CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for May 31

Exercise 1. Let $R = \mathbb{Z}[x]$, and let I be the ideal $(6, x) \subset R$. Let M be the module I/I^2.

(a) Give a projective resolution of M.

(b) Let N be another R-module. Determine $\operatorname{Ext}^2_R(M, N)$, and give a concrete example of an N such that $\operatorname{Ext}^2_R(M, N) \neq 0$.

(c) Give an example of a module N such that $\operatorname{Ext}^1_R(M, N) \neq 0$. (Do this by thinking and not by calculation!)

Exercise 2. Let R be a ring and N a (left-) R-module. Let $n \geq 1$, and suppose $\operatorname{Ext}^n_R(M, N) = 0$ for all R-modules M.

(a) Show that also $\operatorname{Ext}^{n+1}_R(M, N) = 0$ for all M. [*Hint:* Consider a short exact sequence $0 \longrightarrow M' \longrightarrow F \longrightarrow M \longrightarrow 0$ with F a free R-module.]

(b) Show that the functor $\operatorname{Ext}^{n-1}_R(-, N)\colon R\text{-}\mathsf{Mod}^{\mathrm{op}} \to \mathsf{Ab}$ is right exact.

Remark: The *injective dimension* $\operatorname{inj.dim}_R(N) \in \mathbb{N} \cup \{\infty\}$ of the module N is defined to be the smallest integer i (possibly $i = \infty$) such that $\operatorname{Ext}^n_R(M, N) = 0$ for all $n > i$ and all M. This is one of the many homological invariants that one may associate with a module and that attempt to capture meaningful properties of it.

Exercise 3. Let

$$\mathscr{E}: \qquad 0 \longrightarrow N \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0$$

be an extension of R-modules. Prove that the extension

$$\mathscr{E}': \qquad 0 \longrightarrow N \xrightarrow{-\alpha} E \xrightarrow{\beta} M \longrightarrow 0$$

is the inverse of $\mathscr{E}$ for the Baer sum and that the split extension

$$0 \longrightarrow N \xrightarrow{i} M \oplus N \xrightarrow{\mathrm{pr}} M \longrightarrow 0$$

is the neutral element for the Baer sum.

Exercise 4. Let $f\colon R \to S$ be a homomorphism of rings. If M is a (left-) S-module, we may view M as an R-module via the homomorphism f; we denote this R-module by $f^*(M)$. (The notation ${}_RM$ is also sometimes used for this.)

(a) Show that $f^*\colon S\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}$ is a faithful exact functor. Is f^* always full?

(b) Let $\alpha\colon P_\bullet \to M$ be a projective resolution of M. Show that $\alpha\colon f^*(P_\bullet) \to f^*(M)$ is a resolution of $f^*(M)$ by R-modules. Show, by means of a concrete example, that the $f^*(P_i)$ need not be projective R-modules.

(c) Let $\beta\colon Q_\bullet \to f^*(M)$ be a projective resolution of $f^*(M)$. Show (with $P_\bullet$ as in (b)) that there is a morphism of complexes $g\colon Q_\bullet \to f^*(P)$ such that $\alpha \circ g = \beta$, and that g is unique up to homotopy.

We can now define a homomorphism

$$f^*\colon \mathrm{Ext}^i_S(M,N) \to \mathrm{Ext}^i_R\big(f^*(M), f^*(N)\big)$$

by considering (with notation as above) the morphism of cochain complexes

$$\mathrm{Hom}_R(P_\bullet, N) \to \mathrm{Hom}_S\big(Q_\bullet, f^*(N)\big)$$

given by $h \mapsto f^*(h) \circ g$, and then applying $\mathscr{H}^i$.

(d) Let k be a field, and take $f\colon R \to S$ to be the homomorphism $k \to k[x]$. Show that $f^*\colon \mathrm{Ext}^1_{k[x]}(M,N) \to \mathrm{Ext}^1_k\big(f^*(M), f^*(N)\big)$ is the zero map, and conclude that f^* is in general not injective.

(e) Take $f\colon R \to S$ to be the canonical homomorphism $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Show that again $f^*\colon \mathrm{Ext}^1_{\mathbb{Z}/2\mathbb{Z}}(M,N) \to \mathrm{Ext}^1_{\mathbb{Z}/4\mathbb{Z}}\big(f^*(M), f^*(N)\big)$ is the zero map, and conclude that f^* is in general not surjective.

Exercise 5. Let k be a field, let G be a finite group, and let $k[G]$ be the group ring of G over k. Recall that the category of $k[G]$-modules is isomorphic to the category of k-linear representations of G. A representation $\rho\colon G \to \mathrm{GL}(V)$, with V a k-vector space, is said to be *irreducible* if $V \neq 0$ and if V does not have any G-stable subspaces other than 0 and V itself. This is of course equivalent to V being a *simple* R-module (i.e., $V \neq 0$ and V has no non-trivial $k[G]$-submodules). A representation is said to be *semisimple*, or *completely reducible*, if it is isomorphic to a direct sum of irreducible representations.

A famous theorem of Maschke says that when the characteristic of k does not divide the order of the group G, every finite-dimensional representation of G is semisimple. This result plays a very important role in representation theory. The goal of this exercise is to show that this result is false if $\mathrm{char}(k)$ divides $\#G$. In that case the characteristic of k of course needs to be positive; so from now on we assume $\mathrm{char}(k) = p > 0$ and $p | \#G$.

(a) As a first example, show that for $G = \mathbb{Z}/p\mathbb{Z}$ we have $k[G] \cong k[t]/(t^p)$, and use this to give an explicit example of a $k[G]$-module E that is not semisimple. [For the first assertion see the exercises of 1 March.]

(b) Still with $G = \mathbb{Z}/p\mathbb{Z}$, show that there exists $k[G]$-modules M and N such that the module E that you have found in (a) is an extension

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

that corresponds to a non-zero class in $\mathrm{Ext}^1_{k[G]}(M, N)$.

We now turn to the general case of a group G with $p|\#G$. Let $\epsilon\colon k[G] \to k$ be the homomorphism given by $\sum_{g\in G} c_g \cdot g \mapsto \sum_{g\in G} c_g$. The kernel $I = \mathrm{Ker}(\epsilon)$ is called the *augmentation ideal* of $k[G]$.

(c) Show that I is a $k[G]$-submodule of $k[G]$.

Let $V \subset k[G]$ be any other $k[G]$-submodule with $V \neq I$. We are going to show that $I \cap V \neq 0$. Define

$$\gamma = \sum_{g\in G} 1 \cdot g \qquad \in k[G]\,.$$

(d) Let $v = \sum_{g\in G} c_g \cdot g$ be an element of V that is not in I. Show that $\gamma \cdot v$ is a non-zero element of $V \cap I$.

(e) Conclude that for any $k[G]$-submodule $V \subset k[G]$ we have $V \cap I \neq (0)$, and use this to conclude that the $k[G]$-module $k[G]$ is not semisimple.