Chapter IV. Quotients by group schemes.

When we work with group schemes the question naturally arises if constructions from group theory can also be carried out in the context of group schemes. For instance, we have seen that if $f\colon G \to G'$ is a homomorphism then we can form the kernel group scheme, $\mathrm{Ker}(f)$. In this example the geometry and the group theory go hand in hand: there is an obvious scheme-theoretic candidate for the kernel, namely the inverse image of the identity section of G', and this candidate also represents the kernel as a functor.

The present chapter is devoted to the formation of quotients, which is more delicate. (Nog aanvullen)

The reader who wants to go on as quickly as possible with the general theory of abelian varieties, may skip most of this chapter. The only results that are directly relevant for the next chapters are the formation of quotients modulo finite group schemes, Thm. (4.16), Example (4.40), and the material in § 4.

§ 1. *Categorical quotients.*

(4.1) Definition. (i) Let G be a group scheme over a basis S. A *(left) action of G on an S-scheme X* is given by a morphism $\rho\colon G \times_S X \to X$ such that the composition

$$X \xrightarrow{\sim} S \times_S X \xrightarrow{e_G \times \mathrm{id}_X} G \times_S X \xrightarrow{\rho} X$$

is the identity on X, and such that the diagram

$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{\mathrm{id}_G \times \rho} & G \times_S X \\ {\scriptstyle m \times \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \rho} \\ G \times_S X & \xrightarrow{\rho} & X \end{array} \qquad (1)$$

is commutative. In other words: for every S-scheme T, the morphism ρ induces a left action of the group $G(T)$ on the set $X(T)$. We usually denote this action on points by $(g, x) \mapsto g \cdot x$.

(ii) Given an action ρ as in (i), we define the "graph morphism"

$$\Psi = \Psi_\rho := (\rho, \mathrm{pr}_2)\colon G \times_S X \longrightarrow X \times_S X\,;$$

on points this is given by $(g, x) \mapsto (g \cdot x, x)$. The action ρ is said to be *free*, or *set-theoretically free* if Ψ is a monomorphism of schemes, and is said to be *strictly free*, or *scheme-theoretically free*, if Ψ is an immersion.

(iii) If T is an S-scheme and $x \in X(T)$ then the *stabilizer of* x, notation G_x, is the subgroup scheme of G_T that represents the functor $T' \mapsto \{g \in G(T') \mid g \cdot x = x\}$ on T-schemes T'. (See also (4.2), (iii) below.)

`QuoGrSch, 15 september, 2011 (812)`

(4.2) Remarks. (i) In some literature the same terminology is used in a slightly different meaning (cf. GIT, for example).

(ii) The condition that an action ρ is free means precisely that for all T and all $x \in X(T)$ the stabilizer G_x is trivial.

(iii) With notations as in the definition, we have a diagram with cartesian squares

$$\begin{array}{ccccc} G_x & \hookrightarrow & G_T = G \times_S T & \xrightarrow{\mathrm{id}_G \times x} & G \times_S X \\ \downarrow & & \downarrow{\scriptstyle a_x} & & \downarrow{\scriptstyle \Psi} \\ T & \xrightarrow{(x,\mathrm{id}_T)} & X_T = X \times_S T & \xrightarrow{\mathrm{id}_X \times x} & X \times_S X \end{array} ,$$

where the morphism a_x is given by $a_x = (\rho \circ (\mathrm{id}_G \times x), \mathrm{pr}_2)$; on points: $a_x(g) = g \cdot x$. That the functor $T' \mapsto \{g \in G(T') \mid g \cdot x = x\}$ is indeed representable by a subgroup scheme $G_x \subset G_T$ is seen from this diagram, arguing as in (3.13).

(4.3) Examples. If G is a group scheme over S and $H \subset G$ is a subgroup scheme then the group law gives an action of H on G. The graph morphism Ψ: $H \times_S G \to G \times_S G$ is the restriction to $H \times_S G$ of the universal right translation τ: $G \times_S G \to G \times_S G$. Since τ is an isomorphism, the action is strictly free.

More generally, if f: $G \to G'$ is a homomorphism of group schemes then we get a natural action of G on G', given on points by $(g, g') \mapsto f(g) \cdot g'$. The action is free if and only if $\mathrm{Ker}(f)$ is trivial, but if this holds the action need not be strictly free. For instance, with $S = \mathrm{Spec}(\mathbb{Q})$ as a base scheme, take $G = \mathbb{Z}_S$ to be the constant group scheme defined by the (abstract) group $\mathbb{Z}$, and take $G' = \mathbb{G}_{a,S}$. We have a natural homomorphism f: $\mathbb{Z}_S \to \mathbb{G}_{a,S}$ which, for $\mathbb{Q}$-schemes T, is given on points by the natural inclusion $\mathbb{Z} \hookrightarrow \Gamma(T, O_T)$. This homomorphism f is injective, hence it gives a free action of $\mathbb{Z}_S$ on $\mathbb{G}_{a,S}$. The graph morphism can be described as the morphism

$$\Psi\colon \coprod_{n \in \mathbb{Z}} \mathbb{A}^1 \longrightarrow \mathbb{A}^2$$

that maps the nth copy of $\mathbb{A}^1$ to the line $L \subset \mathbb{A}^2$ given by $x - y = n$. But this Ψ is not an immersion (the image is not a subscheme of $\mathbb{A}^2$), so the action is not strictly free.

(4.4) The central issue of this chapter is the following question. *Given a group scheme G acting on a scheme X, does there exist a good notion of a quotient space $G\backslash X$?* As particular instances of this question we have: *given a homomorphism of group schemes f: $G \to G'$, can we form a cokernel of f?*, and *if $N \subset G$ is a normal subgroup scheme, can we define a quotient group scheme G/N?*

Let us first look at an elementary example. Take an integer $N \geqslant 2$, and consider the endomorphism f: $\mathbb{G}_m \to \mathbb{G}_m$ over $S = \mathrm{Spec}(\mathbb{Z})$ given on points by $q \mapsto q^N$. The kernel of f is μ_N, by definition of the latter. As a morphism of schemes, f is faithfully flat, and if k is any algebraically closed field then f is surjective on k-valued points. Therefore we would expect that the cokernel of f is trivial, i.e., $\mathrm{Coker}(f) = S$. But clearly, the "cokernel functor" C: $T \mapsto \mathbb{G}_m(T)/f(\mathbb{G}_m(T))$ is non-trivial. E.g., $C(\mathbb{Q})$ is an infinite group. Moreover, from the fact that $C(\mathbb{Q}) \neq \{1\}$ but $C(\overline{\mathbb{Q}}) = \{1\}$ it follows that C is not representable by a scheme. So, in contrast with (3.13) where we defined kernels, the geometric and the functorial point of view do not give to the same notion of a cokernel.

The first notion of a quotient that we shall define is that of a categorical quotient. Though we are mainly interested in working with schemes, it is useful to extend the discussion to a more general setting.

(4.5) Definition. Let C be a category with finite products. Let G be a group object in C. Let X be an object of C. Throughout, we simply write $X(T)$ for $h_X(T) = \mathrm{Hom}_C(T, X)$.

(i) A *(left) action* of G on X is a morphism $\rho\colon G \times X \to X$ that induces, for every object T, a (left) action of the group $G(T)$ on the set $X(T)$.

(ii) Let an action of G on X be given. A morphism $q\colon X \to Y$ in C is said to be *G-invariant* if $q \circ \rho = q \circ \mathrm{pr}_X\colon G \times X \to Y$. By the Yoneda lemma this is equivalent to the requirement that for every $T \in C$, if x_1, $x_2 \in X(T)$ are two points in the same $G(T)$-orbit then $q(x_1) = q(x_2)$ in $Y(T)$.

(iii) Let $f, g\colon W \rightrightarrows X$ be two morphisms in C. We say that a morphism $h\colon X \to Y$ is a *difference cokernel* of the pair (f, g) if $h \circ f = h \circ g$ and if h is universal for this property; by this we mean that for any other morphism $h'\colon X \to Y'$ with $h' \circ f = h' \circ g$ there is a unique $\alpha\colon Y \to Y'$ such that $h' = \alpha \circ h$.

(iv) Let $\rho\colon G \times X \to X$ be a left action. A morphism $q\colon X \to Y$ is called a *categorical quotient* of X by G if q is a difference cokernel for the pair $(\rho, \mathrm{pr}_X)\colon G \times X \rightrightarrows X$. In other words, q is a categorical quotient if q is G-invariant and if every G-invariant morphism $q'\colon X \to Y'$ factors as $q' = \alpha \circ q$ for a unique $\alpha\colon Y \to Y'$. The morphism $q\colon X \to Y$ is called a *universal categorical quotient* of X by G if for every object S of C the morphism $q_S\colon X_S \to Y_S$ is a categorical quotient of X_S by G_S in the category $C_{/S}$.

In practice the morphism q is often not mentioned, and we simply say that an object Y is the categorical quotient of X by G. Note that if a categorical quotient $q\colon X \to Y$ exists then it is unique up to unique isomorphism.

(4.6) Examples. As in (4.4), let $S = \mathrm{Spec}(\mathbb{Z})$ and let $G = \mathbb{G}_{m,S}$ act on $X = \mathbb{G}_{m,S}$ by $\rho(g, x) = g^N \cdot x$. If $k = \overline{k}$ then $X(k)$ consists of a single orbit under $G(k)$; this readily implies that $X \to S$ is a categorical quotient of X by G. In fact, if we work a little harder we find that $X \to S$ is even a universal categorical quotient; see Exercise (4.1).

As a second example, let $k = \overline{k}$ and consider the action of $G = \mathbb{G}_{m,k}$ on $X = \mathbb{A}^1_k$ given on points by $\rho(g, x) = g \cdot x$. There are two orbits in $X(k)$, one given by the origin $0 \in X(k)$, the other consisting of all points $x \neq 0$. Suppose we have a G-invariant morphism $q\colon X \to Y$ for some k-scheme Y. It maps $X(k) \setminus \{0\}$ to a point $y \in Y(k)$. Because $X(k) \setminus \{0\}$ is Zariski dense in X we find that q is the constant map with value y. This proves that the structural morphism $X \to \mathrm{Spec}(k)$ is a categorical quotient of X by G. We conclude that it is not possible to construct a quotient scheme Y such that the two orbits $\{0\}$ and $\mathbb{A}^1 \setminus \{0\}$ are mapped to different points of Y.

(4.7) Remark. Let G be an S-group scheme acting on an S-scheme X. Suppose there exists a categorical quotient $q\colon X \to Y$ in $\mathsf{Sch}_{/S}$. To study q we can take Y to be our base scheme. More precisely, $G_Y := G \times_S Y$ acts on X over Y and q is also a categorical quotient of X by G_Y in the category $\mathsf{Sch}_{/Y}$. Taking Y to be the base scheme does not affect the (strict) freeness of the action. To see this, note that the graph morphism $\Psi\colon G \times_S X \to X \times_S X$ factors through the subscheme $X \times_Y X \hookrightarrow X \times_S X$ and that the resulting morphism

$$G_Y \times_Y X = G \times_S X \to X \times_Y X$$

is none other than the graph morphism of G_Y acting on X over Y. Hence the action of G on X over S is (strictly) free if and only if the action of G_Y on X over Y is (strictly) free.

§ 2. *Geometric quotients, and quotients by finite group schemes.*

We first give, in its simplest form, a result about the existence of quotients under finite groups. This result will be generalized in (4.16) below. Here we consider an action of an abstract group Γ on a scheme X; this means that for every element $\gamma \in \Gamma$ we have a morphism $\rho(\gamma)\colon X \to X$, satisfying the usual axioms for a group action. Such an action is the same as an action of the constant group scheme Γ on X; hence we are in a special case of the situation considered in (4.1).

(4.8) Proposition. *Let Γ be a finite (abstract) group acting on an affine scheme $X = \mathrm{Spec}(A)$. Let $B := A^\Gamma \subseteq A$ be the subring of Γ-invariant elements, and set $Y := \mathrm{Spec}(B)$.*

(i) *The natural morphism $q\colon X \to Y$ induces a homeomorphism $\Gamma\backslash|X| \xrightarrow{\sim} |Y|$, i.e., it identifies the topological space $|Y|$ with the quotient of $|X|$ under the acion of Γ.*

(ii) *The map $q^\sharp\colon O_Y \to q_*O_X$ induces an isomorphism $O_Y \xrightarrow{\sim} (q_*O_X)^\Gamma$, where the latter denotes the sheaf of Γ-invariant sections of q_*O_X.*

(iii) *The ring A is integral over B; the morphism $q\colon X \to Y$ is quasi-finite, closed and surjective.*

Proof. Write $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$. Define the map $N\colon A \to A^\Gamma = B$ by

$$N(a) = \gamma_1(a) \cdots \gamma_r(a)\,.$$

If $\mathfrak{p}$ and $\mathfrak{p}'$ are prime ideals of A which lie in the same Γ-orbit then $\mathfrak{p} \cap A^\Gamma = \mathfrak{p}' \cap A^\Gamma$. Conversely, if $\mathfrak{p} \cap A^\Gamma = \mathfrak{p}' \cap A^\Gamma$ then $N(x) \in \mathfrak{p}'$ for every $x \in \mathfrak{p}$, so $\mathfrak{p} \subset \gamma_1(\mathfrak{p}') \cup \cdots \cup \gamma_r(\mathfrak{p}')$. This implies (see Atiyah-Macdonald [1], Prop. 1.11) that $\mathfrak{p} \subseteq \gamma_i(\mathfrak{p}')$ for some i, and by symmetry we conclude that $\mathfrak{p}$ and $\mathfrak{p}'$ lie in the same Γ-orbit. Hence $\Gamma\backslash|X| \xrightarrow{\sim} |Y|$ as sets, and q is quasi-finite.

For $a \in A$, let $\chi_a(T) := \bigl(T - \gamma_1(a)\bigr)\bigl(T - \gamma_2(a)\bigr) \cdots \bigl(T - \gamma_r(a)\bigr) \in A[T]$. Then it is clear that $\chi_a(T)$ is a monic polynomial in $B[T]$ and that $\chi_a(a) = 0$. This shows that A is integral over B. That the map q is closed and surjective then follows from Atiyah-Macdonald [1], Thm. 5.10; see also (4.21) below.

Finally we remark that for every $f \in A^\Gamma$ we have a natural isomorphism $(A^\Gamma)_f \xrightarrow{\sim} (A_f)^\Gamma$. As the special open subsets $D(f) := Y \setminus Z(f)$ form a basis for the topology on Y, property (ii) follows. □

(4.9) Remarks. (i) The morphism $q\colon X \to Y$ need not be finite. It may happen that A is noetherian but that $B := A^\Gamma$ is not, and that A is not finitely generated as a B-module. Examples of this kind can be found in Nagata [1], ??. However, if either the action on Γ on X is free, or X is of finite type over a locally noetherian base scheme S and Γ acts by automorphisms of X over S, then q is a finite morphism. See (4.16) below.

(ii) It is not hard to show that $q\colon X \to Y$ is a categorical quotient of X by G. (See also Proposition (4.13) below.) More generally, if $X \to S$ is a morphism such that Γ acts by automorphisms of X over S then also Y has a natural structure of an S-scheme, and q is a categorical quotient in $\mathsf{Sch}_{/S}$. In general, q is not a universal categorical quotient. As an example, let k be a field of characteristic p, take $S = \mathrm{Spec}\bigl(k[\varepsilon]\bigr)$ and $X = \mathbb{A}^1_S = \mathrm{Spec}(A)$, with

$A = k[x,\varepsilon]$. We let the group $\Gamma := \mathbb{Z}/p\mathbb{Z}$ act on X (over S); on rings we give the action of $n \bmod p$ by $x \mapsto x + n\varepsilon$ and $\varepsilon \mapsto \varepsilon$. The ring A^Γ of invariants is generated as a k-algebra by ε, $x\varepsilon, \ldots, x^{p-1}\varepsilon$ and x^p. But on the special fibre $\mathbb{A}^1_k$ the action is trivial. As

$$A^\Gamma \otimes_{k[\varepsilon]} k = k[\varepsilon, x\varepsilon, \ldots, x^{p-1}\varepsilon, x^p] \otimes_{k[\varepsilon]} k = k[x^p]$$

is a proper subring of

$$\big(A \otimes_{k[\varepsilon]} k\big)^\Gamma = k[x]\,,$$

we see that $Y := \mathrm{Spec}(A^\Gamma)$ is not a universal categorical quotient of X in $\mathsf{Sch}_{/S}$.

(4.10) Suppose given an action of a group scheme G on a scheme X, over some basis S, say. We should like to decide if there exists a categorical quotient of X by G in $\mathsf{Sch}_{/S}$, and if yes then we should like to construct this quotient. Properties (a) and (b) in the above proposition point to a general construction. Namely, if $|X|$ is the topological space underlying X then we could try to form a quotient of $|X|$ modulo the action of G and equip this space with the sheaf of G-invariant functions on X.

Another way to phrase this is the following. The category of schemes is a full subcategory of the category LRS of locally ringed spaces, which in turn is a subcategory (not full) of the category RS of all ringed spaces. If G is an S-group scheme acting on an S-scheme X then we shall show that there exists a categorical quotient $(G\backslash X)_{\mathrm{rs}}$ in the category $\mathsf{RS}_{/S}$. It is constructed exactly as just described: form the quotient "$G\backslash|X|$" and equip this with the sheaf "$(q_*O_X)^G$", where $q\colon |X| \to G\backslash|X|$ is the natural map. Then the question is whether $(G\backslash X)_{\mathrm{rs}}$ is a scheme and, if so, if this scheme is a "good" scheme-theoretic quotient of X modulo G.

Before we give more details, let us note that in general $(G\backslash X)_{\mathrm{rs}}$ cannot be viewed as a categorical quotient in the sense of Definition (4.5). Namely, because $\mathsf{Sch}_{/S}$ is not a full subcategory of $\mathsf{RS}_{/S}$, products in the two categories may be different. Hence if G is an S-group scheme then it is not clear if the ringed space $\big(|G|, O_G\big)$ inherits the structure of a group object in $\mathsf{RS}_{/S}$. The assertion that $(G\backslash X)_{\mathrm{rs}}$ is a quotient of X by G will therefore be interpret as saying that the morphism q is a difference cokernel of the pair of morphisms $(\rho, \mathrm{pr}_X)\colon G \times_S X \rightrightarrows X$ in $\mathsf{RS}_{/S}$.

(4.11) Lemma. *Let $\rho\colon G \times_S X \to X$ be an action of an S-group scheme G on an S-scheme X. Consider the continuous maps*

$$|\mathrm{pr}_X|\colon |G \times_S X| \longrightarrow |X| \qquad \textit{and} \qquad |\rho|\colon |G \times_S X| \longrightarrow |X|\,.$$

Given P, $Q \in |X|$, write $P \sim Q$ if there exists a point $R \in |G \times_S X|$ with $|\mathrm{pr}_X|(R) = P$ and $|\rho|(R) = Q$. Then $\sim$ is an equivalence relation on $|X|$.

Proof. See Exercise (4.2).

We refer to the equivalence classes under $\sim$ as the *G-equivalence classes in $|X|$*.

(4.12) Definition. Let $\rho\colon G\times_S X \to X$ be an action of an S-group scheme G on an S-scheme X. Let $|X|/{\sim}$ be the set of G-equivalence classes in $|X|$, equipped with the quotient topology. Write $q\colon |X| \to |X|/{\sim}$ for the canonical map. Let $U = q^{-1}(V)$ for some open subset $V \subset |X|/{\sim}$. If $f \in q_*O_X(V) = O_X(U)$ then we can form the elements $\mathrm{pr}_X^\sharp(f)$ and $\rho^\sharp(f)$ in $O_{G\times_S X}(G \times_S U)$. We say that f is G-invariant if $\mathrm{pr}_X^\sharp(f) = \rho^\sharp(f)$. The G-invariant functions f form a subsheaf of rings $(q_*O_X)^G \subset q_*O_X$.

We define

$$(G\backslash X)_{\mathrm{rs}} := \big(|X|/{\sim}, (q_*O_X)^G\big)\,,$$

and write $q\colon X \to (G\backslash X)_{\mathrm{rs}}$ for the natural morphism of ringed spaces.

If $(G\backslash X)_{\mathrm{rs}}$ is a scheme and q is a morphism of schemes then we say that it is a *geometric quotient* of X by G. If moreover for every S-scheme T we have that $(G\backslash X)_{\mathrm{rs}} \times_S T \cong (G_T\backslash X_T)_{\mathrm{rs}}$ then we say that $(G\backslash X)_{\mathrm{rs}}$ is a *universal geometric quotient*.

The phrase "if a geometric quotient of X by G exists" is used as a synonym for "if $(G\backslash X)_{\mathrm{rs}}$ is a scheme and $q\colon X \to (G\backslash X)_{\mathrm{rs}}$ is a morphism of schemes".

The stalks of the sheaf $(q_*O_X)^G$ may not be local rings; for an example see ??. This is the reason why we work in the category of ringed spaces rather than the category of locally ringed spaces. Further we note that the formation of $(G\backslash X)_{\mathrm{rs}}$ does not, in general, commute with base change; see (ii) of (4.9). However, if $U \subset S$ is a Zariski open subset then $(G_U\backslash X_U)_{\mathrm{rs}}$ is canonically isomorphic to the restriction of $(G\backslash X)_{\mathrm{rs}}$ to U.

(4.13) Proposition. *In the situation of* (4.12), $q\colon X \to (G\backslash X)_{\mathrm{rs}}$ *is a difference cokernel of the pair of morphisms* $(\rho, \mathrm{pr}_X)\colon G \times_S X \rightrightarrows X$ *in the category* $\mathsf{RS}_{/S}$. *By consequence, if a geometric quotient of X by G exists then it is also a categorical quotient in* $\mathsf{Sch}_{/S}$.

Proof. The first assertion is an immediate consequence of how we constructed $(G\backslash X)_{\mathrm{rs}}$. If $(G\backslash X)_{\mathrm{rs}}$ is a geometric quotient then it is also a difference cokernel of (ρ, pr_X) in the category $\mathsf{Sch}_{/S}$ because the latter is a subcategory of $\mathsf{RS}_{/S}$. This gives the second assertion. $\square$

(4.14) Example. Let k be a field, and consider the k-scheme $\mathrm{M}_{2,k}$ $(=\mathbb{A}^4_k)$ of 2×2-matrices over k. The linear algebraic group $\mathrm{GL}_{2,k}$ acts on $\mathrm{M}_{2,k}$ by conjugation: if $g \in \mathrm{GL}_2(T)$ for some k-scheme T then g acts on $\mathrm{M}_2(T)$ by $A \mapsto g \cdot A \cdot g^{-1}$. Write $\rho\colon \mathrm{GL}_{2,k} \times \mathrm{M}_{2,k} \to \mathrm{M}_{2,k}$ for the morphism giving this $\mathrm{GL}_{2,k}$-action.

The trace and determinant give morphisms of schemes trace: $\mathrm{M}_{2,k} \to \mathbb{A}^1_k$ and det: $\mathrm{M}_{2,k} \to \mathbb{A}^1_k$. Now consider the morphism

$$p = (\mathrm{trace}, \det)\colon \mathrm{M}_{2,k} \to \mathbb{A}^2_k\,.$$

Clearly p is a GL_2-invariant morphism, i.e., $p \circ \mathrm{pr}_2\colon \mathrm{GL}_{2,k} \times \mathrm{M}_{2,k} \to \mathrm{M}_{2,k} \to \mathbb{A}^2_k$ is the same as $p \circ \rho$. It can be shown that the pair $(\mathbb{A}^2_k, p)$ is a (universal) categorical quotient of $\mathrm{M}_{2,k}$ by $\mathrm{GL}_{2,k}$, see GIT, Chap. 1, § 2 and Appendix 1C.

On the other hand, it is quite easy to see that $\mathbb{A}^2_k$ is not a geometric quotient. Indeed, if this were the case then on underlying topological spaces the map p should identify $\mathbb{A}^2_k$ as the set of $\mathrm{GL}_{2,k}$-orbits in $\mathrm{M}_{2,k}$. But the trace and the determinant are not able to distinguish a matrix

$$J_\lambda := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

from its semi-simple part

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

To give another explanation of what is going on, let us look at $\overline{k}$-valued points, where $\overline{k}$ is an algebraic closure of k. The theory of Jordan canonical forms tells us that the $\mathrm{GL}_2(\overline{k})$-orbits in $\mathrm{M}_2(\overline{k})$ are represented by the diagonal matrices $\mathrm{diag}(\lambda_1, \lambda_2)$ together with the matrices J_λ. For

$\tau, \delta \in \overline{k}$, write $N(\tau, \delta) \subset \mathrm{M}_{2,\overline{k}}$ for the 2-dimensional subvariety given by the conditions trace $= \tau$ and det $= \delta$. By direct computation one readily verifies that (i) the orbit of a diagonal matrix $A = \mathrm{diag}(\lambda, \lambda)$ is the single closed point A; (ii) the orbit of a diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2)$ with $\lambda_1 \neq \lambda_2$ equals $N(\lambda_1 + \lambda_2, \lambda_1\lambda_2)$; (iii) the orbit of a matrix J_λ equals $N(2\lambda, \lambda^2) \setminus \{\mathrm{diag}(\lambda, \lambda)\}$; in particular, this orbit is not closed in $\mathrm{M}_{2,\overline{k}}$.

From the observation that there are non-closed orbits in $\mathrm{M}_{2,\overline{k}}$, it immediately follows that there does not exist a geometric quotient. (Indeed, the orbits in $\mathrm{M}_2(\overline{k})$ would be the pre-images of the $\overline{k}$-valued points of the geometric quotient. Cf. the second example in (4.6).) This suggests that the points in a subvariety of the form $N(2\lambda, \lambda^2) \subset \mathrm{M}_{2,k}$ are the "bad" points for the given action of GL_2. Indeed, it can be shown that on the open complement $U \subset \mathrm{M}_{2,k}$ given by the condition $4\det - \mathrm{trace}^2 \neq 0$, the map $p = (\mathrm{trace}, \det)\colon U \to D(4y - x^2) \subset \mathbb{A}^2$ (taking coordinates x, y on $\mathbb{A}^2$ and writing $D(f)$ for the locus where a function f does not vanish) makes $D(4y - x^2) \subset \mathbb{A}^2_k$ a geometric quotient of U.

The notion of a geometric quotient plays a central role in geometric invariant theory. There, as in the above simple example, one studies which points, or which orbits under a given group action are so "unstable" that they obstruct the formation of a good quotient. (Which are the "bad" points may depend on further data, such as the choice of an ample line bundle on the scheme in question.) We refer the reader to the book GIT.

We now turn to the promised generalization of Proposition (4.8). First we need a lemma.

(4.15) Lemma. *Let $\varphi\colon A \to C$ be a homomorphism of commutative rings that makes C a projective A-module of rank $r > 0$. Let $\mathrm{Norm}_{C/A}\colon C \to A$ be the norm map. Let $\psi\colon \mathrm{Spec}(C) \to \mathrm{Spec}(A)$ be the morphism of affine schemes given by φ. If $Z \subset \mathrm{Spec}(C)$ is the zero locus of $f \in C$ then $\psi(Z) \subset \mathrm{Spec}(A)$ is the zero locus of $\mathrm{Norm}_{C/A}(f)$.*

Proof. The assumptions imply that φ is injective. As C is integral over A the map ψ is surjective; see also (4.21) below. Let $\mathfrak{p} \in \mathrm{Spec}(A)$; write $\psi^{-1}\{\mathfrak{p}\} = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$. By definition, $N := \mathrm{Norm}_{C/A}(f)$ is the determinant of the endomorphism $\lambda_f\colon c \mapsto fc$ of C as a module over A.

Write $W \subset \mathrm{Spec}(A)$ for the zero locus of N. Write $a_\mathfrak{p}$ for the image of an element $a \in A$ in $A_\mathfrak{p}$; similar notation for elements of C. Then we have

$$\begin{aligned} \mathfrak{p} \notin W &\iff N_\mathfrak{p} \in A_\mathfrak{p}^* \\ &\iff \lambda_{f,\mathfrak{p}}\colon C_\mathfrak{p} \to C_\mathfrak{p} \text{ is an isomorphism} \\ &\iff f_\mathfrak{p} \in C_\mathfrak{p}^* \\ &\iff f \notin \mathfrak{q}_i \text{ for all } i = 1, \ldots, n \\ &\iff \mathfrak{q}_i \notin Z \text{ for all } i = 1, \ldots, n\,, \end{aligned}$$

which proves the lemma. □

(4.16) Theorem. (Quotients by finite group schemes.) *Let G be a finite locally free S-group scheme acting on an S-scheme X. Assume that for every closed point $P \in |X|$ the G-equivalence class of P is contained in an affine open subset.*

(i) *The quotient $Y := (G\backslash X)_{\mathrm{rs}}$ is an S-scheme, which therefore is a geometric quotient of X by G. The canonical morphism $q\colon X \to Y$ is quasi-finite, integral, closed and surjective. If S is locally noetherian and X is of finite type over S then q is a finite morphism and Y is of finite type over S, too.*

(ii) *The formation of the quotient Y is compatible with* flat *base change (terminology: Y is a uniform quotient). In other words, let $h\colon S' \to S$ be a flat morphism. Let a prime $'$ denote a base change via h, e.g., $X' := X \times_S S'$. Then $Y' \cong (G'\backslash X')_{\mathrm{rs}}$.*

(iii) *If G acts freely then $q\colon X \to Y$ is finite locally free and the morphism*

$$G \times_S X \longrightarrow X \times_Y X$$

induced by $\Psi = (\rho, \mathrm{pr}_X)$ is an isomorphism. Moreover, Y is in this case a universal geometric quotient: for any morphism $h\colon S' \to S$, indicating base change via h by a prime $'$, we have $Y' \cong (G'\backslash X')_{\mathrm{rs}}$.

(4.17) Remarks. (i) The condition that every G-equivalence class is contained in an affine open subset is satisfied if X is quasi-projective over S. Indeed, given a ring R, a positive integer N, and a finite set V of closed points of $\mathbb{P}^N_R$, we can find an affine open subscheme $U \subset \mathbb{P}^N_R$ such that $V \subset U$.

(ii) In the situation of the theorem we find that a free action is automatically strictly free. Indeed, by (iii) the graph morphism Ψ gives an isomorphism of $G \times_S X$ with the subscheme $X \times_Y X \subset X \times_S X$; hence Ψ is an immersion.

We break up the proof of the theorem into a couple of steps, (4.18)–(4.26).

(4.18) *Reduction to the case that S is affine.* Suppose $S = \cup_\alpha U_\alpha$ is a covering of S by Zariski open subsets. As remarked earlier, the restriction of $(G\backslash X)_{\mathrm{rs}}$ to $U = U_\alpha$ is naturally isomorphic to $(G_U\backslash X_U)_{\mathrm{rs}}$. If we can prove the theorem over each of the open sets U_α then the result as stated easily follows by gluing. In the rest of the proof we may therefore assume that $S = \mathrm{Spec}(Q)$ is affine and that the affine algebra R of G is free of some rank r as a Q-module.

(4.19) *Reduction to the case that X is affine.* If $P \in |X|$, let us write $G(P)$ for its G-equivalence class; note that this is a finite set. Note further that

$$G(P) = \rho\bigl(\mathrm{pr}_X^{-1}\{P\}\bigr) = \mathrm{pr}_X\bigl(\rho^{-1}\{P\}\bigr)\,,$$

by definition of G-equivalence. (Strictly speaking we should write $|\rho|$ and $|\mathrm{pr}_X|$.)

Say that a subset $V \subset |X|$ is G-stable if it contains $G(P)$ whenever it contains P. If V is open then there is a maximal open subset $V' \subseteq V$ which is G-stable. Namely, if $Z := |X| \setminus V$ then $Z' := \mathrm{pr}_X\bigl(\rho^{-1}\{Z\}\bigr)$ is closed (since $\mathrm{pr}_X\colon G \times_S X \to X$ is proper), and $V' := |X| \setminus Z'$ has the required property.

We claim that X can be covered by G-stable affine open subsets. It suffices to show that every closed point $P \in X$ has a G-stable affine open neighbourhood. By assumption there exists an affine open $V \subset X$ with $G(P) \subset V$. Then also $G(P) \subset V'$. As $G(P)$ is finite there exists an $f \in \Gamma(V, O_V)$ such that, writing $D(f) \subset V$ for the open subset where f does not vanish, $G(P) \subset D(f) \subseteq V'$. In total this gives

$$G(P) \subset D(f)' \subseteq D(f) \subseteq V' \subseteq V\,.$$

Our claim is proven if we can show that $D(f)'$ is affine. Write f' for the image of f in $\Gamma(V', O_{V'})$, so that $Z := V' \setminus D(f)$ is the zero locus of f'. As V' is G-stable we have $\rho^{-1}(V') = G \times_S V'$, which gives an element $\rho^\sharp(f') \in \Gamma(G \times_S V', O_{G\times_S V'})$. The zero locus of $\rho^\sharp(f')$ is of course just $\rho^{-1}(Z) \subset G \times_S V'$. As G is finite locally free, the morphism pr_X makes

$\Gamma\big(G \times_S V', O_{G\times_S V'}\big)$ into a projective module of finite rank over $\Gamma\big(V', O_{V'}\big)$. This gives us a norm map

$$\text{Norm: } \Gamma\big(G \times_S V', O_{G\times_S V'}\big) \longrightarrow \Gamma\big(V', O_{V'}\big)\,.$$

Let $F := \text{Norm}\big(\rho^\sharp(f')\big)$. By Lemma (4.15), the zero locus of F is the image of $\rho^{-1}(Z)$ under the projection to V'. But the complement of this locus in V' is precisely $D(f)'$. Hence if F' is the image of F in $\Gamma\big(D(f), O_{D(f)}\big)$ then $D(f)'$ is the open subset of $D(f)$ where F' does not vanish. As this subset is affine open, our claim is proven.

Except for the last assertion of (i), the proof of the theorem now reduces to the case that X is affine. Namely, by the previous we can cover X by G-stable affine open subsets, and if the theorem is true for each of these then by gluing we obtain the result for X. The last assertion of (i) will be dealt with in (4.23).

(4.20) From now on we assume that $X = \text{Spec}(A) \to S = \text{Spec}(Q)$. Further we assume that $G = \text{Spec}(R)$ for some Q-Hopf algebra R which is free of rank r as a module over Q. Much of what we are going to do is a direct generalization of the arguments in (4.8); that proof may therefore serve as a guide for the arguments to follow.

The action of G on X is given by a Q-algebra homomorphism $\sigma\colon A \to R\otimes_Q A$. Write $j\colon A \to R\otimes_Q A$ for the map given by $a \mapsto 1\otimes a$. (In other words, we write σ for $\rho^\sharp$ and j for $\text{pr}_X^\sharp$.) Define a subring $B := A^G \subset A$ of G-invariants by

$$B := \{a \in A \mid \sigma(a) = j(a)\}\,.$$

We are going to prove that $Y := \text{Spec}(B)$ is the geometric quotient of X under the given action of G.

As a first step, let us show that A is integral over B. For $a \in A$, multiplication by $\sigma(a)$ is an endomorphism of $R\otimes_Q A$, and we can form its characteristic polynomial

$$\chi(t) = t^r + c_{r-1}t^{r-1} + \cdots + c_1 t + c_0 \in A[t]\,.$$

We have cartesian squares

$$\begin{array}{ccc} R\otimes_Q A & \xrightarrow{\tilde m\otimes \text{id}_A} & R\otimes_Q R\otimes_Q A \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle j_{2,3}} \\ A & \xrightarrow{j} & R\otimes_Q A \end{array} \quad\text{and}\quad \begin{array}{ccc} R\otimes_Q A & \xrightarrow{\text{id}_R\otimes\sigma} & R\otimes_Q R\otimes_Q A \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle j_{2,3}} \\ A & \xrightarrow{\sigma} & R\otimes_Q A\,, \end{array} \qquad (2)$$

where the map $j_{2,3}$ is given by $r\otimes a \mapsto 1\otimes r\otimes a$. We view $R\otimes_Q R\otimes_Q A$ as a module over $R\otimes_Q A$ via $j_{2,3}$. It follows from the left-hand diagram that $j\big(\chi(t)\big)$, the polynomial obtained from $\chi(t)$ by applying j to its coefficients, is the characteristic polynomial of $\tilde m\otimes\text{id}_A\big(\sigma(a)\big)$. The right-hand diagram tells us that $\sigma\big(\chi(t)\big)$ is the characteristic polynomial of $\text{id}_R\otimes\sigma\big(\sigma(a)\big)$. But the commutativity of diagram (1) in Definition (4.1) gives the identity $\tilde m\otimes\text{id}_A\big(\sigma(a)\big) = \text{id}_R\otimes \sigma\big(\sigma(a)\big)$. Hence $j\big(\chi(t)\big) = \sigma\big(\chi(t)\big)$, which means that $\chi(t)$ is a polynomial with coefficients c_i in the ring B of G-invariants.

The Cayley-Hamilton theorem tells us that

$$\sigma(a)^r + j(c_{r-1})\sigma(a)^{r-1} + \cdots + j(c_1)\sigma(a) + j(c_0) = 0\,.$$

As $j(c_i) = \sigma(c_i)$ for all i we can rewrite this as

$$\sigma\big(\chi(a)\big) = \sigma(a)^r + \sigma(c_{r-1})\sigma(a)^{r-1} + \cdots + \sigma(c_1)\sigma(a) + \sigma(c_0) = 0\,. \qquad (3)$$

But σ is an injective map, because we have the relation $(\tilde{e}\otimes \mathrm{id}_A)\circ\sigma = \mathrm{id}_A$, which translates the fact that the identity element of G acts as the identity on X. Hence (3) implies that $\chi(a) = 0$. This proves that A is integral over B.

(4.21) The fact that A is integral over B has the following consequences.

(i) If $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ are prime ideals of A with $\mathfrak{p}_1 \cap B = \mathfrak{p}_2 \cap B$ then $\mathfrak{p}_1 = \mathfrak{p}_2$. Geometrically this means that all fibres of $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ have dimension 0.

(ii) The natural map q: $X = \mathrm{Spec}(A) \to Y = \mathrm{Spec}(B)$ is surjective.

(iii) The map q is closed, i.e., if $C \subset X$ is closed then $q(C) \subset Y$ is closed too.

Properties (i) and (ii) can be found in many textbooks on commutative algebra, see for instance Atiyah-Macdonald [1], Cor. 5.9 and Thm. 5.10. For (iii), suppose $C \subset X$ is the closed subset defined by an ideal $\mathfrak{a} \subset A$. We may identify C with $\mathrm{Spec}(A/\mathfrak{a})$. The composite map $C = \mathrm{Spec}(A/\mathfrak{a}) \hookrightarrow \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ factors through the closed subset $\mathrm{Spec}(B/\mathfrak{b}) \subset \mathrm{Spec}(B)$, where $\mathfrak{b} = \mathfrak{a} \cap B$. Note that $A/\mathfrak{a}$ is again integral over its subring $B/\mathfrak{b}$. Applying (ii) with A and B replaced by $A/\mathfrak{a}$ and $B/\mathfrak{b}$, we find that $C = \mathrm{Spec}(A/\mathfrak{a}) \to \mathrm{Spec}(B/\mathfrak{b})$ is surjective. Hence $q(C)$ is the closed subset of B defined by $\mathfrak{b}$.

Define a map N: $A \to B$ by

$$N(a) = \mathrm{Norm}_{R\otimes_Q A/A}\bigl(\sigma(a)\bigr)\,.$$

Note that $N(a) = (-1)^n c_0$, where c_0 is the constant coefficient of the characteristic polynomial $\chi(t)$ considered in (4.20); hence $N(a)$ is indeed an element of B. The relation $\chi(a) = 0$ gives

$$N(a) = (-1)^{n+1}\cdot a \cdot \bigl(a^{n-1} + c_{n-1}a^{n-2} + \cdots + c_1\bigr)\,.$$

In particular, if $a \in \mathfrak{a}$ for some ideal $\mathfrak{a} \subset A$ then $N(a) \in \mathfrak{a} \cap B$.

(4.22) Recall that $Y := \mathrm{Spec}(B)$. We are going to prove that $Y = (G\backslash X)_{\mathrm{rs}}$. Note that the natural map $|X| \to |Y|$ is surjective, by (ii) in (4.21).

By definition, two prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ of A are in the same G-equivalence class if there exists a prime ideal $\mathfrak{Q}$ of $R\otimes_Q A$ with $\sigma^{-1}(\mathfrak{Q}) = \mathfrak{p}$ and $j^{-1}(\mathfrak{Q}) = \mathfrak{p}'$. If such a prime ideal $\mathfrak{Q}$ exists then it is immediate that $\mathfrak{p} \cap B = \mathfrak{p}' \cap B$, so G-equivalent points of X are mapped to the same point of Y.

Conversely, suppose $\mathfrak{p} \cap B = \mathfrak{p}' \cap B$. There are finitely many prime ideals $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_n$ of $R\otimes_Q A$ with the property that $j^{-1}(\mathfrak{Q}_i) = \mathfrak{p}'$. (The morphism pr_X: $G\times_S X \to X$ is finite because G is finite.) Set $\mathfrak{q}_i = \sigma^{-1}(\mathfrak{Q}_i)$. Note that $\mathfrak{q}_i \cap B = \mathfrak{p} \cap B$. Our goal is to prove that $\mathfrak{p} = \mathfrak{q}_i$ for some i. By property (i) above it suffices to show that $\mathfrak{p} \subseteq \mathfrak{q}_i$ for some i. Suppose this is not the case. Then there exists an element $a \in \mathfrak{p}$ that is not contained in $\mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_n$. (Use Atiyah-Macdonald [1], Prop. 1.11, and cf. the proof of Prop. (4.8) above.) Lemma (4.15), applied with $f = \sigma(a) \in R\otimes_Q A$, tells us that the prime ideals of A containing $N(a)$ are all of the form $j^{-1}(\mathfrak{r})$ with $\mathfrak{r}$ a prime ideal of $R\otimes_Q A$ that contains $\sigma(a)$. But $a \in \mathfrak{p}$, hence $N(a) \in \mathfrak{p} \cap B = \mathfrak{p}' \cap B$. Hence one of the prime ideals $\mathfrak{Q}_i$ contains $\sigma(a)$, contradicting our choice of a.

We have now proven that the map $X \to Y$ identifies $|Y|$ with the set $|X|/{\sim}$ of G-equivalence classes in X. Further, by (iii) in (4.21) the quotient map $|X| \to |Y|$ is closed, so the topology on $|Y|$ is the quotient topology. If $V = D_Y(f) \subset Y$ is the fundamental open subset given by $f \in B$ then $q^{-1}(V) = D_X(f)$, and we find

$$O_Y(V) = B_f = (A^G)_f \xrightarrow{\sim} (A_f)^G = \bigl(O_X(q^{-1}(V))\bigr)^G = \bigl((q_*O_X)(V)\bigr)^G\,.$$

As the fundamental open subsets form a basis for the topology on Y, it follows that $q^\sharp\colon O_Y \to q_*O_X$ induces an isomorphism $O_Y \xrightarrow{\sim} (q_*O_X)^G$.

(4.23) Let us now prove the last assertion of part (i) of the theorem. As before we may assume that $S = \mathrm{Spec}(Q)$ is affine. Let $q\colon X \to Y := (G\backslash X)_{\mathrm{rs}}$ be the quotient morphism, which we have already shown to exist. Let $U = \mathrm{Spec}(A)$ be a G-stable affine open subset of X, and let $B = A^G$. By construction, $q(U) = \mathrm{Spec}(B)$ is an open subset of Y, and $q^{-1}\bigl(q(U)\bigr) = U$. If X is locally of finite type over S then A is a finitely generated Q-algebra, a fortiori also of finite type as a B-algebra. But A is also integral over B. It follows that A is finitely generated as a B-module (see e.g. Atiyah-Macdonald [1], Cor. 5.2). Hence q is a finite morphism.

If S is locally noetherian then we may assume, arguing as in (4.18), that Q is a noetherian ring. Choose generators $a_1, \ldots, a_n$ for A as a Q-algebra. We have seen that for each i we can find a monic polynomial $f_i \in B[T]$ with $f_i(a_i) = 0$. Let $B' \subset B$ be the Q-subalgebra generated by the coefficients of the polynomials f_i. Then A is integral over B', and by the same argument as above it follows that A is finitely generated as a B'-module. Because B' is finitely generated over Q it is a noetherian ring. But then $B \subset A$ is also finitely generated as a B'-module, hence finitely generated as a Q-algebra. This shows that Y is locally of finite type over S.

So far we have used only that X is locally of finite type over S. Assume, in addition, that the morphism $f\colon X \to S$ is quasi-compact. Let $g\colon Y \to S$ be the structural morphism of Y. It remains to be shown that g is quasi-compact. But this is clear, for if $V \subset S$ is a quasi-compact open subset then $g^{-1}(V) = q\bigl(f^{-1}(V)\bigr)$, which is quasi-compact because $f^{-1}(V)$ is.

(4.24) Proof of (ii) of the theorem. Let $S' \to S$ be a flat morphism. We want to show that $Y' := Y \times_S S'$ is a geometric quotient of X' by G'. Arguing as in (4.18) one reduces to the case that $S' \to S$ is given by a flat homomorphism of rings $Q \to Q'$. Note that every G'-equivalence class in X' is again contained in an affine open subset. As in (4.19) one further reduces to the case that X, X', Y and Y' are all affine. With notations as above we have $Y = \mathrm{Spec}(B)$, where $B = \mathrm{Ker}(j - \sigma)$. We want to show, writing a prime $'$ for extension of scalars to Q', that $B \otimes_Q Q' = \mathrm{Ker}(j' - \sigma'\colon A' \otimes_{Q'} R' \to A' \otimes_{Q'} A')$. But this is obvious from the assumption that $Q \to Q'$ is flat.

(4.25) We now turn to part (iii) of the theorem. As before, everything reduces to the situation where S, G, X and Y are all affine, with algebras Q, R, A and $B = A^G$, respectively, and that R is free of rank r as a module over Q. We view $R \otimes_Q A$ as an A-module via j. Let

$$\varphi\colon A \otimes_B A \to R \otimes_Q A$$

be the homomorphism given by $\varphi(a_1 \otimes a_2) = \sigma(a_1) \cdot j(a_2) = \sigma(a_1) \cdot (1 \otimes a_2)$.

Assume that G acts freely on X. This means that the morphism $\Psi\colon G \times_S X \to X \times_S X$ is a monomorphism in the category of schemes. The corresponding map on rings is given by $\Psi^\sharp = \varphi \circ q$, where $q\colon A \otimes_Q A \twoheadrightarrow A \otimes_B A$ is the natural map. Since a morphism of affine schemes is a monomorphism if and only if the corresponding map on rings is surjective, it follows that φ is surjective.

Let $\mathfrak{q}$ be a prime ideal of B and write $A_{\mathfrak{q}} = (B - \mathfrak{q})^{-1}A \cong A \otimes_B B_{\mathfrak{q}}$. Note that $A_{\mathfrak{q}}$ is a semi-local ring, because $X \to Y$ is quasi-finite. Let $\mathfrak{r} \subset A_{\mathfrak{q}}$ be its radical. We claim that $A_{\mathfrak{q}}$ is free of rank $r = \mathrm{rank}(G)$ as a module over $B_{\mathfrak{q}}$. If this holds for all $\mathfrak{q}$ then A is a projective B-module of rank r; use Bourbaki [2], Chap. II, § 5, Thm. 2. Furthermore, φ is then a surjective map between projective A-modules of the same rank and is therefore an isomorphism.

We first prove that $A_{\mathfrak{q}}$ is $B_{\mathfrak{q}}$-free of rank r in the case where the residue field k of $B_{\mathfrak{q}}$ is infinite. Consider the $B_{\mathfrak{q}}$-submodule

$$N := \{\sigma(a) \mid a \in A_{\mathfrak{q}}\} \subset M := R \otimes_Q A_{\mathfrak{q}} .$$

Because $\varphi_{\mathfrak{q}}\colon A_{\mathfrak{q}} \otimes_{B_{\mathfrak{q}}} A_{\mathfrak{q}} \to R \otimes_Q A_{\mathfrak{q}}$ is surjective, N spans M as an $A_{\mathfrak{q}}$-module. Therefore $N/\mathfrak{r}N$ spans $M/\mathfrak{r}M \cong (A_{\mathfrak{q}}/\mathfrak{r})^r$ as a module over $A_{\mathfrak{q}}/\mathfrak{r}$, which is a product of fields. Using that k is an infinite subfield of $A_{\mathfrak{q}}/\mathfrak{r}$ it follows that $N/\mathfrak{r}N$ contains a basis of $M/\mathfrak{r}M$ over $A_{\mathfrak{q}}/\mathfrak{r}$; see Exercise (4.3). Applying the Nakayama lemma, it follows that N contains a basis of M over $A_{\mathfrak{q}}$, i.e., we have elements $a_1, \dots a_r \in A_{\mathfrak{q}}$ such that the elements $\varphi_{\mathfrak{q}}(a_i \otimes 1) = \sigma(a_i)$ form an $A_{\mathfrak{q}}$-basis of $R \otimes_Q A_{\mathfrak{q}}$. Hence for every $a \in A_{\mathfrak{q}}$ there are unique coordinates $x_1, \dots, x_r \in A_{\mathfrak{q}}$ such that

$$\begin{aligned} \sigma(a) &= x_1 \cdot \sigma(a_1) + \cdots + x_r \cdot \sigma(a_r) \\ &= (1 \otimes x_1) \cdot \sigma(a_1) + \cdots + (1 \otimes x_r) \cdot \sigma(a_r) . \end{aligned} \tag{4}$$

We view $R'' := R \otimes_Q R \otimes_Q A_{\mathfrak{q}}$ as a module over $R \otimes_Q A_{\mathfrak{q}}$ via the homomorphism $j_{2,3}$ given by $r \otimes a \mapsto 1 \otimes r \otimes a$. The diagrams (2) tell us that the elements

$$\gamma_i := (\tilde{m} \otimes \mathrm{id}_A)\big(\sigma(a_i)\big) = (\mathrm{id}_R \otimes \sigma)\big(\sigma(a_i)\big)$$

form an $R \otimes_Q A_{\mathfrak{q}}$-basis of R''. Applying $\tilde{m} \otimes \mathrm{id}_A$ and $\mathrm{id}_R \otimes \sigma$ to (4) gives

$$\begin{array}{ccc} (\tilde{m} \otimes \mathrm{id}_A)\big(\sigma(a)\big) & = & (1 \otimes 1 \otimes x_1) \cdot \gamma_1 + \cdots + (1 \otimes 1 \otimes x_r) \cdot \gamma_r \\ \Vert & & \\ (\mathrm{id}_R \otimes \sigma)\big(\sigma(a)\big) & = & \big(1 \otimes \sigma(x_1)\big) \cdot \gamma_1 + \cdots + \big(1 \otimes \sigma(x_r)\big) \cdot \gamma_r . \end{array}$$

Hence the coordinates x_i lie in B, and (4) becomes $\sigma(a) = \sigma(x_1 a_1 + \cdots + x_r a_r)$. But we have seen in (4.20) that σ is injective, hence $a = x_1 a_1 + \cdots + x_r a_r$. This proves that the elements $a_1, \dots, a_r$ span $A_{\mathfrak{q}}$ as a $B_{\mathfrak{q}}$-module. On the other hand, since the map $a \mapsto \sigma(a)$ is $B_{\mathfrak{q}}$-linear, the elements $a_1, \dots, a_r$ are linearly independent over $B_{\mathfrak{q}}$. Hence $A_{\mathfrak{q}}$ is free of rank r over $B_{\mathfrak{q}}$.

Finally we consider the case that $B_{\mathfrak{q}}$ has a finite residue field. By what was explained in Remark (4.7) we may assume that $S = Y$. Because $B \to B_{\mathfrak{q}}$ is flat we may, by (ii) of the theorem, further reduce to the case where $B = B_{\mathfrak{q}}$. Let $h\colon B \to B'$ be a faithfully flat homomorphism, where B' is a local ring with infinite residue field; for instance we could take B' to be a strict henselization of $B = B_{\mathfrak{q}}$. In order to show that $A = A_{\mathfrak{q}}$ is free of rank r over B, it suffices to show that $A' := A \otimes_B B'$ is free of rank r over B', see EGA IV, 2.5.2. But, again by (ii), $\mathrm{Spec}(B')$ is the quotient of $\mathrm{Spec}(A')$ under the G-action obtained by base-change. Hence we are reduced to the case treated above.

(4.26) As the final step in the proof we show that if G acts freely, Y is a universal geometric quotient. Consider a morphism $h\colon S' \to S$. Let us indicate base change via h by a $'$, so $X' := X \times_S S'$, etc. Then G' acts again freely on X', and it is easy to see that every G'-equivalence class of closed points in $|X'|$ is contained in an affine open subset. (Since this statement only involves the fibres of X' we may assume that S' is affine, in which case the morphism $X' \to X$ is affine.) Hence there exists a geometric quotient, say $q_Z\colon X' \to Z$. As Z is a categorical quotient of X' by G', the morphism $q'\colon X' \to Y'$ factors as $q' = f \circ q_Z$ with $f\colon Z \to Y'$. We want to show that f is an isomorphism.

As before we may assume that G is free of rank r over S. Then X' is free of rank r over Z but at the same time it is free of the same rank r over Y'. But then Z has to be locally free of rank 1 over Y', so $f\colon Z \xrightarrow{\sim} Y'$. This completes the proof of Theorem (4.16). $\square$

§ 3. *FPPF quotients.*

Consider an action of an S-group scheme G on an S-scheme X. In general there is not a simple procedure to construct a "good" quotient of X by G in the category $\mathsf{Sch}_{/S}$. Of course we have the notion of a categorical quotient, but this is only a "best possible approximation in the given category", and its definition gives no clues about whether there exists a categorical quotient and, if so, how to describe it.

Most approaches to the formation of quotients follow the same pattern:
(a) replace the category $\mathsf{Sch}_{/S}$ of S-schemes by some "bigger" category, in which the formation of quotients is easier;
(b) form the quotient $Y := G\backslash X$ in this bigger category;
(c) study under which assumptions the quotient Y is (representable by) a scheme.
Thus, for instance, in our discussion of geometric quotients the "bigger" category that we used was the category of ringed spaces over S.

The approach usually taken in the theory of group schemes is explained with great clarity in Raynaud [2]. The idea is that one chooses a Grothendieck topology on the category of S-schemes and that all objects in question are viewed as sheaves on the resulting site. The quotient spaces that we are interested in exist as sheaves—this usually involves a sheafification—and their construction has good functorial properties. Then it remains to be investigated under what conditions the quotient sheaf is representable by a scheme. For the choice of the topology, a couple of remarks have to be taken into account. First, we want our original objects, schemes, to be sheaves rather than presheaves; this means that the topology should be no finer than the canonical topology (see Appendix ??). On the other hand, the finer the topology, the weaker the condition that a sheaf is representable. Finally the topology has to be accessible by the methods of algebraic geometry. In practice one usually works with the étale topology, the fppf topology or the fpqc topology. We shall mostly work with the fppf topology. See (4.36) below for further discussion.

From a modern perspective, perhaps the most natural choice for the "bigger category" in which to work, is the category of algebraic stacks. An excellent reference for the foundations of this theory is the book by Laumon and Moret-Bailly [1]. For general results about the formation of quotients as algebraic spaces we recommend the papers by Keel and Mori [1] and Kollár [1]. However, at this stage in our book we shall not assume any knowledge of algebraic spaces or stacks (though algebraic spaces will be briefly mentioned in our discussion of Picard functors in Chap. 6).

Finally let us remark that we shall almost exclusively deal with quotients modulo a group action, and not with more general equivalence relations or groupoids. It should be noted that even if one is interested only in group quotients, the proofs often involve more general groupoids.

(4.27) We shall use some notions that are explained in more detail in Appendix ??.

Let S be a scheme. We write $(S)_{\mathrm{FPPF}}$ for the big fppf site of S, i.e., the category $\mathsf{Sch}_{/S}$ of S-schemes equipped with the fppf topology. We write $\mathrm{FPPF}(S)$ for the category of sheaves on

$(S)_{\mathrm{FPPF}}$.

The fppf topology is coarser than the canonical topology; this means that for every S-scheme X the presheaf $h_X = \mathrm{Hom}_S(-,X)$ is a sheaf on $(S)_{\mathrm{FPPF}}$. As explained in A?? this is essentially a reformulation of results in descent theory. Via $X \mapsto h_X$ we can identify $\mathsf{Sch}_{/S}$ with a full subcategory of $\mathrm{FPPF}(S)$. We shall usually simply write X for h_X.

Denote by $\mathsf{ShGr}_{/S}$ and $\mathsf{ShAb}_{/S}$ the categories of sheaves of groups, respectively sheaves of abelian groups, on $(S)_{\mathrm{FPPF}}$. The category $\mathsf{ShAb}_{/S}$ is abelian; $\mathsf{ShGr}_{/S}$ is not abelian (excluding $S = \emptyset$) but we can still speak about exact sequences. Unless specified otherwise, we shall from now on view the category of S-group schemes as a full subcategory of $\mathsf{ShGr}_{/S}$. For example, we shall say that a sequence of S-group schemes

$$G' \xrightarrow{\varphi} G \xrightarrow{\psi} G''$$

is exact if it is exact as a sequence in $\mathsf{ShGr}_{/S}$, i.e., if $\mathrm{Ker}(\psi)$ represents the fppf sheaf associated to the presheaf $T \mapsto \mathrm{Im}\big(\varphi(T)\colon G'(T) \to G(T)\big)$.

(4.28) Definition. Let G be an S-group scheme acting, by $\rho\colon G \times_S X \to X$, on an S-scheme X. We write $(G\backslash X)_{\mathrm{fppf}}$, or simply $G\backslash X$, for the fppf sheaf associated to the presheaf

$$T \mapsto G(T)\backslash X(T)\,.$$

If $G\backslash X$ is representable by a scheme Y then we refer to Y (or to the quotient morphism $q\colon X \to Y$) as the *fppf quotient of X by G*.

We often say that "an fppf quotient exists" if $(G\backslash X)_{\mathrm{fppf}}$ is representable by a scheme. Note that the sheaf $G\backslash X$ is a categorical quotient of X by G in $\mathrm{FPPF}(S)$, so we are indeed forming the quotient in a "bigger" category. Note further that if $(G\backslash X)_{\mathrm{fppf}}$ is representable by a scheme Y then by the Yoneda lemma we have a morphism of schemes $q\colon X \to Y$.

As we are mainly interested in the formation of quotients of a group scheme by a subgroup scheme, we shall mostly restrict our discussion of fppf quotients to the case that the action is free.

(4.29) Example. Consider the situation as in (iii) of Theorem (4.16). So, G is finite locally free over S, acting freely on X, and every orbit is contained in an affine open set. Let $q_Y\colon X \to Y$ be the universal geometric quotient, as we have proven to exist. We claim that Y is also an fppf quotient. To see this, write $Z := (G\backslash X)_{\mathrm{fppf}}$ and write $q_Z\colon X \to Z$ for the quotient map. As Z is a categorical quotient in $\mathrm{FPPF}(S)$, the morphism q_Y, viewed as a morphism of fppf sheaves, factors as $q_Y = r \circ q_Z$ for some $r\colon Z \to Y$. To prove that r is an isomorphism it suffices to show that it is both a monomorphism and an epimorphism.

By (iii) of (4.16), the morphism q_Y is fppf. By A?? this implies it is an epimorphism of sheaves. But then r is an epimorphism too. On the other hand, suppose T is an S-scheme and suppose $a, b \in Z(T)$ map to the same point in $Y(T)$. There exists an fppf covering $T' \to T$ such that a and b come from points $a', b' \in X(T')$. But we know that $\Psi = (\rho, \mathrm{pr}_X)\colon G \times_S X \to X \times_Y X$ is an isomorphism, so there is a point $c \in G \times_S X(T')$ with $\rho(c) = a'$ and $\mathrm{pr}_X(c) = b'$. By construction of $Z := (G\backslash X)_{\mathrm{fppf}}$ this implies that $a = b$. Hence r is a monomorphism.

(4.30) The formation of fppf quotients is compatible with base change. To explain this in more detail, suppose $j\colon S' \to S$ is a morphism of schemes. Then j gives rise to an inverse image functor $j^*\colon \mathrm{FPPF}(S) \to \mathrm{FPPF}(S')$ which is exact. Concretely, if $f\colon T \to S'$ is an S'-scheme then $j \circ f\colon T \to S$ is an S-scheme, and if F is an fppf sheaf on S then we have $j^*F(f\colon T \to S') = F(j \circ f\colon T \to S)$. In particular, on representable sheaves j^* is simply given by base-change: $j^*X = X \times_S S'$. Writing $X' = X \times_S S'$ and $G' = G \times_S S'$, we conclude that $j^*(G\backslash X) = (G'\backslash X')$ as sheaves on $(S')_{\mathrm{FPPF}}$. Hence if $q\colon X \to Y$ is an fppf quotient over S then $Y' := Y \times_S S'$ is an fppf quotient of X' by G'. Put differently: *An fppf quotient, if it exists, is automatically a* universal *fppf quotient.*

(4.31) Proposition. *Let G be an S-group scheme acting freely on an S-scheme X. Suppose the fppf sheaf $(G\backslash X)_{\mathrm{fppf}}$ is representable by a scheme Y. Write $q\colon X \to Y$ for the canonical morphism. Then q is an fppf covering and the morphism $\Psi\colon G \times_S X \to X \times_Y X$ is an isomorphism. This gives a commutative diagram with cartesian squares*

$$\begin{tikzcd} G \times_S X \arrow[r, "\sim"] \arrow[d, "\mathrm{pr}_2"'] & X \times_Y X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & X \arrow[d, "q"] \\ X \arrow[r, equal] & X \arrow[r, "q"] & Y \end{tikzcd} .$$

In particular, X is a G-torsor over Y in the fppf topology which becomes trivial over the covering $q\colon X \to Y$.

Proof. By construction, the projection $X \to Y$ is an epimorphism of fppf sheaves. This implies that it is an fppf covering; see A??. Further, $\Psi\colon G \times_S X \to X \times_Y X$ is an isomorphism of fppf sheaves, again by construction of $Y = G\backslash X$. By the Yoneda lemma (3.3), Ψ is then also an isomorphism of schemes. □

(4.32) In the situation of the proposition, a necessary condition for $(G\backslash X)_{\mathrm{fppf}}$ to be representable by a scheme is that the action of G on X is *strictly* free. Indeed, this is immediate from the fact that $X \times_Y X$ is a subscheme of $X \times_S X$. But the good news contained in (4.31) is that if an fppf quotient exists, it has very good functorial properties. Let us explain this in some more detail.

We say that a property P of morphisms of schemes is *fppf local on the target* if the following two conditions hold:

(a) given a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

we have $P(f) \Rightarrow P(f')$ (we say: "P is stable under base change");

(b) if furthermore $g\colon S' \to S$ is an fppf covering then $P(f) \Leftrightarrow P(f')$.

Many properties that play a role in algebraic geometry are fppf local on the target. More precisely, it follows from the results in EGA IV, § 2 that this holds for the property P of a morphism of schemes of being flat, smooth, unramified, étale, (locally) of finite type or finite presentation, (quasi-) separated, (quasi-) finite, (quasi-) affine, or integral.

(4.33) Corollary. *Let P be a property of morphisms of schemes which is local on the target for the fppf topology. If $q\colon X \to Y$ is an fppf quotient of X under the free action of an S-group* *scheme G, then*

$$\begin{matrix} q\colon X \to Y \\ \textit{has property } P \end{matrix} \iff \begin{matrix} \mathrm{pr}_2\colon G \times_S X \to X \\ \textit{has property } P \end{matrix} \Longleftarrow \begin{matrix} \pi\colon G \to S \\ \textit{has property } P \end{matrix}$$

where moreover the last implication is an equivalence if $X \to S$ is an fppf covering.

Proof. Clear, as $q\colon X \to Y$ is an fppf covering and $G \times_S X \xrightarrow{\sim} X \times_Y X$. $\square$

In the applications we shall see that this is a most useful result. After all, it tells us that an fppf quotient morphism $q\colon X \to Y$ inherits many properties from the structural morphism $\pi\colon G \to S$. To study π we can use the techniques discussed in Chapter 3. To give but one example, suppose $S = \mathrm{Spec}(k)$ is the spectrum of a field and that G and X are of finite type over k. As before we assume that G acts freely on X. Then the conclusion is that an fppf quotient morphism $q\colon X \to Y$ is smooth if and only if G is a smooth k-group scheme. By (3.17) it suffices to test this at the origin of G, and if moreover $\mathrm{char}(k) = 0$ then by (3.20) G is automatically smooth over k.

(4.34) At this point, let us take a little step back and compare the various notions of a quotient that we have encountered.

Consider a base scheme S, an S-group scheme G acting on an S-scheme X, and suppose $q\colon X \to Y$ is a morphism of S-schemes. Then q realizes Y as

—*a categorical quotient of X by G* if q is universal for G-equivariant morphisms from X to an S-scheme with trivial G-action;

—*a geometric quotient of X by G* if $|Y| = |X|/\sim$ and $O_Y = (q_*O_X)^G$, i.e., Y represents the quotient of X by G formed in the category of ringed spaces;

—*an fppf quotient of X by G* if Y represents the fppf sheaf associated to the presheaf $T \mapsto G(T)\backslash X(T)$, i.e., Y represents the quotient of X by G formed in the category of fppf sheaves.

Further we have defined what it means for Y to be a universal categorical or geometric quotient. As remarked earlier, an fppf quotient is automatically universal.

The following result is due to Raynaud [1] and gives a comparison between fppf and geometric quotients.

(4.35) Theorem. *Let G be an S-group scheme acting on an S-scheme X.*

(i) *Suppose there exists an fppf quotient Y of X by G. Then Y is also a geometric quotient.*

(ii) *Assume that X is locally of finite type over S, and that G is flat and locally of finite presentation over S. Assume further that the action of G on X is strictly free. If there exists a geometric quotient Y of X by G then Y is also an fppf quotient. In particular, the quotient morphism $q\colon X \to Y$ is an fppf morphism and Y is a universal geometric quotient.*

Proof. For the proof of (ii) we refer to Anantharaman [1], Appendix I. Let us prove (i). Suppose that $q\colon X \to Y$ is an fppf quotient. Write $r\colon X \to Z := (G\backslash X)_{\mathrm{rs}}$ for the quotient of X by G in the category of ringed spaces over S. Since r is a categorical quotient in $\mathsf{RS}_{/S}$ we have a unique morphism of ringed spaces $s\colon Z \to Y$ such that $q = s \circ r$. Our goal is to prove that s is an isomorphism. First note that q, being an fppf covering, is open and surjective. Since also r is surjective, this implies that the map s is open and surjective.

Next we show that s is injective. Suppose A and B are points of $|X|$ that map to the same point C in $|Y|$. We have to show that $\rho^{-1}\{A\} \cap \mathrm{pr}_X^{-1}\{B\}$ is non-empty, for then A and B map to the same point of Z, and the injectivity of s follows. Choose a field extension $\kappa(C) \subset K$

and K-valued points $a \in X(K)$ and $b \in X(K)$ with support in A and B, respectively, such that $q(a) = q(b)$. By construction of the fppf quotient, there exists a K-algebra L of finite type and an L-valued point $d \in G \times_S X(L)$ with $\rho(d) = a$ and $\mathrm{pr}_X(d) = b$. But then the image of $d\colon \mathrm{Spec}(L) \to G \times_S X$ is contained in $\rho^{-1}\{A\} \cap \mathrm{pr}_X^{-1}\{B\}$.

Finally, let U an open part of Y. There is a natural bijection between $\Gamma(U, O_Y)$ and the morphisms $U \to \mathbb{A}^1_S$ over S. Write $V := q^{-1}(U)$ and $W := \rho^{-1}(V) = \mathrm{pr}_X^{-1}(V)$. By the Yoneda lemma the morphisms $U \to \mathbb{A}^1_S$ as schemes are the same as the morphisms as fppf sheaves. By construction of the fppf quotient we therefore find that $\Gamma(U, O_Y)$ is in bijection with the set of morphisms $f\colon V \to \mathbb{A}^1_S$ over S such that $f \circ \rho = f \circ \mathrm{pr}_X\colon W \to \mathbb{A}^1_S$. Writing $f := q \circ \rho = q \circ \mathrm{pr}_X$, this shows that O_Y is the kernel of $q_*O_X \rightrightarrows f_*O_{G\times_S X}$, which, by definition, is the subsheaf of G-invariant sections in q_*O_X. This proves that s is an isomorphism of ringed spaces, so that Y is also a geometric quotient of X by G. $\square$

To summarize, we have the following relations between the various notions:*

$$\begin{array}{ccccc} \text{fppf quotient} & \Longrightarrow & \begin{array}{c}\text{universal}\\ \text{geometric quotient}\end{array} & \Longrightarrow & \begin{array}{c}\text{universal}\\ \text{categorical quotient}\end{array} \\ & \nwarrow & \Downarrow & & \Downarrow \\ & & \text{geometric quotient} & \Longrightarrow & \text{categorical quotient}\end{array}$$

where the implication "geometric ⇒ fppf" is valid under the assumptions as in (ii) of the theorem.

(4.36) The sheaf-theoretic approach that we are discussing here of course also makes sense for other Grothendieck topologies on $\mathsf{Sch}_{/S}$, such as the étale topology. Thus, for instance, suppose $q\colon X \to Y$ is an fppf quotient of X by the action of an S-group scheme G. One may ask if q is also an étale quotient. But for this to be the case, q has to be an epimorphism of étale sheaves, which means that étale-locally on Y it admits a section. If this is not the case then q will not be a quotient morphism for the étale topology.

To give a simple geometric example, suppose $q\colon X \to Y$ is a finite morpism of complete non-singular curves over a field such that the extension $k(Y) \subset k(X)$ on function fields is Galois with group G. Then q is an fppf quotient of X by G, but it is an étale quotient only if there is no ramification, i.e., if q is étale.

Conversely, if étale-locally on Y the morphism q has a section then q is an epimorphism of étale sheaves and one shows without difficulty that q is an étale quotient of X by G. (Note that q is assumed to be an fppf quotient morphism, so we already know it is faithfully flat, and in particular also surjective.) But as the simple example just given demonstrates, for a general theory of quotients we obtain better results if we use a finer topology, such as the fppf topology.

(4.37) Working with sheaves of groups has the advantage that many familiar results from ordinary group theory readily generalize. For instance, if H is a normal subgroup scheme of G then the fppf quotient sheaf G/H is naturally a sheaf of groups, and the canonical map $q\colon G \to G/H$ is a homomorphism. Hence if G/H is representable then it is a group scheme and the sequence $0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$ is exact. In this case, if $f\colon G \to G'$ is a homomorphism of S-group schemes such that $f_{|H}$ is trivial then f factors uniquely as $f = f' \circ q$, where $f'\colon G/H \to G'$ is again a homomorphism of group schemes.

* schuine pijl moet gestreepte pijl worden

To conclude our general discussion of fppf quotients, let us now state two existence results. For some finer results see Raynaud [1] and [2], SGA 3, Exp. V and VI, and Anantharaman [1].

(4.38) Theorem. *Let G be a proper and flat group scheme of finite type over a locally noetherian basis S. Let $\rho\colon G \times_S X \to X$ define a strictly free action of G on a quasi-projective S-scheme X. Then the fppf quotient $G\backslash X$ is representable by a scheme.*

A proof of this result can be found in SGA 3, Exp. V, § 7.

(4.39) Theorem. *Let G be a flat group scheme of finite type over a locally noetherian base scheme S. Let $H \subset G$ be a closed subgroup scheme which is flat over S. Suppose that we are in one of the following cases:*

(a) $\dim(S) \leqslant 1$*;*

(b) *G is quasi-projective over S and H is proper over S;*

(c) *H is finite locally free over S such that every fibre $H_s \subset G_s$ is contained in an affine open subset of G.*

Then the fppf quotient sheaf G/H is representable by an S-scheme. If H is normal in G then G/H has the structure of an S-group scheme such that the natural map $q\colon G \to G/H$ is a homomorphism.

For the proof of this result in case (a) see Anantharaman [1], § 4. In case (b) the assertion follows from (4.38), and case (c) is an application of Thm. (4.16); cf. Example (4.29).

(4.40) Example. Let X be an abelian variety over a field k. If $H \subset X$ is a closed subgroup scheme then by Thm. (4.38) there exists an fppf quotient $q\colon X \to Y := X/H$. By Thm. (4.35) q is also a geometric quotient, and from this it readily follows that Y is again an abelian variety.

§ 4. *Finite group schemes over a field.*

Now that we have some further techniques at our disposal, let us return to the study of group schemes. As an application of the above, we sketch the proof of a useful general result.

(4.41) Theorem. *If k is a field then the category of commutative group schemes of finite type over k is abelian.*

Proof (sketch). Write $\boldsymbol{C}$ for the category of commutative group schemes of finite type over k. We view $\boldsymbol{C}$ as a full subcategory of the category $\mathsf{ShAb}_{/k}$ of fppf sheaves of abelian groups on $\mathrm{Spec}(k)$, which is an abelian category. Clearly $\boldsymbol{C}$ is an additive subcategory, and by (3.13) it is stable under the formation of kernels.

Let $f\colon G_1 \to G_2$ be a morphism in $\boldsymbol{C}$. In the category $\mathsf{ShAb}_{/k}$ we can form the quotients $q_1\colon G_1 \to G_1/\mathrm{Ker}(f)$ and $q_2\colon G_2 \to G_2/G_1$, and we have an isomorphism $\alpha\colon G_1/\mathrm{Ker}(f) \xrightarrow{\sim} \mathrm{Ker}(q_2)$. First one shows that the quotient morphism q_1 exists as a homomorphism of group schemes; see also (4.39) below. Let $\bar{G}_1 := G_1/\mathrm{Ker}(f)$, and let $\bar{f}\colon \bar{G}_1 \to G_2$ be the homomorphism induced by f. Note that $\bar{f}$ is a monomorphism. Now one proves that the quotient sheaf $G_2/\bar{G}_1$ is also representable by a k-scheme of finite type; for the details of this see SGA 3, Exp VI_{A}, Thm. 3.2. But the natural map of sheaves $G_2/G_1 \to G_2/\bar{G}_1$ is an isomorphism, so it follows that G_2/G_1 is a group scheme. In particular, $\boldsymbol{C}$ is stable under the formation of cokernels, and since

$\boldsymbol{C}$ is a full subcategory of $\mathsf{ShAb}_{/k}$ we have an isomorphism $\alpha\colon G_1/\mathrm{Ker}(f) \xrightarrow{\sim} \mathrm{Ker}(q_2)$ in $\boldsymbol{C}$. $\square$

We now focus on finite group schemes.

(4.42) Definition. Let G be a finite group scheme over a field k. We say that G is
— *étale* if the structural morphism $G \to \mathrm{Spec}(k)$ is étale;
— *local* if G is connected.

Next suppose that G is commutative. Recall that we write G^D for the Cartier dual of G. We say that G is
— *étale-étale* if G and G^D are both étale;
— *étale-local* if G is étale and G^D is local;
— *local-étale* if G is local and G^D is étale;
— *local-local* if G and G^D are both local.

Let us note that if $k \subset K$ is a field extension and if G is étale (resp. local) then G_K is étale (resp. local), too. For étaleness this is clear; for the property of being local this is just Prop. (3.17), part (i).

(4.43) Examples. If $\mathrm{char}(k) = 0$ then it follows from Thm. (3.20) that every finite commutative k-group scheme is étale-étale. If $\mathrm{char}(k) = p > 0$ then all four types occur:

type:	étale-étale	étale-local	local-étale	local-local
example:	$(\mathbb{Z}/m\mathbb{Z})$ with $p \nmid m$	$(\mathbb{Z}/p^n\mathbb{Z})$	μ_{p^n}	α_{p^n}

(4.44) Lemma. *Let G_1 and G_2 be finite group schemes over a field k, with G_1 étale and G_2 local. Then the only homomorphisms $G_1 \to G_2$ and $G_2 \to G_1$ are the trivial ones.*

Proof. Without loss of generality we may assume that $k = \bar{k}$. Then $G_{2,\mathrm{red}} \subset G_2$ is a connected étale subgroup scheme; hence $G_{2,\mathrm{red}} \cong \mathrm{Spec}(k)$. Now note that any homomorphism $G_1 \to G_2$ factors through $G_{2,\mathrm{red}}$. Similarly, any homomorphism $G_2 \to G_1$ factors through $G_1^0 \cong \mathrm{Spec}(k)$. $\square$

Note that the assertion about homomorphisms from an étale to a local group scheme does not generalize to arbitrary base schemes. For instance, if we take $S = \mathrm{Spec}\big(k[\varepsilon]\big)$ as a base scheme then the group $\mathrm{Hom}_S\big((\mathbb{Z}/p\mathbb{Z}), \mu_p\big)$ is isomorphic to the additive group k, letting $a \in k$ correspond to the homomorphism $(\mathbb{Z}/p\mathbb{Z})_S \to \mu_{p,S}$ given on points by $(n \bmod p) \mapsto (1 + a\varepsilon)^n$.

(4.45) Proposition. *Let G be a finite group scheme over a field k. Then G is an extension of an étale k-group scheme $G_{\text{ét}} = \varpi_0(G)$ by the local group scheme G^0; so we have an exact sequence*

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow G_{\text{ét}} \longrightarrow 1\,. \tag{5}$$

If k is perfect then this sequence splits (i.e., we have a homomorphic section $G \leftarrow G_{\text{ét}}$) and G is isomorphic to a semi-direct product $G^0 \rtimes G_{\text{ét}}$. In particular, if k is perfect and G is commutative then $G \cong G^0 \times G_{\text{ét}}$.

Note that the étale quotient $G_{\text{ét}}$ is nothing but the group scheme $\varpi_0(G)$ of connected components introduced in (3.28). In the present context it is customary to think of $G_{\text{ét}}$ as a "building block" for G, and it is more customary to use a notation like $G_{\text{ét}}$.

Proof. Define $G_{\text{ét}} := \varpi_0(G)$, and consider the homomorphism $q\colon G \to G_{\text{ét}}$ as in Prop. (3.27).

As shown there, q is faithfully flat, and the kernel of q is precisely the identity component G^0. Hence we have the exact sequence (5).

Let us now assume that k is perfect. Then $G_{\mathrm{red}} \subset G$ is a closed subgroup scheme (Exercise 3.2) which by (ii) of Prop. (3.17) is étale over k. We claim that the composition $G_{\mathrm{red}} \hookrightarrow G \twoheadrightarrow G_{\text{ét}}$ is an isomorphism. To see this we may assume that $k = \bar{k}$. But then G, as a scheme, is a finite disjoint union of copies of G^0. If there are n components then G_{red} and $G_{\text{ét}}$ are both isomorphic to the disjoint union of n copies of $\mathrm{Spec}(k)$, and it is clear that $G_{\mathrm{red}} \to G_{\text{ét}}$ is an isomorphism of group schemes. The inverse of this isomorphism gives a splitting of (5). $\square$

Combining this with Lemma (4.44) we find that the category $\boldsymbol{C}$ of finite commutative group schemes over a perfect field k decomposes as a product of categories:

$$\boldsymbol{C} = \boldsymbol{C}_{\text{ét,ét}} \times \boldsymbol{C}_{\text{ét,loc}} \times \boldsymbol{C}_{\text{loc,ét}} \times \boldsymbol{C}_{\text{loc,loc}}\,.$$

As remarked above, $\boldsymbol{C} = \boldsymbol{C}_{\text{ét,ét}}$ if $\mathrm{char}(k) = 0$.

(4.46) Lemma. *Let S be a connected base scheme. If $0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$ is an exact sequence of finite locally free S-group schemes then* $\mathrm{rank}(G_2) = \mathrm{rank}(G_1) \cdot \mathrm{rank}(G_3)$.

Proof. Immediate from the fact that G_2 is a G_1-torsor over G_3 for the fppf topology, as this implies that O_{G_2} is locally free as an O_{G_3}-module, of rank equal to $\mathrm{rank}(G_1/S)$. $\square$

(4.47) Proposition. *Let k be a field of characteristic $p > 0$. Let G be a finite* connected *k-group scheme. Then the rank of G is a power of p.*

Proof. Let $F_{G/k}\colon G \to G^{(p)}$ be the relative Frobenius homomorphism. Write $G[F] := \mathrm{Ker}(F_{G/k})$. The strategy of the proof is to use the short exact sequence $1 \longrightarrow G[F] \longrightarrow G \longrightarrow G/G[F] \longrightarrow 1$ and induction on the rank of G. The main point is then to show that the affine algebra of $G[F]$ is of the form $k[X_1, \dots, X_d]/(X_1^p, \dots, X_d^p)$ with $d = \dim_k(T_{G,e})$. To prove this we use certain differential operators.

Write $G = \mathrm{Spec}(A)$, and let $I \subset A$ be the augmentation ideal. We have an isomorphism $I/I^2 \xrightarrow{\sim} \Omega_{A/k} \otimes_A k$, sending the class of $\xi \in I$ to $d\xi \otimes 1$. Further, (3.15) tells us that $\Omega_{A/k} \cong (\Omega_{A/k} \otimes_A k) \otimes_k A$. In total we find

$$\mathrm{Der}_k(A) = \mathrm{Hom}_A(\Omega_{A/k}, A) \cong \mathrm{Hom}_k(I/I^2, A)\,,$$

where the derivation $D_\varphi\colon A \to A$ corresponding to $\varphi\colon I/I^2 \to A$ satisfies $D_\varphi(\xi) = \varphi(\xi) \bmod I$ for all $\xi \in I$.

Choose elements $x_1, \dots, x_d \in I$ whose classes form a k-basis for I/I^2. By the previous remarks, there exist k-derivations $D_i\colon A \to A$ such that $D_i(x_j) = \delta_{i,j} \bmod I$ for all i and j. We claim that for all non-negative numbers $m_1, \dots, m_d$ and $n_1, \dots, n_d$ with $m_1 + \cdots + m_d = n_1 + \cdots + n_d$ we have

$$D_d^{m_d} D_{d-1}^{m_{d-1}} \cdots D_1^{m_1}\bigl(x_1^{n_1} \cdots x_d^{n_d}\bigr) \equiv \begin{cases} n_1!\, n_2! \cdots n_d! \bmod I, & \text{if } m_i = n_i \text{ for all } i; \\ 0 \bmod I, & \text{otherwise.} \end{cases} \qquad (6)$$

To see this, note that for every $D \in \mathrm{Der}_k(A)$ the product rule implies that $D(I^r) \subseteq I^{r-1}$. With this remark, (6) follows by induction on the number $m_1 + \cdots + m_d$.

By Nakayama's lemma the x_i generate I, so we have

$$A \cong k[X_1, \dots, X_d]/(f_1, \dots, f_q)$$

via $x_i \leftarrow\!\shortmid X_i$. Let $J = (f_1, \dots, f_q) \subset A$. We claim that $J \subseteq (X_1^p, \dots, X_d^p)$. To see this, suppose we have a polynomial relation between the x_i such that there are no terms x_i^a with $a \geqslant p$. Write this relation as

$$0 = h_0 + h_1(x_1, \dots, x_d) + \cdots + h_r(x_1, \dots, x_d)\,,$$

where h_j is a homogeneous polynomial of degree j. Let j be the smallest integer such that $h_j \neq 0$. Suppose $x_1^{n_1} \cdots x_d^{n_d}$ (with $n_1 + \cdots + n_d = j$) is a monomial occurring with non-zero coefficient. Applying the differential operator $D_d^{n_d} D_{d-1}^{n_{d-1}} \cdots D_1^{n_1}$ and using (6) we obtain the relation $n_1! n_2! \cdots n_d! \in I$. This contradicts the fact that k is a field of characteristic p and that all n_i are $< p$. Hence $J \subseteq (X_1^p, \dots, X_d^p)$, as claimed.

Let $F_{G/k}\colon G \to G^{(p)}$ be the relative Frobenius homomorphism. On rings it is given by

$$k[X_1, \dots, X_d]/(f_1^{(p)}, \dots, f_q^{(p)}) \longrightarrow k[X_1, \dots, X_d]/(f_1, \dots, f_q)\,, \qquad X_i \mapsto X_i^p\,.$$

As the zero section of $G^{(p)}$ is given by sending all X_i to 0 we find that the affine algebra of $G[F] := \mathrm{Ker}(F_{G/k})$ is

$$A_{G[F]} = k[X_1, \dots, X_d]/(X_1^p, \dots, X_d^p, f_1, \dots, f_q) = k[X_1, \dots, X_d]/(X_1^p, \dots, X_d^p)\,.$$

In particular, $G[F]$ has rank p^d. Further, $\mathrm{rank}(G) = \mathrm{rank}\big(G[F]\big) \cdot \mathrm{rank}\big(G/G[F]\big) = p^d \cdot \mathrm{rank}\big(G/G[F]\big)$ by Lemma (4.46). As $G = G^0$ we have $d > 0$ if $G \neq \{1\}$; now the proposition follows by induction on $\mathrm{rank}(G)$. $\square$

(4.48) Corollary. *If* $\mathrm{char}(k) = p$ *then a finite commutative* k*-group scheme is étale-étale if and only if* $p \nmid \mathrm{rank}(G)$.

Proof. In the "if" direction this is a direct consequence of the proposition combined with (4.45) and Lemma (4.46). Conversely, suppose G is étale-étale. We may assume that $k = \overline{k}$, in which case G is a constant group scheme. If $p \mid \mathrm{rank}(G)$ then G has a direct factor $(\mathbb{Z}/p^n\mathbb{Z})$. But then G^D has a factor μ_{p^n} and is therefore not étale. $\square$

Exercises.

(4.1) Let S be a base scheme. Fix an integer $N \geqslant 2$. Take $G = X = \mathbb{G}_{m,S}$, and let $g \in G$ act on X as multiplication by g^N.

(i) Let T be an S-scheme. Let x_1 and x_2 be T-valued points of X; they correspond to elements $\gamma_1,\ \gamma_2 \in \Gamma(T, O_T)^*$. Let $c := \gamma_1/\gamma_2$, and define a scheme T', affine over T, by $T' := \mathrm{Spec}\,\big(O_T[t]/(t^N - c)\big)$. Show that the images of x_1 and x_2 in $X(T')$ lie in the same orbit under $G(T')$.

(ii) Show that $T' \to T$ is an epimorphism of schemes over S. (By definition this means that for every S-scheme Z the induced map $Z(T) \to Z(T')$ is injective.)

(iii) Suppose that $q\colon X \to Y$ is a G-invariant morphism of S-schemes. Show that for every S-scheme T the image of $q(T)\colon X(T) \to Y(T)$ consists of a single point. Conclude that $X \to S$ is a universal categorical quotient of X by G.

(iv) Show that the endomorphism $\mathbb{G}_m \to \mathbb{G}_m$ given by $g \mapsto g^N$ is faithfully flat and of finite presentation. Use this to show that the fppf sheaf $G\backslash X$ is represented by the scheme S.

(4.2) Let ρ: $G \times_S X \to X$ be an action of an S-group scheme G on an S-scheme X. Define the relation $P \sim Q$ on $|X|$ as in (4.11). The goal of this exercise is to show that $\sim$ is an equivalence relation.

(i) Let $\Psi = \Psi_\rho$ be the graph morphism, as defined in (4.1). Write

$$\widetilde{\Psi}\colon |G \times_S X| \to |X| \times_{|S|} |X|$$

for the composition of the map $|\Psi|$: $|G \times_S X| \to |X \times_S X|$ and the canonical (surjective) map $|X \times_S X| \to |X| \times_{|S|} |X|$. Show that $P \sim Q$ precisely if $(P, Q) \in \mathrm{Im}\widetilde{\Psi}$.

(ii) Write $e(S) \subset G$ for the image of the identity section. Show that the projection $e(S) \times_S X \to X$ is an isomorphism. Conclude that $\sim$ is reflexive.

(iii) Let s: $X \times_S X \longrightarrow X \times_S X$ be the morphism reversing the factors. Find a morphism f: $G \times_S X \to G \times_S X$ such that $s \circ \Psi = \Psi \circ f$. Conclude that $\sim$ is symmetric.

(iv) Show that $\sim$ is transitive. [*Hint:* use that the natural map

$$\left|(G \times_S X) \underset{\rho, X, \mathrm{pr}_X}{\times} (G \times_S X)\right| \longrightarrow |G \times_S X| \underset{|\rho|, |X|, |\mathrm{pr}_X|}{\times} |G \times_S X|$$

is surjective.]

(4.3) Let k be an infinite field. Let Λ be a k-algebra which is a product of fields. Suppose M is a free Λ-module of finite rank. Let $N \subset M$ be a k-submodule such that N spans M as a Λ-module. Show that N contains a Λ-basis for M. Show by means of an example that the condition that k is infinite is essential.

(4.4) Let π: $G \to S$ be a locally free group scheme of rank r over a reduced, irreducible base scheme S. The goal of this exercise is to show that G is annihilated by its rank, i.e., the morphism $[r]_G$: $G \to G$ given on points by $g \mapsto g^r$ equals the zero morphism $[0]_G = e \circ \pi$: $G \to S \to G$.

(i) Suppose S is the spectrum of a field k. Reduce the problem to the case that $G = G^0$. [*Hint:* Use (4.45) and Lemma (4.46). For étale group schemes reduce the problem to Lagrange's theorem in group theory.]

(ii) Suppose $S = \mathrm{Spec}(k)$ with $\mathrm{char}(k) = p$. Suppose further that $G = G^0 = \mathrm{Spec}(A)$. By (4.47) we have $\mathrm{rank}(G) = p^n$ for some n. If $I \subset A$ is the augmentation ideal, show that $I^{p^n} = (0)$. Now use the result of Exercise (3.7) to derive that $[p^n](I) = (0)$. Conclude that $[p^n]_G = [0]_G$.

(iii) Prove the stated result over an arbitrary reduced and irreducible basis. [*Hint:* use that the generic fibre of G is Zariski dense in G.]

[*Remark:* for *commutative* finite locally free group schemes the result holds without any restriction on the basis. This was proven by Deligne; see Tate-Oort [1]. It is an open problem if the result is also valid over arbitrary base schemes for non-commutative G.]

(4.5) Let S be a locally noetherian scheme. Let G be a finite locally free S-group scheme acting on an S-scheme X of finite type. Assume that for every closed point $P \in |X|$ the G-equivalence class of P is contained in an affine open subset. Write q: $X \to Y$ for the quotient morphism. If $x \in |X|$ then we write $\hat{O}_{X,x}$ for the completed local ring of X at the point x; likewise for other schemes.

(i) Let $y \in |Y|$. Show that the scheme $\hat{F}_y := \prod_{x \in q^{-1}(y)} \mathrm{Spec}(\hat{O}_{X,x})$ inherits a G-action, and that $\mathrm{Spec}(\hat{O}_{Y,y})$ is the quotient of $\hat{F}_y$ modulo G. [*Hint:* First reduce to the case that $S = Y$; then apply a flat base change.]

(ii) Suppose $S = \mathrm{Spec}(k)$ is the spectrum of a field. Let $x \in X(k)$ be a k-rational point with image $y \in Y(k)$ under q. Show that q induces an isomorphism $\hat{O}_{Y,y} \xrightarrow{\sim} (\hat{O}_{X,x})^{G_x}$.

(4.6) Let $X \to S$ be a morphism of schemes. Let G be a finite group that acts on X over S.

(i) For $g \in G$, define a scheme X^g and a morphism $i_g\colon X^g \to X$ by the fibre product square

$$\begin{array}{ccc} X^g & \xrightarrow{i_g} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \Delta_X} \\ X & \xrightarrow[\psi_g]{} & X \times_S X \end{array}$$

where the morphism $\psi_g\colon X \to X \times_S X$ is given by $x \mapsto (g \cdot x, x)$. Show that i_g is an immersion and that it is a closed immersion if X/S is separated.

(ii) Define $X^G \hookrightarrow X$ as the scheme-theoretic intersection of the subschemes X^g, for $g \in G$. (In other words, if $G = \{g_1, \ldots, g_n\}$ then $X^G := X^{g_1} \times_X X^{g_2} \times_X \cdots \times_X X^{g_n}$.) Show that X^G is a subscheme of X, and that it is a closed subscheme if X/S is separated. Further show that for any S-scheme T we have $X^G(T) = X(T)^G$. The subscheme $X^G \hookrightarrow X$ is called the *fixed point subscheme* of the given action of G.