Intro. to algebraic curves — exercise sheet 5

Deadline: 14.30 Thursday 8 December 2015

These exercises are to be handed in to Johan Commelin (`j.commelin@math.ru.nl`), either in his pigeon hole (Huygens building, opposite to room HG03.708), or electronically. Handing in by email is possible only if you write your solutions using TEX or LATEX; in that case, send the pdf output. You are allowed to collaborate with other students but what you write and hand in should be your own work. If different students hand in the same work, we will not accept their work.

1. Let C be a compact Riemann surface of genus 1. Let $\mathcal{O} \in C$ be a point.
   (a) Compute $\ell(n \cdot \mathcal{O})$ for $1 \leq n \leq 6$. Let $(1, x)$ be a basis of $\mathcal{L}(2 \cdot \mathcal{O})$, and let $(1, x, y)$ be a basis of $\mathcal{L}(3 \cdot \mathcal{O})$. Deduce that the meromorphic functions x and y satisfy an equation of the form
   $$y^2 + a_1xy + a_3y = a_0x^3 + a_2x^2 + a_4x + a_6$$
   with $a_i \in \mathbb{C}$. Show that x and y can be chosen in such a way that $a_0 = 1$ and $a_1 = a_3 = a_2 = 0$. We thus get an equation
   $$y^2 = x^3 + bx + c$$
   for some $b, c \in \mathbb{C}$.
   (b) Show that the linear system $|3 \cdot \mathcal{O}|$ is base-point free, and conclude that C is an algebraic curve (*i.e.,* the zero-locus of homogeneous polynomials in $\mathbb{P}^n$).
2. Let $C_0 \subset \mathbb{C}^2$ be the curve given by the equation $u^2 = t^4 + 1$. Let $C \subset \mathbb{P}^2$ be its projective closure, and $\tilde{C}$ the normalisation of C. Observe that $g(\tilde{C}) = 1$.
   Find a Weierstrass equation for $\tilde{C}$. (*Hint:* using the previous exercise might not be the fastest route.)
3. Let $C_0 \subset \mathbb{C}^2$ be the curve given by the equation $y^5 = x(x-1)(x-2)$. Let $C \subset \mathbb{P}^2$ be its projective closure, and $\tilde{C}$ the normalisation of C. Let $f\colon \tilde{C} \to \mathbb{P}^1$ be the morphism that on C_0 is given by $(x, y) \mapsto (x : 1)$.
   (a) Describe precisely which are the ramification points of f, and for each such point give the ramification index.
   (b) Compute $g(\tilde{C})$.
   (c) Determine the divisors of x, y and $\mathrm{d}x$.
   (d) Show that $\mathrm{d}x/y^2$ is a holomorphic 1-form on $\tilde{C}$.
   (e) Write down an explicit basis for the $\mathbb{C}$-vector space of holomorphic 1-forms on $\tilde{C}$.