Coverings in Algebra and Topology—Third homework assignment

May 30, 2017

- Hand in by Sunday, June 18, at the latest. Send your solutions to Salvatore Floccari by email (s.floccari@math.ru.nl), or give them to him in person, or drop them in his pigeon hole. Keep a copy of your work.
- Give complete arguments. You are allowed to use what has been discussed in the course, or what you can find in the book by Szamuely (or similar literature). What counts is that you convince us that you have fully understood how things works.
- What you hand in should be your own work.

Exercise 1. Let X be a compact Riemann surface. Let g be its genus. Let E be a compact Riemann surface of genus 1, and let $f\colon X \to E$ be a non-constant holomorphic map.

(i) If $g \geq 2$, show that f has at least 1 branch point.

(ii) If $g = 1$, show that f has no branch points. Deduce from the fact that the fundamental group of E is abelian that f is a Galois cover.

Next we consider a non-constant holomorphic map $g\colon X \to \mathbb{P}^1$ of degree n that branches over $0 = (0:1)$ and $\infty = (1:0)$ and nowhere else.

(iii) Compute the fundamental group of $\pi_1(\mathbb{P}^1 \setminus \{0, \infty\})$.

(iv) Show that there exists an isomorphism $\alpha\colon \mathbb{P}^1 \xrightarrow{\sim} X$ such that $g \circ \alpha\colon \mathbb{P}^1 \to \mathbb{P}^1$ is given by $z \mapsto z^n$. (Or if you prefer: by $(x:y) \mapsto (x^n : y^n)$.)

Exercise 2. Let X be the quotient space obtained by pasting the edges of a polygonal region in the plane together in pairs.

(i) Show that X is homeomorphic to exactly one of the spaces in the following list:

$$S^2\,, \quad P^2\,, \quad T_n\,, \quad K\,, \quad T_n \# P^2\,, \quad T_n \# K\,,$$

where K is the Klein bottle and $n \geq 1$.

(ii) Show that X is homeomorphic to exactly one of the spaces in the following list:

$$S^2\,, \quad P^2\,, \quad T_n\,, \quad K_m\,, \quad P^2 \# K_m\,,$$

where K_m is the m-fold connected sum of K with itself and m, $n \geq 1$.

You are allowed to use everything that was discussed in the lectures or in the exercise classes.

Exercise 3. Let $X \subset \mathbb{P}^2$ be a non-singular algebraic curve given by the equation $F = 0$, where

$$F = X_0^4 + X_1^4 + X_0 X_2^3 - X_0^3 X_2 \,.$$

This curve X is non-singular, which means that there are no points $P \in X$ with $\frac{\partial F}{\partial X_j}(P) = 0$ for all $j \in \{0,1,2\}$. (You don't need to verify this.) It has as consequence—as explained in the lectures—that X has the structure of a compact Riemann surface.

(i) Determine the genus of X and, for K a canonical divisor on X, determine $\deg(K)$ and $\ell(K)$.

Let $U \subset \mathbb{P}^2$ be the affine open part given by $X_0 \neq 0$. Let $f \in \mathbb{C}[x_1, x_2]$ be the polynomial given by

$$f(x_1, x_2) = F(1, x_1, x_2) = 1 + x_1^4 + x_2^3 - x_2 \,.$$

By construction $(X \cap U) \subset U \cong \mathbb{A}^2$ is given by $f = 0$, and by differentiating this relation we find that

$$\frac{\partial f}{\partial x_1} \cdot \mathrm{d}x_1 + \frac{\partial f}{\partial x_2} \cdot \mathrm{d}x_2 = 0$$

on $X \cap U$.

Let $i \colon X \hookrightarrow \mathbb{P}^2$ be the inclusion map. Recall that to a line $M \subset \mathbb{P}^2$ we have associated a divisor $i^*(M)$ on X that has support $X \cap M$. (In general the points of $X \cap M$ occur in $i^*(M)$ with multiplicities ≥ 1.)

(ii) Let $L \subset \mathbb{P}^2$ be the line given by $X_0 = 0$ (i.e., the complement of U). Consider the 1-form α on X given by

$$\alpha = \Big(\frac{\partial f}{\partial x_2}\Big)^{-1} \cdot \mathrm{d}x_1 \,.$$

Show that $\mathrm{div}(\alpha)$ is equal to the divisor $i^*(L)$.

(iii) Conclude that the canonical divisors on X are precisely the divisors obtained as $i^*(M)$, for $M \subset \mathbb{P}^2$ a line.

(iv) Prove that the Riemann surface X is not hyperelliptic, i.e., there is no non-constant holomorphic map $X \to \mathbb{P}^1$ of degree 2.