Intro. to algebraic curves — exercise sheet 3

Deadline: 14.30 Thursday 22 October 2015

These exercises are to be handed in to Johan Commelin (`j.commelin@math.ru.nl`), either in his pigeon hole (Huygens building, opposite to room HG03.708), or electronically. Handing in by email is possible only if you write your solutions using TeX or LaTeX; in that case, send the pdf output. You are allowed to collaborate with other students but what you write and hand in should be your own work. If different students hand in the same work, we will not accept their work.

Let X be a compact Riemann surface. Recall that $\mathbb{C}(X)$ is the field of meromorphic functions on X. A *divisor* on X is a finite formal sum of points of X with coefficients in $\mathbb{Z}$. (In other words, $\sum_{P\in X} n_P \cdot P$, where $n_P \in \mathbb{Z}$, and $n_P \neq 0$ for finitely many $P \in X$.) The divisors on X form a group $\mathrm{Div}(X)$. There is a natural homomorphism to $\mathbb{Z}$, called the degree map:

$$\deg\colon \mathrm{Div}(X) \to \mathbb{Z}, \qquad \sum_{P\in X} n_P \cdot P \mapsto \sum_{P\in X} n_P.$$

There is also a natural homomorphism

$$\mathrm{div}\colon \mathbb{C}(X)^* \to \mathrm{Div}(X), \qquad f \mapsto \mathrm{div}(f) = \sum_{P\in X} \mathrm{ord}_P(f)\cdot P.$$

Elements in the image of div are called *principal divisors*. The cokernel of div (so: divisors modulo principal divisors) is called the *divisor class group* of X; notation: $\mathrm{Cl}(X)$.

1. Let $f\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Assume that f has degree 1. Give a precise proof that f is an isomorphism (*i.e.,* biholomorphic map).
   NB: Some of you proved this last week. You will have to prove it again. Hint: first show that f is an immersion.
2. Let $f \in \mathbb{C}(X)$ be a non-constant meromorphic function on a compact Riemann surface X. Assume f has exactly 1 pole; and assume this pole has order 1. Show that X is isomorphic to $\mathbb{P}^1$.
3. Show that the degree map deg factors via the divisor class group $\mathrm{Cl}(X)$. We will write $\mathrm{Cl}^n(X)$ for the divisor classes in $\mathrm{Cl}(X)$ of degree n.
4. Show that the map $X \to \mathrm{Cl}^1(X)$ given by $P \mapsto [P]$ is injective if and only if $X \ncong \mathbb{P}^1$.

Let $C \subset \mathbb{P}^2$ be a closed plane curve. Let $C^{\mathrm{reg}} \subset C$ be the smooth (*i.e.,* regular) locus of C. Let $\sigma_1\colon \tilde{C}_1 \to C$ and $\sigma_2\colon \tilde{C}_2 \to C$ be two normalisations of C. The goal of the following exercises is to show that $\tilde{C}_1$ and $\tilde{C}_2$ are isomorphic.

For $i = 1, 2$, let $\tilde{C}_i^\circ \subset \tilde{C}_i$ be the inverse image of C^{reg} under σ_i. By definition of a normalisation, $\sigma_i \colon \tilde{C}_i^\circ \to C^{\text{reg}}$ is an isomorphism, so we have a well-defined isomorphism $\phi \colon \tilde{C}_1^\circ \to \tilde{C}_2^\circ$ given by $\phi = \sigma_2^{-1} \circ \sigma_1$.

5. Let $P \in C$ be a singular point. Show that there are finitely many points $Q_1, \ldots, Q_n$ in the inverse image $\sigma_2^{-1}(P)$.

In general, if Y is a Riemann surface then by a disk in Y we mean an open subset $U \subset Y$ that is isomorphic, as a complex manifold, to the open unit disk $\Delta \subset \mathbb{C}$.

6. Let $U_i \subset \tilde{C}_2$ $(i = 1, \ldots, n)$ be disjoint open disks around Q_i. Let $V \subset C$ be an open neighbourhood of P, such that $\sigma_2^{-1}(V) \subset \bigcup_i U_i$. Let $\tilde{P} \in \tilde{C}_1$ be a point mapping to P. Let $W \subset \tilde{C}_1$ be an open disk around $\tilde{P}$, such that $W \subset \sigma_1^{-1}(V)$, and such that $W \cap \sigma_1^{-1}(V) = \{\tilde{P}\}$. Let W^* be $W - \{\tilde{P}\}$. Show that there is a unique $i \in \{1, \ldots, n\}$, with $\phi(W^*) \subset U_i$.
7. Using the Riemann extension theorem, prove that $\phi \colon \tilde{C}_1^\circ \to \tilde{C}_2$ extends to a holomorphic map $\tilde{C}_1 \to \tilde{C}_2$.
8. Prove that $\tilde{C}_1$ and $\tilde{C}_2$ are isomorphic.