The André-Oort conjecture

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A brief history of the

Conjecture.

Emmanuel Ullmo, Université Paris-Saclay / IHES

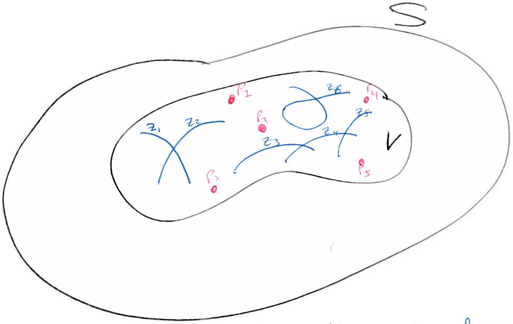
Leiden, November 10, 2015.

Statement of the André-Oort conjecture

The André-Oort Conjecture

Let S be a Shimura variety.
(i) A component of the Zariski closure of a set of special points of S is a special subvariety.
(ii) Let V be a subvariety of S. There exists a finite set $Z_1, \ldots, Z_r$ of special subvarieties of V maximal among special subvarieties of V

A picture of the André-Oort conjecture



Z_i : maximal positive dimensional special subvarieties of V

P_i : maximal 0-dimensional special subvariety of V

History of the formulation of AO

Yves André, interested by questions on periods of Shimura varieties, asked the problem for curves in Shimura varieties (1989).
Frans Oort , interested by questions on Jacobians with complex multiplication, asked the question for subvarieties of the moduli space A_g of principally polarized Abelian varieties of dimension g (1997).
AO is an analogue of the **Manin-Mumford** conjecture for torsion points on Abelian varieties. Both **AO** and **MM** are contained in the **Zilber-Pink** conjecture.

The Manin-Mumford conjecture

Several proofs of **MM** appeared including :

Raynaud (1983) : p-adic methods.
Hindry(1988) : Galois theory+ Intersection theory.
Ullmo-Zhang (Bogomolov conjecture-1998) : Equidistribution of points with small heights.
Hrushowski (2001) : Model Theory.
Pink-Roessler (2002) : reinterpretation in classical Algebraic Geometry.
Pila-Zannier (2008) : o-minimal theory+ functional transcendence.

Main Results on **AO**

Theorem 1: Edixhoven-Hindry's strategy

(i) The André-Oort conjecture holds under the GRH for a general Shimura variety.
(ii) Let Σ be a set of CM points of a Shimura variety S, contained in a single Hecke orbit. A component of the Zariski closure of Σ is special.

Proved by **Klingler-Ullmo-Yafaev** (2006/ published in 2014)+ contributions of **Edixhoven** and **Clozel** (1998-2005).

Theorem 2: Pila-Zannier's strategy

(i) The André-Oort conjecture holds for A_g.
(ii) The André-Oort conjecture holds under the GRH for a general Shimura variety.
(iii) "Good lower bounds" for the size of Galois orbits of CM points implies **AO***.*

Proved by **Pila-Tsimerman**(2008/2015)+ **Klingler-Ullmo-Yafaev** (2011-2014)+**Andreatta-Goren-Howard-Madapusi Pera**(2015)/ **Yuan-Zhang**(2015)+**Daw-Orr**(2015). Extension of these results for mixed Shimura varieties **Gao** (2014). (**Pila**'s lecture)

Definitions and main properties of Shimura varieties

Shimura datum : $(G_{\mathbb{Q}}, X)$ with $G_{\mathbb{Q}}$ reductive, with X a $G(\mathbb{R})$-conjugacy class of morphisms from the Deligne torus $\mathbb{S}$ to $G_{\mathbb{R}}$; $X^+ = G^{ad}(\mathbb{R})/K_\infty$ Hermitian symmetric and $K \subset G(\mathbb{A}_f)$ open compact.

$$\mathrm{Sh}_K(G,X) = G(\mathbb{Q})_+ \backslash \left[X^+ \times G(\mathbb{A}_f)/K \right] = \cup_{\alpha \in G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K} \Gamma_\alpha \backslash X^+.$$

Here $\Gamma_\alpha = G(\mathbb{Q})_+ \cap \alpha K \alpha^{-1}$ is a congruence lattice.
Each component $\Gamma_\alpha \backslash X^+$ is an hermitian locally symmetric space. It's a quasi-projective (Baily-Borel), projective if $G_{\mathbb{Q}}$ is $\mathbb{Q}$-anisotropic, smooth if Γ_α is torsion free and endowed with a canonical probability measure μ_α.

Definitions and main properties of Shimura varieties

$\mathrm{Sh}_K(G, X)$ has a canonical model over a number field $E(G, X)$ (the reflex field) and $S = \Gamma\backslash X^+$ (with $\Gamma = G(\mathbb{Q})_+ \cap K$) is defined over a finite abelian extension of $E(G, X)$. Moreover S is a moduli space for interesting objects (eg. Abelian varieties+extra structures, Hodge classes).

Main example : $A_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$ with

$$\mathbb{H}_g = \mathrm{Sp}_{2g}(\mathbb{R})/U(g) = \{M = M^t \in M_g(\mathbb{C}),\ \mathrm{Im}(M) > 0\}.$$

Main working example : $S = (\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})^g$.

* **Edixhoven** proved that GHR implies **AO** for $g = 2$ (1998),for arbitrary g (2005).

***Pila** proved **AO** for $S = (\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})^g$ (2011).

Special Points

Special points : Let $T_{\mathbb{Q}} \subset G_{\mathbb{Q}}$ such that there exist $x \in X$ factorizing through $T_{\mathbb{R}}$. Let $K_T := K \cap T(\mathbb{A}_f)$. Then $(T, \{x\}) \subset (G, X)$ and

$$\mathrm{Sh}_{K_T}(T, \{x\}) = \cup_{\alpha \in T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K_T}[x, \alpha K] \subset \mathrm{Sh}_K(G, X).$$

is a finite set of special points. When $S = A_g$, special points corresponds to Abelian varieties with complex multiplication.
Bi-algebraic nature of special points : Let $\pi : X^+ \to S = \Gamma \backslash X^+$. There is a realization of X^+ in some algebraic variety V defined over $\overline{\mathbb{Q}}$ such that if $\pi(x) \in S(\overline{\mathbb{Q}})$ is special then $x \in X^+(\overline{\mathbb{Q}})$. If S is of abelian type this condition characterizes special points.

Special Points

Galois action on special points : Let $E = E(T, \{x\})$ be the reflex field. Then $Gal(\overline{\mathbb{Q}}/E)$ acts on $\mathrm{Sh}_{K\cap T(\mathbb{A}_f)}(T, \{x\})$ through an algebraic morphism of tori $r : R_E = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E} \to T$ inducing

$$r : \mathrm{Gal}(\overline{\mathbb{Q}}/E) \to Gal(\overline{\mathbb{Q}}/E)^{ab} \simeq \pi_0(E^*\backslash\mathbb{A}_E^*) \to T(\mathbb{Q})\backslash T(\mathbb{A}_f).$$

Then $\sigma.[x, \alpha K] = [x, r(\sigma)\alpha K]$ for $\sigma \in Gal(\overline{\mathbb{Q}}/E)$.
Main Problem : Lower bounds for $|r(oQ/E)|$ in terms of $|K_T^m/K_T|$ and the discriminant of L_T the splitting field of T. This problem is now solved in general under the GRH , and for A_g combining some results concerning the Colmez conjecture for the Faltings height of CM Abelian varieties (**Andreatta**'s lecture) and the isogeny theorem of Masser-Wüstholz (1993-...) (**Tsimerman**'s lecture).

Special and Weakly Special Subvarieties

Algebraic groups : A special subvariety of $S = \Gamma\backslash X^+$ is a connected component of the image of $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H)$ induced by a Shimura sub-datum (H, X_H) of (G, X) where H is a reductive subgroup of G and X_H the $H(\mathbb{R})$-conjugacy class of some $x \in X$ factorizing through $H_{\mathbb{R}}$.
Moduli interpretation : A special subvariety of S is the locus of "extra symmetries" (endomorphism, level, prescribe Hodge class).
Differential Geometry Assume $G = G^{ad}$. A weakly special subvariety of S is a totally geodesic subvariety of S. **Moonen** (1998), proved that a weakly special variety Z is a special variety or the image of $X_1^+ \times \{x_2\} \subset X_1^+ \times X_2^+$ in S for a sub-Shimura datum $(G_1 \times G_2, X_1 \times X_2)$ of (G, X).
Bi-algebraic description
X^+ has realizations as a subvariety of an algebraic variety $\widehat{X^+}$ (ex bounded realizations, Borel, Siegel, ...). X^+ is real semi-algebraic and complex analytic. A irreducible algebraic subvariety of X^+ is defined as an analytic component of $X^+ \cap V$ for an algebraic subvariety V of $\widehat{X^+}$.
Let $\pi : X^+ \longrightarrow S = \Gamma\backslash X^+$. A subvariety Z of S is weakly special, if and only if the components of $\pi^{-1}(Z)$ are algebraic (**U**-**Yafaev**-2011).

Strategy of Proof of **AO**

Step 1 :

Let V be a subvariety of S. Let (x_n) be a sequence of distinct special points of V. For all n big enough, there exists a positive dimensional special subvariety Z_n of V containing x_n.
(i) Edixhoven-Hindry's strategy : Galois orbits of CM points-Intersection theory-Hecke operators-characterisation of special subvarieties.
(ii)Pila-Zannier's strategy : Galois orbits of CM points, o-minimal theory, Hyperbolic Ax-Lindemann conjecture.

Step 2 :

Prove that the positive dimensional special subvarieties of V are not Zariski dense if V is not special.
(i)Edixhoven-Hindry's strategy : Equidistribution of special subvarieties (ergodic theory, Margulis-Ratner). Galois/Ergodic Alternative.
(ii) Pila-Zannier's strategy : o-minimal theory+ Hyperbolic Ax-Lindemann conjecture.

The hyperbolic Ax-Lindemann conjecture

Theorem 3 (Hyperbolic Ax-Lindemann)

Let $\pi : X^+ \to S = \Gamma\backslash X^+$ be the uniformizing map.
(i) Let Y be an algebraic subvariety of X^+. Then an irreducible component of the Zariski closure of $\pi(X)$ is weakly special.
(ii) Let V be an algebraic subvariety of S. Let Y be a maximal irreducible algebraic subvariety of $\pi^{-1}(V)$. Then $\pi(Y)$ is weakly special.

Pila for $S = (\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})^g$, **U-Yafaev** for S projective, **Pila-Tsimerman** for $S = A_g$, **Klingler-U-Yafaev** in general.
Tool : o-minimal theory : o-minimality of $\mathbb{R}_{an,exp}$ (**Wilkie**(1996), **Van den Dries-Miller** (1994)) + **Pila-Wilkie** counting theorem (2006). **Peterzil-Starchenko** (2013) Hyperbolic geometry. See **Klingler's** lecture.

Step1-Pila-Zannier's strategy

There exist $\alpha > 0$ and $\beta > 0$ with the following properties. Let x be a CM point of $V \subset A_g$. Let $\tilde{x} \in \mathcal{F} \cap \pi^{-1}(x) \subset X^+$. Let $d_x = |disc(Z(EndA_x))|$.
principle of proof :
(i) $|Gal(\overline{\mathbb{Q}}/\mathbb{Q}).x| \gg d_x^{\alpha}$ for some $\alpha > 0$.
(ii) $H(\tilde{x}) \ll d_x^{\beta}$ for some $\beta > 0$ (**Pila-Tsimerman** for A_g, **Orr-Daw** in general).
(iii) **Pila-Wilkie** counting theorem+definability of the restriction of π to $\mathcal{F}$ (**Peterzil-Starchenko**)+(i)+(ii) implies that $\tilde{x} \subset Y \subset \pi^{-1}(V)$ with Y algebraic.
(iv) The Hyperbolic Ax-Lindemann theorem implies that $\pi(Y) \subset V$ is special and contains x.

Step 2 : Equidistribution of special subvarieties

A special subvariety Z is said to be non factor if Z is not of the form $Z = S_1 \times \{x\} \subset S_1 \times S_2$ for a product of special subvarieties $S_1 \times S_2$.

Theorem 4 (equidistribution of special subvarieties)

Let Z_n be a sequence of special subvarieties of S. Let μ_n be the associated sequence of canonical probability measures.
(i) Assume that the Z_n are non factor. Up to a subsequence, there exists a special subvariety Z containing Z_n for all $n >> 0$ such that $(\mu_n)_{n\in\mathbb{N}}$ weakly converges to the canonical probability measure μ_Z.
(ii) Let V be a subvariety of S containing a Zariski dense set of non factor special subvarieties. Then V is special.

Clozel-U (2005-2007).
Tools : Ergodic theory, **Ratner** (1991), **Dani-Margulis**(1991) and **Mozes-Shah** (1995).

Step 2 : By the Pila-Zannier strategy

Theorem 5 (Non density of weakly special subvarieties)

Let V be a Hodge generic subvariety of S. If $S = S_1 \times S_2$ is a product of special subvarieties, assume that V is not of the form $V = S_1 \times V'$ with V' subvariety of S_2. Then the positive dimensional weakly special subvarieties of V are not Zariski dense in V.

Ullmo (2014)

Tools : o-minimal theory+Hyperbolic Ax-Lindemann.

Remark

When S is projective a direct simple proof can be given by ergodic theory. Work in progress with **Daw** *by the ergodic approach for general S.*

Characterization of Special Varieties.

Monodromy Principle : If V is Hodge generic and irreducible, for general Hecke operators $T_g.V$ is irreducible. (Tools : Theorem of **Deligne** and **André**.)

Ergodic/density Principle : For a general Hecke operator T_g and a point $x \in S$ the orbits of $T_g.x$ are dense in S (and even equidistributed in S for the canonical probability measure μ_S.)

Characterisation by Hecke operators : Let V be Hodge generic and irreducible. Assume that $V \subset T_g.V$ for a sufficiently general Hecke operator. Then $V = S$.

Remark

This is central in Edixhoven-Hindry's strategy and used in the proof of the hyperbolic Ax-Lindemann conjecture.

Edixhoven-Hindry's strategy step 1

Theorem 6: (Lower bounds for Galois orbits of special subvarieties)

Assume the GRH for CM fields. There exists $B > 0$, such that for any $N > 0$ there exists $c_N > 0$ such that the following holds. Let (G, X) be a Shimura datum with G semisimple of adjoint type.
Let (H, X_H) be a Shimura subdatum of (G, X) Let T be the connected centre of H . Let Z be a geometric component of $\mathrm{Sh}_{K_H}(H, X_H)$. Let $K_T = T(\mathbb{A}_f) \cap K$, K_T^m be the maximal compact open subgroup of $T(\mathbb{A}_f)$ and L_T be the splitting field of T.

$$\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Z) \geq c_N \prod_{\{p : K^m_{T,p} \neq K_{T,p}\}} \max(1, B|K^m_{T,p}/K_{T,p}|) \cdot (\log(|\mathrm{disc}(L_T)|))^N. \quad (1)$$

The Galois-ergodic Alternative-sketch of the Hindry-Edixhoven's strategy

Let V be a subvariety of S. Assume that V is Hodge generic. Let Σ be an infinite set of maximal special subvarieties of V. We may assume that the dimension of the $Z \in \Sigma$ is fixed.
ergodic argument :If $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Z)$ is bounded when Z varies in Σ then check that you can apply the theorem about equidistribution of non factor special subvarieties to construct a special subvariety Z' of V containing strictly some element of Σ.
If there is a sequence $(Z_n)_{n\in\mathbb{N}}$ with $Z_n \in \Sigma$ and $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Z_n)$ tending to ∞. Using GRH, find a Hecke operator T_{g_n} with $\deg(T_{g_n})$ small compared to $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Z_n)$ and such that $Z_n^\sigma \subset V \cap T_{g_n}.V$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$.
case 1-monodromy argument If V is a component of $V \cap T_{g_n}.V$ apply the monodromy principle to conclude that V is special.
case 2-Galois/geometric argument If not replace V by a component V_2 of $V \cap T_{g_n}.V$ containing Z_n and S by the smallest special subvariety S_2 containing V'. After several steps $\dim(V_k) = \dim(Z_n) + 1$. For some degree reasons using theorem **6** you have to be in case 1

Thank you and congratulations to Frans !