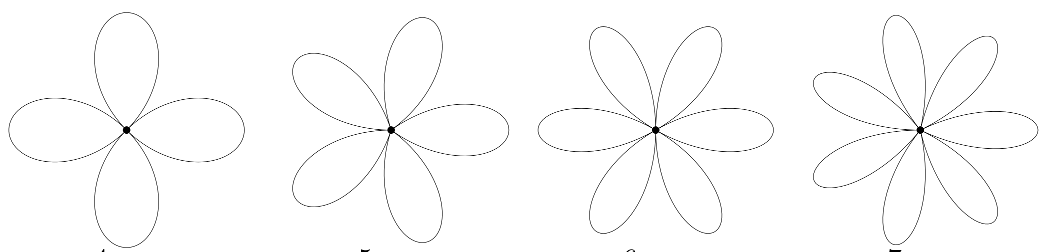
Coverings in Algebra and Topology—Second homework assignment

April 11, 2017

- Hand in by Tuesday May 2 at the latest. Send your solutions to Salvatore Floccari by email (s.floccari@math.ru.nl), or give them to him in person, or drop them in his pigeon hole. Keep a copy of your work.
- Give complete arguments. You are allowed to use what has been discussed in the course, or what you can find in the book by Szamuely (or similar literature). What counts is that you convince us that you have fully understood how things works.
- What you hand in should be your own work.

Exercise 1. Before doing this exercise, study Proposition 1.26 in the book *Algebraic Topology* by A. Hatcher. (A pdf file of this book can be found on the course website.) You will need this result to do part (ii) of this exercise.

(i) Let $X \subset \mathbf{R}^2$ be a union of $n \geq 1$ "circles" that all meet in the origin (and nowhere else).



$n = 4$ $n = 5$ $n = 6$ $n = 7$

Prove that, for $x \in X$ a basepoint, $\pi_1(X, x)$ is a free group on n generators.

(ii) Now let X be the figure "∞", i.e., the space X from part (i) with $n = 2$. Let G be a finite group that can be generated by at most 2 elements. Prove that there exists a connected Galois cover $p: Y \to X$ with Galois group G.

Exercise 2. Throughout this exercise, X is a connected and locally simply connected topological space. Let $p: Y \to X$ be a connected topological covering such that for all points $x \in X$ the fibre $p^{-1}(x)$ is a finite set of cardinality d. (By Corollary 2.1.4 in Szamuely's book, if $\#p^{-1}(x) = d$ for one point $x \in X$ then this holds true for all $x \in X$.) In this situation we call d the *degree* of the covering p, notation $d = \deg(p)$.

(i) If $\deg(p) = 2$, show that $p: Y \to X$ is Galois.

(ii) Give an explicit example of a connected covering $p: Y \to X$ with $\deg(p) = 3$ such that p is not Galois. Prove that your example has the desired property.

(iii) If $\deg(p) = 3$ and p is not Galois, show that $\mathrm{Aut}(Y/X) = \{\mathrm{id}_Y\}$.

Exercise 3. Let X be a connected and locally simply connected topological space, $x \in X$ a basepoint, and suppose $\pi_1(X, x)$ is finite. Prove that every continuous map $f \colon X \to S^1$ is homotopic to a constant map. [*Hint:* use the universal coverings.]