Point-counting and the André-Oort conjecture

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Multiplicative Manin-Mumford

Consider a subvariety $V \subset \mathbb{G}_{\mathrm{m}}^n(\mathbb{C}) = (\mathbb{C}^\times)^n$.

Torsion points: $(\zeta_1, \ldots, \zeta_n)$, each ζ_i a root of unity.
"Torsion cosets": coset of a subtorus by a torsion point = irreducible component of an algebraic subgroup.

Theorem (MMM; Laurent 1984, (Mann 1965))

Such V contains only finitely many maximal torsion cosets.

Raised by Lang in 60's and initial case due to Ihara/Serre/Tate.

If one takes $e : \mathbb{C}^n \to (\mathbb{C}^\times)^n$ given by

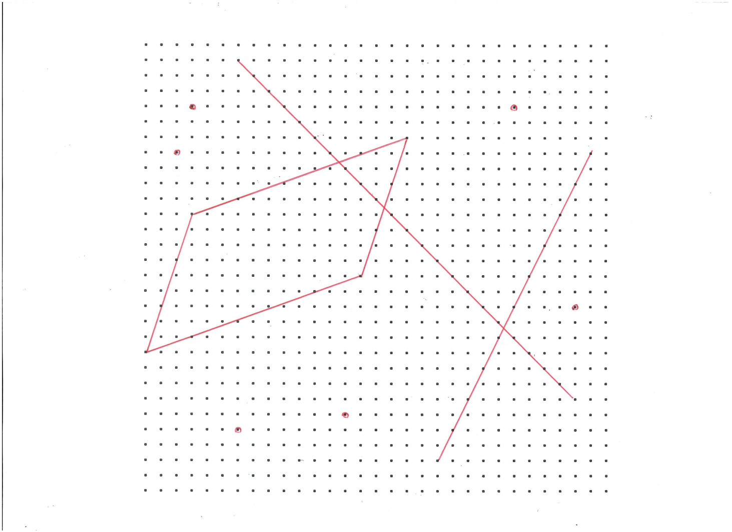
$$e(z_1, \ldots, z_n) = (\ldots, exp(2\pi i z_j), \ldots)$$

then one can instead study **rational points** on

$$Z = \{z \in \mathbb{C}^n \cap [0,1)^n : e(z) \in V\}.$$

This set is analytic but in general far from algebraic.

Induced structure on torsion points



Point-counting

Sarnak gave a proof of MMM (unpublished) by studying rational points on Z. Raised questions about rational points on smooth convex and transcendental plane curves.

Bombieri-P, 1989: estimates for curves variously "sufficiently smooth", C^∞, real analytic, algebraic. Same idea gives:

Theorem

Let $f : [0,1] \to \mathbb{R}$ be real analytic (on some open nbd of $[0,1]$), not algebraic, with graph Z. Let $\epsilon > 0$. Then

$$N(Z, T) \leq c(\epsilon, f) T^\epsilon.$$

Uniform estimates for algebraic curves of given degree.

Heath-Brown's p-adic determinant method (2002): projective varieties of all dimension.

Method of proof

Fix $d \geq 1$, set $D = (d+1)(d+2)/2$.
Let $M_d = \{(a,b) \in \mathbb{N}^2 : 0 \leq a, b \leq a+b \leq d\}$. $\#M_d = D$.
Let $(x_i, y_i) \in Z(\mathbb{Q}, T), i = 1, \ldots D$.
For $\mu = (a,b) \in M_d$ let $\phi_\mu(x) = x^a f(x)^b$. Let

$$\Delta = \det\big(\phi_\mu(x_i)\big)$$

Then Δ is an **alternant**. H. A. Schwarz (1888):

$$\Delta = VM,$$

where V is Vandermonde,

$$V = \prod_{1 \leq i < j \leq D} (x_j - x_i), \quad M = \det\Big(\frac{\phi_\mu^{(i-1)}(x_{\mu,i})}{(i-1)!}\Big),$$

where $x_{\mu,i}$ are suitable intermediate points.

Method of proof, concluded

If $x_i \in I_r(x_0) = (x_0 - r, x_0 + r)$ then

$$|\Delta| \leq r^{D(D-1)/2} c_1(f, d),$$

here $c_1(f, d) = c_1(f, f', \ldots, f^{(D-1)}, d)$. But $N\Delta \in \mathbb{Z}$ with

$$|N| \leq T^{2dD}.$$

So all these points will lie on **one** curve Γ degree d if

$$r \leq c(f, d) T^{-8/(d+3)},$$

and $8/(d+3) < \epsilon$ for suitable $d = d_\epsilon$.
Then $C(f, d) T^\epsilon$ such intervals cover $[0, 1]$.
For Γ degree d, $\#(Z \cap \Gamma) \leq c_2(f, d)$. □

Point-counting in higher-dimensional real sets

Heath-Brown's (2002) paper provoked:

What about higher-dimensional real analytic sets?
E.g. compact analytic sets?

Leads to bounded subanalytic sets and to the more general **"definable sets in o-minimal structure"**.

And this generality turned out to be crucial in applications.

Such sets can contain "many" rational points even if they are non-algebraic. E.g.

$$Z = \{(x, y, z) : z = x^y, (x, y) \in [1, 2]^2\}.$$

We will exclude such algebraic subsets, and count points in the "transcendental part" of a definable set.

First-order structures and definability

We consider first-order definable sets in some expansion of the real field:

$$\langle \mathbb{R}, 0, 1, +, \times, f_i, i \in I, R_j, j \in J \rangle$$

where $f_i : \mathbb{R}^{n_i} \to \mathbb{R}$, $R_j \subset \mathbb{R}^{m_j}$ are some functions and relations.

A set $Z \subset \mathbb{R}^n$ is **definable** (with parameters) if it has a definition:

$$Z = \{(z_1, \ldots, z_n) \in \mathbb{R}^n : \phi(z_1, \ldots, z_n) \text{ holds } \}$$

where ϕ is a first-order formula

$$\phi = \forall x \exists y \ldots f_1(x) = y \wedge R_2(x, y, \lambda_1) \ldots \vee \ldots \neg \ldots$$

Allow arbitrary fixed real numbers $\lambda_1, \ldots$ ("with parameters").

If $Z \subset \mathbb{R}^2$ is definable then so is the subset where Z is locally the graph of a function, where this function is differentiable etc.

O-minimal structures

The structure $\langle \mathbb{R}, 0, 1, +, \times, f_i, i \in I, R_j, j \in J \rangle$ is **o-minimal** if every definable subset of $\mathbb{R}$ is a finite union of points and intervals.

A model-theory free definition:

A **pre-structure** is a sequence $(\mathcal{S}_i, i = 1, 2, \ldots)$ where $\mathcal{S}_n$ is a collection of subsets of $\mathbb{R}^n$. It is a **structure** over the real field if:
$*$ Each $\mathcal{S}_n$ is a Boolean algebra containing every semi-algebraic set
$*$ Have closure under cartesian products and coordinate projections.
The structure is **o-minimal** if boundary of every $X \in \mathcal{S}_1$ is finite.

However: the "definability" definition is more intuitive:
Exercise: show that the set of points where a definable function is differentiable is definable using this definition.

The origins

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Remarks on Tarski's problem concerning $(\mathbb{R}, +, \cdot, \exp)$

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INTRODUCTION

In his monograph on the elementary theory of the structure $(\mathbb{R}, +, \cdot)$ Tarski asked whether his results could be extended to the structure $(\mathbb{R}, +, \cdot, \exp)$ ([T, p. 45]). (Instead of $\exp(x) = e^x$, Tarski suggested the function $f(x) = 2^x$, but this makes little difference since exp is definable in $(\mathbb{R}, +, \cdot, f)$ as the unique function of the form $x \longmapsto f(ax)$ which is its own derivative; the axioms mentioned in [T, p. 57, note 20] for $\mathrm{Th}(\mathbb{R}, +, \cdot, f)$ are far from adequate, see e.g. [D-W].)

Before we discuss Tarski's question, let us briefly review some aspects of his work on $(\mathbb{R}, +, \cdot)$ and see what use has been made of it:

(1) Decidability of $\mathrm{Th}(\mathbb{R}, +, \cdot)$,

(2) $\mathrm{Th}(\mathbb{R}, +, \cdot)$ = theory of real closed fields,

(3) Elimination of quantifiers for $(\mathbb{R}, <, 0, 1, +, \cdot)$,

(4) Properties of definable subsets of $\mathbb{R}^n$,

(5) Properties of definable functions.

These aspects are closely related in Tarski's work, but it makes sense to discuss them separately. (1) is a nice result in its own right and quite useful in many theoretical decidability questions, but has otherwise not been important in settling open problems, as far as I know. (2) is sometimes useful in proving properties of real closed fields: in certain cases the only known proof consists of first establishing the property for the field of reals by transcendental methods and then invoking (2). (This is called Tarski's principle.) (2) and (3) combined give a trivial and improved solution of Hilbert's 17th problem,

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DEFINABLE SETS IN ORDERED STRUCTURES. I

ANAND PILLAY AND CHARLES STEINHORN[1]

ABSTRACT. This paper introduces and begins the study of a well-behaved class of linearly ordered structures, the $\mathcal{O}$-minimal structures. The definition of this class and the corresponding class of theories, the strongly $\mathcal{O}$-minimal theories, is made in analogy with the notions from stability theory of minimal structures and strongly minimal theories. Theorems 2.1 and 2.3, respectively, provide characterizations of $\mathcal{O}$-minimal ordered groups and rings. Several other simple results are collected in §3. The primary tool in the analysis of $\mathcal{O}$-minimal structures is a strong analogue of "forking symmetry," given by Theorem 4.2. This result states that any (parametrically) definable unary function in an $\mathcal{O}$-minimal structure is piecewise either constant or an order-preserving or reversing bijection of intervals. The results that follow include the existence and uniqueness of prime models over sets (Theorem 5.1) and a characterization of all $\aleph_0$-categorical $\mathcal{O}$-minimal structures (Theorem 6.1).

1. Introduction. The class of linearly ordered structures has long been an important subject of concern to model theorists. Impressive results have been obtained in the study of models of several particular theories that extend the theory of linear order. Among those theories that have been approached successfully are Peano arithmetic, the theory of ordered abelian groups, that of real-closed fields, and that of linear order itself. Yet, very little has been done in the way of developing a general model theory for ordered structures. In this paper, we develop the model theory for a class of linearly ordered structures that we isolate by demanding that a structure in this class satisfy a condition whose effect is that the linear ordering and the algebraic part of the structure behave quite well with respect to one another.

Let L be a finitary first-order language, and $\mathcal{M}$ an L-structure. A set of n-tuples $A \subseteq \mathcal{M}^n$ is said to be *parametrically definable* if there is some L-formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_k)$ and $b_1, \ldots, b_k \in \mathcal{M}$ so that $A = \{(a_1, \ldots, a_n): \mathcal{M} \models \varphi(a_1, \ldots, a_n, b_1, \ldots, b_k)\}$. If A is definable without parameters, we simply say that A is *definable*. Model theorists have enjoyed particular success in their efforts to determine structural properties of models of first-order theories T by restricting their considerations to those T for which the parametrically definable sets of n-tuples in models of T satisfy certain conditions. We isolate the class of linearly ordered structures with which we shall be concerned in this paper by requiring that the parametrically definable subsets of an ordered structure in our class be of a particular simple form, which we now describe.

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The origins, continued

Idea introduced by van den Dries in his work (1982) on the model theory of the real exponential function. He saw: this finiteness condition **would** make the collection of definable sets in $\langle \mathbb{R}, 0, 1, +, \times, \exp \rangle$ well-behaved, like real algebraic sets.

For quantifier-free sets ("exponential-algebraic varieties") finiteness follows from Khovanskii's theory of "fewnomials".

A bit later, van den Dries saw that Gabrielov's theorem on subanalytic functions implies that $\mathbb{R}_{\mathrm{an}}$ is o-minimal. Here

$$\mathbb{R}_{\mathrm{an}} = \langle \mathbb{R}, 0, 1, +, \times, \{f\} \rangle$$

for the set $\{f\}$ of arbitrary $f : [0,1]^m \to \mathbb{R}$ (all $m = 1, 2, \ldots$) real analytic on an open neighbourhood of the box.

The questions was: is $\mathbb{R}_{\exp} = \langle \mathbb{R}, 0, 1, +, \times, \exp \rangle$ o-minimal?
Answer: Yes (Wilkie, 1991).
Later: $\mathbb{R}_{\mathrm{an\ exp}}$ is o-minimal (van den Dries-Miller, 1994)

Cell decomposition

General theory developed by Pillay and Steinhorn.
O-minimal = order-minimal cf minimal, strongly minimal

Cells are defined inductively. A cell in $\mathbb{R}$ is either a point or an open interval. A cell in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ is either
(i) the graph of a definable continuous function on a cell in $\mathbb{R}^{n-1}$
(ii) the region between two such functions on a cell in $\mathbb{R}^{n-1}$

Theorem (Pillay-Steinhorn, 1986)

A definable set in an o-minimal structure is a finite union of cells.

E.g. A definable function is continuous except at finitely many points. (Likewise: differentiable, but needs proof.)

A **definable family** is the family of fibres of a definable set.
E.g. In a definable family, the subset of the parameters where the fibre has given dimension is a definable set.

Rational points in definable sets

For $Z \subset \mathbb{R}^n$, define its **algebraic part** Z^{alg} to be the union of connected positive dimensional semi-algebraic sets contained in Z. Set $Z^{\mathrm{trans}} = Z - Z^{\mathrm{alg}}$ to be the **transcendental part**.

Theorem (P-Wilkie, 2006)

Let $Z \subset \mathbb{R}^n$ be a definable set in an o-minimal structure over $\mathbb{R}$. Let $\epsilon > 0$. Then

$$\# Z^{\mathrm{trans}}(\mathbb{Q}, T) \leq c(Z, \epsilon) T^{\epsilon}$$

Z^{alg} is a coarse analogue of the "special set" in diophantine geometry, the locus where rational points could accumulate.

The theorem is uniform in definable families (in fact this is a crucial element in the proof).

Sketch of proof

For any $r \geq 1$, a definable set $Z \subset (0,1)^n$ of dimension m can be r-**parameterised**:

$$Z = \bigcup_{\text{finite}} Y, \quad Y = \phi\big((0,1)^m\big), \quad \phi : (0,1)^m \to (0,1)^n,$$

with all $|\partial^\alpha \phi_i(x)| \leq 1$ all $|\alpha| \leq r$. **The number of such** ϕ **required is uniformly bounded in definable families**.

Generalises: Yomdin/Gromov for semi-algebraic sets.

For each ϕ: a determinant argument shows:
$Z(\mathbb{Q}, T)$ contained in $\ll_{Z,\epsilon} T^\epsilon$ hypersurfaces of some degree d_ϵ.

These hypersurface intersections $Z \cap V, \deg V = d$ form a **definable family**, and we can repeat the process. . . . □

Extensions, variations

Same theorem holds for points of bounded degree. This is crucial in the application to AO. E.g. for special points in $Y(1)$ the pre-images under j are quadratic points.

For a general o-minimal structure (e.g. for $\mathbb{R}_{\mathrm{an}}$) the T^{ϵ} estimate cannot be much improved.

Conjecture (Wilkie, 2005)

For Z definable in $\mathbb{R}_{\mathrm{exp}}$ have $\#Z^{\mathrm{trans}}(\mathbb{Q}, T) \leq C_Z(\log T)^{c_Z}$.

Partial results: curves and some surfaces, some in $\mathbb{R}_{\mathrm{Pfaff}}$: Butler, Jones-Miller-Thomas, P.

Non-archimedean version: Cluckers-Comte-Loeser (2015).

Refinement: “blocks”

In fact the result proved can be stated in a stronger way. This is necessary to deal with points of bounded degree, and also in some further applications.

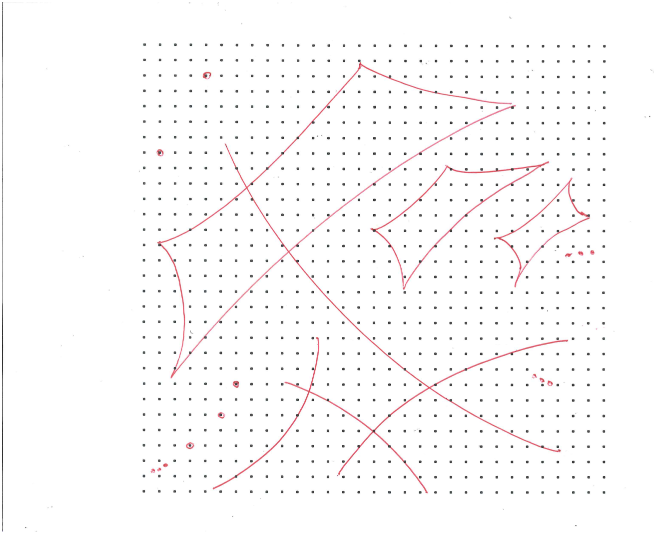
A **block** is a definable set, dimension $k \geq 0$, regular of dimension k, such that locally at every point it coincides with a semi-algebraic set of dimension k.

E.g. a point, or the part of $x^2 + y^2 + z^2 = 1$ lying above $z = e^x$. I.e. a “definable piece of a semi-algebraic set”.

Then: $Z(\mathbb{Q}, T)$ is contained in $c(Z, \epsilon)T^\epsilon$ blocks contained in Z.

In this form, theorem says something even if $Z^{\mathrm{trans}}(\mathbb{Q})$ is empty.

Rational points and "blocks"



In this form, it looks like a weak generalised form of MM/AO.

The strategy to prove MMM/AO

Strategy proposed by Zannier for MM.

To conclude MM/AO one has to:

1. Go from T^ϵ blocks to finitely many.

By opposing the T^ϵ upper bound with a lower bound for Galois conjugates (and an upper bound for the height of the pre-images), show that torsion/special points of high order/discriminant must lie in some positive dimensional blocks.

2. "Straighten out the blocks", from some general pieces of semi-algebraic sets to sets of specific ("special") form.

This is done by functional transcendence: "Ax-Lindemann".

Lindemann, Schanuel, Ax-Schanuel, and "Ax-Lindemann"

Lindemann's (or L-Weierstrass;) theorem (1882/85):
If $a_i \in \overline{\mathbb{Q}}$ are l.i. over $\mathbb{Q}$ then $exp(a_i)$ are a.i. over $\mathbb{Q}$

is a special case of **Schanuel's Conjecture** (1960's):
If $z_1, \ldots, z_n \in \mathbb{C}$ then $\mathrm{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(z_i, e^{z_i}) \geq \mathrm{lin.dim}_{\mathbb{Q}}(z_i)$

"**Ax-Schanuel**" affirms SC for functions: If $\mathbb{Q} \subset K$ is a differential field, commuting derivations D_i, constant field $C = \bigcap \ker D_i$:

Theorem (Ax, 1971)

If $y_j, x_j \in K^\times, j = 1, \ldots, n$ with $D_i y_j = y_j D_i x_j$ for all i, j, then

$$\mathrm{tr.deg.}_C C(x_j, y_j) \geq n + \mathrm{rank}_K(D_i x_j)$$

unless x_i are l. dep. $/\mathbb{Q}$ mod C, (non-trivial $\sum q_i x_i \in C, q \in \mathbb{Q}$).

"Ax-Lindemann" is the part of Ax-Schanuel corresponding to Lindemann's theorem.

Ax-Lindemann for the exponential function

Consider algebraic subvariety $W \subset \mathbb{C}^n$ of dimension k.

Have coordinate functions $z_i, exp(z_i)$ on W locally.

Then by Ax-Schanuel:

$$\mathrm{tr.deg.}(z_i, \exp(z_i)) \geq n + k$$

Therefore: $\exp(z_1), \ldots, \exp(z_n)$ are a.i. over $\mathbb{C}$ unless the z_i are l.i. over $\mathbb{Q}$ mod $\mathbb{C}$.

I.e. $\exp(W)$ is Zariski dense in $(\mathbb{C}^\times)^n$ unless W is contained in a coset of a proper rational subspace (="weakly special subvariety").

Equivalently: For $V \subset (\mathbb{C}^\times)^n$, a maximal algebraic subvariety of $\exp^{-1}(V)$ is "weakly special".

Singular Moduli

Singular moduli are the "special values" of the j-function.

Definition

A **singular modulus** is a complex number $j(\tau)$ where $j : \mathbb{H} \to \mathbb{C}$ is the modular function, and $\tau \in \mathbb{H}$ is quadratic ($[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$).

$$\Sigma = \{\sigma = j(\tau) : \tau \in \mathbb{H}, [\mathbb{Q}(\tau) : \mathbb{Q}] = 2\}$$

Schneider: These are the only points with $\tau, j(\tau) \in \overline{\mathbb{Q}}$.

$j(\tau)$ is the j-invariant of $E_\tau = \mathbb{C}/\Lambda_\tau$, lattice $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$
Singular moduli are the j-invariants of elliptic curves with CM.

They are algebraic integers, and $[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = \mathrm{Cl}(\mathcal{O}_{D(\tau)})$.

E.g. $j(\frac{1+\sqrt{-163}}{2}) = -2^{18}3^3 5^3 23^3 29^3$, $\quad j(\sqrt{-5}) = (50 + 26\sqrt{5})^3$.

Special points in $\mathbb{C}^k$: tuples of singular moduli

Modular André-Oort Conjecture

Concerns subvarieties of $\mathbb{C}^k$, moduli of k-tuples of elliptic curves.

Modular relations: of the form $\Phi_N(x_h, x_\ell) = 0$
Special points in $\mathbb{C}^k$: $(\sigma_1, \ldots, \sigma_k)$, each σ_i a singular modulus.
Special subvariety in $\mathbb{C}^k$: is a component of a subvariety defined by a system of modular relations.

A special point is a special subvariety, and special subvarieties contain (Zariski-)dense sets of torsion points.

Look for special points on $V \subset \mathbb{C}^k$.

Theorem (André, $k = 2$, 1998; Edixhoven, on GRH, 2005; P 2011)

Such V contains only finitely many maximal special subvarieties.

Or: all special pts $\in V$ lie in finitely many special subvrieties $\subset V$.
Or: A variety with a Zariski-dense set of special points is special.

Modular Ax-Lindemann

Concerns: $j : \mathbb{H}^n \to \mathbb{C}^n$, and $V \subset \mathbb{C}^n$.

By "algebraic" $W \subset \mathbb{H}^n$ mean a component of some $W' \cap \mathbb{H}^n$.

Weakly special subvariety of $\mathbb{H}^n$: defined by relations of the form $z_i = g_{ij} z_j$ with $g_{ij} \in \mathrm{GL}_2^+(\mathbb{Q})$, or $z_k = c \in \mathbb{C}$.

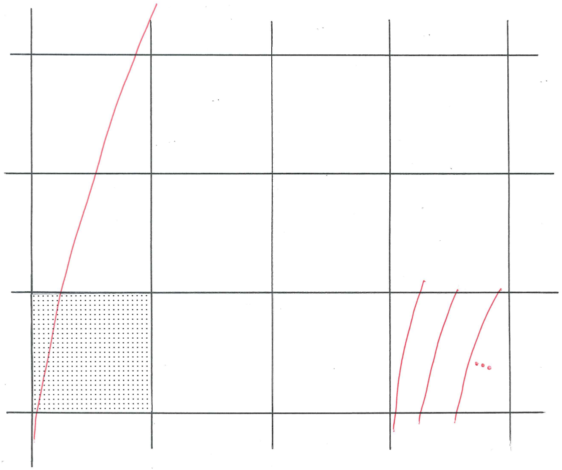
Strongly special subvariety in $\mathbb{C}^n$: no constant coordinate.

Theorem (P, 2011)

Let $V \subset \mathbb{C}^n$. A maximal algebraic $W \subset j^{-1}(V)$ is weakly special.

I.e. if no relation $z_i = g_{ij} z_j$ with $g_{ij} \in \mathrm{GL}_2^+(\mathbb{Q})$, or $z_k = c \in \mathbb{C}$ holds on W then $j(W)$ is Zariski-dense in $\mathbb{C}^n$.

Sketch of idea of proof of Modular Ax-Lindemann



Implication of Modular Ax-Lindemann

Mobius subvariety of $\mathbb{H}^n$. Like weakly specials, but allow relations $z_i = \gamma_{ij} z_j$ with $\gamma_{ij} \in \mathrm{SL}_2(\mathbb{R})$. These form a definable family, even finitely many definable families of "strongly mobius subvarieties" and their "translates".

Let $V \subset \mathbb{C}^n$, $Z = j^{-1}(V) \cap F^n$.

Then: the set of translates of strongly mobius subvarieties on some subset of coordinates which are "maximal among translates" of such which intersect Z in their full dimension is a definable set.

By Ax-Lindemann, everything in this set is given by elements of $\mathrm{GL}_2^+(\mathbb{Q})$. **Therefore this set is finite**.

I.e. all maximal weakly special subvarieties in V are "translates" of finitely many. This allows inductive proof of AO.

Same holds for any definable family of V, like V of fixed degree.

Immediately implies: finiteness of max strongly special sbvrts of V.

Three applications of o-minimality

Ax-Lindemann (proved via o-minimality and point-counting)

Given V:

Maximal weakly specials in $j^{-1}(V)$ come in finitely many families (by Ax-Lindemann and o-minimality).

Take T of highest dimension of these. Have $V_T \subset \mathbb{C}^m$ parameterising translates of T which belong to V.

Now if the special point $x \in V_T$ parameterising $T_x \subset V$ has too high discriminant, we get too many rational points in Z_T. By Counting Theorem, we get something weakly special in V_T giving a bigger weakly special in V. Contradiction.

Having dealt with all T of highest dimension, get finitely many T_i, and repeat argument with T' of next lowest dimension, taking conjugates over V and the T_i.

The general picture

A Shimura variety X has a transcendental uniformisation $\pi : \mathbb{H} \to X$ with semi-algebraic symmetric hermitian domain $\mathbb{H}$. Ullmo has shown that four ingredients imply AO:

1. Definability of π on a fundamental domain:
(Mixed) $\mathcal{A}_g$: Peterzil-Starchenko; General case: KUY.

2. Upper bound for height of pre-image of a special point in a fundamental domain (for $\Gamma : X = \Gamma\backslash\mathbb{H}$):
$\mathcal{A}_g$: Tsimerman; General case: Daw-Orr.

3. Lower bound for Galois orbit of special point:
$Y(1)$: Landau-Siegel; $\mathcal{A}_g$: Tsimerman; General case??.

4. Ax-Lindemann
P, P-Tsimerman, UY, $\mathcal{A}_g$: PT; General: KUY;
Mixed Ax-L, and mixed AO: Gao

Only for $Y(1)^2$ is AO effective (Kühne, Bilu-Masser-Zannier)

The Zilber-Pink Conjecture

Encompasses ML (MM), MMM, AO.

Has three independent sources:
* Zilber: model theory of exponentiation
* Bombieri-Masser-Zannier: Problems of Schinzel, developments
* Pink: combining ML (and MM) and AO

Is open in general (even for $\mathbb{G}_{\mathrm{m}}^n$).

Concerns: A mixed Shimura variety X, its special subvarieties, and a subvariety $V \subset X$.

Governs: Intersections between V and special subvarieties $T \subset X$ which are **atypical in dimension**.

Asserts: a finiteness property, although there are countably many special subvarieties.

Sources for ZP

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EXPONENTIAL SUMS EQUATIONS AND THE SCHANUEL CONJECTURE

BORIS ZILBER

Abstract

A uniform version of the Schanuel conjecture is discussed that has some model-theoretical motivation. This conjecture is assumed, and it is proved that any 'non-obviously-contradictory' system of equations in the form of exponential sums with real exponents has a solution.

1. *Introduction*

In [8] we started a model-theoretical study of the formal theory of exponentiation (pseudo-exponentiation). The crux of the analysis is the observation that the Schanuel conjecture on the degree of algebraic independence between complex numbers and their exponentials (see [5]) is responsible for very basic geometric properties of fields allowing a function ex satisfying $\mathrm{ex}(x+y) = \mathrm{ex}(x) \cdot \mathrm{ex}(y)$. It turns out that in the class of fields with pseudo-exponentiation, given an uncountable cardinal, there is a unique 'right' one of the given cardinality. This model we call *canonical*. The mere fact of the canonicity tempts us to conjecture that the classical exponentiation on the complex numbers is formally equivalent to the pseudo-exponentiation in a canonical model, or, even more concretely, the structure of complex numbers in the language $(+, \cdot, \exp)$ is the canonical model of the field with pseudo-exponentiation of cardinality continuum. This is a very strong conjecture, which presumes among other properties the Schanuel conjecture.

As a matter of fact, the analysis in [8] shows that the 'other properties' could be reduced to a unique and natural one: the property of *exponential-algebraic closedness*, meaning that any 'non-obviously-inconsistent' system of equations has a solution in the field. In this paper this condition is explained in precise technical terms (*normal* and *free* system of equations).

On the other hand, the further logical analysis [9] of pseudo-exponentiation shows that the Schanuel conjecture is much better motivated under an extra conjecture of Diophantine type (the conjecture on intersections with tori). We show that this conjecture holds in a 'function field case', using a result of J. Ax.

In a more general form, corresponding to the more general versions of the Schanuel conjecture, the Diophantine conjecture implies both the Mordell–Lang and the Manin–Mumford conjectures (now proved, see [6]). In the first part of this paper we discuss the conjectures and show that the Schanuel conjecture plus the conjecture on intersections with tori is in effect equivalent to a uniform version of the Schanuel conjecture.

In the second half of the paper we attack the problem of exponential-algebraic closedness for a special class of equations, given by exponential sums with real

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Anomalous Subvarieties—Structure Theorems and Applications

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When a fixed algebraic variety in a multiplicative group variety is intersected with the union of all algebraic subgroups of fixed dimension, a key role is played by what we call the anomalous subvarieties. These arise when the algebraic variety meets translates of subgroups in sets larger than expected. We prove a Structure Theorem for the anomalous subvarieties, and we give some applications, emphasizing in particular the case of codimension two. We also state some related conjectures about the boundedness of absolute height on such intersections as well as their finiteness.

1 Introduction

For $n \geq 1$ let $\mathcal{X}$ be an algebraic subvariety of the group variety $\mathbf{G}_m^n$ defined by the non-vanishing of the coordinates $x_1, \ldots, x_n$ in affine n-space. In this paper we are interested in the intersection of $\mathcal{X}$ with varying algebraic subgroups of $\mathbf{G}_m^n$ restricted only by dimension. Recall that every such subgroup is defined by monomial equations $x_1^{a_1} \ldots x_n^{a_n} = 1$, and its dimension is $n - h$, where h is the rank of the subgroup of $\mathbf{Z}^n$

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A Common Generalization of the Conjectures of André-Oort, Manin-Mumford, and Mordell-Lang

Richard PINK*

April 17, 2005

1 Introduction

Let S be a mixed Shimura variety over the field of complex numbers $\mathbb{C}$. By definition an irreducible component of a mixed Shimura subvariety of S, or of its image under a Hecke operator, is called a *special subvariety* of S.

Consider any irreducible closed subvariety $Z \subset S$. Since any intersection of special subvarieties is a finite union of special subvarieties, there exists a unique smallest special subvariety containing Z. We call it the *special closure of* Z and denote it by S_Z. We call the dimension of S_Z the *amplitude of* Z, and the codimension of Z in S_Z the *defect of* Z. The defect measures how far Z is away from being special; in particular Z is special if and only if its defect is zero. Moreover Z is called *Hodge generic* if S_Z is an irreducible component of S, that is, if Z is not contained in any special subvariety of codimension > 0.

For any point $s \in S$ the amplitude and the defect of $\{s\}$ coincide and are called the *amplitude of* s. The points of amplitude zero in S are precisely the special points in S. Moreover s is called *Hodge generic* if $\{s\}$ is Hodge generic.

Conjecture 1.1 *Consider a mixed Shimura variety S over $\mathbb{C}$, an integer d, and a subset $\Xi \subset S$ of points of amplitude $\leq d$. Then any irreducible component Z of the Zariski closure of Ξ has defect $\leq d$.*

Clearly this is equivalent to the following contrapositive version:

Conjecture 1.2 *Consider a mixed Shimura variety S over $\mathbb{C}$ and an irreducible closed subvariety Z. Then the intersection of Z with the union of all special subvarieties of S of dimension $< \dim S_Z - \dim Z$ is not Zariski dense in Z.*

Moreover, since these conjectures are invariant under Hecke operators, one may assume that S_Z is an irreducible component of a mixed Shimura subvariety S' of S. Replacing S by S' then leads to the following equivalent formulation:

Conjecture 1.3 *Consider a mixed Shimura variety S over $\mathbb{C}$ and a Hodge generic irreducible closed subvariety Z. Then the intersection of Z with the union of all special subvarieties of S of codimension $> \dim Z$ is not Zariski dense in Z.*

The aim of this note is to propose and explain these conjectures, and to relate them to other conjectures and known results. Conjecture 1.1 for $d = 0$ is precisely the André-Oort conjecture, which has been established in special cases or under additional assumptions by Moonen [13] [14], André [1], Edixhoven [8], [9], [10],

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Statement of ZP

Let X be a mixed Shimura variety, $\mathcal{S}$ its collection of special subvarieties, and $V \subset X$.

Definition

A subvariety $A \subset V$ is called **atypical** if there is $T \in \mathcal{S}$ with $A \subset V \cap T$ such that

$$\dim A > \dim V + \dim T - \dim X.$$

Denote by $\mathrm{Atyp}(V)$ the union of atypical subvarieties of V.

Conjecture (Zilber-Pink)

$\mathrm{Atyp}(V)$ *is a finite union.*

ZP implies AO

Proposition

ZP implies AO, MM, MMM, ML.

Proof. Let $V \subset X$ where X is a mixed Shimura variety.

By ZP, V has finitely many maximal atypical subvarieties $A_i \subset V \cap T_i$, for some special T_i.

For a proper $V \subset X$ to contain any special subvariety S is atypical.

So $S \subset A_i$ for some i, but is atypical in $A_i \subset T_i$, unless $A_i = T_i$.

Continue, finding finitely many such $S = A_{ijk\ldots} = T_{ijk\ldots}$.

ZP implies MM, MMM likewise, and ML "similarly". □

ZP for curves in $(\mathbb{C}^\times)^k$

Let $V \subset (\mathbb{C}^\times)^k$ be a curve. An atypical component of V is either:
* a point $P \in V$ lying in a special T of codimension 2
* V if it is contained in a proper special subvariety of $(\mathbb{C}^\times)^k$

Theorem (Bombieri-Masser-Zannier 1999/2008, Maurin 2008)

ZP holds for curves in $(\mathbb{C}^\times)^k$: if $V/\mathbb{C}$ is not contained in a proper special subvariety (= proper algebraic subgroup) then its intersection with the **union** *of all subgroups of codimension at least 2 is a finite set.*

Example (BMZ): There are only finitely many $t \in \mathbb{C}$ for which there are **two or more** independent multiplicative relations among

$$2, 3, 5, t, 1-t, 1+t$$

Abelian varieties: Remond, Viada, others, Habegger+P.
Partial “modular” result for curves in $\mathbb{C}^k$: Habegger+P.

Partial Modular ZP for curves

A curve $V \subset Y(1)^n$ is **asymmetric** if among positive $\deg_{X_i} V$ there is at most one repeat.

Theorem (Habegger-P, 2013)

Let $V \subset Y(1)^n$ be asymmetric and defined over $\overline{\mathbb{Q}}$. Then ZP holds.

Example: $V = \{t, t^2, 1-t, 2, 3 : t \in \mathbb{C}\}$ (2, 3 are not in same Hecke orbit).

Further (P, 2015): ZP holds for a curve $V \subset Y(1)^n$ **not** defined over $\overline{\mathbb{Q}}$ (because modular curves have high gonality).

Via: o-minimality and point-counting. The rational points are now $\mathrm{GL}_2^+(\mathbb{Q})$-points on some definable subset of $\mathrm{GL}_2^+(\mathbb{R})^n$.