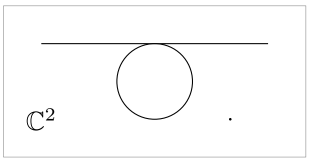
Intro. to algebraic curves — exercise sheet 1

Deadline: 14.30 Thursday 24 September 2015

These exercises are to be handed in to Johan Commelin (`j.commelin@math.ru.nl`), either in his pigeon hole (Huygens building, opposite to room HG03.708), or electronically. Handing in by email is possible only if you write your solutions using TEX or LATEX; in that case, send the pdf output. You are allowed to collaborate with other students but what you write and hand in should be your own work. If different students hand in the same work, we will not accept their work.

1. Let $\mathbb{H}$ denote the upper half plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Show that every holomorphic map $\mathbb{C} \to \mathbb{H}$ is constant.
2. Let X be a complex manifold. Let $\mathcal{O}_X(X)$ be the ring of holomorphic functions on X. For a subset $F \subset \mathcal{O}_X(X)$ of holomorphic functions on X, write $Z(F)$ for the vanishing locus of F, *i.e.*, $Z(F) = \{x \in X \mid f(x) = 0 \text{ for all } f \in F\}$. For a subset $S \subset X$, write $I(S)$ for the set of holomorphic functions vanishing on S, *i.e.*, $I(S) = \{f \in \mathcal{O}_X(X) \mid f(s) = 0 \text{ for all } s \in S\}$.
   (a) Show that $I(S)$ is an ideal of $\mathcal{O}_X(X)$.
   (b) Prove that $Z(\_)$ and $I(\_)$ are inclusion-reversing.
   (c) For two subsets $S_1, S_2 \subset X$, prove the relation
   $$I(S_1 \cup S_2) = I(S_1) \cap I(S_2).$$
   (d) Let X be $\mathbb{C}^2$. Let S_1 be the subset defined by $y = 1$, Let S_2 be the unit circle. Let S_3 be the singleton $(2, -1)$ Let S be the union $S_1 \cup S_2 \cup S_3$. Find an explicit, finite set $F \subset \mathcal{O}_X(X)$, such that $Z(F) = S$.



3. Let $C \subset \mathbb{C}^3$ be the subset $\{(t^3, t^4, t^5) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$.
   (a) Find three holomorphic functions $f, g, h \colon \mathbb{C}^3 \to \mathbb{C}$, such that $C = Z(f, g, h)$.
   (b) Find the set S of singular points of C.
   (c) For every $x \in C - S$, find an open subset $U \subset \mathbb{C}^3$, with $x \in U$, and holomorphic functions $f, g \colon U \to \mathbb{C}$, such that $C \cap U = Z(f, g) \cap U$.
   (d) For $s \in S$, compute the dimension of the tangent space of C at s.
4. Let $f \colon \mathbb{C} \to \mathbb{C}^2$ be defined by $t \mapsto (t^2, t^3)$.
   (a) Show that f defines a bijective bicontinuous morphism of $\mathbb{C}$ onto the curve $y^2 = x^3$.
   (b) Show that f is not an immersion.