# Arithmetic of K3 surfaces (open problems and conjectures)

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Surface: smooth, projective, geometrically integral scheme of finite type over a field, of dimension $2$.

K3 surface : a surface $X$ with $\dim H^1(X, \mathcal{O}_X) = 0$ and trivial canonical sheaf $\omega_X \cong \mathcal{O}_X$.

**Examples**:

- A smooth quartic surface in $\mathbb{P}^3$.
- Smooth double cover of $\mathbb{P}^2$, ramified over a smooth sextic.
- Kummer surface: minimal nonsingular model of $A/[-1]$, with $A$ an abelian surface over a field of characteristic not equal to $2$.

**Geometry**. K3 surfaces (like abelian surfaces) are between Fano (del Pezzo) surfaces, with $\omega_X^{-1}$ ample, and surfaces of general type, with $\omega_X$ ample.

**Arithmetic**.

**Theorem** (Segre, Manin, Kollár).
Let $X/k$ be a del Pezzo surface with $\omega_X^{-1}$ very ample.
Then $X$ is unirational if and only if $X(k) \neq \emptyset$.

**Conjecture** (Colliot-Thélène).
Let $X$ be a del Pezzo surface over a global field $k$.
If $X(k) \neq \emptyset$, then $X(k)$ is Zariski dense in $X$.

**Conjecture** (Bombieri–Lang).
Let $X$ be a surface of general type over a finitely generated field $k$.
Then $X(k)$ is not Zariski dense. If $\operatorname{char} k = 0$, then $X_{\overline{k}}$ contains only finitely many curves of genus at most $1$, and $X$ contains only finitely many $k$-rational points outside those curves.

Let $X$ be a K3 surface over a number field $k$.

1. Is $X(k_v) \neq \emptyset$ for every completion $k_v$ of $k$? Yes
2. Is $X(k) \neq \emptyset$? Yes
3. Is $\#X(k) = \infty$ ?
4. Is $X(k)$ Zariski dense in $X$? No
5. Is $X(k)$ dense in $X(k_\infty)$?
6. How does the number of rational points of height bounded by $B$ grow as $B \to \infty$?
7. Is $X(k)$ dense in $X(\mathbb{A}_k)$, with $\mathbb{A}_k$ the adeles of $k$? Yes

Yes / No: Failure of the Hasse principle

Yes / No: Possible? (Open problem 1)

$\downarrow$

Weak Approximation holds

Possible?

(Open problem 2)

**Example**. Let $X \subset \mathbb{P}^3$ be given by

$$x^4 + 2y^4 = z^4 + 4w^4.$$

**Question** (Swinnerton-Dyer, 2002).
Does $X$ have more than two rational points?

**Answer** (Elsenhans–Jahnel, 2004).

$$1484801^4 + 2 \cdot 1203120^4 \ = \ 1169407^4 + 4 \cdot 1157520^4.$$

**Open problem 3.**
Does $X$ have more than ten rational points?

**Theorem** (Noam Elkies, 1988).

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

The set of rational points on the surface

$$\mathbb{P}^3 \supset X : x^4 + y^4 + z^4 = t^4.$$

is Zariski dense.

**Theorem** (Logan, McKinnon, vL, 2010).
Take $a, b, c, d \in \mathbb{Q}^*$ with $abcd \in (\mathbb{Q}^*)^2$. Let $X \subset \mathbb{P}^3$ be given by

$$ax^4 + by^4 + cz^4 + dw^4.$$

If $P \in X(\mathbb{Q})$ has no zero coordinates and $P$ does not lie on one of the $48$ lines (no two terms sum to $0$), then $X(\mathbb{Q})$ is Zariski dense.

**Open problem 4.** Can the conditions on $P$ be left out?

**Conjecture** (vL). Every $t \in \mathbb{Q}$ can be written as

$$t = \frac{x^4 - y^4}{z^4 - w^4}.$$

**Definition.** Let $X$ be any variety over any field $k$. Then rational points are potentially dense on $X$ if there exists a finite field extension $\ell$ of $k$ such that $X(\ell)$ is Zariski dense in $X_\ell$.

**Conjecture** (Campana, 2004). Let $X$ be a K3 surface over a number field $k$. Then rational points are potentially dense on $X$.

Let $X$ be a K3 surface over $\mathbb{C}$.

**Facts.**

Hodge diagram:
$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\boxed{U^3 \oplus E_8(-1)^2} =: \Lambda$

even, unimodular

The exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X(\mathbb{C})} \to \mathcal{O}^*_{X(\mathbb{C})} \to 1$$

of sheaves on $X(\mathbb{C})$ (together with Serre's GAGA) yields

$$\begin{tikzcd}
\mathrm{H}^1(X, \mathcal{O}_X) \arrow[r] \arrow[d, phantom, "\wr\|"] & \mathrm{H}^1(X, \mathcal{O}_X^*) \arrow[r] \arrow[d, phantom, "\wr\|"] & \boxed{\mathrm{H}^2(X(\mathbb{C}), \mathbb{Z})} \arrow[r] \arrow[dr, hook] & \mathrm{H}^2(X, \mathcal{O}_X) \\
0 & \mathrm{Pic}\, X \arrow[rr, hook] \arrow[dr, hook] & & \mathrm{H}^2(X(\mathbb{C}), \mathbb{C}) \\
& & \mathrm{H}^{1,1}(X(\mathbb{C})) \cap \mathrm{H}^2(X(\mathbb{C}), \mathbb{Z}) \arrow[ur, hook] &
\end{tikzcd}$$

**Definition.** A polarised K3 surface is a K3 surface $X$ together with a primitive ample line bundle $H$. Its degree is $H^2 = 2d$. The Picard number of $X$ is $\rho(X) = \mathrm{rk}\,\mathrm{Pic}\,X \in \{1, \ldots, 20\}$.

**Facts over $\mathbb{C}$.** For each $d \geq 1$, there is a coarse moduli space $M_d$ of polarised complex K3 surfaces of degree $2d$. It is irreducible, quasi-projective, and $\dim M_d = 19$.

There is a countable union of divisors in $M_d$, such that for every polarised K3 surface $(X, H)$ in the complement we have $\rho(X) = 1$.

**Theorem** (Bogomolov, Tschinkel, 2000). There is a set $\mathcal{S}$ of eight lattices of rank $3$ or $4$, such that rational points are potentially dense on every K3 surface $X$ over a number field satisfying

(a) $\rho(\overline{X}) = 2$ and $\overline{X}$ does not contain a $(-2)$-curve, or

(b) $\rho(\overline{X}) \geq 3$ and $\operatorname{Pic} X$ not isomorphic to a lattices in $\mathcal{S}$.

**Proof (sketch)**. Such surfaces have an infinite automorphism group or an elliptic fibration. We find a rational curve and move it around using either one. □

**Open problem 5a.** Is there a K3 surface $X$ over a number field with $\rho(\overline{X}) = 1$ on which rational points are potentially dense?

**Open problem 5b.** Is there a K3 surface $X$ over a number field $k$ with $\rho(X) = 1$ for which $X(k)$ is Zariski dense?

**Open problem 2.** Is there a K3 surface $X$ over a number field $k$ with $X(k)$ neither empty nor Zariski dense?

**Question**.
Is there a K3 surface $X$ over a number field with $\rho(\overline{X}) = 1$?

**Ineffective answers**.
Terasoma (1985): Yes, for degrees $4$, $6$, and $8$ over $\mathbb{Q}$.
Ellenberg (2004): Yes, for any degree $2d$ over some number field.

**Theorem** (vL,2004) The K3 surface $X$ in $\mathbb{P}^3(x, y, z, w)$ given by

$$wf = 3pq - 2zg$$

with $f \in \mathbb{Z}[x, y, z, w]$ and $g, p, q \in \mathbb{Z}[x, y, z]$ equal to

$$g = xy^2 + xyz - xz^2 - yz^2 + z^3, \quad f = x^3 - x^2y - x^2z + x^2w - xy^2 - xyz +$$
$$p = z^2 + xy + yz, \qquad 2xyw + xz^2 + 2xzw + y^3 + y^2z - y^2w +$$
$$q = z^2 + xy, \qquad yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3,$$

has geometric Picard number $\rho(\overline{X}) = 1$ and infinitely many rational points.

**Proof**. Take $\mathfrak{p} \in \{2, 3\}$, and write $k_\mathfrak{p}$ for the residue field of $\mathbb{Z}_\mathfrak{p}$.
The equation $wf = 3pq - 2zg$ defines a scheme $\mathfrak{X}_\mathfrak{p}$ in $\mathbb{P}^3$ over $\mathbb{Z}_\mathfrak{p}$.
The morphism $\mathfrak{X}_\mathfrak{p} \to \operatorname{Spec} \mathbb{Z}_\mathfrak{p}$ is proper and smooth.
Write $X_\mathfrak{p} = \mathfrak{X}_\mathfrak{p} \times_{\mathbb{Z}_\mathfrak{p}} k_\mathfrak{p}$ for the reduction. By properness, we obtain

$$\operatorname{Pic} X \xleftarrow{\cong} \operatorname{Pic} \mathfrak{X}_\mathfrak{p} \to \operatorname{Pic} X_\mathfrak{p}.$$

The composition $\operatorname{Pic} X \to \operatorname{Pic} X_\mathfrak{p}$ respects intersection numbers, so it is injective (numerical and linear equivalence agree on K3's).

The direct limit of the analog over all finite extensions of $\mathbb{Q}$ yields

$$\operatorname{Pic} \overline{X} \hookrightarrow \operatorname{Pic} \overline{X}_\mathfrak{p}$$

with $\overline{X} = X_{\overline{\mathbb{Q}}}$ and $\overline{X}_\mathfrak{p} = \mathfrak{X}_{\overline{k_\mathfrak{p}}}$.

For a prime $\ell \neq \mathfrak{p}$ and $n > 0$ an integer, the Kummer sequence

$$1 \to \mu_{\ell^n} \to \mathbb{G}_{\mathrm{m}} \xrightarrow{\ell^n} \mathbb{G}_{\mathrm{m}} \to 1$$

is exact on the étale site of $\overline{X}_{\mathfrak{p}}$ and yields

$$\operatorname{Pic} \overline{X}_{\mathfrak{p}} \xrightarrow{\ell^n} \operatorname{Pic} \overline{X}_{\mathfrak{p}} \to \mathrm{H}^2_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mu_{\ell^n}),$$

so an injection

$$\operatorname{Pic} \overline{X}_{\mathfrak{p}} / \ell^n \operatorname{Pic} \overline{X}_{\mathfrak{p}} \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mu_{\ell^n}).$$

Because $\operatorname{Pic} \overline{X}_{\mathfrak{p}}$ is finitely generated and free,
the inverse limit gives a Galois invariant injection

$$\operatorname{Pic} \overline{X}_{\mathfrak{p}} \hookrightarrow \varprojlim_{n} \mathrm{H}^2_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mu_{\ell^n}) =: \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1)).$$

$$\operatorname{Pic} \overline{X} \hookrightarrow \operatorname{Pic} \overline{X}_{\mathfrak{p}} \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$$

So $\rho(\overline{X})$ is bounded from above by the number of eigenvalues $\lambda$ of Frobenius acting on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$ for which $\lambda$ is a root of unity.

The Lefschetz formula

$$\# X(\mathbb{F}_{\mathfrak{p}^n}) = \sum_{i=0}^{4} (-1)^i \operatorname{Tr}\left(\text{Frob}^n \text{ on } \mathrm{H}^i_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell)\right)$$

yields traces of powers of Frobenius on $\mathrm{H}^i_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell)$ (without twist).

Expressing the elementary symmetric polynomials in the eigenvalues in terms of the power sums (the traces), gives the characteristic polynomial of Frobenius acting on $\mathrm{H}^i_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell)$.

Scaling its roots by $\mathfrak{p}$ gives the eigenvalues of Frobenius acting on $\mathrm{H}^i_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mathbb{Z}_\ell(1))$.

The nonreal eigenvalues of Frobenius on $\mathrm{H}^i_{\text{ét}}(\overline{X}_\mathfrak{p}, \mathbb{Z}_\ell(1))$ come in conjugate pairs, so an even number of those is not a root of unity.

The second Betti number $b_2 = 22$ is even, so this leaves an even number of eigenvalues that **are** roots of unity.

For $\mathfrak{p} \in \{2, 3\}$, we find $\rho(\overline{X}_\mathfrak{p}) = 2$ . If $\rho(\overline{X}) = 2$, then $\operatorname{Pic} \overline{X} \subset \operatorname{Pic} \overline{X}_\mathfrak{p}$ has finite index, so in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ we have

$$\operatorname{disc} \operatorname{Pic} \overline{X}_2 = \operatorname{disc} \operatorname{Pic} \overline{X} = \operatorname{disc} \operatorname{Pic} \overline{X}_3.$$

The reduction of $wf = 3pq - 2zg$ modulo $2$ is $wf = pq$, so $X_2$ contains the conics $C_1, C_2$ given by $w = p = 0$ and $w = q = 0$. The sublattice $\langle C_1, C_2 \rangle \subset \operatorname{Pic} \overline{X}_2$ has finite index and discriminant $-12$, so $\operatorname{disc} \operatorname{Pic} \overline{X}_2 = -12 \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$.

The reduction modulo $3$ is $wf = zg$, so $X_3$ contains the line $L$ given by $w = z = 0$. The sublattice $\langle L, H \rangle \subset \operatorname{Pic} \overline{X}_3$ has discriminant $-9$, so $\operatorname{disc} \operatorname{Pic} \overline{X}_3 = -9 \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$.

Contradiction, so $\rho(\overline{X}) = 1$. $\square$

**Remarks** for $X$ a K3 surface over a number field, $\mathfrak{p}$ a prime of good reduction, and $X_\mathfrak{p}$ the reduction.

1. This method works as soon as $\rho = \rho(\overline{X})$ is odd and there is a pair $S$ of two primes $\mathfrak{p}$ with $\rho(\overline{X}_\mathfrak{p}) = \rho + 1$, and the discriminants of $\operatorname{Pic} \overline{X}_\mathfrak{p}$ for $\mathfrak{p} \in S$ are different in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$.
2. (Kloosterman, 2005) The Artin-Tate formula (known in odd characteristic, by Nijgaard, Ogus, Maulik, Madapusi Pera, Charles) allows us to compute the discriminants up to squares without knowing explicit generators of a finite-index subgroup of $\operatorname{Pic} \overline{X}_\mathfrak{p}$.
3. (Elsenhans–Jahnel) Various tricks make the method more powerful. Very useful result is that, under mild conditions, the reduction map $\operatorname{Pic} \overline{X} \hookrightarrow \operatorname{Pic} \overline{X}_\mathfrak{p}$ has torsion-free cokernel.

**Question**.
Does there always exist a prime $\mathfrak{p}$ with $\rho(\overline{X}_\mathfrak{p}) \leq \rho(\overline{X}) + 1$?

**Answer**. No!

Let $X$ be a K3 surface over a number field $k \subset \mathbb{C}$. Let $T$ be the orthogonal complement of $\operatorname{Pic} X_{\mathbb{C}}$ in $\mathrm{H}^2(X(\mathbb{C}), \mathbb{Q})$. The algebra $E = \operatorname{End}_H(T)$ of endomorphisms respecting the Hodge structure is either a totally real field or a CM field (Zarhin, 1983).

**Theorem** (Charles, 2011).

1. If $E$ is a CM field or $\dim_E(T)$ is even, then there are infinitely many primes $\mathfrak{p}$ of good reduction with $\rho(\overline{X}_{\mathfrak{p}}) = \rho(\overline{X})$.
2. Otherwise, for any odd prime $\mathfrak{p}$ of good reduction, we have $\rho(\overline{X}_{\mathfrak{p}}) \geq \rho(\overline{X}) + [E : \mathbb{Q}]$; equality holds for infinitely many $\mathfrak{p}$.

**Corollary** (Charles, 2011). There is an algorithm (i.e., a Turing machine) that, given a projective K3 surface $X$ over a number field, either returns $\rho(\overline{X})$ or does not terminate. If $X \times X$ satisfies the Hodge conjecture for codimension-2 cycles, then the algorithm terminates on $X$.

There is also an algorithm that terminates unconditionally.

**Theorem** (Poonen, Testa, vL, 2012).
There is an algorithm that, given a K3 surface over a finitely generated field $k$ of characteristic not $2$, computes $\operatorname{Pic} \overline{X}$.

**Proof sketch**.
We can compute the $\operatorname{Gal}(\overline{k}/k)$-module $\mathrm{H}^i_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell^n\mathbb{Z})$ for any $\ell \neq \operatorname{char} k$, and any $i, n \geq 0$ (Madore–Orgogozo, 2013).

Use this to approximate $\mathrm{H}^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))^{\operatorname{Gal}(\overline{k}/k)}$, which by Tate's conjecture equals $\rho(\overline{X})$. This yields an upper bound for $\rho(\overline{X})$.

To find a lower bound for $\rho(\overline{X})$, we simply search for divisors (for example, by enumeration).

In order to compute not only the rank, but also the group $\operatorname{Pic} \overline{X}$ itself, we use Hilbert schemes to compute the saturation of an already known subgroup.

# Batyrev–Manin conjectures

Let $X$ be a variety over a number field $k$ and $h\colon X(k) \to \mathbb{R}$ a height function associated to an ample line bundle (not logarithmic). For any bound $B \in \mathbb{R}$ and any open $U \subset X$ we set

$$N_{U,h}(B) = \#\{P \in U(k) \ : \ h(P) \leq B\}.$$

**Conjecture** (Batyrev–Manin, 1990).
Suppose $X$ is a Fano variety over a number field $k$, and $h$ the height associated to an ample line bundle $\mathcal{L}$ with $\mathcal{L}^{\otimes a} \cong \omega_X^{-1}$ for some $a > 0$. Set $b = \mathrm{rk}\,\mathrm{Pic}\,X$. Then there is an open subset $U \subset X$ and a constant $c$ with

$$N_{U,h}(B) \sim cB^a(\log B)^{b-1}.$$

This conjecture is proved in many cases for surfaces.
False in higher dimension, but no counterexamples to lower bound.

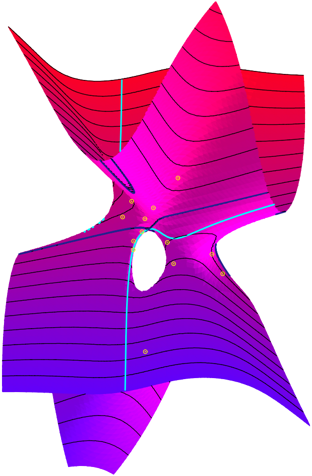
**Question**. What about K3 surfaces? Just take $a = 0$ ?

**Conjecture** (Batyrev–Manin, 1990).
Suppose $X$ is a K3 surface over a number field $k$, and $h$ the height associated with an ample line bundle. Then for every $\varepsilon > 0$, there is an open subset $U \subset X$ such that

$$N_{U,h}(B) \sim \mathcal{O}(B^{\varepsilon}).$$

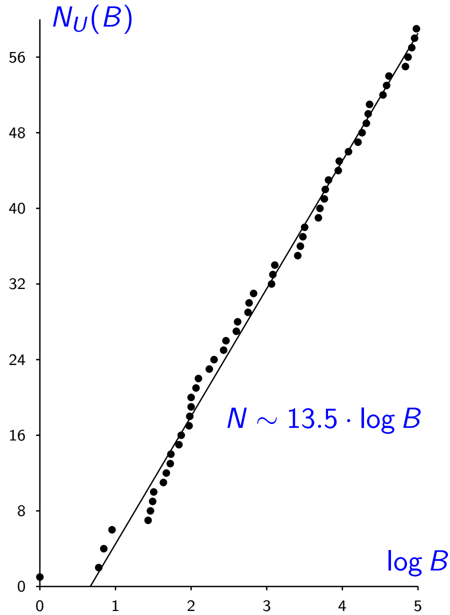
**Remark**. A rational curve of degree $d$ gives contribution $B^{2/d}$, so we need to leave out those with $d < 2\varepsilon^{-1}$.

**Question.** What about lower bounds for K3 surfaces?



$$S\colon x^3 - 3x^2y^2 + 4x^2yz - x^2z^2 + x^2z - xy^2z - xyz^2 + x + y^3 + y^2z^2 + z^3 = 0$$

$\rho(\overline{S}) = 1$



# Suggestion by Swinnerton-Dyer

Define the height-zeta function

$$Z(U,s) = \sum_{P \in U(k)} h(P)^{-s}.$$

From

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^s \, \frac{ds}{s} = \begin{cases} 1 & \text{if } x > 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 0 & \text{if } x < 1 \end{cases} \qquad (c > 0)$$

we get

$$\begin{aligned} N_U(x) &= \sum_{P \in U(K)} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(xh(P)^{-1}\right)^s \frac{ds}{s} \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z(U,s)\, x^s \, \frac{ds}{s} \qquad (c >> 0). \end{aligned}$$

Assuming $Z(U,s)$ is nice, including analytic on $\Re(s) > a - 2\epsilon$, except for a pole of order $b$ at $a$, we can write

$$
\begin{aligned}
N_U(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z(U,s)\,x^s\,\frac{ds}{s} \qquad (c >> 0)\\
&= \operatorname{res}_{s=a}\left[Z(U,s)s^{-1}\exp(s\log x)\right] + \frac{1}{2\pi i}\int_{a-\epsilon-i\infty}^{a-\epsilon+i\infty} Z(U,s)\,x^s\,\frac{ds}{s}.
\end{aligned}
$$

The integral is smaller than the residue, the main term, which is

$$x^a p(\log x)$$

for some polynomial $p$ of degree $\begin{cases} b-1 & \text{if } a \neq 0,\\ b & \text{if } a = 0.\end{cases}$

**Question**. For $X$ a K3 surface: $N(U,B) \sim c(\log B)^{\mathrm{rk}\,\mathrm{Pic}\,X}$ ?

**Question**. For $X$ a K3 surface: $N(U, B) \sim c(\log B)^{\mathrm{rk}\,\mathrm{Pic}\,X}$ ?

Could go wrong if

1. $X$ admits an elliptic fibration (in particular, if $\mathrm{rk}\,\mathrm{Pic}\,X \geq 5$);
2. $\#\,\mathrm{Aut}(X) = \infty$.

In these cases, we may get even more rational points.

**Conjecture** (vL, based on an idea by Swinnerton-Dyer).
Suppose $X$ is a K3 surface over a number field $k$ with $\rho(X) = 1$. There is an open subset $U \subset X$ and a constant $c$ such that

$$N_U(B) \sim c \log B$$

as $B \to \infty$. Moreover, if $X(k) \neq \emptyset$, then $c \neq 0$.

**Conjecture**.
Suppose $X$ is a K3 surface over a number field $k$ with $\rho(X) = b$. There is an open subset $U \subset X$ and a constant $c$ such that

$$N_U(B) \geq c(\log B)^b.$$

for $B >> 0$. Moreover, if $X(k) \neq \emptyset$, then $c \neq 0$.

# Brauer-Manin obstruction

For a variety $X$ we define the Brauer group $\mathrm{Br}\, X = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_\mathrm{m})$.
Every morphism $X \to Y$ induces a homomorphism $\mathrm{Br}\, Y \to \mathrm{Br}\, X$.
For every point $P$ over a field $k$ we have $\mathrm{Br}(P) = \mathrm{Br}(k)$.

Let $X$ be a smooth and projective variety over a number field $k$.
Let $\Omega$ be the set of all places of $k$. Then $X(\mathbb{A}_k) = \prod_{v \in \Omega} X(k_v)$.

$$\begin{tikzcd}
X(k) \arrow[rr] \arrow[d] & & X(\mathbb{A}_k) \arrow[d] \arrow[dr, "\phi"] & \\
\mathrm{Hom}(\mathrm{Br}\, X, \mathrm{Br}(k)) \arrow[rr] & & \mathrm{Hom}(\mathrm{Br}\, X, \bigoplus_v \mathrm{Br}(k_v)) \arrow[r] & \mathrm{Hom}(\mathrm{Br}\, X, \mathbb{Q}/\mathbb{Z})
\end{tikzcd}$$

**Corollary.** If $X(\mathbb{A}_k)^{\mathrm{Br}} := \phi^{-1}(0)$ is empty, then $X(k) = \emptyset$.
If $X(\mathbb{A}_k)^{\mathrm{Br}} \neq X(\mathbb{A}_k)$, then obstruction to weak approximation.

Conjecture (Colliot-Thélène).
This Brauer-Manin obstruction is the only obstruction to the Hasse principle and weak approximation for rationally connected varieties.

**Notation**.
$\mathrm{Br}_0(X) = \mathrm{im}(\mathrm{Br}\, k \to \mathrm{Br}\, X)$
$\mathrm{Br}_1(X) = \ker(\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X})$

Hochschild–Serre:

$$0 \to \mathrm{Pic}\, X \to (\mathrm{Pic}\, \overline{X})^{G_k} \to \mathrm{Br}\, k \to \mathrm{Br}_1(X) \to \mathrm{H}^1(k, \mathrm{Pic}\, \overline{X}) \to \mathrm{H}^3(k, \mathbb{G}_\mathrm{m})$$

For a number field $k$, we have $\mathrm{H}^3(k, \mathbb{G}_\mathrm{m}) = 0$, so

$$\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \cong \mathrm{H}^1(k, \mathrm{Pic}\, \overline{X}),$$

the algebraic part of the Brauer group.

**Theorem** (Skorobogatov–Zarhin, 2008). If $X$ is a K3 surface over a number field $k$, then $\mathrm{Br}\, X / \mathrm{Br}_0\, X$ is finite.

**Theorem** (Hassett–Várilly-Alvarado, 2012). There is a K3 surface $X$ of degree $2$ over $\mathbb{Q}$ with $\rho(\overline{X}) = 1$ and a Brauer–Manin obstruction to the Hasse principle.

**Open Problem 6.** Is the Brauer–Manin obstruction the only obstruction to the Hasse principle and weak approximation for K3 surfaces over number fields?

**Open Problem 7.** Does the odd part of the Brauer–Manin group ever obstruct the Hasse principle?