CATEGORIES AND HOMOLOGICAL ALGEBRA

Exercises for April 19

Exercise 1. Let R be a ring and let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \tag{$*$}$$

be a short exact sequence of (left) R-modules. A homomorphism $s\colon M'' \to M$ with $g \circ s = \mathrm{id}_{M''}$ is called a *section* of g. A homomorphism $r\colon M \to M'$ with $r \circ f = \mathrm{id}_{M'}$ is called a *retraction* of f.

(a) If $s\colon M'' \to M$ is a section of g, prove that for $m \in M$ there is a unique $m' \in M'$ with $f(m') = m - sg(m)$. Show that the map $m \mapsto m'$ defines a retraction of f.

(b) If $r\colon M \to M'$ is a retraction of f, show that $m - fr(m) \in \mathrm{Ker}(r)$ for all $m \in M$. Using this, show that g induces an isomorphism $\mathrm{Ker}(r) \xrightarrow{\sim} M''$ and that the inverse of this isomorphism gives a section of g.

(c) Prove that the following properties are equivalent:

  (1) There exists a section of g.

  (2) There exists a retraction of f.

  (3) There exists an isomorphism $\phi\colon M \xrightarrow{\sim} M' \oplus M''$ such that $\phi \circ f\colon M' \to M' \oplus M''$ is the inclusion map and $g \circ \phi^{-1}\colon M' \oplus M'' \to M''$ is the projection map.

The short exact sequence $(*)$ is said to be *split exact* if the equivalent properties in (c) hold.

Exercise 2. Let G be the group $\mathbb{Z}$, and consider the group ring $R = \mathbb{Z}[G]$ (which is isomorphic to $\mathbb{Z}[t, t^{-1}]$). Let $M = \mathbb{Z} \oplus \mathbb{Z}$ with $n \in G$ acting on M by $(a, b) \mapsto (a + nb, b)$.

(a) Verify that this defines an action of G on M, making M into a left R-module.

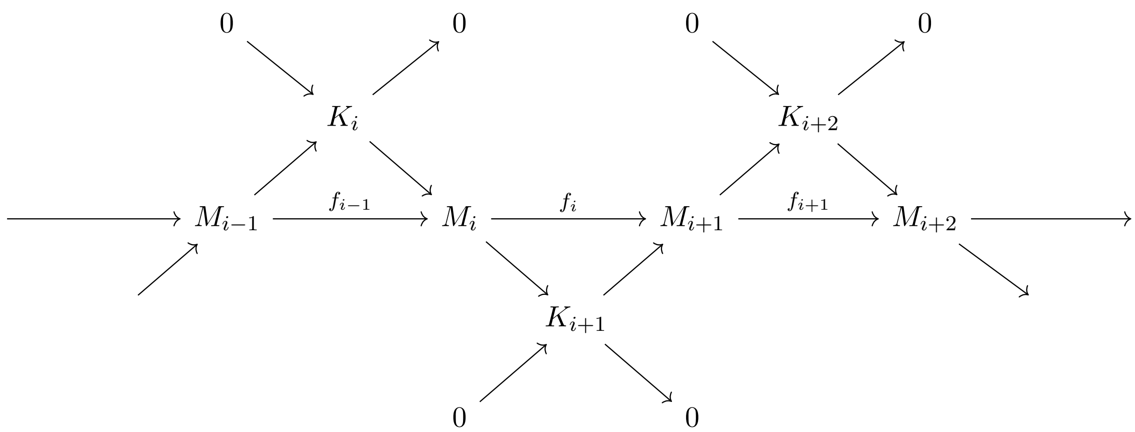
(b) Let $M' = (\mathbb{Z} \oplus 0) \subset M$. Show that M' is an R-submodule of M.

(c) Let $M'' = M/M'$, and consider the exact sequence

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

where i is the inclusion map and p is the projection map. Show that this sequence is split exact as a sequence of abelian groups, but that it is not split as a sequence of R-modules.

Exercise 3. Let $\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \to \cdots$ be an exact sequence of R-modules. Show that there exists a commutative diagram



such that all diagonal sequences are short exact.

Exercise 4. Give examples of short exact sequences of abelian groups

$$0 \to M' \to M \to M'' \to 0 \qquad \text{and} \qquad 0 \to N' \to N \to N'' \to 0$$

such that:

(a) $M' \cong N'$ and $M \cong N$ but $M'' \not\cong N''$;

(b) $M' \cong N'$ and $M'' \cong N''$ but $M \not\cong N$;

(c) $M \cong N$ and $M'' \cong N''$ but $M' \not\cong N'$.

Exercise 5. Let R be a ring, and let

$$\begin{array}{ccc} M_1 & \xrightarrow{\alpha} & M_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ N_1 & \overset{\beta}{\hookrightarrow} & N_2 \end{array}$$

be a commutative diagram of R-modules with β injective. Show that this gives rise to an exact sequence

$$\mathrm{Ker}(f_1) \xrightarrow{\bar{\alpha}} \mathrm{Ker}(f_2) \longrightarrow L \longrightarrow \mathrm{Coker}(f_1) \xrightarrow{\bar{\beta}} \mathrm{Coker}(f_2)\,,$$

where L is some R-module (that you need to construct) and $\bar{\alpha}$ and $\bar{\beta}$ are the maps induced by α and β, respectively.

Exercise 6. Let R be a ring.

(a) If $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ is a short exact sequence of R-modules with P a projective module, show that the sequence is split exact. [*Hint:* Apply $\mathrm{Hom}_R(P, -)$ to the sequence.]

(b) If M and N are R-modules show that $(M \oplus N)$ is projective if and only if both M and N are projective.

(c) Show that every free R-module is projective.

(d) Let P be an R-module. Prove that P is projective if and only if it is a direct summand of a free R-module (i.e., there is an R-module P' such that $P \oplus P'$ is a free R-module). [*Hint:* For "if" use the previous two items; for "only if" choose a free module N and a surjective homomorphism $N \to P$ and then apply (a).]