Fundamental groups and moduli of curves in positive characteristic

On the occasion of Professor Oort's "sanju" (傘寿)

November 10, 2015

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(RIMS, Kyoto Univ.)

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$k = \overline{k}$, of char. $p \geq 0$
A: abelian variety over k, $g := \dim(A)$
$\Lambda_g := \pi_1^{\mathrm{top}}((\mathbf{R}/\mathbf{Z})^{2g}) = \mathbf{Z}^{2g}$

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Then

$\pi_1(A) \simeq \widehat{\Lambda_g} = \widehat{\mathbf{Z}}^{2g}$ $(p = 0)$
$\widehat{\Lambda_g} \twoheadrightarrow \pi_1(A) = \pi_1(A)^{(p')} \times \pi_1(A)^{(p)} \simeq \widehat{\Lambda_g}^{(p')} \times (\mathbf{Z}_p)^f \simeq (\widehat{\mathbf{Z}}^{(p')})^{2g} \times (\mathbf{Z}_p)^f$
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where

$\widehat{\Gamma}$: profinite completion of a group Γ
$G^{(p)}$: maximal pro-p quotient of a profinite group G
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Thus (setting $f = -$ for $p = 0$)

$$\{\pi_1(A) \mid k,\ A\text{: as above}\}/\simeq \ \overset{1:1}{\longleftrightarrow} \{(p, g, f)\}$$

$\mathcal{A}_{g,k}$: coarse moduli space of g-dim. (polarized) abelian varieties over k
$(\dim(\mathcal{A}_{g,k}) = g(g+1)/2)$
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$\mathcal{S}_{g,f,k}$ is locally closed of codim. $g-f$ ([Norman-Oort1980], cf. Pries's talk).

§2. Geometric fundamental groups of curves.

$k = \overline{k}$, of char. $p \geq 0$
X^{cpt}: proper smooth connected curve over k, $g :=$ genus(X^{cpt})
$S \subset (X^{\mathrm{cpt}})^{\mathrm{cl}}$, $r := \sharp(S) < \infty$
$X := X^{\mathrm{cpt}} \smallsetminus S$ (or, more precisely, (X^{cpt}, S)), called (g, r)-curve

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$\Pi_{g,r} := \pi_1^{\mathrm{top}}(\Sigma_{g,r})$
$\simeq \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_r \mid [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g]\gamma_1 \ldots \gamma_r = 1 \rangle$
($\Sigma_{g,r}$: compact orientable topological surface of genus g minus r points)
For $r > 0$, $\Pi_{g,r} \simeq F_b$, $b = 2g + r - 1$

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$\pi_1(X) \simeq \widehat{\Pi_{g,r}}$ $(p = 0)$
$\widehat{\Pi_{g,r}} \twoheadrightarrow \pi_1^t(X) \twoheadrightarrow \pi_1(X)^{(p')} \times \pi_1(X^{\mathrm{cpt}})^{(p)} \simeq \widehat{\Pi_{g,r}}^{(p')} \times \widehat{F_f}^{(p)}$ $(p > 0)$

where

f: p-rank of $J = \mathrm{Jac}(X^{\mathrm{cpt}})$ $(0 \leq f \leq g)$

Thus, for $p = 0$,

$$\begin{aligned}
&\{\pi_1(X) \mid k,\, X\text{: as above}\}/\simeq \\
&\xleftrightarrow{1:1} \{\{(0,0),(0,1)\}, \{(g,0)\}(g>0), \{(g,r) \mid r>0, 2g+r-1=b\}(b>0)\} \\
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$$\begin{array}{l} (g,r)=(0,0) \implies \pi_1(X)=\pi_1^t(X)=\{1\} \\ (g,r)=(0,1) \implies \pi_1^t(X)=\{1\} \\ (g,r)=(0,2) \implies \pi_1^t(X) \simeq \widehat{\mathbf{Z}}^{(p')} \\ (g,r)=(1,0) \implies \pi_1(X)=\pi_1^t(X) \simeq (\widehat{\mathbf{Z}}^{(p')})^2 \times (\mathbf{Z}_p)^f \end{array}$$

Expectation

Assume $p > 0$. Then:

(i) $\pi_1(X)$ determines the $\simeq$-class of X, if $r > 0$.

(ii) $\pi_1(X)$ determines the $\sim$-class of X, unless $(g, r) = (0, 0), (1, 0)$.

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Here,

$$(X/k) \simeq (X'/k') \overset{\text{def}}{\iff} X \simeq X' \text{ as schemes.}$$

$$(X/k) \sim (X'/k')$$
$\overset{\text{def}}{\iff} \exists X_1/k_1,\ k \hookrightarrow k_1 \hookleftarrow k'$, s.t. $X_1 \underset{k_1}{\simeq} X \times_k k_1$ & $X_1 \underset{k_1}{\simeq} X' \times_{k'} k_1$
$\iff \exists X_0/k_0,\ k \hookleftarrow k_0 \hookrightarrow k'$, s.t. $X_0 \times_{k_0} k \underset{k}{\simeq} X$ & $X_0 \times_{k_0} k' \underset{k'}{\simeq} X'$
$\iff$ the images (which are scheme-theoretic points) in $\mathcal{M} := \coprod_{(g,r)} \mathcal{M}_{g,[r]}$ of the classifying morphisms for X/k and X'/k' coincide, where $\mathcal{M}_{g,[r]}$ is the coarse moduli space of (g, r)-curves over $\mathbf{Z}$. ($\mathcal{M}_{0,[0]} = \mathcal{M}_{0,[1]} = \mathcal{M}_{0,[2]} = \mathcal{M}_{0,[3]} = \mathrm{Spec}(\mathbf{Z})$, $\mathcal{M}_{1,[0]} = \mathcal{M}_{1,[1]} = \mathrm{Spec}(\mathbf{Z}[j])$)

Remark: $(X/k) \simeq (X'/k') \iff (X/k) \sim (X'/k')$ & $k \simeq k'$.

$\mathcal{M}_{g,[r],\mathbf{F}_p}$: coarse moduli space of (g,r)-curves over $\mathbf{F}_p$ ($\mathbf{F}_0 := \mathbf{Q}$)
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π_1^t: $|\mathcal{M}_{g,[r],\mathbf{F}_p}| \to \{\text{profinite groups}\}/\simeq$, $\; x \mapsto \pi_1^t(X_{\overline{x}})$
(When $p=0$ or $r=0$, $\pi_1 = \pi_1^t$.)

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$p>0$: Expectation (iii) $\implies$ π_1^t is injective!

§3. Previous results.

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<u>Theorem 1</u>
(i) ([T1999]) $\pi_1(X)$ determines p, g, r, f, inertia groups, $\pi_1^t(X)$, $\pi_1(X^{\mathrm{cpt}})$, unless $(g, r) = (0, 0)$.
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For (i), this is elementary by comparing the Riemann-Hurwitz formula for genus and the Deuring-Shafarevich formula for p-rank.

For (ii), we prove a certain limit formula to the effect that (for $r > 1$)

average of p-rank of cyclic covers $\to g$ (covering degree $\to \infty$)

by establishing a ramified version of Raynaud's theory of vector bundles B and theta divisors Θ (cf. Pries's talk). $\square$

Theorem 2
Assume $g = 0$ and that either X is defined over $\overline{\mathbf{F}_p}$ or $r \leq 4$. Then:
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Applying this to a certain cyclic cover of genus 0, we recover the coordinates themselves. $\square$

Theorem 3 ([Raynaud2002], [Pop-Saïdi2003], [T2004])
Assume that X is defined over $\overline{\mathbf{F}_p}$. Then:
(i) $\pi_1(X)$ determines the $\simeq$-class ($= \sim$-class) of X up to finite possibilities, unless $(g, r) = (0, 0), (1, 0)$.
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Proof. (i) is reduced to (ii). For (ii), suppose that there are infinitely many closed points s of $\mathcal{M}_{g,r,\mathbf{F}_p}$ whose corresponding curves admit the same π_1^t. Then, considering the generic point η of an irreducible component of the Zariski closure of these points and resorting to the fact that π_1^t is (topologically) finitely generated, we show that the specialization map

$$\pi_1^t(X_{\overline{\eta}}) \twoheadrightarrow \pi_1^t(X_{\overline{s}})$$

is an isomorphism. So, we have only to show that, in fact, this map is never an isomorphism.

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Then the main ingredients of the proof are:

— Raynaud's theory of Θ

— (a variant of) the Anderson-Indik theorem to show that Θ contains many prime-to-p torsion points over s

— Hrushovski's theorem to show that Θ contains few torsion points over η

— the local Torelli (cf. [Oort-Steenbrink1980])

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In general, we face the annoying possibility that an irreducible component of Θ is a translate of abelian subvariety (for s and for η) or essentially descends to $\overline{\mathbf{F}_p}$ (for η). To treat it, we also need

— to analyze the gonality of cyclic covers, and

— to establish the local Torelli for (generalized) Prym varieties. $\square$

Theorem 4 ([Saïdi2005])
$S \subset \mathcal{M}_{g,[r],k}$: connected subvariety, dim > 0, proper over k
$\implies \pi_1^t$ is non-constant on $|S| \subset |\mathcal{M}_{g,r,k}|$.

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Now we can resort to the (local) Torelli for Prym varieties. $\square$

§4. Recent results.

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Next, we take a model $\mathcal{X} \to \mathcal{S} \to T = \mathrm{Spec}(R)$ of this family, where R is a finitely generated $\mathbf{F}_p$-subalgebra of k. By (the proof of) Theorem 3, we can choose a closed point $t \in T$ and construct a cover of this model for which the intersection of Θ with prime-to-p torsion is different over the generic point of $\mathcal{S}_t$ and over a closed point of $\mathcal{S}_t$.

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Now, by an elementary scheme-theoretic argument, we can conclude that such a difference is also available for the generic fiber of $\mathcal{S} \to T$. $\square$

Theorem 6 (T, in progress)
Assume $p \neq 2$, $(g, r) = (1, 1)$ and that X and its 2-torsion points are defined over $\mathbf{F}_p$. Then:
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This is done by using a certain property typical of elliptic curves (which is unfortunately unavailable for hyperelliptic curves). $\square$

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— and to Professor Oort!!!