CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for March 22

Names for some categories:

Category	Objects	Morphisms
Set	Sets	Maps
Ab	Abelian groups	Group homomorphisms
Ring	Rings	Ring homomorphisms
CRing	Commutative rings	Ring homomorphisms
Field	Fields	Homomorphisms of fields
Top	Topological spaces	Continuous maps
Haus	Hausdorff topological spaces	Continuous maps
${}_R\mathsf{Mod}$	Left R-modules	R-linear maps

Exercise 1. Let $h\colon R \to S$ be a homomorphism of commutative rings. We have an associated functor $h^*\colon {}_S\mathsf{Mod} \to {}_R\mathsf{Mod}$: if N is an S-module then h^*N is the same abelian group, now viewed as an R-module via the rule $r \cdot n = h(r) \cdot n$. Prove that the functor $S \otimes_R -\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ is a left adjoint of h^*.

Exercise 2.

(a) Let TAb denote the category of torsion abelian groups, and let $i\colon \mathsf{TAb} \to \mathsf{Ab}$ be the inclusion-functor into the category of abelian groups. Prove that i admits a right adjoint and describe it explicitly.

(b) Let TFAb denote the category of torsion-free abelian groups, and let $j\colon \mathsf{TFAb} \to \mathsf{Ab}$ be the inclusion-functor into the category of abelian groups. Prove that j admits a left adjoint and describe it explicitly.

(c) Prove that the homomorphism $\mathbb{Z} \to \mathbb{Z}$ given by $n \mapsto 2n$ is an epimorphism in the category TFAb, and conclude from this that j does not admit a right adjoint.

Exercise 3. Let $\mathsf{IntDom} \subset \mathsf{CRing}$ be the full subcategory of integral domains (commutative rings with 1 that have no zero divisors). Let Field be the category of fields. It is tempting to think that the inclusion functor $i\colon \mathsf{Field} \to \mathsf{IntDom}$ has a left adjoint, given by sending a domain R to its fraction field $Q(R)$. Explain why this does not work.

Exercise 4. Let X be a topological space. We introduce a relation $\sim$ on X by declaring $x \sim y$ if for every continuous map $f\colon X \to H$ with H a Hausdorff space we have $f(x) = f(y)$.

(a) Show that $\sim$ is an equivalence relation on X and that the quotient space $X/{\sim}$ is Hausdorff.

(b) Show that there is a functor $h\colon \mathsf{Top} \to \mathsf{Haus}$ that on objects is given by $h(X) = X/{\sim}$.

(c) Prove that the functor h is left adjoint to the inclusion functor $\mathsf{Haus} \to \mathsf{Top}$.

(The space $X/{\sim}$ is called the *maximal Hausdorff quotient of* X.)

Exercise 5. If C is a category and S is an object of C we may form a new category $\mathsf{C}_{/S}$ of "objects over S". (This $\mathsf{C}_{/S}$ is also sometimes denoted by $(\mathsf{C} \downarrow S)$ and is called a *slice category*.) The objects of $\mathsf{C}_{/S}$ are morphisms $a\colon X \to S$. The morphisms from $(a\colon X \to S)$ to $(b\colon Y \to S)$ are the morphisms $f\colon X \to Y$ in C for which $b \circ f = a$.

(a) Show that $\mathrm{id}_S\colon S \to S$ is a final object of $\mathsf{C}_{/S}$.

(b) What is $\mathsf{C}_{/S}$ if S is a final object of C ?

Suppose $a\colon X \to S$ and $b\colon Y \to S$ are in $\mathsf{C}_{/S}$ and admit a product in this category. This product is then denoted by $X \times_S Y$ (or if it is necessary to indicate the structural maps a and b by the more elaborate notation $X \times_{a,S,b} Y$) and is called a *fibre product* of X and Y over S.

(c) Taking $\mathsf{C} = \mathsf{Set}$, show that for any $X \xrightarrow{a} S \xleftarrow{b} Y$ the fibre product $X \times_S Y$ exists and is given by

$$X \times_S Y = \big\{(x, y) \in X \times Y \;\big|\; a(x) = b(y)\big\}$$

with $X \times_S Y \to S$ given by $(x, y) \mapsto a(x) = b(y)$.

(d) In a general category $\mathsf{C}_{/S}$, if the fibre product $X \times_S Y$ exists, express $\mathrm{Hom}_{\mathsf{C}}(-, X \times_S Y)$ in terms of the functors h_X, h_Y and h_S. (Note: we here view $X \times_S Y$ as an object of C, simply by forgetting its structural morphism to S.)

Exercise 6. Let C be a category with the property that for any two objects X and Y, there exists a coproduct $X \coprod Y$. Let $\Delta\colon \mathsf{C} \to \mathsf{C} \times \mathsf{C}$ be the diagonal functor, sending an object X to (X, X) and a morphism $f\colon X \to Y$ to $(f, f)\colon (X, X) \to (Y, Y)$. Prove that Δ admits a left adjoint, and describe it explicitly.