta)$. □

As another corollary we obtain the following elegant result.

**(13.23) Theorem.** *The Fourier transform of* $X$ *induces an isomorphism of rings*

$$\tau = \tau_{\mathrm{CH}}\colon \big(\mathrm{CH}^*_{\mathbb{Q}}(X), *\big) \xrightarrow{\sim} \big(\mathrm{CH}^*_{\mathbb{Q}}(X^t), \cdot\big)\,,$$

*where* $\cdot$ *and* $*$ *denote the intersection product and the convolution product, respectively.*

This theorem should justify the name Fourier transform. Just like the Fourier transform for functions on the real line which transform the convolution product into the usual product our Fourier transform interchanges the Pontryagin product, which one can see as a sort of convolution product, with the usual intersection product.

§ 4. *Decomposition of the diagonal.*

**(13.24)** For any reasonable cohomology theory with a Künneth formula, Poincaré duality and a cycle class map we have for an abelian variety $X$ of dimension $g$

$$H^{2g}(X \times_k X) = \bigoplus_{i=0}^{2g} H^{2g-i}(X) \otimes_L H^i(X) = \bigoplus_{i=0}^{2g} H^i(X)^\vee \otimes_L H^i(X) = \bigoplus_{i=0}^{2g} \mathrm{End}_L\bigl(H^i(X)\bigr)\,.$$

The diagonal class $cl(\Delta_X) \in H^{2g}(X \times_k X)$ corresponds to the element $\oplus \mathrm{id}_{H^i(X)}$. Hence we can write

$$cl(\Delta_X) = \gamma_0 + \gamma_1 + \cdots + \gamma_{2g}\,,$$

with $\gamma_i \in \mathrm{End}_L\bigl(H^i(X)\bigr)$. The classes $\gamma_i$ are called the Künneth components of the diagonal. Standard conjectures, as discussed for instance in Kleiman [1], predict that these classes are algebraic. That is, there should exist codimension $g$ cycles $D_i$ on $X \times_k X$ such that $[\Delta_X] = D_0 + D_1 + \cdots + D_{2g}$ and $cl(D_i) = \gamma_i$. The main result of this section establishes the existence of such algebraic classes.

Throughout this section, let $S$ be a smooth connected quasi-projective scheme of dimension $d$ over a field $k$. We consider an abelian scheme $f\colon X \to S$ of relative dimension $g$. Recall that if $\xi \in \mathrm{CH}^*_{\mathbb{Q}}(X \times_S X)$ then we define its transpose ${}^{\mathrm{t}}\xi \in \mathrm{CH}^*_{\mathbb{Q}}(X \times_S X)$ by ${}^{\mathrm{t}}\xi := s_*(\xi)$, where $s\colon X \times_S X \to X \times_S X$ is the automorphism switching the two factors.

If $x \in X(S)$ is a section of $f$, we define the graph class $[\Gamma_x]$ of $x$ by

$$[\Gamma_x] := x_*[S] = \bigl[x(S)\bigr] \in \mathrm{CH}^g_{\mathbb{Q}}(X)\,.$$

In particular, $[\Gamma_e]$ is the identity element of $\mathrm{CH}^*_{\mathbb{Q}}(X)$ for the Pontryagin product.

Further, let $i_x := x \times 1_{X^t}\colon S \times_S X^t \to X \times_S X^t$, and consider the pull-back $i_x^*(\ell) \in \mathrm{CH}^1_{\mathbb{Q}}(X^t)$ of the class of the Poincaré bundle. The following two formulas, due to Beauville, give relations between $i_x^*(\ell)$ and the graph classes $[\Gamma_x]$.

**(13.25) Lemma.** *For all $x \in X(S)$ we have*

$$\tau\bigl([\Gamma_x]\bigr) = \exp(i_x^*\ell) \qquad \textit{and} \qquad \tau^t(i_x^*\ell) = (-1)^{g+1} \sum_{j=1}^{g+d} \frac{(-1)^j}{j} \cdot \bigl([\Gamma_x] - [\Gamma_e]\bigr)^{*j}\,.$$

*Proof.* We have $\tau\bigl([\Gamma_x]\bigr) = p_{X^t\,*}(p_X^* x_*[S] \cdot e^\ell) = p_{X^t\,*} i_{x*}\bigl([X^t] \cdot i_x^* e^\ell\bigr) = e^{i_x^*\ell}$. This proves the first relation. Further, in $\mathrm{CH}^*_{\mathbb{Q}}(X^t)$ we have the identity

$$i_x^*\ell = \log\bigl(1 - (1 - e^{i_x^*\ell})\bigr) = -\sum_{j=1}^{\infty} \frac{1}{j}(1 - e^{i_x^*\ell})^j\,.$$

Note that for dimension reasons a term of the form $\bigl(1 - \exp(i_x^*\ell)\bigr)^j$ vanishes for $j > \dim X^t = g + d$. By our first identity and Cor. (13.22) we have

$$\tau^t\bigl((1 - e^{i_x^*\ell})^j\bigr) = \tau^t \circ \tau\Bigl(\bigl([\Gamma_e] - [\Gamma_x]\bigr)^{*j}\Bigr) = (-1)^g(-1)^j\bigl([\Gamma_x] - [\Gamma_e]\bigr)^{*j}\,,$$

and combining this with the previous formula this gives the second relation. □

**(13.26) Lemma.** *For $x$, $y \in X(S)$ we have $[\Gamma_x] * [\Gamma_y] = [\Gamma_{x+y}]$.*

*Proof.* By the Theorem of the Square, $i_{x+y}^*\ell = i_x^*\ell + i_y^*\ell$. This implies that $\tau\bigl([\Gamma_x] * [\Gamma_y]\bigr) = \tau\bigl([\Gamma_x]\bigr)\tau\bigl([\Gamma_y]\bigr) = e^{i_x^*\ell}e^{i_y^*\ell} = e^{i_x^*\ell + i_y^*\ell} = \tau\bigl([\Gamma_{x+y}]\bigr)$. Now apply Thm. (13.23). $\square$

These formulas can be used to deduce a vanishing property. Let $I(X/S)$ be the $\mathbb{Q}$-subspace of $\mathrm{CH}^g_{\mathbb{Q}}(X)$ generated by the elements $[\Gamma_x] - [\Gamma_e]$ for all $x \in X(S)$. By Lemma (13.26), $I(X/S)$ is a subring of $\mathrm{CH}^g_{\mathbb{Q}}(X)$ with respect to the ring structure defined by the Pontryagin product.

**(13.27) Proposition.** *Let $d = \dim(S)$ and $g = \dim(X/S)$. Then $I(X/S)^{*(g+d+1)} = 0$.*

*Proof.* The Fourier transform of a product

$$\bigl([\Gamma_{x_1}] - [\Gamma_e]\bigr) * \bigl([\Gamma_{x_2}] - [\Gamma_e]\bigr) * \cdots * \bigl([\Gamma_{x_n}] - [\Gamma_e]\bigr)$$

equals $(\exp(i_{x_1}^*\ell) - 1) \cdot (\exp(i_{x_2}^*\ell) - 1) \cdots (\exp(i_{x_n}^*\ell) - 1)$, and for dimension reasons this expression vanishes if $n > \dim(X^t) = g + d$. By Thm. (13.23) the result follows. $\square$

In view of Lemma (13.25) we now put

$$\log\bigl([\Gamma_x]\bigr) := (-1)^{g+1} \cdot \tau^t(i_x^*\ell)\,.$$

This is a well-defined element of $I(X/S)$.

**(13.28) Corollary.** *The map $X(S) \to I(X/S)$ given by $x \mapsto \log\bigl([\Gamma_x]\bigr)$ is a group homomorphism.*

*Proof.* This follows from the identity of formal power series $\log\bigl((1+x)(1+y)\bigr) = \log(1+x) + \log(1+y)$. $\square$

**(13.29) Theorem.** (Deninger, Murre) *There is a unique decomposition of the class of the diagonal in $\mathrm{CH}^*_{\mathbb{Q}}(X \times_S X)$,*

$$[\Delta_{X/S}] = \sum_{i=0}^{2g} \pi_i \tag{8}$$

*such that*

$$\pi_i \circ \pi_j = \begin{cases} 0 & \text{if } i \neq j, \\ \pi_i & \text{if } i = j, \end{cases}$$

*and such that*

$$[{}^{\mathrm{t}}\Gamma_{n_X}] \circ \pi_i = n^i \pi_i \qquad \text{for all } n \in \mathbb{Z}\,. \tag{9}$$

*Moreover,*

(i) $\pi_i \circ [{}^{\mathrm{t}}\Gamma_{n_X}] = n^i \pi_i$ *for all $n \in \mathbb{Z}$;*
(ii) ${}^{\mathrm{t}}\pi_i = \pi_{2g-i}$*;*
(iii) *if $f\colon X \to Y$ is a homomorphism then $[{}^{\mathrm{t}}\Gamma_f] \circ \pi_{i,Y} = \pi_{i,X} \circ [{}^{\mathrm{t}}\Gamma_f]$.*

*Proof.* First we prove unicity. Suppose $\{\pi_i'\}$ is another collection of elements satisfying (8) and (9). Then $\sum_{i=0}^{2g} n^i(\pi_i - \pi_i') = 0$ for every integer $n$; hence $\pi_i = \pi_i'$ for every $i$.

Let us consider $X \times_S X$ as an abelian scheme over $X$ via $p_1\colon X \times_S X \to X$. We also consider the convolution product on $\mathrm{CH}^*_{\mathbb{Q}}(X \times_S X)$ relative to the base scheme $X$. If $n \in \mathbb{Z}$ then the morphism $X \to X \times_S X$ given by $x \mapsto (x, nx)$ is a section of $X \times_S X$ over $X$; its graph class is none other than the class $[\Gamma_{n_X}] \in \mathrm{CH}^g_{\mathbb{Q}}(X \times_S X)$ of the graph of $n_X$. If there is no risk of confusion we simply write $[\Gamma_n]$ for this class. In particular, $[\Gamma_{\mathrm{id}}] = [\Gamma_1] = [\Delta]$ and $[\Gamma_e] = [\Gamma_0] = [X \times e(S)]$. (Here the "$e$" in $\Gamma_e$ has to be interpreted as the identity section of $X \times_S X$ over $X$.)

For $i \leqslant 2g$, define $\pi_i \in \mathrm{CH}^*_{\mathbb{Q}}(X \times_S X)$ by

$$\pi_i := \frac{1}{(2g-i)!}\log\bigl([\Gamma_{\mathrm{id}}]\bigr)^{*(2g-i)} = \frac{1}{(2g-i)!}\Bigl(\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\bigl([\Gamma_{\mathrm{id}}]-[\Gamma_e]\bigr)^{*j}\Bigr)^{*(2g-i)} .$$

Note that $\pi_i = 0$ for $i < -d$ and $\pi_{2g} = \bigl[X \times e(S)\bigr]$. By the identity $\exp\bigl(\log(1+x)\bigr) = 1+x$ of formal power series we have

$$[\Delta] = [\Gamma_{\mathrm{id}}] = \sum_{i=-d}^{2g} \pi_i \,. \tag{10}$$

By Lemmas (13.6) and (13.17) we have $[\Gamma_n]\circ(\alpha * \beta) = \bigl([\Gamma_n]\circ\alpha\bigr) * \bigl([\Gamma_n]\circ\beta\bigr)$. Combining this with (13.26) and (13.28) we get

$$\begin{aligned}[\Gamma_n]\circ\pi_i &= \frac{1}{(2g-i)!}\log\bigl([\Gamma_n]\bigr)^{*(2g-i)}\\ &= \frac{1}{(2g-i)!}\log\bigl([\Gamma_{\mathrm{id}}]^{*n}\bigr)^{*(2g-i)}\\ &= \frac{1}{(2g-i)!}\bigl(n\,\log([\Gamma_{\mathrm{id}}])\bigr)^{*(2g-i)} = n^{2g-i}\pi_i \,.\end{aligned} \tag{11}$$

So we have $[\Gamma_n] = [\Gamma_n]\circ\Delta = [\Gamma_n]\circ\sum_{i=-d}^{2g}\pi_i = \sum_{i=-d}^{2g} n^{2g-i}\pi_i$; hence $n^{2g-j}\pi_j = [\Gamma_n]\circ\pi_j = \sum_{i=-d}^{2g} n^{2g-i}\pi_i\circ\pi_j$. As this holds for every integer $n$, it follows that

$$\pi_i\circ\pi_j = \begin{cases} 0 & \text{if } i \neq j,\\ \pi_j & \text{if } i = j.\end{cases}$$

From the relation $[\Gamma_n] = \sum n^{2g-j}\pi_j$ we get that $\pi_i\circ[\Gamma_n] = n^{2g-i}\pi_i$. Furthermore, we have $[\Gamma_n]\circ[{}^{\mathrm{t}}\Gamma_n] = n^{2g}\Delta$, and so $n^{2g-i}\pi_i\circ[{}^{\mathrm{t}}\Gamma_n] = \pi_i\circ[\Gamma_n]\circ[{}^{\mathrm{t}}\Gamma_n] = n^{2g}\pi_i$. We find that $[\Gamma_n]\circ{}^{\mathrm{t}}\pi_i = {}^{\mathrm{t}}\bigl(\pi_i\circ[{}^{\mathrm{t}}\Gamma_n]\bigr) = n^i\cdot{}^{\mathrm{t}}\pi_i$. Now remark that the relations (10) and (11) uniquely determine the collection $\{\pi_i\}$—the argument is the same as for the unicity with respect to the relations (8) and (9). But what we have shown means that the collection of elements $\{{}^{\mathrm{t}}\pi_{2g-i}\}$ satisfies (10) and (11) too, and (ii) follows. This also implies that $\pi_i = 0$ for $i < 0$, so (10) reduces to (8). Further, (9) and (i) follow by transposition from the relations that we have already proven.

To prove (iii) we let $c_{ij} = \pi_{j,X}\circ[{}^{\mathrm{t}}\Gamma_f]\circ\pi_{i,Y}$. Then

$$\begin{aligned} n^i c_{ij} &= \pi_{j,X}\circ[{}^{\mathrm{t}}\Gamma_f]\circ n^i\pi_{i,Y}\\ &= \pi_{j,X}\circ[{}^{\mathrm{t}}\Gamma_f]\circ[{}^{\mathrm{t}}\Gamma_n]\circ\pi_{i,Y}\\ &= \pi_{j,X}\circ[{}^{\mathrm{t}}\Gamma_n]\circ[{}^{\mathrm{t}}\Gamma_f]\circ\pi_{i,Y} = n^j c_{ij}\,,\end{aligned}$$

which implies that $c_{ij} = 0$ unless $i = j$. Hence

$$\begin{aligned}[{}^{\mathrm{t}}\Gamma_f]\circ\pi_{i,Y} &= [\Delta_X]\circ[{}^{\mathrm{t}}\Gamma_f]\circ\pi_{i,Y}\\ &= c_{ii}\\ &= \pi_{i,X}\circ[{}^{\mathrm{t}}\Gamma_f]\circ[\Delta_Y] = \pi_{i,X}\circ[{}^{\mathrm{t}}\Gamma_f]\,.\end{aligned}$$

This completes the proof of the theorem. $\square$

**(13.30) Example.** As remarked in the proof, we have $\pi_{2g} = \big[X \times e(S)\big]$. Combining this with (ii) gives that $\pi_0 = \big[e(S) \times X\big]$.

Next consider an elliptic curve $E$ over a field $k$. By formula (8) and the previous remark, we should have

$$\pi_1 = \big[\Delta_E\big] - \big[\{0\} \times E\big] - \big[E \times \{0\}\big] .$$

On the other hand, we have defined $\pi_1 \in \mathrm{CH}^*_{\mathbb{Q}}(E \times_k E)$ to be

$$\log\big([\Gamma_{\mathrm{id}}]\big) = \big([\Delta] - [E \times \{0\}]\big) - \frac{1}{2} \cdot \big([\Delta] - [E \times \{0\}]\big)^{*2} ,$$

where the Pontryagin is computed on $E \times_k E$, viewed as an abelian scheme over $E$ via the first projection. Using Lemma (13.26) we find

$$\pi_1 = 2 \cdot \big[\Delta_E\big] - \frac{3}{2} \cdot \big[E \times \{0\}\big] - \frac{1}{2} \cdot \big[\Gamma_2\big] ,$$

where $\Gamma_2 \subset E \times_k E$ is the graph of multiplication by 2. To see that the two ansers for $\pi_1$ agree we should check that

$$\big[\Gamma_2\big] + \big[E \times \{0\}\big] - 2 \cdot \big[\Delta_E\big] - 2 \cdot \big[\{0\} \times E\big] = 0 \tag{12}$$

in $\mathrm{CH}^1_{\mathbb{Q}}(E \times_k E)$. This is indeed the case, for if $E$ is given by a Weierstrass equation $f(X,Y) = 0$ for some cubic $f(X,Y) \in k[X,Y]$ then

$$(P,Q) \mapsto \frac{x_Q - x_{2P}}{\big(\partial f/\partial X\big)(P) \cdot (x_Q - x_P) + \big(\partial f/\partial Y\big)(P) \cdot (y_Q - y_P)} \tag{13}$$

is a rational function on $E \times E$ whose divisor is precisely the left hand side of (12). (Note that the restriction of the LHS of (12) to $\{P\} \times E$ equals $[2P] + [0] - 2[P]$. This is the divisor of the rational function $l_1/l_2$ where $l_1$ is the linear form that defines the line through $2P$ and 0, and where $l_2$ is the linear form that defines the tangent space at $P$. Working this out in coordinates, $l_1$ and $l_2$ give precisely the numerator and denominator in (13).)

**(13.31)** The interpretation of Thm. (13.29) is that the motive of $X$ decomposes as a direct sum of $2g$ submotives—this point of view shall be further discussed in § 4 below. Let us now already make the connection with cohomology theory. For this, consider any Weil cohomology $X \mapsto H^\bullet(X)$, defined for varieties over a ground field $k$, with coefficients in a field $L$ of characteristic 0. In particular, we have a Künneth formula, Poincaré duality, and a cycle class map $cl$: $\mathrm{CH}^*_{\mathbb{Q}}(X) \to H^\bullet(X)$ mapping $\mathrm{CH}^i_{\mathbb{Q}}(X)$ into $H^{2i}(X)$.

Let $g = \dim(X)$. By the Künneth decomposition and Poincaré duality we have

$$H^{2g}(X \times_k X) = \bigoplus_{i=0}^{2g} H^{2g-i}(X) \otimes_L H^i(X) = \bigoplus_{i=0}^{2g} H^i(X)^\vee \otimes_L H^i(X) = \bigoplus_{i=0}^{2g} \mathrm{End}_L\big(H^i(X)\big) .$$

The diagonal class $cl(\Delta_X) \in H^{2g}(X \times_k X)$ corresponds to the element $\oplus \mathrm{id}_{H^i(X)}$. Hence we can write

$$cl(\Delta_X) = \gamma_0 + \gamma_1 + \cdots + \gamma_{2g} ,$$

with $\gamma_i \in \mathrm{End}_L\bigl(H^i(X)\bigr)$. The classes $\gamma_i$ are called the Künneth components of the diagonal. Standard conjectures, as discussed for instance in Kleiman [1], predict that these classes are algebraic. That is, there should exist codimension $g$ cycles $D_i$ on $X \times_k X$ such that $[\Delta_X] = D_0 + D_1 + \cdots + D_{2g}$ and $cl(D_i) = \gamma_i$. For abelian varieties, this is exactly what Theorem (13.29) achieves, as we shall now prove.

**(13.32) Corollary.** *Let $k$ be a field, and let $X \mapsto H^\bullet(X)$ be any Weil cohomology for $k$-varieties, with coefficients in a field of characteristic $0$. Then for any abelian variety $X$ the Künneth components of the diagonal are algebraic; more precisely, the classes $\pi_i$ in (8) satisfy $cl(\pi_i) = \gamma_i$. Further we have $H^\bullet(X) \cong \wedge^\bullet H^1(X)$, and $n_X$ induces multiplication by $n^i$ on $H^i(X)$.*

*Proof.* Let $g := \dim(X)$. We make $H^\bullet := H^\bullet(X)$ into a graded bialgebra by taking $m^*$ as co-multiplication and $e^*$ as augmentation, cf. (6.14) where we used a similar construction for the cohomology of the structure sheaf. By the Borel-Hopf Theorem (6.12) we have $H^\bullet = H_1^\bullet \otimes \cdots \otimes H_r^\bullet$, with $H_i^\bullet$ generated by a single element $x_i$ of degree $d_i > 0$. Note that the degrees $d_i$ are odd. Indeed, if $d_i$ were even then $x_i^q \neq 0$ for all $q > 0$, which is absurd; see the restrictions discussed in (iv) of (6.11), and see Exercise (6.4). It follows that the elements $x_i$, which are primitive in the sense of (6.16), satisfy $x_i^2 = 0$; see again Exercise (6.4). This means that $H^\bullet$ is a product of exterior algebras; more precisely: if $V_j \subset H^\bullet$ is the span of the elements $x_i$ for which $d_i = j$ then we have

$$H^\bullet \cong \bigotimes_{j \text{ odd}} \bigl(\wedge^\bullet V_j\bigr)$$

as graded bialgebras. In particular, if $r_j := \dim(V_j)$ then

$$H^{2g} = \bigl(\wedge^{r_1} V_1\bigr) \otimes \bigl(\wedge^{r_3} V_3\bigr) \otimes \cdots \otimes \bigl(\wedge^{r_{2g-1}} V_{2g-1}\bigr)\,, \tag{14}$$

and by comparison of the degrees this gives the relation

$$2g = r_1 + 3r_3 + 5r_5 + \cdots + (2g-1)r_{2g-1}\,. \tag{15}$$

We are going to show that $r_j = 0$ for $j > 1$.

We have $cl(\Delta_X) = \sum_{i=0}^{2g} cl(\pi_i)$, and the elements $cl(\pi_i) \in \mathrm{End}_L(H^\bullet)$ are projectors. Let us provisionally write $H^\bullet\{i\}$ for the image of $cl(\pi_i)$. It follows from (9) that $H^\bullet\{i\} \subset H^\bullet$ is precisely the subspace on which $n_X$ induces multiplication by $n^i$.

Suppose $h \in H^\bullet$ is a primitive element in the sense of (6.16). As $2_X$ equals the composition $m \circ \Delta\colon X \to X \times_k X \to X$, we find that $2_X^*(h) = \Delta^* m^*(h) = \Delta^*(h \otimes 1 + 1 \otimes h) = 2h$. Hence for every $n$ which is a power of 2 we have $n_X^*(h) = nh$, and this suffices to conclude that $h \in H^\bullet\{1\}$. But the elements of $V := V_1 \oplus V_3 \oplus \cdots \oplus V_{2g-1}$ are all primitive; hence $V \subset H^\bullet\{1\}$. This implies that

$$\bigl(\wedge^{r_1} V_1\bigr) \otimes \bigl(\wedge^{r_3} V_3\bigr) \otimes \cdots \bigl(\wedge^{r_{2g-1}} V_{2g-1}\bigr) \subseteq H^\bullet\{s\} \quad \text{with} \quad s = r_1 + r_3 + \cdots + r_{2g-1}\,.$$

On the other hand, we know that $n_X$ acts as multiplication by $n^{2g}$ on $H^{2g}$, as $H^{2g}$ is spanned by the cohomology class of a point. So it follows from (14) that $s = 2g$, and comparison with (15) gives that $r_1 = 2g$ and $r_j = 0$ for $j > 1$. Hence $H^\bullet = \wedge^\bullet H^1$ with $H^1 = V_1 \subset H^\bullet\{1\}$, so $n_X$ induces multiplication by $n^i$ on $H^i$. This last property also implies that $cl(\pi_i) = \gamma_i$. $\square$

**(13.33)** Let $X$ be an abelian variety over a field $k$. We now study the effect of $n_X$ on $\mathrm{CH}^i_{\mathbb{Q}}(X)$. The elements $\pi_l$ of (13.29) give rise to a collection of orthogonal idempotents in $\mathrm{End}_{\mathbb{Q}}\bigl(\mathrm{CH}^i_{\mathbb{Q}}(X)\bigr)$.

Accordingly, we can decompose $\mathrm{CH}^i_{\mathbb{Q}}(X)$ as a direct sum of subspaces. To make this more precise, let us define

$$\mathrm{CH}^{i,j}_{\mathbb{Q}}(X) := \bigl\{\alpha \in \mathrm{CH}^i_{\mathbb{Q}} \bigm| n_X^*(\alpha) = n^{2i-j}\alpha \quad \text{for all } n\bigr\}\,.$$

It follows from (9) that $\mathrm{CH}^{i,j}_{\mathbb{Q}}(X)$ is precisely the subspace of $\mathrm{CH}^i_{\mathbb{Q}}(X)$ that is cut out by the idempotent $\pi_{2i-j}$.

For example, for $i = 1$ we have $\mathrm{CH}^1(X) = \mathrm{Pic}(X)$. We know that

$$\mathrm{Pic}^0(X) = \bigl\{[L] \in \mathrm{Pic}(X) \bigm| n^*[L] = [L^{\otimes n}] \quad \text{for all } n\bigr\}\,,$$

and we may also consider the symmetric line bundles

$$\begin{aligned}\mathrm{Pic}^{\mathrm{symm}}(X) &:= \bigl\{[L] \in \mathrm{Pic}(X) \bigm| L \text{ is symmetric}\bigr\}\\ &= \bigl\{[L] \in \mathrm{Pic}(X) \bigm| n^*[L] = [L^{\otimes n^2}] \quad \text{for all } n\bigr\}\,,\end{aligned}$$

where the last equality follows from Cor. (2.12). After tensoring with $\mathbb{Q}$ we can invert 2 and we have a direct sum decomposition

$$\begin{aligned}\mathrm{CH}^1_{\mathbb{Q}}(X) &= \bigl(\mathrm{Pic}^0(X) \otimes \mathbb{Q}\bigr) \oplus \bigl(\mathrm{Pic}^{\mathrm{symm}}(X) \otimes \mathbb{Q}\bigr)\\ &= \mathrm{CH}^{1,1}_{\mathbb{Q}}(X) \oplus \mathrm{CH}^{1,0}_{\mathbb{Q}}(X)\,.\end{aligned}$$

(Cf. the comments after Cor. (2.12).) It is this decomposition that we shall now generalize.

**(13.34) Lemma.** *Let $x \in \mathrm{CH}^i_{\mathbb{Q}}(X)$, and write $\tau_{\mathrm{CH}}(x) = \sum_{j=0}^g \xi_j$ with $\xi_j \in \mathrm{CH}^j_{\mathbb{Q}}(X^t)$. Then $\xi_j \in \mathrm{CH}^{j,g-i+j}_{\mathbb{Q}}(X^t)$.*

*Proof.* Recall that we write $\ell \in \mathrm{CH}^1_{\mathbb{Q}}(X \times X^t)$ for the class of the Poincaré bundle. We have $(\mathrm{id} \times n)^*\ell = n \cdot \ell$, and, by definition, $\tau_{\mathrm{CH}}(x) = p_{X^t,*}\bigl(p_X^*(x) \cdot \exp(\ell)\bigr)$. Hence

$$\xi_j = p_{X^t*}\Bigl(p_X^*(x) \cdot \frac{\ell^{g-i+j}}{(g-i+j)!}\Bigr)\,,$$

so

$$n^*(\xi_j) = p_{X^t*}(\mathrm{id} \times n)^*\Bigl(p_X^*(x) \cdot \frac{\ell^{g-i+j}}{(g-i+j)!}\Bigr) = p_{X^t*}\Bigl(p_X^*(x) \cdot \frac{(n \cdot \ell)^{g-i+j}}{(g-i+j)!}\Bigr) = n^{g-i+j}\xi_j\,,$$

which is what we want. $\square$

**(13.35) Proposition.** *For $\alpha \in \mathrm{CH}^i_{\mathbb{Q}}(X)$ and $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$, the following are equivalent:*

(i) *$\alpha \in \mathrm{CH}^{i,j}_{\mathbb{Q}}(X)$;*
(ii) *$n^*(\alpha) = n^{2i-j}\alpha$;*
(iii) *$n_*(\alpha) = n^{2g-2i+j}\alpha$;*
(iv) *$\tau_{\mathrm{CH}}(\alpha) \in \mathrm{CH}^{g-i+j}_{\mathbb{Q}}(X^t)$;*
(v) *$\tau_{\mathrm{CH}}(\alpha) \in \mathrm{CH}^{g-i+j,j}(X^t)$.*

*Proof.* That (i) implies (ii) is just the definition of $\mathrm{CH}^{i,j}_{\mathbb{Q}}$. For the implication (ii) $\Rightarrow$ (iii) we use that $n_*n^*$ is multiplication by $n^{2g}$ on $\mathrm{CH}^*_{\mathbb{Q}}(X)$. To see that (iii) implies (iv) we use (ii) of Cor. (13.22), which gives

$$n^*\tau(\alpha) = \tau(n_*\alpha) = n^{2g-2i+j}\tau(\alpha)\,. \tag{16}$$

Since $|n| > 1$ this implies, by the preceding lemma, that $\tau(\alpha) \in \mathrm{CH}^{g-i+j}_{\mathbb{Q}}(X^t)$. The implication (iv) $\Rightarrow$ (v) is again the preceding lemma.

We now have shown that (i) implies (v). Next assume that (v) holds, and apply (i) $\Rightarrow$ (v) to the class $\tau_{\mathrm{CH}}(\alpha)$ on the dual abelian variety. We get that $\tau^t\tau(\alpha) \in \mathrm{CH}^{i,j}(X)$. By Cor. (13.22) this means that $(-1)^*\alpha \in \mathrm{CH}^{i,j}_{\mathbb{Q}}(X)$, which implies that $\alpha = (-1)^*(-1)^*\alpha \in \mathrm{CH}^{i,j}_{\mathbb{Q}}(X)$. $\square$

**(13.36) Corollary.** *The Fourier transform gives a bijection*

$$\tau_{\mathrm{CH}}\colon \mathrm{CH}^{i,j}_{\mathbb{Q}}(X) \xrightarrow{\sim} \mathrm{CH}^{g-i+j,j}_{\mathbb{Q}}(X^t)\,.$$

**(13.37) Theorem.** *We have*

$$\mathrm{CH}^{i}_{\mathbb{Q}}(X) = \bigoplus_{j=i-g}^{i} \mathrm{CH}^{i,j}_{\mathbb{Q}}(X)\,.$$

*If $\xi \in \mathrm{CH}^{i,j}_{\mathbb{Q}}(X)$ and $\eta \in \mathrm{CH}^{r,s}_{\mathbb{Q}}(X)$ then $\xi\cdot\eta \in \mathrm{CH}^{i+r,j+s}$ and $\xi * \eta \in \mathrm{CH}^{i+r-g,j+s}_{\mathbb{Q}}$.*

*Proof.* It follows from (13.36) that $\mathrm{CH}^{i,j}_{\mathbb{Q}}(X)$ vanishes if $j > i$ or $j < i-g$, since then $g-i+j$ lies outside the range $[0,g]$. It is clear that $\xi\cdot\eta$ lies in $\mathrm{CH}^{i+r,j+s}$, and the last assertion follows from this using Thm. (13.23) and Cor. (13.36). $\square$

§ 5. *Motivic decomposition.*

**(13.38)** We now give a brief introduction to Chow motives. For more explanation we refer to Manin [1], Scholl [??], ...

Let $S$ be a smooth quasi-projective scheme over a field $k$. For simplicity we shall assume $S$ to be connected. The category $\mathscr{M}(S)$ of relative Chow motives has as its objects pairs $(f\colon X \to S, p)$ with $f$ a smooth morphism, and with $p \in \mathrm{CH}^*_{\mathbb{Q}}(X \times_S X)$ an idempotent (meaning that $p\circ p = p$). If there is no risk of confusion we use the shorter notation $(X,p)$. The morphisms are given by

$$\mathrm{Hom}_{\mathscr{M}(S)}\big((X,p),(Y,q)\big) = \big\{q\circ\alpha\circ p \bigm| \alpha \in \mathrm{CH}^*_{\mathbb{Q}}(X\times_S Y)\big\}\,,$$

and composition of morphisms is given by composition of correspondences.

The set of morphisms $\mathrm{Hom}_{\mathscr{M}(S)}\big((X,p),(Y,q)\big)$ carries a natural grading: if $X = \amalg_j X_j$ is the decomposition of $X$ into connected components, with $X_j$ of relative dimension $d(X_j/S)$ over $S$ then we set

$$\mathrm{Hom}^i\big((X,p),(Y,q)\big) := \big\{q\circ\alpha\circ p \bigm| \alpha \in \oplus_j \mathrm{CH}^{d(X_j/S)+i}_{\mathbb{Q}}(X_j\times_S Y)\big\}\,.$$

Composition of morphisms respects this grading: if $\alpha \in \mathrm{Hom}^i$ and $\beta \in \mathrm{Hom}^j$ then $\alpha\circ\beta \in \mathrm{Hom}^{i+j}$.

The category $\mathscr{M}^0_+(S)$ of *effective Chow motives* is a variant of $\mathscr{M}(S)$. The objects are pairs $(X,p)$ in $\mathscr{M}(S)$, but we require $p$ to be of degree 0 and morphisms are also of degree 0; in other words, $\mathrm{Hom}_{\mathscr{M}^0_+(S)} = \mathrm{Hom}^0_{\mathscr{M}(S)}$. There is a natural contravariant functor $R\colon \mathscr{V}(S) \to \mathscr{M}^0_+(S)$ sending $X/S$ to $(X,[\Delta_{X/S}])$, and sending a morphism $f\colon X \to Y$ over $S$ to $[{}^{\mathrm{t}}\Gamma_f]$.

In $\mathscr{M}^0_+(S)$ we have direct sums, given by taking disjoint unions; so,

$$(X,p) \oplus (Y,q) = (X \amalg Y, p \amalg q)\,.$$

For instance, if $p \in \mathrm{CH}^*_{\mathbb{Q}}(X)$ is a projector then so is $q := [\Delta_{X/S}] - p$, and we have $R(X) \cong (X,p) \oplus (X,q)$.

Since we want to keep track of "Tate twists", we introduce a third category, denoted by $\mathscr{M}^0(S)$. Its objects are triples $(X,p,m)$ with $(X,p)$ in $\mathscr{M}^0_+(S)$ and $m \in \mathbb{Z}$ an integer. The morphisms are given by

$$\mathrm{Hom}_{\mathscr{M}^0(S)}\big((X,p,m),(Y,q,n)\big) = \mathrm{Hom}^{n-m}_{\mathscr{M}(S)}\big((X,p),(Y,q)\big)\,.$$

We view $\mathscr{M}^0_+(S)$ as a full subcategory of $\mathscr{M}^0(S)$ by sending $(X,p)$ to $(X,p,0)$.

**(13.40)** The category $\mathscr{M}^0(S)$ is an additive $\mathbb{Q}$-linear category in which every projector has a kernel and a cokernel. Such a category is called pseudo-abelian. We have a tensor product, given by

$$(X,p,m) \otimes (Y,q,n) = (X \times_S Y, p \times_S q, m+n)\,.$$

The object $1_S := \big(S,[S],0\big)$ is an identity for the tensor product. As an immediate consequence of the definitions we have the Künneth formula

$$R(X \times_S Y) = R(X) \otimes R(Y)\,.$$

An object $M = (X,p,m)$ has a dual $M^\vee$ in $\mathscr{M}^0(S)$. Namely, if $X$ is of pure relative dimension $n$ over $S$ then we set $M^\vee := (X, {}^{\mathrm{t}}p, n-m)$; to extend this to the general case we first decompose $X$ into connected components. We have a canonical isomorphism

$$\mathrm{Hom}(A \otimes B, C) = \mathrm{Hom}(A, B^\vee \otimes C)\,,$$

functorial in $A$, $B$ and $C$ in $\mathscr{M}^0(S)$. (In the terminology of tensor categories, as in Deligne and Milne [1], this makes $\mathscr{M}^0(S)$ into a rigid tensor category.)

We define Tate twisting in $\mathscr{M}^0(S)$ by

$$(X,p,m)\big(n\big) := (X,p,m+n)\,.$$

In particular, for $X/S$ of relative dimension $n$ we have the relation

$$R(X)^\vee = R(X)\big(n\big)\,,$$

which may be thought of as the motivic analogue of Poincaré duality. (Note, however, that in the present context this relation is a tautology.)

**(13.41)** As an example of a Chow motive we have the *Lefschetz motive* $\mathbb{L}_S$. Take the projective line over $S$, take a section $e\colon S \to \mathbb{P}^1_S$, and consider the projector $[\Gamma_e] := \big[\mathbb{P}^1_S \times_S e(S)\big] \in \mathrm{CH}^1_{\mathbb{Q}}(\mathbb{P}^1_S \times_S \mathbb{P}^1_S)$, which is independent of the choice of $e$. Then we define

$$\mathbb{L}_S := (\mathbb{P}^1_S, [\Gamma_e], 0)\,.$$

One can check that $R(\mathbb{P}^1_S) \cong 1_S \oplus \mathbb{L}_S$. This is reminiscent of the splitting $\mathbb{P}^1 = \{\infty\} \amalg \mathbb{A}^1$, and indeed we can think of $\mathbb{L}_S$ as a "motivic form" of the affine line.

For $M \in \mathscr{M}^0(S)$ we have the relation $M(-1) \cong M \otimes \mathbb{L}$; see Exercise (13.3). It easily follows from this that for all $n \in \mathbb{Z}$ we have

$$M(n) \cong M \otimes \mathbb{L}^{\otimes n}\,, \tag{17}$$

where for $n = -\nu \leqslant 0$ we define $\mathbb{L}^{\otimes n}$ to be $\big(\mathbb{L}^\vee\big)^{\otimes \nu}$.

Using the Lefschetz motive we can say how to form direct sums in $\mathscr{M}^0(S)$. On the full subcategory $\mathscr{M}^0_+(S)$ the direct sum is as described in (13.39). We extend this to the whole of $\mathscr{M}^0(S)$ by using the relation (17). Thus, given $M = (X, p, m)$ and $N = (Y, q, n)$, choose $r \geqslant \max(m, n)$, and use that $M \cong M'(r)$ and $N \cong N'(r)$ with $M' = (X/S, p, 0) \otimes \mathbb{L}^{\otimes r-m}$ and $N'(Y/S, q, 0) \otimes \mathbb{L}^{r-n}$. Then $M'$ and $N'$ are in $\mathscr{M}^0_+(S)$ and $(M' \oplus N')\big(r\big)$ is a direct sum of $M$ and $N$.

**(13.42)** A *multiplicative structure* on a relative motive $M$ in $\mathscr{M}(S)$ is a morphism $M \otimes M \to M$ in $\mathscr{M}(S)$. A morphism $\varphi\colon M \to N$ in $\mathscr{M}(S)$ is compatible with multiplicative structures on $M$ and $N$ if it fits in a commutative diagram

$$\begin{array}{ccc} M \otimes M & \xrightarrow{\varphi\otimes\varphi} & N \otimes N \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & N \end{array} .$$

For example, the relative motive $R(X/S)$ carries a canonical multiplicative structure coming from the diagonal embedding $\Delta\colon X \to X \times_S X$ via

$$R(X/S) \otimes R(X/S) = R(X \times_S X/S) \xrightarrow{[{}^{\mathrm{t}}\Gamma_\Delta]} R(X/S)\,.$$

Another example is given by an abelian scheme $A/S$. The multiplication $m\colon A \times_S A \to A$ induces the convolution multiplicative structure

$$R(A/S) \otimes_S R(A/S) \xrightarrow{[\Gamma_m]} R(A/S)\,.$$

The relations obtained in Thm. (13.21) may now be reformulated by saying that the Fourier transform $\tau$ yields an isomorphism $R(A/S) \xrightarrow{\sim} R(A^t/S)$, compatible with the canonical multiplicative structure on $R(A/S)$ and the convolution structure on $R(A^t/S)$. The inverse isomorphism is given by $(-1)^g[{}^{\mathrm{t}}\Gamma_{-\mathrm{id}_X}] \circ T^t$.

**(13.43)** We shall need the exterior powers $\wedge^i M$ of a motive $M = (X, p, m)$ in $\mathscr{M}^0(S)$. Recall that for cycles we have an exterior product: if $\xi \in \mathrm{CH}^*_{\mathbb{Q}}(X)$ and $\eta \in \mathrm{CH}^*_{\mathbb{Q}}(Y)$ then we have a well-defined cycle class $\xi \times \eta \in \mathrm{CH}^*_{\mathbb{Q}}(X \times_S Y)$.

Let $S_i$ be the symmetric group on $i$ letters, acting on $X^i = X \times_S \cdots \times_S X$ by permuting the factors. Define $s_i \in \mathrm{CH}^*_{\mathbb{Q}}(X^i \times_S X^i)$ by

$$s_i := \frac{1}{i!} \sum_{\sigma \in S_i} [{}^{\mathrm{t}}\Gamma_\sigma]\,,$$

and let

$$s_{i,M} := s_i \circ (p \times \cdots \times p) = (p \times \cdots \times p) \circ s_i \circ (p \times \cdots \times p) = (p \times \cdots \times p) \circ s_i\,.$$

We now define
$$\wedge^i M := \big(X^i, s_{i,M}, mi\big)\,.$$
Note that $s_{i,M} \in \mathrm{CH}^*_{\mathbb{Q}}(X^i \times_S X^i)$ can be viewed both as a morphism $\wedge^i M \to M^{\otimes i}$ and as a morphism $M^{\otimes i} \to \wedge^i M$.

We say that $M$ has finite dimension if there exists an integer $d$ such that $\wedge^i M = 0$ for all $i > d$. For a finite-dimensional $M$ we put
$$\wedge^* M = \oplus_{i=0}^{d} \wedge^i M\,.$$
The exterior algebra $\wedge^* M$ carries a canonical multiplicative structure induced by the composite maps
$$s_{i+j} \circ (s_i \times s_j)\colon \wedge^i M \otimes_S \wedge^j M \longrightarrow M^{\otimes i+j} \longrightarrow \wedge^{i+j} M\,.$$

**(13.44) Remark.** In order to get some feeling for these notions, it helps to think about realisations of motives. For instance, suppose $S = \mathrm{Spec}(k)$ and suppose we have a Weil cohomology $X \mapsto H^\bullet(X)$ for $k$-varieties, with coefficients in some field $L$. Then this gives a (covariant!) functor $h$ from $\mathscr{M}^0(k)$ into the category of finite dimensional, augmented, graded-commutative $L$-algebras, referred to as a realisation functor. Via this functor we recognize several notions defined above as being "motivic analogues" of familiar notions in cohomology. For instance, the canonical multiplicative structure on $R(X)$ may be thought of as the motivic analogue of cup-product.

There is a subtle point in this last remark, though. If we have two motives $M_1 = (X_1, p_1, m_1)$ and $M_2 = (X_2, p_2, m_2)$ then there is an obvious isomorphism
$$\psi\colon M_1 \otimes M_2 \xrightarrow{\sim} M_2 \otimes M_1\,,$$
obtained from the isomorphism $X_1 \times X_2 \xrightarrow{\sim} X_2 \times X_1$ that reverses the two factors. However, with this identification the multiplicative structure on an exterior algebra $\wedge^* M$ is *commutative* rather than graded-commutative. Also, the canonical multiplicative structure on $R(X)$ is commutative, unlike cup-product, which is graded-commutative. Though this does not make any difference for the results discussed in this section, let us point out that, in a suitable sense, the above isomorphism $\psi$ is not the right identification to use. A modified version of $\psi$ would give a theory in which $\wedge^* M$ and $R(X)$ are graded-commutative, as it should be. However, to define the right identification $M_1 \otimes M_2 \cong M_2 \otimes M_1$ we need the algebraicity of the Künneth components of the diagonal, which, as already mentioned, is not known in general. See ?? for further discussion.

**(13.45)** Let $X/S$ be an abelian scheme of relative dimension $g$. Define
$$R^i(X) := (X, \pi_i, 0)\,,$$
with $\pi_i$ as in (8). Then Theorem (13.29) yields a canonical decomposition
$$R(X) = \mathop{\oplus}_{i=0}^{2g} R^i(X)$$
such that $[{}^{\mathrm{t}}\Gamma_n]$ acts on $R^i(X)$ by $n^i$. Poincaré duality tells us that $R^{2g-i}(X)^\vee = R^i(X)\big(g\big)$.

Our goal is to prove a theorem of Künnemann, which asserts that $R^i(X)$ is isomorphic to $\wedge^i R^1(X)$. As a preparation we first give another description of the motive $\wedge^i R^1(X)$. Since we shall need the projectors $\pi_i$ for various abelian schemes, we shall from now on often write $\pi_{i,X}$ for the elements obtained in (8).

By definition we have $\wedge^i R^1(X) = \big(X^i, s_i \circ (\pi_{1,X} \times \cdots \times \pi_{1,X}), 0\big)$. By the motivic Künneth formula we have

$$\pi_{i,X^i} = \sum_{n_1+\cdots+n_i=i} \pi_{n_1,X} \times \cdots \times \pi_{n_i,X}\,,$$

where the indices $n_i$ run from 0 to $2g$, satisfying the condition on their sum. To filter out the term $\pi_{1,X} \times \cdots \times \pi_{1,X}$ we use the action of $-\mathrm{id}_X$. Note that for $[X] \in \mathrm{CH}^0_{\mathbb{Q}}(X)$ we have

$$\mathrm{id}_X^*[X] - (-\mathrm{id}_X)^*[X] = 0\,. \tag{18}$$

Therefore, for $a = (a_1, \ldots, a_i) \in \{\pm 1\}^i$, let $\mathrm{sgn}(a) := a_1 a_2 \cdots a_i \in \{\pm 1\}$ and define

$$\lambda_i := (1/2^i) \sum_{a \in \{\pm 1\}^i} \mathrm{sgn}(a)[{}^{\mathrm{t}}\Gamma_a] \quad \in \mathrm{CH}^{gi}_{\mathbb{Q}}(X^i \times_S X^i)\,,$$

where of course $\Gamma_a$ denotes the graph of the automorphism $(a_1, \ldots, a_i)$ of $X^i$. Now observe that

$$\begin{aligned} \lambda_i \circ \pi_{i,X^i} &= \lambda_i \circ \Big( \sum_{n_1+\cdots+n_i=i} \pi_{n_1,X} \times \cdots \times \pi_{n_i,X} \Big) \\ &= \lambda_i \circ (\pi_{1,X} \times \cdots \times \pi_{1,X}) \\ &= \pi_{1,X} \times \cdots \times \pi_{1,X}\,. \end{aligned} \tag{19}$$

Indeed, if $n_j > 1$ for some index $j$ then there is also an index $l \in \{1, \ldots, i\}$ with $n_l = 0$; but then it easily follows from (18) that the term $\lambda_i \circ (\pi_{n_1,X} \times \cdots \times \pi_{n_i,X})$ vanishes. We are left with the term corresponding to $(n_1, \ldots, n_i) = (1, \ldots, 1)$, which is preserved because each $[{}^{\mathrm{t}}\Gamma_a]$ acts on it as the identity.

**(13.46) Lemma.** *We have $\wedge^i R^1(X) = (X^i, \lambda_i \circ s_i \circ \pi_{i,X^i}, 0)$ in $\mathscr{M}^0(S)$.*

*Proof.* One easily checks that the elements $s_i$ and $\lambda_i$ are projectors and that they commute. Now (19) gives

$$\wedge^i R^1(X) = \big(X^i, s_i \circ (\pi_{1,X} \times \ldots \times \pi_{1,X}), 0\big) = (X^i, s_i \circ \lambda_i \circ \pi_{i,X^i}, 0) = (X^i, \lambda_i \circ s_i \circ \pi_{i,X^i}, 0)\,,$$

which is what we want. □

**(13.47) Theorem.** (Künnemann) *There is an isomorphism of motives with multiplicative structures*

$$\wedge^* R^1(X) \xrightarrow{\sim} R(X)\,.$$

*Proof.* Let $\Sigma^i \colon X^i \to X$ and $\Delta^i \colon X \to X^i$ be the homomorphisms given by $\Sigma^i(x_1, \ldots, x_i) = x_1 + \cdots + x_i$ and $\Delta^i(x) = (x, \ldots, x)$. We have the relations

$$[{}^{\mathrm{t}}\Gamma_{\Delta^i}] \circ s_i = [{}^{\mathrm{t}}\Gamma_{\Delta^i}] \qquad \text{and} \qquad s_i \circ [{}^{\mathrm{t}}\Gamma_{\Sigma^i}] = [{}^{\mathrm{t}}\Gamma_{\Sigma^i}]\,. \tag{20}$$

Let us also note that we have the relations $\pi_{i,X^i} \circ s_i = s_i \circ \pi_{i,X^i}$ and $\pi_{i,X^i} \circ \lambda_i = \lambda_i \circ \pi_{i,X^i}$, as follows from (iii) of Thm. (13.29).

Define morphisms

$$\Phi_i := [{}^{\mathrm{t}}\Gamma_{\Delta^i}]\circ(\lambda_i\circ s_i\circ\pi_{i,X^i}) = [{}^{\mathrm{t}}\Gamma_{\Delta^i}]\circ\lambda_i\circ\pi_{i,X^i}\in \mathrm{Hom}_{\mathscr{M}^0(S)}\big(\wedge^i R^1(X), R(X)\big)\,,$$

and

$$\Psi_i := \frac{1}{i!}(\lambda_i\circ s_i\circ\pi_{i,X^i})\circ[{}^{\mathrm{t}}\Gamma_{\Sigma^i}] = \frac{1}{i!}\lambda_i\circ\pi_{i,X^i}\circ[{}^{\mathrm{t}}\Gamma_{\Sigma^i}]\in \mathrm{Hom}_{\mathscr{M}^0(S)}\big(R(X),\wedge^i R^1(X)\big)\,.$$

The theorem will result from the following more precise claims:
(i) $\Phi_i\circ\Psi_i = \pi_{i,X}$,
(ii) $\Psi_i\circ\Phi_i = \lambda_i\circ s_i\circ\pi_{i,X^i} = \mathrm{id}_{\wedge^i R^1(X)}$.
To prove (i) we write

$$\begin{aligned}\Phi_i\circ\Psi_i &= (1/i!)\cdot[{}^{\mathrm{t}}\Gamma_{\Delta^i}]\circ\lambda_i\circ\pi_{i,X^i}\circ\lambda_i\circ\pi_{i,X^i}\circ[{}^{\mathrm{t}}\Gamma_{\Sigma^i}]\\ &= (1/i!)\cdot[{}^{\mathrm{t}}\Gamma_{\Delta^i}]\circ\lambda_i\circ[{}^{\mathrm{t}}\Gamma_{\Sigma^i}]\circ\pi_{i,X} && \text{by (iii) of (13.29)}\\ &= (1/2^i i!)\cdot\sum_{a\in\{\pm1\}^i}\mathrm{sgn}(a)[{}^{\mathrm{t}}\Gamma_{\Sigma^i\circ a\circ\Delta^i}]\circ\pi_{i,X}\\ &= (1/2^i i!)\cdot\sum_{a\in\{\pm1\}^i}\mathrm{sgn}(a)[{}^{\mathrm{t}}\Gamma_{a_1+\cdots+a_i}]\circ\pi_{i,X}\\ &= (1/2^i i!)\cdot\sum_{a\in\{\pm1\}^i}\mathrm{sgn}(a)(a_1+\cdots+a_i)^i\cdot\pi_{i,X} && \text{by (9).}\end{aligned}$$

Now use that

$$\sum_{a\in\{\pm1\}^i}\mathrm{sgn}(a)(a_1+\cdots+a_i)^k = \begin{cases}0 & \text{if } 0\leqslant k<i,\\ 2^i i! & \text{if } k=i,\end{cases}$$

as is easily shown by induction on $i$.

To prove (ii) we must show that $(1/i!)\cdot\lambda_i\circ\pi_{i,X^i}\circ[{}^{\mathrm{t}}\Gamma_{\Sigma^i}]\circ[{}^{\mathrm{t}}\Gamma_{\Delta^i}]\circ\lambda_i\circ\pi_{i,X^i} = \lambda_i\circ s_i\circ\pi_{i,X^i}$. What we shall actually prove is that

$$(1/i!)\cdot{}^{\mathrm{t}}s_i\circ{}^{\mathrm{t}}\lambda_i\circ[\Gamma_{\Delta^i\circ\Sigma^i}]\circ{}^{\mathrm{t}}\pi_{i,X^i}\circ{}^{\mathrm{t}}\lambda_i = {}^{\mathrm{t}}\lambda_i\circ{}^{\mathrm{t}}s_i\circ{}^{\mathrm{t}}\pi_{i,X^i}\,. \tag{21}$$

After transposition, using (20) and using that $s_i$, $\lambda_i$ and $\pi_{i,X^i}$ are mutually commuting projectors, this gives the desired relation.

Write $\mathrm{pr}_l\colon X^i\to X$ for the projection on the $l$th factor and $j_l\colon X\to X^i$ for the inclusion of the $l$th factor.

As before, we view $X^i\times_S X^i$ as an abelian scheme over $X^i$ via the first projection. We know that ${}^{\mathrm{t}}\pi_{i,X^i} = \pi_{2gi-i,X^i}$, and by construction the latter equals $(1/i!)\cdot\log\big([\Gamma_{\mathrm{id}}]\big)^{*i}$. (This takes place on $X^i$.) Recall that when we write "$\Gamma_{\mathrm{id}}$" we may interpret this as the graph class associated to the section $\xi\mapsto(\xi,\xi)$ of $X^i\times_S X^i$ over $X^i$. Likewise, we have meaningfully defined graph classes $[\Gamma_{j_k\circ\mathrm{pr}_l}]$.

With these remarks, the LHS of (21) equals

$$\begin{aligned}&(1/i!)^2\cdot{}^{\mathrm{t}}s_i\circ{}^{\mathrm{t}}\lambda_i\circ[\Gamma_{\Delta^i\circ\Sigma^i}]\circ\log\big([\Gamma_{\mathrm{id}}]\big)^{*i}\circ{}^{\mathrm{t}}\lambda_i\\ &= (1/i!)^2\cdot{}^{\mathrm{t}}s_i\circ{}^{\mathrm{t}}\lambda_i\circ\log\big([\Gamma_{\Delta^i\circ\Sigma^i}]\big)^{*i}\circ{}^{\mathrm{t}}\lambda_i && \text{using Exercise (13.4)}\\ &= (1/i!)^2\cdot{}^{\mathrm{t}}s_i\circ{}^{\mathrm{t}}\lambda_i\circ\Big(\sum_{k,l=1}^{i}\log\big([\Gamma_{j_k\circ\mathrm{pr}_l}]\big)\Big)^{*i}\circ{}^{\mathrm{t}}\lambda_i\\ &= (1/i!)^2\cdot{}^{\mathrm{t}}s_i\circ{}^{\mathrm{t}}\lambda_i\circ\Big(\sum_{k_1,\ldots,k_i=1}^{i}\ \sum_{l_1,\ldots,l_i=1}^{i}\log\big([\Gamma_{j_{k_1}\circ\mathrm{pr}_{l_1}}]\big)*\cdots*\log\big([\Gamma_{j_{k_i}\circ\mathrm{pr}_{l_i}}]\big)\Big)\circ{}^{\mathrm{t}}\lambda_i\,.\end{aligned}$$

We claim that this equals

$$(1/i!)^2 \cdot {}^{\mathrm{t}}s_i \circ {}^{\mathrm{t}}\lambda_i \circ \Big( \sum_{\sigma \in S_i} \sum_{\tau \in S_i} \log\big([\Gamma_{j_{\sigma(1)} \circ \mathrm{pr}_{\tau(1)}}]\big) * \cdots * \log\big([\Gamma_{j_{\sigma(i)} \circ \mathrm{pr}_{\tau(i)}}]\big) \Big) . \tag{22}$$

Indeed, expanding $\lambda_i$ we have

$$\begin{aligned} &{}^{\mathrm{t}}\lambda_i \circ \Big( \log\big([\Gamma_{j_{k_1} \circ \mathrm{pr}_{l_1}}]\big) * \cdots * \log\big([\Gamma_{j_{k_i} \circ \mathrm{pr}_{l_i}}]\big) \Big) \circ {}^{\mathrm{t}}\lambda_i \\ &= 2^{-2i} \cdot \sum_{a,b \in \{\pm 1\}^i} \mathrm{sgn}(a)\mathrm{sgn}(b) \cdot \log\big([\Gamma_{a \circ j_{k_1} \circ \mathrm{pr}_{l_1} \circ b}]\big) * \cdots * \log\big([\Gamma_{a \circ j_{k_i} \circ \mathrm{pr}_{l_i} \circ b}]\big) . \end{aligned} \tag{23}$$

If $(a_1, \ldots, a_i)$ is not a permutation of $(1, \ldots, i)$, choose $j \in \{1, \ldots, i\} \setminus \{n_1, \ldots, n_i\}$; then the corresponding terms with $a_j = -1$ and $a_j = 1$ cancel out. Likewise, if there is an index $j$ in $\{1, \ldots, i\} \setminus \{b_1, \ldots, b_i\}$ then the terms with $b_j = -1$ and $b_j = 1$ cancel out. Hence we may assume that $(k_1, \ldots, k_i) = \big(\sigma(1), \ldots, \sigma(i)\big)$ and $(l_1, \ldots, l_i) = \big(\tau(1), \ldots, \tau(i)\big)$. If for $\alpha \in \{\pm 1\}^i$ we group the $2^i$ terms of (23) with $a_{\sigma(i)} \cdot b_{\tau(i)} = \alpha_i$ for all $i$ then we find that (23) equals

$$\begin{aligned} &2^{-i} \cdot \sum_{\alpha \in \{\pm 1\}^i} \mathrm{sgn}(\alpha) \cdot \Big( \log\big([\Gamma_{\alpha \circ j_{\sigma(1)} \circ \mathrm{pr}_{\tau(1)}}]\big) * \cdots * \log\big([\Gamma_{\alpha \circ j_{\sigma(i)} \circ \mathrm{pr}_{\tau(i)}}]\big) \Big) \\ &= {}^{\mathrm{t}}\lambda_i \circ \Big( \log\big([\Gamma_{j_{\sigma(1)} \circ \mathrm{pr}_{\tau(1)}}]\big) * \cdots * \log\big([\Gamma_{j_{\sigma(i)} \circ \mathrm{pr}_{\tau(i)}}]\big) \Big) , \end{aligned}$$

proving our claim.

Next we remark that we may reorder the log-factors in (22), and since ${}^{\mathrm{t}}s_i \circ \log\big([\Gamma_{j_{\sigma(l)} \circ \mathrm{pr}_l}]\big) = \log\big([\Gamma_{j_l \circ \mathrm{pr}_l}]\big)$ for all $\sigma$ and $l$, we finally get that the LHS of (21) equals

$$\begin{aligned} &(1/i!) \cdot {}^{\mathrm{t}}\lambda_i \circ {}^{\mathrm{t}}s_i \circ \Big( \sum_{\sigma \in S_i} \log\big([\Gamma_{j_{\sigma(1)} \circ \mathrm{pr}_1}]\big) * \cdots * \log\big([\Gamma_{j_{\sigma(i)} \circ \mathrm{pr}_i}]\big) \Big) \\ &= {}^{\mathrm{t}}s_i \circ {}^{\mathrm{t}}\lambda_i \circ \Big( \log\big([\Gamma_{j_1 \circ \mathrm{pr}_1}]\big) * \cdots * \log\big([\Gamma_{j_i \circ \mathrm{pr}_i}]\big) \Big) . \end{aligned} \tag{24}$$

The RHS of (21) equals

$$\begin{aligned} &(1/i!) \cdot {}^{\mathrm{t}}\lambda_i \circ {}^{\mathrm{t}}s_i \circ \log\big([\Gamma_{\mathrm{id}}]\big)^{*i} \\ &= (1/i!) \cdot {}^{\mathrm{t}}\lambda_i \circ {}^{\mathrm{t}}s_i \circ \Big( \log\big([\Gamma_{j_1 \circ \mathrm{pr}_1}]\big) + \cdots + \log\big([\Gamma_{j_i \circ \mathrm{pr}_i}]\big) \Big)^{*i} \\ &= (1/i!) \cdot {}^{\mathrm{t}}s_i \circ {}^{\mathrm{t}}\lambda_i \circ \Big( \sum_{n_1, \ldots, n_i = 1}^{i} \log\big([\Gamma_{j_{n_1} \circ \mathrm{pr}_{n_1}}]\big) * \cdots * \log\big([\Gamma_{j_{n_i} \circ \mathrm{pr}_{n_i}}]\big) \Big) . \end{aligned}$$

With the same argument as above we see that the only non-trivial contributions come from the terms with $(n_1, \ldots, n_i)$ a permutation of $(1, \ldots, i)$. Hence we get

$$(1/i!) \cdot {}^{\mathrm{t}}s_i \circ {}^{\mathrm{t}}\lambda_i \circ \Big( \sum_{\sigma \in S_i} \log\big([\Gamma_{j_{\sigma(1)} \circ \mathrm{pr}_{\sigma(1)}}]\big) * \cdots * \log\big([\Gamma_{j_{\sigma(i)} \circ \mathrm{pr}_{\sigma(i)}}]\big) \Big) ,$$

and after reordering the log-factors we see that this equals (24), proving relation (ii).

To finish the proof of the theorem we have to check that the maps $\sum_i \Phi_i \colon \wedge^* R^1(X) \to R(X)$ and $\sum_i \Psi_i \colon R(X) \to \wedge^* R^1(X)$ respect the multiplicative structures. This is a straightforward verification that we leave as an exercise. $\square$

**(13.48) Remark.** Passing to cohomology this gives another proof of Thm. (13.32).

**Exercises.**

**(13.1)** Let $X$ be an abelian variety. Write $\tau = \tau_{\mathrm{CH}}$. If $\alpha \in \mathrm{CH}^*_{\mathbb{Q}}(X)$ is a symmetric element, meaning that $(-1_X)^*\alpha = \alpha$, prove that $\tau(\alpha)$ is symmetric too, and that $\tau(\alpha) \in \oplus_j \mathrm{CH}^{g-i+2j}_{\mathbb{Q}}(X)$. Similarly, if $\alpha$ is anti-symmetric, meaning that $(-1_X)^*\alpha = -\alpha$, prove that $\tau(\alpha)$ is also anti-symmetric, and that $\tau(\alpha) \in \oplus_j \mathrm{CH}^{g-i+2j+1}_{\mathbb{Q}}(X)$.

**(13.2)** Let $\Theta$ be a divisor on an abelian variety $X$ giving a principal polarization. Let $\theta \in \mathrm{CH}^1_{\mathbb{Q}}(X)$ be its class. Prove that $\tau(e^\theta) = e^{-\theta}$.

**(13.3)** Consider the category $\mathscr{M}^0(S)$ as in (13.39). Let $\mathbb{L} = (\mathbb{P}^1_S, [\Gamma_e], 0)$ be the Lefschetz motive as defined in (13.41).

(i) Let $q := [\Delta] - [\Gamma_e]$, with $\Delta \subset \mathbb{P}^1_S \times_S \mathbb{P}^1_S$ the diagonal. Show that $(\mathbb{P}^1_S/S, q, 0) \cong 1_S$. Conlude that $R(\mathbb{P}^1_S/S) \cong 1_S \oplus \mathbb{L}$.

(ii) For $M$ in $\mathscr{M}^0(S)$ and $\mathbb{L}$, prove that $M(-1) \cong M \otimes \mathbb{L}$.

**(13.4)** Let $f\colon X \to Y$ be a homomorphism of abelian schemes over a basis $S$ as in (13.38). For $x \in X(S)$, view $\log\big([\Gamma_x]\big)$ as a correspondence from $S$ to $X$. Show that $[\Gamma_f] \circ \log\big([\Gamma_x]\big) = \log\big([\Gamma_{f(x)}]\big)$. Using Lemmas (13.6) and (13.17), generalize this to the identity $[\Gamma_f] \circ \log\big([\Gamma_x]\big)^{*n} = \log\big([\Gamma_{f(x)}]\big)^{*n}$ for all $n \geqslant 0$.

**Notes.** Pontryagin introduced the Pontryagin product in his investigations of the homology of Lie groups in 1935; see Pontryagin [1], [2]. The Fourier transform can be defined in various contexts. It first occurred in a paper of Lieberman (see Kleiman [1], Appendix) at the level of cohomology. Mukai introduced it in the derived category of $O_X$-modules and established many properties of it. Beauville studied the Fourier transform on the Chow rings of an abelian variety and especially the action of multiplication by $n$. Deninger and Murre [1] used work of Beauville to give a canonical decomposition of the Chow motive of an abelian variety which is the analogue of the well-known cohomological decomposition $H(X) \cong \oplus_{i=0}^{2g} H^i(X)$. It is based on the decomposition of $\mathrm{CH}^*_{\mathbb{Q}}(X \times X)$ into eigenspaces of the endomorphism $(1_X \times n_X)^*$ for any integer $n$. If $|n| > 1$ then the components of the diagonal for this decomposition yield pairwise orthogonal idempotents in the ring of correspondences and this gives a decomposition of the Chow motive of an abelian variety. Shermenev had given such a decomposition earlier, but his decomposition was not canonical. Künneman used these idempotents to prove that the Chow motive $R(X)$ is the exterior algebra $\wedge^1 R^1$ generalizing the result for cohomology. Proposition (13.27) is due to Bloch; the proof using the Fourier transform stems from Beauville.

Inventarisatie van wat we nodig hebben in dit hoofdstuk:

— $\xi^t\colon X^t \to S$ dan $R^g\xi^t_*\mathscr{P} \cong e^t_* \det(\mathbb{E}^t)^{-1}$;

— resultaten over Chern klassen van de Hodge bundel; i.h.b. dat $\mathrm{Td}(\mathbb{E}) = \exp(\lambda_1/2)$ en dat $\lambda_1 = \lambda_1^t$;

— Thm of the Cube voor abelse schema's;

— Thm of the Square voor abelse schema's.