NOTES ON MUMFORD-TATE GROUPS

(preliminary and incomplete version)

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by

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§1. Hodge structures and their Mumford-Tate groups.

(1.1) *Representations of algebraic tori.* Let T be a torus over a field k. Choose a separable closure k^s. The character group $X^*(T)$ and the cocharacter group $X_*(T)$ are defined by

$$X^*(T) := \mathrm{Hom}(T_{k^s}, \mathbb{G}_{m,k^s})\,, \qquad X_*(T) := \mathrm{Hom}(\mathbb{G}_{m,k^s}, T_{k^s})\,.$$

If r is the rank of T then $X^*(T)$ and $X_*(T)$ are free abelian groups of rank r which come equipped with a continuous action of $\mathrm{Gal}(k^s/k)$. There is a natural perfect pairing $X^*(T) \times X_*(T) \longrightarrow \mathrm{End}(\mathbb{G}_{m,k^s}) = \mathbb{Z}$.

The functor

$$X_*(\quad)\colon \begin{pmatrix}\text{algebraic tori}\\ \text{over } k\end{pmatrix} \xrightarrow{\text{eq}} \begin{pmatrix}\text{free abelian group of finite rank}\\ +\text{ continuous action of } \mathrm{Gal}(k^s/k)\end{pmatrix}$$

is an equivalence of categories. Similarly, the functor $X^*(\;)$ gives an anti-equivalence of categories.

Let now $\rho\colon T \to \mathrm{GL}(V)$ be a representation of T on a finite dimensional k-vector space. If $k = k^s$, so that $T \cong \mathbb{G}_m^r$ then the situation is clear: the space V decomposes as a direct sum of character spaces and this completely determines the representation. Thus, a representation of $\mathbb{G}_m^r$ on a vector space V corresponds to a $\mathbb{Z}^r$-grading

$$V = \bigoplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^r} V^{n_1,\ldots,n_r}\,.$$

Sign convention: we write $V^{n_1,\ldots,n_r}$ for the subspace of V where $(z_1,\ldots,z_r) \in \mathbb{G}_m^r$ acts as multiplication by $z_1^{-n_1}\cdots z_r^{-n_r}$. (Note the minus signs.) This is nowadays the standard sign convention in Hodge theory, see for instance [23], Remark 3.3.

Over an arbitrary field k, all we have to do is to require that the actions of $\mathrm{Gal}(k^s/k)$ on $X^*(T)$ and on $V \otimes_k k^s$ "match". Thus, let T be a k-torus and write $\mathbf{Rep}_k(T)$ for the category of finite dimensional k-representations of T. Then we have an equivalence of categories

$$\mathbf{Rep}_k(T) \longrightarrow \begin{pmatrix}\text{finite dimensional } k\text{-vector spaces } V\ +\\ X^*(T)\text{-grading } V\otimes_k k^s = \bigoplus_{\chi\in X^*(T)} V_{k^s}(\chi)\\ \text{s.t. } {}^\sigma(V_{k^s}(\chi)) = V_{k^s}({}^\sigma\chi) \text{ for all } \sigma\in\mathrm{Gal}(k^s/k)\end{pmatrix}.$$

(1.2) *The Deligne torus.* Define the torus $\mathbb{S}$ by

$$\mathbb{S} := \operatorname*{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}\,,$$

where "Res" denotes restriction of scalars à la Weil. Thus, $\mathbb{S}$ is an algebraic torus over $\mathbb{R}$; its character group is generated by two characters z and $\bar{z}$ such that the induced maps on points

$$\mathbb{C}^* = \mathbb{S}(\mathbb{R}) \subset \mathbb{S}(\mathbb{C}) \longrightarrow \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$$

are the identity, resp. complex conjugation. In other words: $X^*(\mathbb{S}) = \mathbb{Z}\cdot z + \mathbb{Z}\cdot\bar{z}$ with complex conjugation $\iota \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acting by ${}^\iota z = \bar{z}$, ${}^\iota\bar{z} = z$. By what was explained in (1.1), this uniquely determines $\mathbb{S}$ as an $\mathbb{R}$-torus.

Define the *weight cocharacter* $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ to be the cocharacter given on points by the natural inclusion $\mathbb{R}^* = \mathbb{G}_{m,\mathbb{R}}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$. The *norm character* $\mathrm{Nm}\colon \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$ is defined by $\mathrm{Nm} = z\bar{z}$. The kernel of Nm is the *circle group* $\mathbb{U}_1 = \{z \in \mathbb{C}^* \mid |z| = 1\}$, viewed as an $\mathbb{R}$-torus. Finally we define the cocharacter $\mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_\mathbb{C}$ to be the unique cocharacter such that $\bar{z}\circ\mu$ is trivial and $z\circ\mu = \mathrm{id} \in \mathrm{End}(\mathbb{G}_{m,\mathbb{C}})$.

(1.3) *Hodge structures.* A $\mathbb{Q}$-Hodge structure of weight n ($n \in \mathbb{Z}$) consists of a finite dimensional $\mathbb{Q}$-vector space V together with a homomorphism of algebraic groups over $\mathbb{R}$

$$h\colon \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$$

such that $h \circ w \colon \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}(V)_\mathbb{R}$ is given by $z \mapsto z^{-n} \cdot \mathrm{id}_V$. We leave it to the reader to connect this to the more traditional definition, using (1.1) above. By our sign convention, a point $(z_1, z_2) \in \mathbb{C}^* \times \mathbb{C}^* = \mathbb{S}(\mathbb{C})$ acts on $V^{p,q}$ as multiplication by $z_1^{-p} z_2^{-q}$.

We shall use HS as an abbreviation for "Hodge Structure". By a $\mathbb{Q}$-HS we shall mean a direct sum of pure $\mathbb{Q}$-HS. Equivalently: a $\mathbb{Q}$-HS is a finite dimensional $\mathbb{Q}$-vector space V plus a homomorphism $h \colon \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$ such that $h \circ w \colon \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}(V)_\mathbb{R}$ is defined over $\mathbb{Q}$. The category $\mathbb{Q}\mathbf{HS}$ of all such $\mathbb{Q}$-HS is Tannakian.

The automorphism $C := h(i)$ of $V_\mathbb{R}$ is called the Weil operator. Concretely: C acts on $V^{p,q}$ as multiplication by i^{q-p}. (Note that this indeed gives an endomorphism defined over $\mathbb{R}$.)

The Hodge filtration of V, notation $F^\bullet V_\mathbb{C}$, is the one given by $F^m V_\mathbb{C} = \oplus_{p \geq m} V^{p,q}$. If V is pure of weight n then $F^\bullet V_\mathbb{C}$ and $\overline{F^\bullet} V_\mathbb{C}$ are n-opposed in the sense of [16], meaning that $F^p V_\mathbb{C} \oplus \overline{F^q} V_\mathbb{C} \xrightarrow{\sim} V_\mathbb{C}$ for all p, q with $p + q = n + 1$. Conversely, if $F^\bullet V_\mathbb{C}$ is a filtration of $V_\mathbb{C}$ such that $F^\bullet V_\mathbb{C}$ and $\overline{F^\bullet} V_\mathbb{C}$ are n-opposed then we obtain a HS by setting $V^{p,q} := F^p V_\mathbb{C} \cap \overline{F^q} V_\mathbb{C}$.

The Tate structure $\mathbb{Q}(1)$ is defined to be the vector space $\mathbb{Q}(1) := 2\pi i \cdot \mathbb{Q} \subset \mathbb{C}$, with Hodge structure purely of type $(-1, -1)$. The corresponding homomorphism h is the Norm character Nm.

By a homomorphism of Hodge structures $f \colon V_1 \to V_2$ we mean a $\mathbb{Q}$-linear map f such that $f_\mathbb{R}$ is equivariant w.r.t. the given actions of $\mathbb{S}$. (This corresponds to a morphism of type $(0,0)$ in the traditional sense, i.e., a map preserving the Hodge bigrading.)

A $\mathbb{Q}$-HS V of weight n is said to be polarizable if there exists a homomorphism of Hodge structures $\varphi \colon V \otimes V \to \mathbb{Q}(-n)$ such that the bilinear form $V_\mathbb{R} \times V_\mathbb{R} \to \mathbb{R}$ given by $(v, w) \mapsto (2\pi i)^n \cdot \varphi(v \otimes Cw)$ is symmetric and positive definite. This implies that φ is alternating if n is odd, symmetric if n is even. An arbitrary $\mathbb{Q}$-HS is said to be polarizable if all its pure summands are. We write $\mathbb{Q}\mathbf{HS}^{\mathrm{pol}} \subset \mathbb{Q}\mathbf{HS}$ for the full subcategory of polarizable $\mathbb{Q}$-HS.

(1.4) Definition. Let V be a $\mathbb{Q}$-HS; write $h \colon \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$ for the corresponding homomorphism. We define the Mumford-Tate group of V, notation $\mathrm{MT}(V)$, to be the smallest algebraic subgroup $M \subseteq \mathrm{GL}(V)$ (over $\mathbb{Q}$) such that h factors through $M_\mathbb{R}$.

It is immediate from the definition that $\mathrm{MT}(V)$ is connected. If V is pure of weight $n \neq 0$ then by looking at $h \circ w$ we find that $\mathrm{MT}(V)$ contains the torus $\mathbb{G}_{m,\mathbb{Q}} \cdot \mathrm{id}_V$ of homotheties. (By contrast, if V is pure of weight 0 then $\mathrm{MT}(V)$ is contained in $\mathrm{SL}(V)$; see also (1.11) below.)

It is easy to see that $\mathrm{MT}(V)$ can also be described as the smallest algebraic subgroup $M \subset \mathrm{GL}(V)$ such that $h \circ \mu \colon \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GL}(V)_\mathbb{C}$ factors through $M_\mathbb{C}$. This description will become important later.

(1.5) Key Property. Let V be a $\mathbb{Q}$-HS. For $m, n \in \mathbb{Z}_{\geq 0}$, write $T^{m,n} := V^{\otimes m} \otimes (V^*)^{\otimes n}$. Let T be a finite direct sum of spaces of the form $T^{m,n}$, viewed as a $\mathbb{Q}$-HS. Consider the action of $\mathrm{MT}(V)$ on T induced by its action on V. Let $W \subseteq T$ be a $\mathbb{Q}$-subspace. Then

$$W \subseteq T \quad \text{is a } \mathbb{Q}\text{-Hodge substructure} \quad \Longleftrightarrow \quad W \subseteq T \quad \text{is a } \mathrm{MT}(V)\text{-submodule}\,.$$

Proof. The implication "$\Leftarrow$" is obvious. For "$\Rightarrow$", suppose W is a $\mathbb{Q}$-subHS. Consider the algebraic subgroup $M \subseteq \mathrm{GL}(V)$ consisting of those $g \in \mathrm{GL}(V)$ which, under the induced action on T, leave the subspace $W \subseteq T$ stable. The assumption that W is a $\mathbb{Q}$-subHS means that $W_\mathbb{R} \subset T_\mathbb{R}$ is stable under the (induced) action of $\mathbb{S}$. It follows that h factors through M. The definition of $\mathrm{MT}(V)$ then gives the implication "$\Rightarrow$". $\square$

If $M \subseteq \mathrm{GL}(V)$ is an algebraic subgroup then there exists a "tensor construction" T as above and a line $l \subseteq T$ such that M is the stabilizer of this line l. (Chevalley's theorem, see e.g. [23], Prop. 3.1. Exercise: describe such a line for the case $M = \mathrm{SL}(V) \subset \mathrm{GL}(V)$.) It follows that the Key Property characterizes $\mathrm{MT}(V)$ uniquely.

(1.6) Definition. Let V be a $\mathbb{Q}$-HS. A *Hodge class* in V is an element $v \in V$ which is purely of type $(0,0)$ in the Hodge decomposition.

In other literature a Hodge class is sometimes defined to be a rational class which is purely of some type (p,p) in the Hodge decomposition. The connection between these two definitions is given by using a Tate twist: if $v \in V$ is a (p,p)-class then $(2\pi i)^p \cdot v \in V(p) := V \otimes \mathbb{Q}(1)^{\otimes p}$ is a Hodge class in our sense.

Exercise. Show that $v \in V$ is a Hodge class if and only if v is invariant under $\mathrm{MT}(V)$.

(1.7) Example. Suppose V is pure of weight n and $\tilde{\varphi}\colon V \otimes V \to \mathbb{Q}(n)$ is a polarization. Write $\varphi \in (V^*)^{\otimes 2}$ for the element given by $(2\pi i)^{-n} \cdot \tilde{\varphi}$. Then the line spanned by φ is stable under the action of $\mathrm{MT}(V)$, which means that $\mathrm{MT}(V)$ acts on it through some character $\nu\colon \mathrm{MT}(V) \to \mathbb{G}_{m,\mathbb{Q}}$. The conclusion is that

$$\mathrm{MT}(V) \subseteq \begin{cases} \mathrm{GSp}(V,\varphi) & \text{if } n \text{ is odd;} \\ \mathrm{GO}(V,\varphi) & \text{if } n \text{ is even.} \end{cases}$$

Explanation: if V is a vector space over a field k equipped with a symplectic, resp. an orthogonal form $\varphi\colon V \times V \to k$ then we define the group of symplectic similitudes $\mathrm{GSp}(V,\varphi)$, resp. the group of orthogonal similitudes $\mathrm{GO}(V,\varphi)$, by

$$\mathrm{GSp}(V,\varphi) := \{g \in \mathrm{GL}(V) \mid \exists \nu(g) \in k^* : \varphi(gv,gw) = \nu(g)\cdot\varphi(v,w) \quad \text{for all } v,w \in V\}$$

and, similarly,

$$\mathrm{GO}(V,\varphi) := \{g \in \mathrm{GL}(V) \mid \exists \nu(g) \in k^* : \varphi(gv,gw) = \nu(g)\cdot\varphi(v,w) \quad \text{for all } v,w \in V\}\,.$$

Associating $\nu(g)$ to g gives a character $\nu\colon \mathrm{GSp}(V,\varphi) \to \mathbb{G}_{m,k}$, resp. $\nu\colon \mathrm{GO}(V,\varphi) \to \mathbb{G}_{m,k}$, called the multiplier character. The kernel of this multiplier character is the symplectic group $\mathrm{Sp}(V,\varphi)$, resp. the orthogonal group $\mathrm{O}(V,\varphi)$.

If we want to treat the symplectic and the orthogonal case uniformly we shall write $\mathrm{GU}(V,\varphi)$ for the group of automorphisms of V preserving the form φ up to a scalar.

Exercise. Show that the multiplier character $\nu\colon \mathrm{MT}(V) \to \mathbb{G}_{m,\mathbb{Q}}$ is independent of the chosen polarization.

(1.8) Remark. Let T be a tensor construction as in (1.5). Write $r\colon \mathrm{GL}(V) \to \mathrm{GL}(T)$ for the canonical homomorphism. Then $\mathrm{MT}(T)$ equals the image of $\mathrm{MT}(V)$ under r. To see this, let us first remark that $\mathrm{MT}(T)$ is contained in the image of $\mathrm{MT}(V)$; this is immediate from the definitions. Now suppose that $\mathrm{MT}(T)$ is strictly contained in $r(\mathrm{MT}(V))$. Then we can make a tensor construction T', built from T, and a $\mathbb{Q}$-subspace $W \subset T'$ such that W is a $\mathrm{MT}(T)$-submodule but W is not stable under $\mathrm{MT}(V)$. This would contradict the Key Property. (Compare this with [25], Prop. 2.21.)

As examples of this principle, we find that $\mathrm{MT}(V^*)$ is isomorphic to $\mathrm{MT}(V)$ (under the natural isomorphism $g \mapsto (g^*)^{-1}$). Also, $\mathrm{MT}(V^{\oplus n})$ $(n \geq 1)$ is isomorphic to $\mathrm{MT}(V)$ acting diagonally on $V^{\oplus n}$.

(1.9) Example. If $\dim(V) = 0$ then $\mathrm{MT}(V) = \{1\}$. If $\dim(V) = 1$ then V is isomorphic to a Tate structure $\mathbb{Q}(n)$; if $n = 0$ then $\mathrm{MT}(V) = \{1\}$, if $n \neq 0$ then $\mathrm{MT}(V) = \mathbb{G}_{m,\mathbb{Q}}$.

(1.10) *Tannakian formulation.* In the formalism of Tannakian categories, the Mumford-Tate group can be described as follows. Let V be a $\mathbb{Q}$-HS. Write $\langle V\rangle^{\otimes} \subset \mathbb{Q}\mathbf{HS}$ for the Tannakian subcategory generated by V. The forgetful functor defines a fibre functor $\omega\colon \langle V\rangle^{\otimes} \to \mathbf{Vec}_{\mathbb{Q}}$. Then $\mathrm{MT}(V) = \mathrm{Aut}^{\otimes}(\omega)$ and we obtain an equivalence of categories $\langle V\rangle^{\otimes} \xrightarrow{\text{eq}} \mathbf{Rep}_{\mathbb{Q}}(\mathrm{MT}(V))$. This is essentially just a fancy reformulation of the above Key Property.

There are some possible variations on our definition of the Mumford-Tate group.

(1.11) Definition. Let V be a $\mathbb{Q}$-HS of pure weight. We define the *Hodge group* $\mathrm{Hg}(V) \subseteq \mathrm{MT}(V)$, also called the *special Mumford-Tate group* to be the smallest algebraic subgroup $H \subseteq \mathrm{GL}(V)$ such that $h_{|\mathbb{U}_1}\colon \mathbb{U}_1 \to \mathrm{GL}(V)_{\mathbb{R}}$ factors through $H_{\mathbb{R}}$.

If V has weight n and $z \in \mathbb{S}$ then the automorphism $h(z)$ of V has determinant $\mathrm{Nm}(z)^{-n\dim(V)/2}$. (Recall that $h(z)$ is multiplication by $z^{-p}\bar{z}^{-q}$ on $V^{p,q}$.) It follows that $\mathrm{Hg}(V)$ is contained in $\mathrm{SL}(V)$.

The relation between $\mathrm{MT}(V)$ and $\mathrm{Hg}(V)$ is not so difficult to describe. Namely, if V is pure of weight 0 then we easily find that $\mathrm{MT}(V) = \mathrm{Hg}(V)$. If V is pure of weight $\neq 0$ then $\mathrm{MT}(V)$ contains $\mathbb{G}_{m,\mathbb{Q}}$ (the homotheties) and $\mathrm{MT}(V)$ is the almost direct product (inside $\mathrm{GL}(V)$) of $\mathbb{G}_{m,\mathbb{Q}}$ and $\mathrm{Hg}(V)$.

Now let us look at the key property (1.5). So, let T be a tensor space constructed from V and let $W \subseteq T$ be a $\mathbb{Q}$-subspace. Clearly, if W is a $\mathbb{Q}$-subHS then it is a $\mathrm{MT}(V)$-submodule, hence it is a $\mathrm{Hg}(V)$-submodule. The converse holds *provided T is of pure weight.* This is clear if we think of $\mathrm{MT}(V)$ as being the almost direct product of $\mathbb{G}_{m,\mathbb{Q}}$ and $\mathrm{Hg}(V)$: the assumption that T is of pure weight means that $\mathbb{G}_{m,\mathbb{Q}}$ acts on T by homotheties, so that the $\mathrm{MT}(V)$-submodules of V are the same as the $\mathrm{Hg}(V)$-submodules. To summarize:

(1.12) Key Property. (Hodge group version.) Let V be a $\mathbb{Q}$-HS. For $m, n \in \mathbb{Z}_{\geq 0}$, write $T^{m,n} := V^{\otimes m} \otimes (V^*)^{\otimes n}$. Let T be a finite direct sum of spaces of the form $T^{m,n}$, viewed as a $\mathbb{Q}$-HS. Assume that T is of pure weight. Consider the action of $\mathrm{Hg}(V)$ on T induced by its action on V. Let $W \subseteq T$ be a $\mathbb{Q}$-subspace. Then

$$W \subseteq T \quad \text{is a } \mathbb{Q}\text{-Hodge substructure} \quad \Longleftrightarrow \quad W \subseteq T \quad \text{is a } \mathrm{Hg}(V)\text{-submodule}\,.$$

Loosely speaking we might say that $\mathrm{Hg}(V)$ contains the same information as $\mathrm{MT}(V)$, except that it is not able to "see" the weight of a Hodge structure. For instance, if T is of pure weight and $w \in T$ then w is a $\mathrm{Hg}(V)$-invariant if and only if it is a Hodge (p,p)-class for some p. This also explains why we have to require that T is of pure weight. Namely, suppose we can make two tensor spaces T_1 and T_2 and non-zero classes $w_1 \in T_1$ of type (p_1, p_1), and $w_2 \in T_2$ of type (p_2, p_2). Now set $T := T_1 \oplus T_2$, and let $l \subset T$ be the line spanned by (w_1, w_2). Clearly, if $p_1 \neq p_2$ then l is *not* a $\mathbb{Q}$-subHS of T. But l *is* invariant under the action of $\mathrm{Hg}(V)$. That l is not stable under the action of $\mathrm{MT}(V)$ is due to the action of the central torus $\mathbb{G}_{m,\mathbb{Q}}$.

(1.13) *The Hodge group of a product.* Let V_1 and V_2 be $\mathbb{Q}$-HS. Write $V := V_1 \oplus V_2$. It readily follows from the definitions that $\mathrm{Hg}(V) \subseteq \mathrm{Hg}(V_1) \times \mathrm{Hg}(V_2)$ and that the two projections $\mathrm{Hg}(V) \to \mathrm{Hg}(V_i)$ are surjective. In general the Hodge group $\mathrm{Hg}(V)$ need not be equal to the product group $\mathrm{Hg}(V_1) \times \mathrm{Hg}(V_2)$. For instance, if $V_1 = V_2$ then $\mathrm{Hg}(V)$ is the diagonal subgroup of $\mathrm{Hg}(V_1) \times \mathrm{Hg}(V_2)$; see (1.8). We shall see more interesting examples of this later.

For the Mumford-Tate group similar statements hold, but note that $\mathrm{MT}(V)$ is almost never equal to $\mathrm{MT}(V_1) \times \mathrm{MT}(V_2)$. This is because the central factor $\mathbb{G}_m$ ("keeping track of the weight") is counted twice in $\mathrm{MT}(V_1) \times \mathrm{MT}(V_2)$, unless one of the V_i has weight 0.

(1.14) *The extended Mumford-Tate group.* Another possible variant is to consider the Tannakian subcategory $\langle V, \mathbb{Q}(1)\rangle^{\otimes} \subset \mathbb{Q}\mathbf{HS}$ generated by V and $\mathbb{Q}(1)$. Let $\omega\colon \langle V, \mathbb{Q}(1)\rangle^{\otimes} \to \mathbf{Vec}_{\mathbb{Q}}$ be the forgetful functor, and write $\widetilde{\mathrm{MT}}(V) := \mathrm{Aut}^{\otimes}(\omega)$. Concretely, this $\widetilde{\mathrm{MT}}(V)$ can be described as the smallest algebraic $\mathbb{Q}$-subgroup $M \subset \mathrm{GL}(V) \times \mathbb{G}_{m,\mathbb{Q}}$ such that $h \times \mathrm{Nm}\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}$ factors through $M_{\mathbb{R}}$. The projection onto $\mathrm{GL}(V)$ gives a surjective homomorphism $\widetilde{\mathrm{MT}}(V) \twoheadrightarrow \mathrm{MT}(V)$, which is an isogeny if V has weight $n \neq 0$ and an isomorphism if V is polarizable of weight ± 1. See also [23] and [48].

One possible reason for working with this "extended" Mumford-Tate group $\widetilde{\mathrm{MT}}(V)$ is that it allows to include arbitrary Tate twists in all considerations. We leave it to the reader to formulate a version of the Key Property for $\widetilde{\mathrm{MT}}(V)$. (Consider tensor spaces of the form $T^{m,n,p} := V^{\otimes m} \otimes (V^*)^{\otimes n} \otimes \mathbb{Q}(p)$.) Note that $\mathrm{MT}(V)$, for V polarizable of weight n, only has a natural action on Tate twists $\mathbb{Q}(r \cdot n)$ for $r \in \mathbb{Z}$. (The action on $\mathbb{Q}(-n)$ is given by the multiplier character ν as in (1.7).)

(1.15) *Polarizable HS.* Let V be a pure $\mathbb{Q}$-HS of weight n, and let $\varphi\colon V \otimes V \to \mathbb{Q}(-n)$ be a polarization. Write $Q\colon V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ for the symmetric positive definite form given by $(v, w) \mapsto (2\pi i)^n \cdot \varphi(v \otimes Cw)$.

If T is a tensor space constructed from V as in (1.5) then T inherits a polarization from V. Similarly, if $W \subseteq V$ is a $\mathbb{Q}$-subHS then the restriction of φ to $W \otimes W$ is a polarization of W. Now consider the $\mathbb{Q}$-subspace

$$W^{\perp} := \{x \in V \mid \varphi(x, w) = 0 \quad \text{for all } w \in W\}\,.$$

Note that $W^{\perp}$ is a $\mathbb{Q}$-subHS of V; writing $V \to V^*(-n)$ for the morphism given by $v \mapsto \varphi(v, -)$ we can in fact also define $W^{\perp}$ as the kernel of the composite morphism $V \to V^*(-n) \twoheadrightarrow (V^*/W^*)(-n)$. As $W_{\mathbb{R}} \subset V_{\mathbb{R}}$ is stable under the Weil operator C, we have

$$\begin{aligned}(W^{\perp} \otimes \mathbb{R}) &= \{x \in V_{\mathbb{R}} \mid \varphi(x, w) = 0 \quad \text{for all } w \in W_{\mathbb{R}}\} \\ &= \{x \in V_{\mathbb{R}} \mid Q(x, w) = 0 \quad \text{for all } w \in W_{\mathbb{R}}\}\,.\end{aligned}$$

It follows that $V = W \oplus W^\perp$ as $\mathbb{Q}$-HS.

(1.16) Theorem. *The category* $\mathbb{Q}\mathbf{HS}^{\mathrm{pol}}$ *is a semi-simple Tannakian category. If V is a polarizable $\mathbb{Q}$-HS then* $\mathrm{Hg}(V)$ *and* $\mathrm{MT}(V)$ *are reductive $\mathbb{Q}$-groups.*

Proof. The first assertion follows from what was said in (1.15). The second claim is a general statement in the theory of Tannakian categories: if G is a connected group scheme over a field k of characteristic 0 then $\mathbf{Rep}_k(G)$ is semi-simple if and only if G is pro-reductive. (See [25], Prop. 2.23.) In fact, in the case that we are interested in it suffices to see that $M := \mathrm{MT}(V)$ is reductive, as $\mathrm{Hg}(V)$ is a normal subgroup of $\mathrm{MT}(V)$. We have $\mathbf{Rep}_{\mathbb{Q}}(M) \xrightarrow{\mathrm{eq}} \langle V\rangle^{\otimes}$, which is semi-simple. In particular, the tautological representation $M \to \mathrm{GL}(V)$ is faithful and semi-simple. It is then a standard result in the theory of algebraic groups (over a field of characteristic 0) that M is reductive. (Loc. cit., Lemmas 2.24, 2.25 and 2.27.) $\square$

Exercise. (i) Let V be a $\mathbb{Q}$-vector space of finite dimension. Show that giving V a $\mathbb{Q}$-HS of type $(-1,0)+(0,-1)$ is equivalent to giving an endomorphism $C \in \mathrm{End}(V_{\mathbb{R}})$ (the Weil operator) with $C^2 = -\mathrm{id}$.

(ii) Let V be a polarizable $\mathbb{Q}$-HS of weight n. Choose a polarization $\varphi\colon V \otimes V \to \mathbb{Q}(-n)$ and write $d \mapsto d^\dagger$ for the associated involution on $D := \mathrm{End}_{\mathbb{Q}\mathbf{HS}}(V)$. (So, $d \mapsto d^\dagger$ is the involution determined by the rule that $\varphi(dv \otimes w) = \varphi(v \otimes d^\dagger w)$ for all $v, w \in V$ and $d \in D$.) Show that the set of polarizations is in natural bijection with an open cone in the vector space $\{d \in D \mid d^\dagger = d\}$.

(iii) Let V_1 and V_2 be polarizable $\mathbb{Q}$-HS of type $(-1,0)+(0,-1)$. Write $C_1 \in \mathrm{End}(V_{1,\mathbb{R}})$ and $C_2 \in \mathrm{End}(V_{2,\mathbb{R}})$ for the Weil operators. Suppose that $\mathrm{End}_{\mathbb{Q}\mathbf{HS}}(V_1) = \mathbb{Q} = \mathrm{End}_{\mathbb{Q}\mathbf{HS}}(V_2)$. Show that

$$\mathrm{Ext}_{\mathbb{Q}\mathbf{HS}}(V_2, V_1) \cong \{A \in \mathrm{Hom}(V_{2,\mathbb{R}}, V_{1,\mathbb{R}}) \mid C_1 A + A C_2 = 0\}/\mathbb{Q}^* .$$

(iv) Assume that $V_1 \not\cong V_2$. Let W be an extension of V_2 by V_1 which corresponds to a non-zero class in $\mathrm{Ext}_{\mathbb{Q}\mathbf{HS}}(V_2, V_1)$. (I.e., $W \not\cong V_1 \oplus V_2$.) Show that W is not polarizable.

(v) Show that the category $\mathbb{Q}\mathbf{HS}$ is not semi-simple, e.g. by proving that there exist examples as in (iv).

(1.17) *Compact real forms.* Let V be a polarizable $\mathbb{Q}$-HS of weight n. There is another way of proving that $H := \mathrm{Hg}(V)$ is reductive. (Although essentially it of course boils down to the same.) Namely, consider the Weil operator $C = h(i) \in H(\mathbb{R})$. As $C^2 = (-\mathrm{id}_V)^n$, the inner automorphism $\sigma := \mathrm{Ad}(C)$ of $H_{\mathbb{R}}$ is an involution. Now consider the inner form $H^{(\sigma)}$ of H defined by this involution σ. Concretely: having a real group $H_{\mathbb{R}}$ means that we have a complex group $H_{\mathbb{C}}$ with a comlex conjugation $x \mapsto \bar{x}$. Then the inner form $H^{(\sigma)}$ is given by the same $\mathbb{C}$-group $H_{\mathbb{C}}$ but with $x \mapsto \sigma(\bar{x})$ as complex conjugation.

Let $\varphi\colon V \otimes V \to \mathbb{Q}(-n)$ be a polarization. Write $\Psi\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ for the symmetric, positive definite hermitian form $(v, w) \mapsto (2\pi i)^n \varphi(v \otimes C\bar{w})$. We claim that the real group $H^{(\sigma)}$ is a subgroup of the unitary group $\mathrm{U}(V_{\mathbb{C}}, \Psi)$ and is therefore compact. This is now a simple computation: let $x \in H^{(\sigma)}(\mathbb{R})$, so that $\bar{x} = \sigma(x) = C^{-1}xC\}$. Then

$$\begin{aligned}\Psi(xv, xw) &= (2\pi i)^n \cdot \varphi(xv \otimes C\bar{x}\bar{w}) \\ &= (2\pi i)^n \cdot \varphi(xv \otimes xC\bar{w}) = (2\pi i)^n \cdot \varphi(v \otimes C\bar{w}) = \Psi(v, w) .\end{aligned}$$

This shows that $H_{\mathbb{R}}$ is an inner form of a compact group. Now

$$H^{(\sigma)} \text{ is compact} \Longrightarrow H^{(\sigma)} \text{ is reductive} \Longleftrightarrow H^{(\sigma)}_{\mathbb{C}} = H_{\mathbb{C}} \text{ is reductive} \Longleftrightarrow H \text{ is reductive} .$$

(1.18) Remark. Let V be a polarizable $\mathbb{Q}$-HS of weight n. From the fact that $C \in \mathrm{Hg}(V)(\mathbb{R})$ defines a Cartan involution we can deduce some further conclusions. For this, decompose $H_{\mathbb{R}} := \mathrm{Hg}(V)_{\mathbb{R}}$ as $H_{\mathbb{R}} = H_0 \cdot H_1 \cdots H_q$, the almost direct product of the connected center H_0 and a number of $\mathbb{R}$-simple factors $H_1, \ldots, H_q$. Write $p_j\colon H_{\mathbb{R}} \twoheadrightarrow H_j'$ for the quotient of $H_{\mathbb{R}}$ modulo $H_0 \cdots H_{j-1} \cdot H_{j+1} \cdots H_q$ and let C_j be the image of C in $H_j'(\mathbb{R})$, which is again a Cartan involution.

The first remark is that each of the factors H_j is absolutely simple. In fact, we have seen that the Cartan involution $\sigma = \mathrm{Ad}(C)$ defines a compact inner form $H_j^{(\sigma)}$. Then the compactness of $H_j^{(\sigma)}$ implies that $H_{j,\mathbb{C}} = H_{j,\mathbb{C}}^{(\sigma)}$ is simple.

Further we conclude that:

(i) the center of $\mathrm{Hg}(V)$ is compact over $\mathbb{R}$,

(ii) if $H_i(\mathbb{R})$ is not compact then $h_{|\mathbb{U}_1}: \mathbb{U}_1 \to \mathrm{Hg}(V)_{\mathbb{R}}$ has a nontrivial component in the factor H_i,

(iii) if $H_i(\mathbb{R})$ is compact then C lies in the (finite) center of H_i.

In general it is not true that $h_{|\mathbb{U}_1}$ has a trivial component in the compact factors. For later use, let us record here, however, that this is the case if V has level ≤ 1. By this we mean that either n is even and V is purely of type $(n/2, n/2)$ or $n = 2p+1$ is odd and V is of type $(p, p+1) + (p+1, p)$; see also (2.22) below. So, we claim that if V has level ≤ 1 then $h_{|\mathbb{U}_1}$ has a trivial component in the factor H_j if and only if H_j is compact. To see this, take a compact factor H_j and consider the action of $\mathbb{U}_1$ on $\mathrm{Lie}(H) \subseteq \mathrm{End}(V)$. The assumption that the level is ≤ 1 implies that in $\mathrm{End}(V)$ only the Hodge types $(-1, 1)$, $(0, 0)$ and $(1, -1)$ occur. Together with (iii) above it follows that $\mathrm{Lie}(H_j) \subset \mathrm{End}(V)_{\mathbb{R}}$ is purely of type $(0, 0)$. This means that $p_j \circ h_{|\mathbb{U}_1}$ factors through the finite center of H'_j, so the H_j-component of $h_{|\mathbb{U}_1}$ is indeed trivial.

Exercise. Construct a polarizable $\mathbb{Q}$-HS V such that $\mathrm{Hg}(V)(\mathbb{R})$ is compact. Also try to construct such an example such that $\mathrm{Hg}(V)$ is *not* a torus.

(1.19) *Albert's classification.* Let D be a simple $\mathbb{Q}$-algebra with a positive (anti-)involution ι: $d \mapsto d^\dagger$. Such algebras have been classified by Albert; the result is explained in [53], §21. (What is a positive involution is also explained for instance in [43], §2.) The result is that D is of one of the following types; here we write $F = \mathrm{Cent}(D)$, $F_0 = \{a \in F \mid a^\dagger = a\}$ and $e_0 = [F_0 : \mathbb{Q}]$, $e = [F : \mathbb{Q}]$, $d^2 = [D : F]$.

Type I(e_0): $e = e_0$, $d = 1$; $D = F = F_0$ is a totally real field. The involution ι is the identity.

Type II(e_0): $e = e_0$, $d = 2$; D is a quaternion algebra over a totally real field $F = F_0$; D splits at all infinite places. The involution ι is different from the canonical involution on D (i.e., the one given by $d \mapsto d^* = \mathrm{tr}_{D/F}(d) - d$); there exists an element $a \in D$ with $a^* = -a$ such that $d^\dagger = ad^*a^{-1}$.

Type III(e_0): $e = e_0$, $d = 2$; D is a quaternion algebra over a totally real field $F = F_0$; D is inert at all infinite places. The involution ι is the canonical involution on D.

Type IV(e_0, d): $e = 2e_0$; F is a CM-field with totally real subfield F_0; D is a division algebra of rank d^2 over F. The involution ι is such that under a suitable isomorphism $D \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} M_d(\mathbb{C}) \times \cdots \times M_d(\mathbb{C})$ (e_0 factors) it coresponds to the involution $(A_1, \ldots, A_{e_0}) \mapsto (A_1^*, \ldots, A_{e_0}^*)$, where $A_i^* := {}^t\overline{A}_i$. In particular, ι is complex conjugation on F.

(1.20) Remark. As we have seen above, the category $\mathbb{Q}\mathbf{HS}^{\mathrm{pol}}$ of polarizable $\mathbb{Q}$-HS is semi-simple. If V is a simple polarizable $\mathbb{Q}$-HS then its endomorphism algebra $D := \mathrm{End}_{\mathbb{Q}\mathbf{HS}}(V)$ is a division algebra over $\mathbb{Q}$. Let n be the weight of $\mathbb{Q}$ and let φ: $V \otimes V \to \mathbb{Q}(-n)$ be a polarization. There is an involution $d \mapsto d^\dagger$ determined by the rule $\varphi(dv \otimes w) = \varphi(v \otimes d^\dagger w)$ for all v, w. By definition of a polarization, the form Q: $V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ given by $Q(v, w) = (2\pi i)^n \cdot \varphi(v \otimes Cw)$ is symmetric and positive definite. Notice that $d \in D$ commutes with the Weil operator C, so that $Q(dv, w) = Q(v, d^\dagger w)$. We conclude that $d \mapsto d^\dagger$ is a positive involution and that D is an algebra of the type considered above.

(1.21) *An upper bound for the Mumford-Tate group.* Let V be a polarizable $\mathbb{Q}$-HS. We can decompose V as $V = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r}$, where $V_1, \ldots, V_r$ are simple, mutually non-isomorphic $\mathbb{Q}$-HS and $m_1, \ldots, m_r \in \mathbb{Z}_{\geq 1}$. Write $D := \mathrm{End}_{\mathbb{Q}\mathbf{HS}}(V)$, $D_i := \mathrm{End}_{\mathbb{Q}\mathbf{HS}}(V_i)$. The D_i are division algebras of the type discussed above and $D = M_{m_1}(D_1) \times \cdots \times M_{m_r}(D_r)$. If V is simple then we shall say it is of type I (type II, etc.) if D is of the corresponding type in the Albert classification.

We have

$$\begin{aligned} D = \mathrm{End}_{\mathbb{Q}\mathbf{HS}}(V) &= \{\text{Hodge classes in } \mathrm{End}_{\mathbb{Q}}(V)\} \\ &= \mathrm{MT}(X)\text{-invariants in } \mathrm{End}_{\mathbb{Q}}(V) \\ &= \mathrm{Hg}(X)\text{-invariants in } \mathrm{End}_{\mathbb{Q}}(V). \end{aligned}$$

This means that $\mathrm{MT}(V)$ is contained in the algebraic group $\mathrm{GL}_D(V)$ of D-linear automorphisms of V. Choosing a polarization $\tilde{\varphi}$ of V and combining the previous with what we found in (1.7) we get

$$\mathrm{MT}(V) \subseteq \mathrm{GU}_D(V, \varphi)\,, \qquad \mathrm{Hg}(X) \subseteq \mathrm{U}_D(V, \varphi)\,,$$

where $\mathrm{GU}_D(V,\varphi)$ and $\mathrm{U}_D(V,\varphi)$ denote the centralizers of D inside $\mathrm{GU}(V,\varphi)$ resp. $\mathrm{U}(V,\varphi)$. Here we recall that $\mathrm{GU}(V,\varphi)$ is our uniform notation for the group $\mathrm{GSp}(V,\varphi)$ (if V has odd weight, so that φ is symplectic), resp. $\mathrm{GO}(V,\varphi)$ (if V has even weight, so that φ is orthogonal).

Exercise. Show that this centralizer $\mathrm{GU}_D(V,\varphi)$ does not depend on the chosen polarization.

(1.22) *Classical groups.* Let us make the group $\mathrm{U}_D(V,\varphi)$ a little more explicit in the situation that D is a field. (In particular, V must be simple.) There are two cases to consider.

First suppose that $D=F$ is a totally real field. Then there is a unique F-bilinear form ψ: $V\times V\to F$ with the property that $\varphi=\mathrm{tr}_{F/\mathbb{Q}}(\psi)$. We have an algebraic group $\mathrm{U}_F(V,\psi)$ over F and the centralizer $\mathrm{U}_F(V,\varphi)$ of F inside $\mathrm{U}(V,\varphi)$ is the group $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{U}_F(V,\psi)$. We shall usually simply write $\mathrm{U}_F(V,\psi)$ for this group, assuming it is understood that we view it as an algebraic group over $\mathbb{Q}$. If V has even weight then ψ is symmetric and we have $\mathrm{U}_F(V,\psi)=\mathrm{O}_F(V,\psi)$; if V has odd weight then $\mathrm{U}_F(V,\psi)=\mathrm{Sp}_F(V,\psi)$.

The other possibility is that $D=F$ is a CM-field. Write $F_0\subset F$ for its totally real subfield and $x\mapsto\bar{x}$ for the complex conjugation on F.

First assume that the weight of V is even, so that φ is symmetric. Then there exists a unique F-hermitian form ψ: $V\times V\to F$ with $\varphi=\mathrm{tr}_{F/\mathbb{Q}}(\psi)$. (So, ψ is F-linear in the first variable, F-anti-linear in the second variable, and $\psi(w,v)=\overline{\psi(v,w)}$ for all v,w.) Now the unitary group $\mathrm{U}_F(V,\psi)$ is an algebraic group over F_0 and the centralizer $\mathrm{Sp}_F(V,\varphi)$ of F inside $\mathrm{Sp}(V,\varphi)$ is the group $\mathrm{Res}_{F_0/\mathbb{Q}}\,\mathrm{U}_F(V,\psi)$. Again we shall usually simply write $\mathrm{U}_F(V,\psi)$ for this group.

If the weight of V is odd then essentially the same works. Imitating the previous would lead us to work with an anti-symmetric F-hermitian form ψ but we can modify this to a symmetric F-hermitian form using an imaginary element in F. More precisely: choose an element $a\in F$ with $\bar{a}=-a$. Then there exists a unique F-hermitian form ψ: $V\times V\to F$ with $\varphi=\mathrm{tr}_{F/\mathbb{Q}}(a\cdot\psi)$. Now again $\mathrm{U}_F(V,\psi)$ is an algebraic group over F_0 and the centralizer $\mathrm{Sp}_F(V,\varphi)$ of F inside $\mathrm{Sp}(V,\varphi)$ is the group $\mathrm{Res}_{F_0/\mathbb{Q}}\,\mathrm{U}_F(V,\psi)$.

If D is not a field the previous still works but has to be phrased in terms of D-hermitian forms. In each case we find that $\mathrm{U}_D(V,\psi)$ is obtained by restriction of scalars from a classical group over the field F_0. (For algebras with involutions, hermitian forms and algebraic groups, see for instance the appendix of [69], [9], section 23, or [41]).

(1.23) *The center of the Mumford-Tate group.* Let V be a polarizable $\mathbb{Q}$-HS. Decompose $V=V_1^{m_1}\oplus\cdots\oplus V_r^{m_r}$ as in (1.21). We have

$$D=[\mathrm{End}_{\mathbb{Q}}(V)]^{\mathrm{MT}(V)}=[\mathrm{End}_{\mathbb{Q}}(V)]^{\mathrm{Hg}(V)}\,. \tag{1.23.1}$$

Write $Z_M=Z(\mathrm{MT}(V))$ for the connected center of the Mumford-Tate group. Then we see from (1.23.1) that Z_M is contained in the algebraic group D^* (viewed as an algebraic group over $\mathbb{Q}$). Now again applying (1.23.1) we find that even $Z_M\subseteq\mathrm{T}_F:=\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m$, where F is the center of D.

By taking a polarization into account we can further sharpen this. First assume that V is simple, say of weight n, and let $d\mapsto d^\dagger$ be the Rosati involution on D associated to a polarization φ: $V\otimes V\to\mathbb{Q}(-n)$. This involution induces complex conjugation on the center F. (The identity if F is totally real.) Write Z_H for the connected center of the Hodge group. If V has weight 0 then $Z_M=Z_H$; otherwise Z_M is isogenous to $\mathbb{G}_{m,\mathbb{Q}}\times Z_H$. The form φ is preserved by the Hodge group: we have $\varphi(hv\otimes hw)=\varphi(v\otimes w)$ for all $v,w\in V$ and $h\in\mathrm{Hg}(V)$. On the other hand, $\varphi(dv\otimes w)=\varphi(v\otimes d^\dagger w)$ for $d\in D$. We conclude that Z_H is contained in U_F^0, where U_F is the $\mathbb{Q}$-group of multiplicative type given by

$$\mathrm{U}_F:=\{x\in\mathrm{T}_F\mid x\bar{x}=1\}\,.$$

On character groups T_F and U_F are described as follows: if Σ_F is the set of embeddings $F\to\overline{\mathbb{Q}}$ then $X^*(\mathrm{T}_F)$ is the free abelian group on Σ_F, with its natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The character group $X^*(\mathrm{U}_F)$ is the quotient of $X^*(\mathrm{T}_F)$ by the submodule generated by all elements $\sigma+\bar{\sigma}$, where the bar denotes complex conjugation.

Now drop the assumption that V is simple. In this case the center F is a product of totally real fields and CM-fields, say $F=F_1\times\cdots\times F_r$. Set $\mathrm{T}_F:=\mathrm{T}_{F_1}\times\cdots\times\mathrm{T}_{F_r}$ and $\mathrm{U}_F:=\mathrm{U}_{F_1}\times\cdots\times\mathrm{U}_{F_r}$. Using what was said in (1.13) we again find that Z_H is contained in U_F^0.

(1.24) Proposition. (See [88], Lemma 1.4.) *Let V be a polarizable $\mathbb{Q}$-HS. Assume V has no simple factors of type* IV. *Then* $\mathrm{Hg}(V)$ *is semi-simple.*

Proof. This is clear from the previous, as the assumption implies that F is a product of totally real fields, so that U_F is finite. □

Next we look at the opposite extreme.

(1.25) Definition. A $\mathbb{Q}$-HS is said to be of *CM-type* if V is polarizable and $\mathrm{MT}(V)$ is a torus.

If V is of CM-type then $\mathrm{Hg}(V)(\mathbb{R})$ is compact, by (1.17). Conversely, if V is a $\mathbb{Q}$-HS such that $\mathrm{Hg}(V)$ is a torus and $\mathrm{Hg}(V)(\mathbb{R})$ is compact then one can show that V is polarizable; see [70], Chap. 1, §6.1.

(1.26) *Description of HS of CM-type.* Let V be a polarizable $\mathbb{Q}$-HS. Decompose $V = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r}$ as in (1.21). By what we have seen in (1.13), V is of CM-type if and only if each V_i is of CM-type.

Now assume that V is simple and of CM-type. Let F be the center of $D := \mathrm{End}_{\mathbb{Q}\mathbf{HS}}(V)$. Let $d := \dim_F(V)$. In (1.23) we have seen that $\mathrm{Hg}(V) \subseteq \mathrm{T}_F$. This gives that $M_d(F) \subseteq D$. But D is a division algebra, so $d = 1$ and for dimension reasons we then can only have $D = F$. By the Albert classification, either F is totally real or F is a CM-field.

If F is totally real then (1.23) gives $\mathrm{Hg}(V) = \{1\}$. Then $F = \mathrm{End}_{\mathbb{Q}}(V)$ and we must have $\dim(V) = 1$ and $F = \mathbb{Q}$. We conclude that $V \cong \mathbb{Q}(n)$ for some n. These are indeed of CM-type.

Next suppose F is a CM-field. If φ is a polarization of V then the associated Rosati involution on F is complex conjugation. We know that $\mathrm{Hg}(V) \subseteq \mathrm{U}_F$. For the Mumford-Tate group this means that $\mathrm{MT}(V) \subseteq \mathrm{GU}_F$, where $\mathrm{GU}_F \subseteq \mathrm{T}_F$ is the subtorus generated by U_F and $\mathbb{G}_{m,\mathbb{Q}} \cdot \mathrm{id}$. On character groups: $X^*(\mathrm{GU}_F)$ is the quotient of $X^*(\mathrm{T}_F)$ (= the free abelian group on Σ_F) by the relations $\sigma + \bar{\sigma} = \tau + \bar{\tau}$ for all $\sigma, \tau \in \Sigma_F$. The homomorphism $h\colon \mathbb{S} \to \mathrm{GU}_F$ can now easily be described on character groups: it is given by a function

$$\Phi\colon \Sigma_F \to \mathbb{Z}^2\,, \qquad \text{say } \sigma \mapsto (m_\sigma, n_\sigma)$$

such that $(m_{\bar{\sigma}}, n_{\bar{\sigma}}) = (n_\sigma, m_\sigma)$ and such that the function $\sigma \mapsto m_\sigma + n_\sigma$ is constant (=the negative weight). Such a function Φ may be seen as a generalization of the classical notion of a CM-type (the case where $m_\sigma, n_\sigma \in \{0, 1\}$ for all σ, with weight equal to 1.) For the Hodge structure V this means the following. As $\dim_F(V) = 1$ we may identify $V = F$ as an F-vector space. Then

$$V \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{\sigma \in \Sigma_F} \mathbb{C}\,.$$

Then the summand $\mathbb{C}^{(\sigma)}$ is of Hodge type $(-m_\sigma, -n_\sigma)$. As $\mathbb{C}^{(\bar{\sigma})}$ is the complex conjugate of $\mathbb{C}^{(\sigma)}$, the conditions on the function Φ ensure that this indeed gives a Hodge structure on V, of weight $m_\sigma + n_\sigma$. By construction this Hodge structure is simple and of CM-type.

(1.27) Question. ("Converse problem.") Which algebraic groups can occur as the Hodge group of a polarizable $\mathbb{Q}$-HS? In (1.16) and (1.18) we have found some conditions that such a group must satisfy; are these sufficient conditions? Although I have no complete answer, it seems to me that this question is manageable. More interesting, and *much* more difficult is the question whether every "reasonable" group occurs as $\mathrm{Hg}(V)$ for V a $\mathbb{Q}$-HS "coming from geometry". Here I have really no idea of what comes out. For further discussion see for instance [35], [34] and [80]. Notice that there are many polarizable $\mathbb{Q}$-HS which do not come from geometry; see [36], second footnote on page 300. (It would be very interesting to work out this footnote in greater detail.)

(1.28) Remark. Let X be a smooth proper variety over $\mathbb{C}$, say of dimension d. For $0 \leq n \leq 2d$, let $\mathrm{Hg}^n(X)$ be the Hodge group of $H^n(X, \mathbb{Q})$. Write $\mathrm{Hg}(X)$ for the Hodge group of $H^\bullet(X, \mathbb{Q}) := \oplus_n H^n(X, \mathbb{Q})$. Then $\mathrm{Hg}(X)$ is a subgroup of $\prod_n \mathrm{Hg}^n(X)$, projecting surjectively to each factor (cf. (1.13)). In some cases, for instance if X is an abelian variety, it is easy to describe the relation between $\mathrm{Hg}(X)$ and the $\mathrm{Hg}^n(X)$. In general this is not so easy, but it helps to consider the decomposition of the cohomology into primitive pieces. There is a very natural generalization of this, which was proposed only recently in the paper [47] by Looijenga and Lunts. The basic remark is this: the primitive decomposition depends on the choice of an ample class in H^2. It can be viewed as giving $H^\bullet(X,\mathbb{Q})$ the structure of a module under $\mathfrak{sl}_2$, where the primitive decomposition corresponds precisely to the decomposition of $H^\bullet(X,\mathbb{Q})$ into irreducible $\mathfrak{sl}_2$-representations. Now in general there is more than one ample class (up to multiples), and a different choice may lead to a different primitive decomposition. Taking all, or several, ample classes in H^2 simultaneously, one constructs a Lie algebra $\mathfrak{g}$ (generally bigger than $\mathfrak{sl}_2$) acting on $H^\bullet(X,\mathbb{Q})$ and decomposing it into a sum of sub-Hodge structures. This decomposition can be finer than the usual primitive decomposition associated to one ample class.

Further reading. There are many further ideas and constructions that we have not yet touched upon. Among them mixed Hodge structures, variation of Hodge structure, period domains, ... For those who want to read more, Deligne's papers [16], [17], [18], [19], [20], [21], [22] and [23] are a must. Other references of great interest are (a fairly random selection): [25] and [68] (Tannakian categories), [14], [32], [33], [60], [86] (variation of Hodge structure), [31] and [13] (overviews of various developments; the first more geometrically oriented, the second more abstract), [2], [12], [61] (Mumford-Tate groups of mixed HS; in [2] we find a very interesting result on the relation with algebraic monodromy groups; the other two references deal with mixed Shimura varieties). For papers containing interesting examples see also the suggestions at the end of the next section.

§2. Mumford-Tate groups of abelian varieties.

(2.1) *Abelian varieties.* In this section we shall mainly look at Mumford-Tate groups of abelian varieties. There are several reasons why Mumford-Tate groups are particularly effective in this case. For one thing, if X is a complex abelian variety then $H^\bullet(X,\mathbb{Q}) \cong \wedge^\bullet H^1(X,\mathbb{Q})$ as Hodge structures, so that the whole cohomology of X, and even of all powers of X, is determined by $H^1(X,\mathbb{Q})$. Now $H^1(X,\mathbb{Q})$ is a polarizable Hodge structure of level 1, by which we mean that $h^{p,q} = 0$ if $|p-q| > 1$. As we shall see later, this puts interesting restrictions on $\mathrm{MT}(V)$. In the sequel, if X is a complex abelian variety we shall write $\mathrm{MT}(X) := \mathrm{MT}(H^1(X,\mathbb{Q}))$ and $\mathrm{Hg}(X) := \mathrm{Hg}(H^1(X,\mathbb{Q}))$.

Another thing that is special about abelian varieties is that we have a very strong "Torelli" result:

(2.2) Theorem. *The functor $X \mapsto H_1(X,\mathbb{Z})$ gives an equivalence of categories*

$$\begin{pmatrix}\text{abelian varieties}\\ \text{over } \mathbb{C}\end{pmatrix} \xrightarrow{\text{eq.}} \begin{pmatrix}\text{polarizable torsion-free } \mathbb{Z}\text{-HS}\\ \text{of type } (-1,0)+(0,-1)\end{pmatrix}.$$

As variants of this equivalence: polarized abelian varieties correspond to polarized HS, abelian varieties up to isogeny correspond to polarizable $\mathbb{Q}$-HS of type $(-1,0)+(0,-1)$, and families of abelian varieties correspond to polarizable variations of Hodge structures.

Notice that the duality $X \mapsto X^t := \mathrm{Pic}^0_{X/\mathbb{C}}$ of abelian varieties corresponds to the duality $V \mapsto V^*(1)$ of Hodge structures. Furthermore, the notions of a polarization correspond: if $\lambda\colon X \to X^t$ is a polarization of the abelian variety X then the induced morphism $\varphi = H_1(\lambda)\colon H_1(X,\mathbb{Z}) \to H_1(X,\mathbb{Z})^*(1)$ is a polarization of the Hodge structure $H_1(X,\mathbb{Z})$ and vice versa. The form $(2\pi i)^{-1}\cdot\varphi$ is usually referred to as the Riemann form of the polarization.

(2.3) *The results of §1 for $V = V_X$.* Let X be a complex abelian variety. A first consequence of (2.2)—and this is really special about abelian varieties—is that the endomorphism algebra of X equals the endomorphism algebra of V_X:

$$\mathrm{End}^0(X) \xrightarrow{\sim} \mathrm{End}_{\mathbb{Q}\mathbf{HS}}(V_X)\,,$$

where we set $\mathrm{End}^0(X) := \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. (That $D := \mathrm{End}^0(X)$ is a semi-simple $\mathbb{Q}$-algebra with a positive involution is in fact true over an arbitrary field. Given $g := \dim(X)$, there are some numerical restrictions on what D can be. See [53], §21, [59], [81].)

Up to isogeny (notation $\sim$) we can decompose X as

$$X \sim Y_1^{m_1} \times \cdots \times Y_r^{m_r} ,$$

where $Y_1, \ldots, Y_r$ are simple, mutually non-isogenous, abelian varieties and $m_1, \ldots, m_r \in \mathbb{Z}_{\geq 1}$. Write $V_i := H^1(Y_i, \mathbb{Q})$ and $D_i := \mathrm{End}^0(Y_i)$. Then $V_X = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r}$ is a decomposition of V_X as a direct sum of simple $\mathbb{Q}$-HS and $D = M_{m_1}(D_1) \times \cdots \times M_{m_r}(D_r)$. If $r = 1$ then we say that X is *elementary*. If X is simple (i.e., $r = 1$ and $m_1 = 1$) then we say that X is of type I (type II, etc.) if D is of the corresponding type in the Albert classsification.

Let $\lambda \colon X \to X^t$ be a polarization of X. As explained above, it corresponds to a polarization $\varphi \colon V_X \otimes V_X \to \mathbb{Q}(-1)$ of V_X. In (1.21) we have seen that

$$\mathrm{MT}(X) \subseteq \mathrm{GSp}_D(V_X, \varphi) , \quad \text{and} \quad D = [\mathrm{End}(V_X)]^{\mathrm{MT}(X)} .$$

From the description given in (1.26) we easily see that X is of CM-type (in the sense of abelian varieties) if and only if V_X is of CM-type, i.e., iff $\mathrm{MT}(X)$ is a torus. This was first proven by Mumford, [52]. If X has no simple factors of type IV then $\mathrm{Hg}(X)$ is semi-simple.

(2.4) Let X be a simple abelian variety over $\mathbb{C}$. Set $g := \dim(X)$. As remarked above, there are some numerical restrictions on what $D := \mathrm{End}^0(X)$ can be. Most of these can be derived by remarking that $V := H_1(X, \mathbb{Q})$ is a $2g$-dimensional $\mathbb{Q}$-vector space on which D acts and such that there exists a symplectic form φ with $\varphi(dv, w) = \varphi(v, d^\dagger w)$ for all $v, w, \in V$ and $d \in D$. Writing $e := [F : \mathbb{Q}]$ and $d^2 := [D : F]$ we find that $e|g$ if X has type I, that $2e|g$ if X has type II or III and that $ed^2|2g$ if X has type IV.

There are some further invariants associated to the action of D on V. The ones we shall need can be described as follows. Write Σ_F for the set of embeddings of F into $\mathbb{C}$. Then $V \otimes_{\mathbb{Q}} \mathbb{C} = V^{-1,0} \oplus V^{0,-1}$ is free of rank $2g/e$ over

$$F \otimes_{\mathbb{Q}} \mathbb{C} = \prod_{\sigma \in \Sigma_F} \mathbb{C} .$$

Moreover, the action of $F \otimes_{\mathbb{Q}} \mathbb{C}$ respects the Hodge decomposition of $V_{\mathbb{C}}$. Therefore we can write

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma_F} V^{-1,0}(\sigma) \oplus V^{0,-1}(\sigma) ,$$

where $V^{i,j}(\sigma) = \{v \in V^{i,j} \mid f(v) = \sigma(f) \cdot v \quad \text{for all } f \in F\}$. We shall write

$$n_\sigma := \dim_{\mathbb{C}} V^{-1,0}(\sigma) .$$

Note that $\dim V^{-1,0}(\sigma) + \dim V^{0,-1}(\sigma) = 2g/e$ for all σ. Also, writing $\bar{\sigma}$ for the complex conjugate of σ, the summand $V^{-1,0}(\bar{\sigma})$ is complex conjugate to $V^{0,-1}(\sigma)$. This gives the relation

$$n_\sigma + n_{\bar{\sigma}} = 2g/e \qquad \text{for all } \sigma \in \Sigma_F .$$

The integers n_σ are often referred to as the multiplicities of the action of F on the tangent space of X; notice that $V^{-1,0}$ is indeed naturally isomorphic to the tangent space of X at the origin.

(2.5) Comment. How succesful we are in describing, or computing, the Mumford-Tate group of an abelian variety strongly depends on how we play the game. By this we mean the following. If we take an "abstract" abelian variety, we can decompose it (up to isogeny) as a product of powers of simple ones. Then, using the Albert classification, we find a finite list of possibilities for the endomorphism algebra. In each of the possible cases we can try to determine the Mumford-Tate group, possibly after taking into account further "discrete" invariants, such as the type of the action on the tangent space. In many cases this leads to interesting results, and we shall see that sometimes it even allows to prove the Hodge conjecture for our abelian variety. Usually, however, the abelian variety itself remains "invisible" in this game; it is not a concrete geometric object. What we are really studying are abstract $\mathbb{Q}$-Hodge structures satisfying certain properties. By contrast, if we start with an abelian variety as coming from some geometric situation, e.g., as the Jacobian of some other variety, then it is often very hard to determine its Mumford-Tate group.

(2.6) Example. Let E be an elliptic curve over $\mathbb{C}$. We know that the Hodge group $\mathrm{Hg}(E)$ is a reductive subgroup of $\mathrm{Sp}(V,\varphi) = \mathrm{SL}_{2,\mathbb{Q}}$. A priori there are therefore only three possibilities: either (a) $\mathrm{Hg}(E) = \{1\}$, or (b) $\mathrm{Hg}(E) = \mathrm{SL}_{2,\mathbb{Q}}$, or (c) $\mathrm{Hg}(E)$ is a maximal torus of $\mathrm{SL}_{2,\mathbb{Q}}$. On the other hand, we know that either $\mathrm{End}^0(E) = \mathbb{Q}$ or $\mathrm{End}^0(E) = k$ is an imaginary quadratic field. This rules out case (a). The remaining options are now easily matched:

(i) if $\mathrm{End}^0(E) = \mathbb{Q}$ then $\mathrm{Hg}(E)$ is semi-simple so we must have $\mathrm{Hg}(E) = \mathrm{SL}_{2,\mathbb{Q}}$.

(ii) if $\mathrm{End}^0(E) = k$ is an imaginary quadratic field then $\mathrm{Hg}(E)$ is a torus contained in $\mathrm{U}_k = \{z \in k^* \mid z\bar{z} = 1\}$; see (1.23). Since U_k has rank 1 we have $\mathrm{Hg}(E) = \mathrm{U}_k$.

(2.7) Example. Let X be a simple abelian surface. For $D := \mathrm{End}^0(X)$ we have the following four possibilities:

(i) $D = \mathbb{Q}$,

(ii) $D = F$ is a real quadratic field,

(iii) D is an indefinite quaternion algebra over $\mathbb{Q}$,

(iv) $D = F$ is a CM-field of degree 4 over $\mathbb{Q}$ which does not contain an imaginary quadratic field.

(Note that for $\dim(X) = 2$ we cannot have that $\mathrm{End}^0(X)$ is an imaginary quadratic field; see [81].)

We claim that in each of these cases we have $\mathrm{Hg}(X) = \mathrm{Sp}_D(V,\varphi)$. We first do case (ii). We know that $\mathrm{Hg}(X)$ is semi-simple and contained in $\mathrm{Sp}_F(V,\psi) = \mathrm{SL}_F(V) \cong \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SL}_{2,F}$. Write H for this group. Then $H_\mathbb{C} \cong \mathrm{SL}_2 \times \mathrm{SL}_2$. Writing $\mathrm{St}^{(1)}$ resp. $\mathrm{St}^{(2)}$ for the standard respresentation of the first (resp. second) factor SL_2 we have $V_X \otimes_\mathbb{Q} \mathbb{C} \cong \mathrm{St}^{(1)} \oplus \mathrm{St}^{(2)}$ as a $H_\mathbb{C}$-module. Now assume that $\mathrm{Hg}(X) \neq \mathrm{Sp}_F(V,\varphi)$. As $\mathrm{Hg}(X)$ is semi-simple we must have $\mathrm{Hg}(X)_\mathbb{C} \cong \mathrm{SL}_2 \hookrightarrow H_\mathbb{C} = \mathrm{SL}_2 \times \mathrm{SL}_2$. Since

$$\mathrm{End}(V_{X,\mathbb{C}})^{\mathrm{Hg}(X)_\mathbb{C}} = [\mathrm{End}(V_X)^{\mathrm{Hg}(X)}] \otimes \mathbb{C} = F \otimes \mathbb{C} \cong \mathbb{C} \times \mathbb{C} \qquad (2.7.1)$$

the projections of $\mathrm{Hg}(X)_\mathbb{C}$ to the two factors SL_2 are both surjective. Now remark that SL_2 has only one irreducible 2-dimensional representation, up to isomorphism. It follows that $V_{X,\mathbb{C}} \cong \mathrm{St}^{\oplus 2}$ as a representation of $\mathrm{Hg}(V)_\mathbb{C}$. As this contradicts (2.7.1) we conclude that $\mathrm{Hg}(X) = \mathrm{Sp}_F(V,\psi) = \mathrm{SL}_F(V)$.

Case (iii) is quite easy. We know that $\mathrm{Hg}(X)$ is semi-simple and contained in $\mathrm{Sp}_D(V,\varphi)$. The latter group can be described as follows. Let D^{opp} be the opposite of the algebra D and write $d \mapsto d^*$ for its canonical involution. Then $\mathrm{Sp}_D(V,\varphi) \cong \mathrm{U}_{D^{\mathrm{opp}}}$, the algebraic $\mathbb{Q}$-group given on points by

$$\mathrm{U}_{D^{\mathrm{opp}}} = \{d \in (D^{\mathrm{opp}})^* \mid dd^* = 1\}\,,$$

which is a $\mathbb{Q}$-form of SL_2. By rank considerations we must have $\mathrm{Hg}(X) = \mathrm{U}_{D^{\mathrm{opp}}}$.

Case (iv) is also not difficult. In this case $\mathrm{Hg}(X)$ is contained in the torus U_F, which has rank 2. Using that F does not contain an imaginary quadratic field, one checks that U_F does not contain any nontrivial $\mathbb{Q}$-subtorus. (This is done using the explicit description of $X^*(\mathrm{U}_F)$ given in (1.23).)

Finally, suppose we are in case (i). Then $\mathrm{Hg}(X)$ is semi-simple, contained in $\mathrm{Sp}_\mathbb{Q}(V,\varphi) \cong \mathrm{Sp}_{4,\mathbb{Q}}$, and $V_{X,\mathbb{C}}$ is an irreducible representation of $\mathrm{Hg}(X)_\mathbb{C}$. Suppose that $\mathrm{Hg}(X) \neq \mathrm{Sp}(V,\varphi)$. By rank considerations we must have $\mathrm{Hg}(X)_\mathbb{C} \cong \mathrm{SL}_2$ and $V_{X,\mathbb{C}} \cong \mathrm{Sym}^3(\mathrm{St})$. (Notice that $\mathrm{SL}_2 \times \mathrm{SL}_2$ does not have an irreducible faithful symplectic representation of dimension 4. Also notice that $-\mathrm{id} \in Z(\mathrm{Hg})$, so $\mathrm{Hg}(X)_\mathbb{C}$ could also not be PSL_2.) If $T \subset \mathrm{Hg}(X)_\mathbb{C}$ is a maximal torus and t is a generator of the character group $X^*(T)$ then the weights of T that occur are

$$\begin{array}{cccc} t^{-3} & t^{-1} & t & t^3 \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

Let $\tilde{T} \subset \mathrm{MT}(X)_\mathbb{C}$ be the maximal torus generated by $\mathbb{G}_m \cdot \mathrm{id}$ and T. (We are still assuming that $\mathrm{Hg}(X)_\mathbb{C} \cong \mathrm{SL}_2$.) Then $X^*(\tilde{T}) \cong \mathbb{Z}^2$ and the 4 weights of $\tilde{T}$ on $V_{X,\mathbb{C}}$ lie on a line (not through the origin). At this point we use a little extra information about the Hodge structure $V_{X,\mathbb{C}}$. Namely, consider the cocharacter $h\circ\mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathrm{MT}(X)_{\mathbb{C}}$. Letting $\mathbb{G}_m$ act on $V_{X,\mathbb{C}}$ through this cocharacter we know that the weights that occur are $z \mapsto z^{-1}$ and $z \mapsto 1$. We can choose our $\tilde{T}$ such that $h\circ\mu$ factors through it. On character groups this gives a $\mathbb{Z}$-linear map $X^*(\tilde{T}) \to X^*(\mathbb{G}_m) = \mathbb{Z}$ such that the 4 weights of $\tilde{T}$ map onto the 2 weights of $\mathbb{G}_m$. This is clearly impossible. We conclude that $\mathrm{Hg}(X) = \mathrm{Sp}(V,\varphi)$.

(2.8) Remark. Let X be a complex abelian variety. Let $\tilde{T} \subset \mathrm{MT}(X)_{\mathbb{C}}$ be a maximal torus. Assume that $\mathrm{End}^0(X) = \mathbb{Q}$, which means precisely that $V_{X,\mathbb{C}}$ is an irreducible representation of $\mathrm{MT}(X)_{\mathbb{C}}$. The weights of $\tilde{T}$ in $V_{X,\mathbb{C}}$ form a finite subset (with multiplicities) $\mathrm{Supp}(V_{X,\mathbb{C}}) \subset X^*(\tilde{T}) \cong \mathbb{Z}^r$. The arguments used in case (i) above can be visualized by saying that there exists a $\mathbb{Z}$-linear map $X^*(\tilde{T}) \to \mathbb{Z}$ such that the image of $\mathrm{Supp}(V_{X,\mathbb{C}})$ consists of two elements. In other words: the weights that occur lie in two parallel hyperplanes inside $X^*(\tilde{T})$. This restriction comes from the fact that only two types occur in the Hodge decomposition of V_X and puts strong restrictions on the representations that can occur. Similar arguments can be applied to arbitrary Hodge structures; we shall study this in §3.

(2.9) *The Hodge conjecture.* We shall use Mumford-Tate groups to prove the Hodge conjecture in certain cases. Let us first set up some notations and recall the statement of the conjecture.

Let X be a nonsingular proper variety over $\mathbb{C}$. We write $\mathscr{B}^n(X) \subseteq H^{2n}(X,\mathbb{Q})(n)$ for the subspace of Hodge classes. Then $\mathscr{B}^\bullet := \oplus_n \mathscr{B}^n(X)$ is a commutative graded $\mathbb{Q}$-algebra, called the *Hodge ring* of X. If Z is an algebraic cycle on X of codimension n then there is an associated cohomology class $cl(Z) \in \mathscr{B}^n(X)$. We in fact have a homomorphism of graded $\mathbb{Q}$-algebras $cl\colon \mathrm{CH}^\bullet_{\mathbb{Q}}(X) \to \mathscr{B}^\bullet(X)$. The (special) Hodge conjecture is the following:

$$\mathrm{HC}(X,n): \qquad \text{the map } cl\colon \mathrm{CH}^n_{\mathbb{Q}}(X) \to \mathscr{B}^n(X) \quad \text{is surjective.}$$

In other words, the conjecture says that every Hodge class in $H^{2n}(X,\mathbb{Q})(n)$ is a $\mathbb{Q}$-linear combination of algberaic classes $cl(Z)$. Let us also write $\mathrm{HC}(X)$ for the statement "$\mathrm{HC}(X,n)$ holds for all n".

Although the Hodge theorem (which gives rise to the Hodge structure on $H^{2n}(X,\mathbb{Q})$) works for compact Kähler manifolds, not necessarily projective, it is known that the Hodge conjecture is definitely *false* for general compact Kähler manifolds. For an example, see [46], ??.

The Hodge conjecture was originally formulated by Hodge with $\mathbb{Z}$-coefficients. In the paper [5] by Atiyah and Hirzebruch it was shown however, that $H^{2n}(X,\mathbb{Z})(n)$ may contain torsion classes which are not algebraic. But not only torsion phenomena force us to work with $\mathbb{Q}$-coefficients: it may also happen that $H^{2n}(X,\mathbb{Z})(n)$ contains elements ξ which are not torsion, such that some multiple $m\cdot\xi$ is the class of an algebraic cycle but ξ itself is not; see for instance [6].

(2.10) *Divisor classes.* It is a theorem of Lefschetz that $\mathrm{HC}(X,1)$ holds for every (proper, nonsingular) X. So, $\mathscr{B}^1(X) \subset H^2(X,\mathbb{Q})(1)$ is generated by the divisor classes. Now set $\mathscr{D}^1(X) := \mathscr{B}^1(X)$, and write $\mathscr{D}^\bullet(X) \subseteq \mathscr{B}^\bullet(X)$ for the $\mathbb{Q}$-subalgebra generated by $\mathscr{D}^1(X)$. In other words: $\mathscr{D}^\bullet(X)$ consists of all $\mathbb{Q}$-linear combinations of cup-products of divisor classes. Clearly all classes in $\mathscr{D}^\bullet(X)$ are algebraic; in particular, if $\mathscr{D}^\bullet(X) = \mathscr{B}^\bullet(X)$ then the Hodge conjecture is true for X.

In general it is quite easy to cook up examples where $\mathscr{D}^\bullet(X) \neq \mathscr{B}^\bullet(X)$, taking X to be a (suitable) hypersurface in some $\mathbb{P}^n$ for instance. For abelian varieties the situation is different. We shall find many cases where $\mathscr{D}^\bullet(X) = \mathscr{B}^\bullet(X)$, and it is not so easy to produce examples where this actually does not hold. We shall further go into this below.

(2.11) *The basic strategy.* Let us now explain a strategy that was already hinted at in (2.5). Take an abelian variety X. We want to prove $\mathrm{HC}(X)$, or at least we want to find out if $\mathscr{D}^\bullet(X) = \mathscr{B}^\bullet(X)$.

Assume we know $D := \mathrm{End}^0(X)$. In any case, knowing X, and possibly the way X decomposes into simple factors, the Albert classification gives us a finite number of possible "types" for D, which we can try to deal with one by one. Now we have

$$\mathrm{MT}(X) \subseteq \mathrm{GSp}_D(V,\varphi)\,, \quad \text{and} \quad D = [\mathrm{End}(V_X)]^{\mathrm{MT}}(X)\,.$$

The first gives an "upper bound" for $\mathrm{MT}(X)$, the second says that $\mathrm{MT}(X)$ cannot be too much smaller than $\mathrm{GSp}_D(V,\varphi)$. Using this, combined with various facts from the theory of reductive groups, we can try to determine (a finite list of possibilities for) $\mathrm{MT}(X)$ and its representation V_X.

Once we know (a candidate for) $\mathrm{MT}(X)$ we can turn things around: the Hodge ring $\mathscr{B}^\bullet(X)$ is the space of $\mathrm{Hg}(X)$-invariants in $\wedge^\bullet V_X$, and is therefore (at least in principle) easy to compute. (Note that for the computation of invariants we can extend scalars to $\mathbb{C}$, which is sometimes helpful.) In any case, the question whether $\mathscr{D}^\bullet(X) = \mathscr{B}^\bullet(X)$ now becomes a problem in invariant theory.

(2.12) *Some results from invariant theory.* Almost all we shall need from invariant theory can be reduced to one of the following cases.

(C) *Symplectic groups.* Let W be a $\mathbb{C}$-vector space equipped with a non-degenerate symplectic form φ. Let $m \in \mathbb{Z}_{\geq 1}$. Then the graded $\mathbb{C}$-algebra

$$[\bigwedge^{\bullet}(W^{\oplus m})]^{\mathrm{Sp}(W,\varphi)}$$

is generated by its elements in degree 2.

(A) *Unitary groups.* Let W be a $\mathbb{C}$-vector space equipped with a non-degenerate hermitian form φ. Let $m \in \mathbb{Z}_{\geq 1}$. Then the graded $\mathbb{C}$-algebra

$$[\bigwedge^{\bullet}(W^{\oplus m})]^{\mathrm{U}(W,\varphi)}$$

is generated by its elements in degree 2.

(2.13) Example. Suppose X is simple and $\mathrm{End}^0(X) = F$ is a totally real field. Suppose furthermore that $\mathrm{MT}(X) = \mathrm{GSp}_F(V,\varphi)$, so $\mathrm{Hg}(X) = \mathrm{Sp}_F(V,\varphi)$. Let $\Sigma_F = \{\sigma_1,\ldots,\sigma_e\}$ be the set of embeddings of F into $\mathbb{C}$ and set $d = \dim_F(V_X)$. Then $V_{X,\mathbb{C}}$ is free of rank d over $F \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}_{(1)} \times \cdots \times \mathbb{C}_{(e)}$, so it decomposes as $V_{X,\mathbb{C}} = W_{(1)} \oplus \cdots \oplus W_{(e)}$. The symplectic form φ on $V_{X,\mathbb{C}}$ decomposes as a sum of symplectic forms $\varphi_{(j)}\colon W_{(j)} \times W_{(j)} \to \mathbb{C}_{(j)}$. (Exercise: show this, using that the Rosati involution on F is trivial.) We then have $\mathrm{Sp}_F(V_X,\varphi) \otimes \mathbb{C} = \mathrm{Sp}(W_{(1)},\varphi_{(1)}) \times \cdots \times \mathrm{Sp}(W_{(e)},\varphi_{(e)})$ and

$$\mathscr{B}^\bullet(X^m) = [\bigwedge^{\bullet}(V_{X,\mathbb{C}}^{\oplus m})]^{\mathrm{Sp}_F(V_X,\varphi)\otimes\mathbb{C}} = [\bigwedge^{\bullet}(W_{(1)}^{\oplus m})]^{\mathrm{Sp}(W_{(1)},\varphi_{(1)})} \otimes \cdots \otimes [\bigwedge^{\bullet}(W_{(e)}^{\oplus m})]^{\mathrm{Sp}(W_{(e)},\varphi_{(e)})}\,.$$

Applying (2.12), case (C), we find that this algebra is generated by its elements in degree 2, which just means that $\mathscr{D}^\bullet(X^m) = \mathscr{B}^\bullet(X^m)$ for all m.

Similar arguments, now working with unitary groups, work in case F is a CM-field and $\mathrm{Hg}(X) = \mathrm{U}_F(V_X,\varphi)$.

Let us now give a number of results on Mumford-Tate groups of abelian varieties and applications to the Hodge conjecture. For the proofs of these results we refer to the literature. First we have a result of Hazama [37] and Murty [54]. The proof heavily uses invariant theory of the kind indicated above.

(2.14) Theorem. *Let X be a complex abelian variety. Set $D := \mathrm{End}^0(X)$, let $V := H_1(X,\mathbb{Q})$ and let φ be the Riemann form of a polarization. Then*

$$\mathscr{B}^\bullet(X^n) = \mathscr{D}^\bullet(X^n) \quad \text{for all } n \geq 1 \quad \Longleftrightarrow \quad \begin{pmatrix} X \text{ has no factors of type III} \\ \text{and} \quad \mathrm{Hg}(X) = \mathrm{Sp}_D(V,\varphi) \end{pmatrix}.$$

The next result, and especially its corollary, is perhaps surprising if you see it for the first time. After all, *a priori* most people would probably not expect simple abelian varieties of dimension 31, say, to be much simpler than ones of dimension 32, say. But if we think of abelian varieties (up to isogeny) just as being special kinds of Hodge structures then it is already much more plausible that numerical conditions on the dimension could make a big difference for what possibilities may occur.

The corollary is due to Tankeev [89], although it seems that several cases were done independently by Ribet and Serre. The theorem as we state it includes generalizations due to Ribet [67]. We refer to this very readable paper for the proof. Some of the technical results needed in the paper are discussed in the next section. (See for instance (3.14) where we prove a special case of statement (i).) In the statement we use the notations as explained in (1.22) and (2.4).

(2.15) Theorem. *Let X be a complex abelian variety. Set $g := \dim(X)$.*

(i) *Suppose* $\mathrm{End}^0(X) = F$ *is a totally real field such that the integer $g/[F:\mathbb{Q}]$ is odd. Then* $\mathrm{Hg}(X) = \mathrm{Sp}_F(V,\psi)$.

(ii) *Suppose* $\mathrm{End}^0(X) = k$ *is an imaginary quadratic field such that n_σ and $n_{\bar\sigma}$ are relatively prime. Then* $\mathrm{Hg}(X) = \mathrm{U}_k(V,\psi)$.

(iii) *Suppose g is prime and* $\mathrm{End}^0(X) = F$ *is a CM-field of degree $2g$ over $\mathbb{Q}$. Then* $\mathrm{Hg}(X) = \mathrm{U}_F(V,\psi) = \mathrm{U}_F$.

(2.16) Corollary. *Let X be a simple abelian variety of prime dimension. Then* $\mathrm{Hg}(X) = \mathrm{Sp}_D(V,\varphi)$ *(notations as in* (2.14)*) and $\mathscr{B}^\bullet(X^n) = \mathscr{D}^\bullet(X^n)$ for all n. In particular, the Hodge conjecture is true for all X^n.*

To get the corollary from the theorem note that if X is simple of prime dimension g, the endomorphism algebra $\mathrm{End}^0(X)$ can only be $\mathbb{Q}$, or a totally real field of degree g over $\mathbb{Q}$, or an imaginary quadratic field, or a CM-field of degree $2g$ over $\mathbb{Q}$. All these cases are covered by (2.15). Note also that X is not of type III, so that (2.14) applies. (In fact, in all cases that occur the implication "$\Leftarrow$" in (2.14) can be proven by hand; (2.14) in its full strength was proven later than (2.15) and (2.16).)

There are many more cases where it is proven that $\mathrm{Hg}(X) = \mathrm{Sp}_D(V,\varphi)$; see for instance [38], [50], [49], [55], [62] or the many references given in [30].

(2.17) *Weil classes.* Let us now explain a construction that leads to examples where $\mathscr{B}^\bullet(X) \neq \mathscr{D}^\bullet(X)$. The first such example (with X an abelian variety) was given by Mumford; see [64]. The construction given here is due to Weil [94], who remarked that there is one ingredient in the example constructed by Mumford which is essential. For more details we refer to [51].

Start with a $\mathbb{Q}$-HS V, a (commutative) field k and a homomorphism $k \to \mathrm{End}_{\mathbb{Q}\mathbf{HS}}(V)$ sending 1 to id_V. Set $d := \dim_k(V)$. Then $\wedge_k^d V$ is a 1-dimensional k-vector space and there is a canonical $\mathbb{Q}$-linear surjection

$$p = p_V \colon \bigwedge_{\mathbb{Q}}^{d} V \twoheadrightarrow \bigwedge_{k}^{d} V\,.$$

It is not difficult to show that $\mathrm{Ker}(p)$ is a $\mathbb{Q}$-subHS of $\wedge_{\mathbb{Q}}^d V$. By the semi-simplicity of $\mathbb{Q}\mathbf{HS}$ is follows that there is a unique $\mathbb{Q}$-subHS

$$W_k \subset \bigwedge_{\mathbb{Q}}^{d} V$$

which maps isomorphically to $\wedge_k^d V$ under p. Alternatively, as we are working over a ground field of characteristic 0 the natural $\mathbb{Q}$-linear map $(\wedge_{\mathbb{Q}}^d V)^* \to [\wedge_{\mathbb{Q}}^d (V^*)]$ is an isomorphism and we may define $W_k \subset \wedge_{\mathbb{Q}}^d V$ by dualizing p_{V^*}. We refer to W_k as the space of *Weil classes* w.r.t. the given action of k.

Let us now specialize this to the case of abelian varieties, taking $V = H^1(X,\mathbb{Q})$. In this case W_k is a subspace of $H^d(X,\mathbb{Q})$ and one can show ([51], sections 2–6) that the following conditions are equivalent:

(i) the space W_k contains a non-zero Hodge class,
(ii) the space W_k consists entirely of Hodge classes,
(iii) for all embeddings $\sigma\colon k \to \mathbb{C}$ we have, using the notations introduced in (2.4), $n_\sigma = n_{\bar\sigma}$,
(iv) the Hodge group $\mathrm{Hg}(X) \subset \mathrm{GL}_k(V)$ is contained in $\mathrm{SL}_k(V)$.

For instance, if X has no factors of type IV then $\mathrm{Hg}(X)$ is semi-simple and (iv) is automatically satisfied. Notice that the four equivalent conditions can only be satisfied if V has even dimension as a k-vector space.

Now suppose that W_k consists of Hodge classes. Then the next question is whether these classes lie in $\mathscr{D}^\bullet(X)$. In loc. cit. we find a complete answer to this questions, purely in terms of the given action of k on X. Here we shall only give some examples.

Exercise. The multiplicative group k^* acts on V. It also acts on W_k. Describe how the two actions are related. Show that (i) and (ii) above are equivalent. Also show that if there is one non-zero class in W_k which lies in $\mathscr{D}^\bullet(X)$ then W_k is fully contained in $\mathscr{D}^\bullet(X)$.

(2.18) Example. Suppose that X is simple and that $\mathrm{End}^0(X) = k$ is an imaginary quadratic field. This implies that X has even dimension, say $\dim(X) = 2d$. Suppose furthermore that the multiplicities of the action of k on the tangent space of X at the origin satisfy $n_\sigma = n_{\bar\sigma} = 2d$. Then $W_k \subset H^{2d}(X, \mathbb{Q})$ is a 2-dimensional subspace consisting of Hodge classes. But the Picard number of X (:= the dimension of $\mathscr{B}^1(X)$) is 1. So W_k could impossibly be contained in $\mathscr{D}^d(X)$. ˚e conclude that W_k consists of exceptional Hodge classes, i.e., Hodge classes which do not lie in $\mathscr{D}^\bullet(X)$.

(2.19) Example. Suppose that X is simple of type III. Recall that this means that $D := \mathrm{End}^0(X)$ is a quaternion algebra over a totally real field F such that D is inert at all infinite places. Notice that for every element $\alpha \in D \setminus F$ the subfield $F(\alpha) \subset D$ is a CM-field. Take $k = F(\alpha)$ for such an α. As remarked above, the semi-simplicity of $\mathrm{Hg}(X)$ implies that W_k consists of Hodge classes. In fact, with a construction as in (1.22) we find that the Hodge group is contained in a special unitary group $\mathrm{SU}_k(V, \psi)$. Next one shows that all divisor classes are invariant under the full unitary group $\mathrm{U}_k(V, \psi)$ and that this group acts on W_k as multiplication by the k-linear determinant $\det_k\colon \mathrm{U}_k(V, \psi) \to k^*$. Again it follows that W_k consists of exceptional Hodge classes. (For more details see e.g., [50], especially section 3.)

(2.20) Example. As a final example, suppose that X is a product of two abelian varieties, say $X = Y_1 \times Y_2$. Set $d_i := 2\dim(Y_i)/[k : \mathbb{Q}]$, so that $d = d_1 + d_2$. Then $W_k = W_k(X) \subset H^d(X, \mathbb{Q})$ may be identified with the subspace $W_k(Y_1) \otimes W_k(Y_2)$ of the Künneth component $H^{d_1}(Y_1, \mathbb{Q}) \otimes H^{d_2}(Y_2, \mathbb{Q}) \subset H^d(X, \mathbb{Q})$. Suppose then that $W_k(X)$ consists of Hodge classes. It follows that these Hodge classes can lie in $\mathscr{D}^\bullet(X)$ only if $W_k(Y_1)$ and $W_k(Y_2)$ are contained in $\mathscr{D}^\bullet(Y_1)$ resp. $\mathscr{D}^\bullet(Y_2)$. But this is clearly only possible if d_1 and d_2 are even.

To make this more concrete, suppose Y_1 is an elliptic curve such that $\mathrm{End}^0(Y_1) = k$ is imaginary quadratic, and suppose Y_2 is a simple abelian threefold such that there exists an embedding $k \to \mathrm{End}^0(Y_2)$. Put $X = Y_1 \times Y_2$. We can choose an embedding $k \to \mathrm{End}^0(X)$ such that $W_k \subset H^4(X, \mathbb{Q})$ consists of Hodge classes. (Exercise: check this, using condition (iii) in (2.17) above. You need to know that the multiplicities of the k-action on Y_2 are either $(n_\sigma, n_{\bar\sigma}) = (2, 1)$ or $(n_\sigma, n_{\bar\sigma}) = (1, 2)$, see [81], Prop. 14.) As $W_k(Y_1) = H^1(Y_1, \mathbb{Q})$ and $W_k(Y_2) \subset H^3(Y_2, \mathbb{Q})$ can obviously not consist of divisor classes we find once again an example where $\mathscr{B}^\bullet(X) \neq \mathscr{D}^\bullet(X)$.

(2.21) Remark. In all the examples discussed here one is now faced with the task of finding algebraic cycles giving rise to the exceptional Hodge classes found. This is usually very hard; the desired cycles are known to exist only in some very special cases. See Shioda [82] (an example of the type discussed in (2.20), with $\dim(X) = 4$), Schoen [71], [73] and van Geemen [93], [92] (examples of the type discussed in (2.18), with $\dim(X) = 4$ and an action of either $\mathbb{Q}(i)$ or $\mathbb{Q}(\zeta_3)$). As already commented on in (2.5), the first difficulty is that the abelian varieties in question are not constructed in a (projective) geometrical way. Indeed, in all examples referred to, the story begins with a more geometrical description of the abelian variety, either as a projective variety defined by some special equation, or as a Prym variety associated to a covering of curves, or through a study of theta functions.

As the last topic in this section, let us indicate how in some cases Mumford-Tate groups can even be used to tackle the general Hodge conjecture. First some preparations.

(2.22) Definition. Let V be a $\mathbb{Q}$-HS. Then the *level* of V is defined to be the minimum of $|p - q|$ for all $(p, q) \in \mathbb{Z}^2$ with $V^{p,q} \neq 0$.

For instance, $H^n(X, \mathbb{Q})$ has level at most n.

(2.23) *Sub-HS defined by algebraic subvarieties.* Let X again be a proper, nonsingular variety over $\mathbb{C}$. If $i\colon Z \hookrightarrow X$ is an algebraic subvariety of codimension p, Deligne's mixed Hodge theory (see [20]) gives the following. Consider a resolution of singularities $\pi\colon \tilde{Z} \to Z$ and write $\tilde{i} = i \circ \pi$. Let $d = \dim(X)$. If

$\tilde{i}_!\colon H^{m-2p}(\tilde{Z},\mathbb{Q})(-p) \to H^m(X,\mathbb{Q})$ denotes the transpose under Poincaré dualtiy of $\tilde{i}^*\colon H^{2d-m}(X,\mathbb{Q}) \to H^{2d-m}(\tilde{Z},\mathbb{Q})$ then we have an exact "Gysin" sequence

$$H^{m-2p}(\tilde{Z},\mathbb{Q})(-p) \xrightarrow{\tilde{i}_!} H^m(X,\mathbb{Q}) \longrightarrow H^m(X-Z,\mathbb{Q})\,.$$

Now define a filtration, the so-called *arithmetic filtration* $F'^{\bullet}$ on $H^m(X,\mathbb{Q})$ by

$$F'^pH^{2n}(X,\mathbb{Q}) := \left\{ \xi\in H^{2n}(X,\mathbb{Q}) \;\middle|\; \begin{array}{l}\text{there exists a Zariski-closed } Z\subset X\\ \text{with } \mathrm{cod}_X(Z)\geq p \text{ such that}\\ \xi \text{ maps to zero in } H^{2n}(X-Z,\mathbb{Q})\end{array}\right\}.$$

Since $\tilde{i}_!$ is a morphism of Hodge structures, $F'^pH^m(X,\mathbb{Q})$ is a sub-$\mathbb{Q}$-HS of $H^m(X,\mathbb{Q})$ which is contained in $F^pH^m(X,\mathbb{C})\cap H^m(X,\mathbb{Q})$. In particular, $F'^pH^m(X,\mathbb{Q})(n)$ has level $\leq m-2p$.

(2.24) *The general Hodge conjecture.* After these preparations we can recall the statement of the general Hodge conjecture. Again we start with a proper nonsingular variety X over $\mathbb{C}$. Then the general Hodge conjecture says:

$$\mathrm{GHC}(X,m,p): \qquad \text{if } V\subset H^m(X,\mathbb{Q}) \text{ is a sub-HS of level} \leq m-2p \text{ then } V\subset F'^pH^m(X,\mathbb{Q}).$$

The strategy that we shall try to explain is based on the following lemma, which we copy from Schoen's paper [72].

(2.25) Lemma. *Let X be a smooth proper variety over $\mathbb{C}$. Let $V\subseteq H^n(X,\mathbb{Q})$ be a $\mathbb{Q}$-subHS contained in $F^kH^n(X,\mathbb{C})$. Suppose there exists a smooth proper variety Y such that (i) $V(k)$ is isomorphic to a $\mathbb{Q}$-subHS of $H^{n-2k}(Y,\mathbb{Q})$ and (ii) the Hodge (p,p)-conjecture is true for $Y\times X$. Then $V\subseteq F'^kH^n(X,\mathbb{Q})$.*

Proof. Set $d=\dim(Y)$, $e=\dim(X)$. We assume that $V\neq(0)$. Choose a morphism of $\mathbb{Q}$-HS $\varphi\colon V(k)\hookrightarrow H^{n-2k}(Y,\mathbb{Q})$. Since $H^{n-2k}(Y,\mathbb{Q})$ is a polarisable $\mathbb{Q}$-HS we can choose a decomposition (as $\mathbb{Q}$-HS) $H^{n-2k}(Y,\mathbb{Q})=\mathrm{Im}(\varphi)\oplus V'$. Consider the composition

$$\xi\colon H^{n-2k}(Y,\mathbb{Q}) \xrightarrow{\mathrm{pr}} \mathrm{Im}(\varphi) \xrightarrow{\varphi^{-1}} V(k) \hookrightarrow H^n(X,\mathbb{Q})(k)\,.$$

By the Künneth formula and Poincaré duality we have,

$$H^{2d+2k}(Y\times X,\mathbb{Q})(d+k) \cong \bigoplus_{i=2k}^{2d+2k} \mathrm{Hom}\big(H^{i-2k}(Y,\mathbb{Q}),H^i(X,\mathbb{Q})(k)\big)\,.$$

Thus we see that ξ gives a Hodge class in $H^{2d+2k}(Y\times X,\mathbb{Q})(d+k)$. By assumption (ii) there exists an algebraic cycle $Z\subset Y\times X$ of codimension $d+k$ such that $cl(Z)$ is an integer multiple of ξ.

Write $p_1:=\mathrm{pr}_Y\colon Y\times X\to Y$ and $p_2=\mathrm{pr}_X\colon Y\times X\to X$. In terms of the cycle Z the map $\xi\colon H^{n-2k}(Y,\mathbb{Q})\to H^n(X,\mathbb{Q})(k)$ is given by

$$\xi\colon \eta\mapsto p_{2,*}(p_1^*(\eta)\cup cl(Z))\,,$$

which can be rewritten as

$$\xi\colon \eta\mapsto D\big(p_{2,*}(p_1^*(\eta)\cap[Z])\big)\,,$$

where $[Z]\in H_{2(e-k)}(Y\times X,\mathbb{Q})$ is the fundamental class of Z and where $D\colon H_{2e-n}(X,\mathbb{Q})\to H^n(X,\mathbb{Q})$ is the Poincaré duality isomorphism. (Here we are being sloppy about Tate twists.)

Write $\overline{Z}=p_2(Z)$. Then we have a diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & Y\times X \\ {\scriptstyle p_2'}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ \overline{Z} & \xrightarrow[j]{} & X \end{array}$$

and $p_{2,*}(p_1^*(\eta) \cap [Z])$ is equal to $(j \circ p_2')_*\big(D'(i^* p_1^*(\eta))\big)$, where $D'\colon H^{n-2k}(Z,\mathbb{Q}) \to H_{2e-n}(Z,\mathbb{Q})$ is Poincaré duality. Hence ξ factors through $H_{2e-n}(Z,\mathbb{Q})$. If $Z \to \overline{Z}$ is not a birational morphism then the map ξ is zero, contradicting our assumption that $V \neq (0)$. Therefore, we have $\dim(\overline{Z}) = \dim(Z) = e - k$ and $\mathrm{codim}_X(\overline{Z}) = k$. This shows that $V \subseteq F'^k H^n(X,\mathbb{Q})$, as claimed. □

Exercise. Do not read any further! Take an abelian variety X, write $V = H^1(X,\mathbb{Q})$ and assume that $\mathrm{MT}(X) = \mathrm{CSp}(V,\varphi)$. (In particular, $\mathrm{End}^0(X) = \mathbb{Q}$.) Try to prove the GHC for all powers of X.

From this lemma a clear strategy emerges for trying to prove GHC(X), at least in some cases. Namely, suppose we know HC(X^n) for all $n \geq 1$, for instance because we know that $\mathrm{Hg}(X) = \mathrm{Sp}_D(V,\varphi)$ and X has no factors of type III. Let $V \subset H^n(X,\mathbb{Q})$ be a $\mathbb{Q}$-subHS contained in $F^k H^n(X,\mathbb{Q})$. Without loss of generality we may assume V to be simple. In order to show that $V \subset F\prime k H^n(X,\mathbb{Q})$ it would suffice, by the lemma, to find a non-zero $\mathrm{Hg}(X)$-equivariant homomorphism $V \to H^{n-2k}(X^m,\mathbb{Q})$ for some m. But just as in (2.11) this becomes a problem in representation theory.

The following theorem summarizes how far this method has been pushed, at present. It collects the main results of the papers [91] by Tankeev, [39] by Hazama and [1] by Abdulali. (In some special cases the result was already known.) There are further examples that can be dealt with, see for instance [72] or the overview paper [30].

(2.26) Theorem. *Let X be a complex abelian variety.*

(i) *Suppose that either* (a) *X has only simple factors of type* I *and* II *and* $\mathrm{Hg}(X) = \mathrm{Sp}_D(V,\varphi)$, *or* (b) *$X$ is a product of elliptic curves. Then GHC(X^n, m, p) holds for all n, m and p.*

(ii) *Suppose that X has no simple factors of type* IV *and that* $\mathrm{Hg}(X) = \mathrm{Sp}_D(V,\varphi)$. *Suppose furthermore that for every simple factor Y of type* III *the integer* $2\dim(Y)/[\mathrm{End}^0(Y):\mathbb{Q}]$ *is odd. If the Hodge conjecture is true for all powers of X then also the general Hodge conjecture is true for all powers of X.*

Further reading. The most relevant papers concerning Mumford-Tate groups of abelian varieties were already mentioned in the text. Good overview papers are [30] and [92]. (Some more recent results can be found in [49].) For more general surveys of the Hodge conjecture and the general Hodge conjecture, see [46], [83] and [87]. These also contain many references to papers that deal with the Hodge conjecture for special classes of varieties.

§3. Levels of Hodge structures and lengths of representations.

(3.1) The material in this section is entirely based on Zarhin's paper [99]. We shall make free use of the theory of semi-simple Lie algebras and their representations. In this we shall follow the notations of Bourbaki [10], [11]. More precisely, we use the following notation.

K	an algebraically closed field of characteristic 0
$\mathfrak{g}$	a semi-simple K-Lie algebra
$\mathfrak{h}$	a Cartan subalgebra of $\mathfrak{g}$;
	set $\mathfrak{h}^* := \mathrm{Hom}(\mathfrak{h}, K)$ and write $\langle\ \ ,\ \ \rangle\colon \mathfrak{h} \times \mathfrak{h}^* \to K$ for the canonical pairing
$R \subset \mathfrak{h}^*$	the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$
$R^\vee \subset \mathfrak{h}$	the dual root system
$B = \{\alpha_1, \ldots, \alpha_\ell\}$	a basis of R
	set $B^\vee := \{\alpha^\vee \mid \alpha \in B\}$, which is a basis of $R^\vee$
$P = P(R) \subset \mathfrak{h}^*$	the weight lattice
$\varpi_1, \ldots, \varpi_\ell$	the fundamental dominant weights
$P_{++} \subset P$	the dominant weights ($\mathbb{Z}_{\geq 0}$-linear combinations of the ϖ_j)
$\geq$	the partial ordering on $P \otimes \mathbb{Q}$ defined by B, i.e., $\lambda_1 \geq \lambda_2$ iff $\lambda_1 - \lambda_2 \in \sum \mathbb{Q}_{\geq 0} \cdot \alpha_i$,
	(Note: P_{++} is in general *strictly* contained in $P_+ := \{\lambda \in P \mid \lambda \geq 0\}$)
$\tilde{\alpha}$, $\tilde{\beta}^\vee$	the maximal roots in R, resp. $R^\vee$

W	the Weyl group of R
$w_0 \in W$	the longest element (w.r.t. the chosen basis B)
$\lambda \mapsto \lambda' := -w_0(\lambda)$	the opposition involution on $\mathfrak{h}^*$
$V(\lambda)$	the irreducible $\mathfrak{g}$-module with highest weight λ (for λ a dominant weight)

(3.2) Let λ be a dominant weight. We can write

$$\lambda = \sum_{\alpha \in B} c_\alpha \cdot \alpha = \sum_{i=1}^{\ell} c_i \cdot \alpha_i = \sum_{i=1}^{\ell} m_i \cdot \varpi_i \,,$$

where $c_i = c_{\alpha_i} \in \mathbb{Q}_{\geq 0}$ and $m_i \in \mathbb{Z}_{\geq 0}$ for all i. If $n_{i,j} := n(\alpha_i, \alpha_j) := \langle \alpha_i, \alpha_j^\vee \rangle$ are the coefficients of the Cartan matrix then $\alpha_i = \sum_{j=1}^{\ell} n_{i,j} \varpi_j$, so that $m_j = \sum_{i=1}^{\ell} c_i n_{i,j}$.

(3.3) Lemma. *We have $c_\alpha + c_{\alpha'} \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in B$.*

Proof. If $w \in W$ then $w(\lambda)$ can be written as

$$w(\lambda) = \lambda - \sum_{\alpha \in B} a_\alpha \cdot \alpha \,, \quad \text{with } a_\alpha \in \mathbb{Z}_{\geq 0} \,.$$

Taking $w = w_0$ gives

$$-\sum_{\alpha \in B} c_{\alpha'} \cdot \alpha = -\lambda' = w_0(\lambda) = \left(\sum_{\alpha \in B} c_\alpha \cdot \alpha \right) - \left(\sum_{\alpha \in B} a_\alpha \cdot \alpha \right) ,$$

hence $c_\alpha + c_{\alpha'} = a_\alpha \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in B$. $\square$

(3.4) Definition. Notation as above. Assume R to be irreducible (corresponding to simple $\mathfrak{g}$). We define

$$\begin{aligned} s(\lambda) &= \sum_{\alpha \in B} \langle \lambda, \alpha^\vee \rangle = \sum_{i,j=1}^{\ell} n_{i,j} c_i = \sum_{i=1}^{\ell} m_i \,, \\ \mathrm{depth}(\lambda) &= \langle \lambda, \tilde{\beta}^\vee \rangle = \max_{\alpha \in R} \langle \lambda, \alpha^\vee \rangle \,, \\ \mathrm{length}(\lambda) &= \min_{\alpha \in B} c_\alpha + c_{\alpha'} \,. \end{aligned}$$

(3.5) Proposition. *The functions s,* depth *and* length *take integral values and have the following properties.*

(i) $\mathrm{length}(\lambda) \geq \mathrm{depth}(\lambda) \geq s(\lambda) \geq 0$,
(ii) $s(\lambda) \geq 1$ *if* $\lambda \neq 0$,
(iii) $s(\lambda') = s(\lambda)$, $\mathrm{depth}(\lambda) = \mathrm{depth}(\lambda')$, $\mathrm{length}(\lambda') = \mathrm{length}(\lambda)$,
(iv) $s(\lambda_1 + \lambda_2) = s(\lambda_1) + s(\lambda_2)$, $\mathrm{depth}(\lambda_1 + \lambda_2) = \mathrm{depth}(\lambda_1) + \mathrm{depth}(\lambda_2)$,
(v) $\mathrm{length}(\lambda_1 + \lambda_2) \geq \mathrm{length}(\lambda_1) + \mathrm{length}(\lambda_2)$, $\mathrm{length}(m\lambda + n\lambda') = (m+n) \cdot \mathrm{length}(\lambda)$ *for* $m, n \in \mathbb{Z}_{\geq 0}$.

Proof. Properties (ii), (iii), (iv) and (v) are clear. That $\mathrm{depth}(\lambda) \geq s(\lambda)$ follows from the fact that $\tilde{\beta}^\vee \geq \alpha_i^\vee$ for all $\alpha_i \in B$, so that $\langle \varpi_i, \tilde{\beta}^\vee \rangle \geq 1$ for all i. Next consider a positive root $\gamma = \sum_{\alpha \in B} e_\alpha \cdot \alpha$ (with $e_\alpha \in \mathbb{Z}_{\geq 0}$ for all α) and write $w_\gamma \in W$ for the associated reflection. As in (3.3) we have

$$-\lambda' = w_0(\lambda) = \lambda - \sum_{\alpha \in B} a_\alpha \cdot \alpha \,, \qquad \text{with} \quad a_\alpha = c_\alpha + c_{\alpha'} \in \mathbb{Z}_{\geq 0}$$

and

$$w_\gamma(\lambda) = \lambda - \langle \lambda, \gamma^\vee \rangle \cdot \gamma = \lambda - \sum_{\alpha \in B} \langle \lambda, \gamma^\vee \rangle \cdot e_\alpha \cdot \alpha \,.$$

As $w_\gamma(\lambda) \geq w_0(\lambda)$ we find the estimate

$$c_\alpha + c_{\alpha'} \geq \langle \lambda, \gamma^\vee \rangle \cdot e_\alpha \qquad \text{for all } \alpha \in B\,.$$

Taking $\gamma = \tilde{\beta}$ we have $e_\alpha \geq 1$ for all $\alpha \in B$ and this gives that $\mathrm{length}(\lambda) \geq \mathrm{depth}(\lambda)$. $\square$

Given an irreducible root system R, and given an expression for λ as a linear combination of basis vectors (or a linear combination of fundamental dominant weights), the numbers $s(\lambda)$, $\mathrm{depth}(\lambda)$ and $\mathrm{length}(\lambda)$ are easily computed. For the length function the result is listed in Table 1 (p. 26), taken from [99].

(3.6) Example. Suppose R is an irreducible root system. In [11], Chap. VIII, §7.3, the notion of a *miniscule weight* is defined. One possible definition is that a dominant weight λ is miniscule if and only if $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\alpha \in R$. The miniscule weights are easily listed (see Table 3, p. 27); they only occur among the fundamental dominant weights, and for R of type E_8, F_4 or G_2 there are no miniscule weights at all. With this terminology we have

$$\begin{aligned} s(\lambda) = 1 &\Longleftrightarrow \lambda \text{ is a fundamental dominant weight}\,, \\ \mathrm{depth}(\lambda) = 1 &\Longleftrightarrow \lambda \text{ is a miniscule weight}\,, \\ \mathrm{length}(\lambda) = 1 &\Longleftrightarrow \lambda \text{ is a miniscule weight and } R \text{ is of classical type.} \end{aligned}$$

(By "R is of classical type" we mean that R is of one of the types A_ℓ, B_ℓ, C_ℓ or D_ℓ.)

For later use, we list in Table 3 (p. 27) all pairs (R, λ) where R is an irreducible root system and λ is a miniscule weight. (We assume that a basis of R is chosen.)

The following proposition gives the properties of $\mathrm{depth}(\lambda)$ and $\mathrm{length}(\lambda)$ that are crucial for the application to Mumford-Tate groups. Recall that an arithmetic progression of rational numbers $q_0, q_1, \ldots, q_r$ is said to have length r (not $r+1$).

(3.7) Proposition. *Notation as in* (3.1). *Assume $\mathfrak{g}$ to be simple, so that R is an irreducible root system. Let λ be a dominant weight and consider the smallest R-saturated subset $\mathscr{X} \subset P(R)$ containing λ. (I.e., $\mathscr{X}$ is the support of the irreducible $\mathfrak{g}$-module $V(\lambda)$ with highest weight λ.)*

(i) *If φ: $P(R) \to \mathbb{Q}$ is a nonzero homomorphism then $\varphi(\mathscr{X})$ contains an arithmetic progression of length equal to* $\mathrm{depth}(\lambda)$.

(ii) *We have*

$$\begin{aligned} \mathrm{length}(\lambda) &= \min \left\{ n \;\middle|\; \begin{array}{c} \textit{there exists a nonzero homomorphism } \varphi\colon P(R) \to \mathbb{Q} \textit{ such that} \\ \varphi(\mathscr{X}) \textit{ is contained in an arithmetic progression of length } n \end{array} \right\} \\ &= \min \left\{ n \;\middle|\; \begin{array}{c} \textit{there exists a nonzero homomorphism } \varphi\colon P(R) \to \mathbb{Q} \\ \textit{such that } \varphi(\mathscr{X}) \textit{ has cardinality } n+1 \end{array} \right\} \end{aligned}$$

Proof. If $\lambda = 0$ the proposition is clear, so we may assume that $V(\lambda)$ is a faithful representation. Let φ: $P(R) \to \mathbb{Q}$ be a nonzero homomorphism. Replacing φ by a nonzero multiple does not change the cardinality of $\varphi(\mathscr{X})$ or the lengths of the arithmetic progressions involved. Possibly after such a replacement there exists a weight $\gamma \in P(R)$ such that φ is given by $\varpi \mapsto \langle \varpi, \gamma^\vee \rangle$. As $\mathscr{X} \subset P(R)$ is stable under the action of the Weyl group W and as every W-orbit in $P(R)$ meets $P_{++}(R)$ we may further assume that $\gamma \in P_{++}(R)$, $\gamma \neq 0$.

For the proof of the proposition we use two facts. First, that $\mathscr{X}$ is saturated implies (by definition) that for every $\alpha \in R$ it contains the arithmetic progression

$$\lambda, \lambda - \alpha, \cdots, \lambda - \langle \lambda, \alpha^\vee \rangle \cdot \alpha$$

of length $\langle \lambda, \alpha^\vee \rangle$. Assertion (i) now follows taking $\alpha = \tilde{\alpha}$, noting that $\langle \tilde{\alpha}, \gamma^\vee \rangle \neq 0$.

The second fact we use is that for every $\mu \in \mathscr{X}$ there exists a sequence $\lambda = \mu_0, \mu_1, \ldots, \mu_n = \mu$ of elements of $\mathscr{X}$ such that each μ_{j+1} is of the form $\mu_{j+1} = \mu_j - \alpha$ for some $\alpha \in R$. (See [11], Chap. 8, §6, Ex. 2.) We apply this with $\mu = w_0(\lambda)$. As in (3.2) above, write $\lambda = \sum_{i=1}^{\ell} c_i \cdot \alpha_i = \sum_{\alpha \in B} c_\alpha \cdot \alpha$. In the proof of (3.3) we have seen that $w_0(\lambda) = \lambda - \sum_{\alpha \in B} a_\alpha \cdot \alpha$, where $a_\alpha = c_\alpha + c_{\alpha'}$. In the chain $\mu_0 = \lambda, \mu_1, \ldots, \mu_n = w_0(\lambda)$ there are therefore precisely a_α indices j such that $\mu_{j+1} = \mu_j - \alpha$. Given $\varphi = \langle -, \gamma^\vee \rangle$, we can choose $\alpha \in B$ with $\langle \alpha, \gamma^\vee \rangle \neq 0$ (as $\gamma \neq 0$), in which case we find that $\varphi(\mathscr{X})$ has at least cardinality $1 + a_\alpha$. This shows that

$$\text{for all nonzero homomorphisms } \varphi\colon P(R) \to \mathbb{Q} \text{ we have } 1 + \mathrm{length}(\lambda) \leq \mathrm{card}(\varphi(\mathscr{X})). \tag{1}$$

Now we choose $\alpha = \alpha_j \in B$ such that $a_\alpha = a_j = \mathrm{length}(\lambda)$. Consider the dual root $\alpha^\vee = \alpha_j^\vee$ and the associated dual fundamental dominant weight $\varpi_j^\vee$. Consider the homomorphism $\varphi = \varpi_j^\vee(-)\colon \sum b_i \cdot \alpha_i \mapsto b_j$. Applying the previous with this φ we find that $\varphi(\mathscr{X})$ is the set $c_\alpha, c_\alpha - 1, \ldots, c_\alpha - a_\alpha$. Thus,

$$\begin{array}{l}\text{there exists a nonzero homomorphisms } \varphi\colon P(R) \to \mathbb{Q} \text{ such that} \\ \varphi(\mathscr{X}) \text{ is an arithmetic progression of length equal to length}(\lambda).\end{array} \tag{2}$$

Combining (1) and (2) the proposition follows. □

We may visualize this as saying that $1 + \mathrm{length}(\lambda)$ is the mimimum number of "layers" in which $\mathscr{X} = \mathrm{Supp}(V(\lambda))$ is contained. We illustrate two examples in Figure 1 (p. 20).

(3.8) We shall apply the above to the study of Mumford-Tate groups. As we shall later use the same arguments in a different context, we use the following notation.

k	a field of characteristic zero
K	an algebraically closed field containing k
G	a connected reductive algebraic group over k
$\rho\colon G \to \mathrm{GL}(V)$	a faithful, finite dimensional representation over k

There is a canonical decomposition (up to permutation of the factors) $G_K = Z(G)_K \cdot G_1 \cdots G_q$ of G_K as an almost direct product of its center $Z(G)_K$ and its simple factors G_j $(1 \leq j \leq q)$. Write $p_j\colon G_K \twoheadrightarrow G_j'$ for the quotient of G_K modulo the subgroup $Z(G)_K \cdot G_1 \cdots G_{j-1} \cdot G_{j+1} \cdots G_q$.

We write $\mathfrak{c} = \mathrm{Lie}(Z(G))_K$ and $\mathfrak{g} = \mathrm{Lie}(G^{\mathrm{der}})_K$, so that $\mathrm{Lie}(G)_K = \mathfrak{c} \times \mathfrak{g}$. We keep the notations introduced in (3.1). Also we write $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_q$, with $\mathfrak{g}_j = \mathrm{Lie}(G_j)$. The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a product $\mathfrak{h} = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_q$, where $\mathfrak{h}_j$ is a Cartan subalgebra of $\mathfrak{g}_j$. The root system R is the direct sum of root systems R_j.

The Lie algebra $\mathfrak{c}$ is canonically isomorphic to $X_*(Z(G)_K) \otimes_{\mathbb{Z}} K$. We write $P_0 := X^*(Z(G)_K) \subset \mathfrak{c}^*$; it is the "toral" analogue of the weight lattice $P(R) \subset \mathfrak{h}^*$.

Suppose we have a cocharacter $\gamma\colon \mathbb{G}_{m,K} \to G_K$. We say that γ has a non-trivial component in the simple factor G_j if the composition $p_j \circ \gamma$ is non-trivial. The K-linear map $\dot\gamma\colon K \to \mathrm{Lie}(G)_K$ induced by γ on tangent spaces can be written as $\dot\gamma = (\dot\gamma_0, \dot\gamma_1, \ldots, \dot\gamma_q)$, where $\dot\gamma_0$ is its component in $\mathfrak{c}$ and $\dot\gamma_j$ $(1 \leq j \leq q)$ is its component in $\mathfrak{g}_j$. We may, and shall, assume that $\dot\gamma$ factors through $\mathfrak{c} \times \mathfrak{h}$. (This is the case if we replace $\mathfrak{h}$ by a conjugate, which we may do.) Dualizing, we obtain a K-linear map $\varphi = \varphi_\gamma := \dot\gamma^*\colon \mathfrak{h}^* \to K$ with the property that $\varphi(P_0 \times P(R)) \subset \mathbb{Z}$.

We shall work in a situation where we know something about the weights of $\rho \circ \gamma\colon \mathbb{G}_{m,K} \to \mathrm{GL}(V)_K$. (That is, we know something about the image under φ of the set of weights of $\mathfrak{h}$ in V_K.) Our goal is to deduce from this information about the simple factors G_j and their action on V_K.

Let $W \subset V_K$ be an irreducible G_K-submodule. As a representation of $\mathrm{Lie}(G)_K$ we can decompose W as

$$W = \chi \boxtimes \rho_0 \boxtimes \cdots \boxtimes \rho_q\,,$$

where χ is a character of $\mathfrak{c}$ and where ρ_j is an irreducible representation of $\mathfrak{g}_j$. Let λ_j be the highest weight of the representation ρ_j w.r.t. the Cartan subalgebra $\mathfrak{h}_j \subset \mathfrak{g}_j$ and the chosen basis of the root system. Let $\mathscr{X}_j \subset P(R_j)$ be the support of ρ_j. The support of W is the set

$$\mathscr{X} := \mathrm{Supp}(W) = \chi + \mathscr{X}_1 + \cdots + \mathscr{X}_q = \{\chi + \mu_1 + \cdots + \mu_q \mid \mu_j \in \mathscr{X}_j\} \subset \mathfrak{c}^* \times \mathfrak{h}_1^* \times \cdots \times \mathfrak{h}_q^*\,.$$

Root system B_2;
$\tilde{\alpha} = 2\varpi_2$, $\tilde{\beta} = \varpi_1$.
Take $\lambda = 3\varpi_1 + \varpi_2$;
$\dim(V(\lambda)) = 64$.
The numbers in brackets are multiplicities of the corresponding weights. (Each weight is conjugate to one of the labeled ones under $W \cong D_4$.)
The $\tilde{\beta}$-string through λ consists of 6 weights.
The minimal number of "layers" in which $\mathrm{Supp}(V(\lambda))$ is contained is 6.

$s(\lambda) = 4$,
$\mathrm{depth}(\lambda) = 5$,
$\mathrm{length}(\lambda) = 5$.

Root system G_2;
$\tilde{\alpha} = \varpi_2$, $\tilde{\beta} = \varpi_1$.
Take $\lambda = \varpi_2$ (adjoint representation);
$\dim(V(\lambda)) = \dim(G_2) = 14$.
The weights of $V(\lambda)$ are the 12 roots (multiplicity 1) and 0 (multiplicity 2).
The $\tilde{\beta}$-string through λ consists of 4 weights.
The minimal number of "layers" in which $\mathrm{Supp}(V(\lambda))$ is contained is 5.

$s(\lambda) = 1$,
$\mathrm{depth}(\lambda) = 3$,
$\mathrm{length}(\lambda) = 4$.

Figure 1. Examples of depths and lengths of dominant weights.

If A and B are finite subsets of $\mathbb{Q}$ then the set $A+B := \{a+b \mid a \in A\,, b \in B\}$ has cardinality at least $\mathrm{card}(A)+\mathrm{card}(B)-1$. Phrased differently: $[\mathrm{card}(A+B)-1] \geq [\mathrm{card}(A)-1]+[\mathrm{card}(B)-1]$. Now we consider the image of $\mathscr{X}$ under φ: $P_0 \times P(R) \to \mathbb{Q}$. Combining the previous with (3.7) we find the following result.

(3.9) Theorem. *Let $N+1$ be the number of weights in W w.r.t. the cocharacter $\rho\circ\gamma$: $\mathbb{G}_{m,K} \to \mathrm{GL}(V)_K$. Suppose that γ has a non-trivial component in the simple factors $G_1, \ldots, G_r$ ($r \leq q$). Then*

$$\mathrm{length}(\lambda_1)+\cdots+\mathrm{length}(\lambda_r) \leq N\,.$$

In particular, if for $1 \leq i \leq r$ we set $M_i := \mathrm{card}\{j \leq r \mid j \neq i, \lambda_j \neq 0\}$ then

$$\mathrm{length}(\lambda_i) \leq N - M_i \leq N\,.$$

(3.10) Let us now specialize the previous to the case where G is a Mumford-Tate group. We consider a polarizable Hodge structure V of pure weight n, given by h: $\mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$.

We apply the previous with

$$k=\mathbb{Q}\,, \qquad K=\mathbb{C}\,, \qquad G=\mathrm{MT}(V)\,,$$

with

$$\rho\colon \mathrm{MT}(V) \to \mathrm{GL}(V) \qquad \text{the tautological representation,}$$

and we take

$$\gamma := h\circ\mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathrm{MT}(V)_{\mathbb{C}}\,.$$

The weights of $\rho\circ\gamma$ in $V_{\mathbb{C}}$ are precisely the cocharacters $z \mapsto z^{-p}$ where p is an integer with $V^{p,n-p} \neq 0$. In particular, the number of such weights is at most the level of V plus 1.

We have $G_{\mathbb{C}} = \mathrm{MT}(V)_{\mathbb{C}}$. Let $W \subset V_{\mathbb{C}}$ be an irreducible $G_{\mathbb{C}}$-submodule. We keep the notations introduced above. In particular we decompose W as $W = \chi \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_q$, where χ is a character of $\mathfrak{c}$ and where ρ_j is an irreducible representation of $\mathfrak{g}_j$. The highest weight of ρ_j we call λ_j.

(3.11) Theorem. *Assumptions and notations as above. Let $N+1$ be the number of integers p such that $V^{p,n-p} \neq 0$. Then $\mathrm{length}(\lambda_j) \leq N \leq \mathrm{level}(V)$ for all j.*

Proof. Suppose G_j is one of the simple factors of $G_{\mathbb{C}}$ in which $\gamma := h\circ\mu$ has a non-trivial component. (Notice that it is equivalent to say that $h_{|\mathbb{U}_1}$ has a non-trivial component in G_j.) That $\mathrm{length}(\lambda_j) \leq N$ is then an immediate application of (3.9).

Next we want to extend this to arbitrary simple factors of $G_{\mathbb{C}}$. We use that there is a $G(\mathbb{C})$-conjugate of γ which is defined over $\overline{\mathbb{Q}}$. So, there exists a δ: $\mathbb{G}_{m,\overline{\mathbb{Q}}} \to G_{\overline{\mathbb{Q}}}$ such that $\delta_{\mathbb{C}}$ is $G(\mathbb{C})$-conjugate to γ. Consider cocharacters of the form ${}^\tau\delta$: $\mathbb{G}_{m,\mathbb{C}} \to G_{\overline{\mathbb{Q}}}$, where $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $G_{j_1}, \cdots, G_{j_t}$ be the simple factors of $G_{\overline{\mathbb{Q}}}$ in which some conjugate ${}^\tau\delta$ has a non-trivial component. Then $G'_{\overline{\mathbb{Q}}} := Z_{\overline{\mathbb{Q}}} \cdot G_{j_1} \cdots G_{j_t}$ is a normal algebraic subgroup of $G_{\overline{\mathbb{Q}}}$ which is defined over $\mathbb{Q}$ and such that $\gamma := h\circ\mu$ factors through $G'_{\mathbb{C}}$. By definition of the Mumford-Tate group this implies that $G'_{\overline{\mathbb{Q}}} = G_{\overline{\mathbb{Q}}}$. In other words, if G_j is any of the simple factors of $G_{\overline{\mathbb{Q}}}$ then we can find $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that ${}^\tau\delta$ has a non-trivial component in G_j. The estimate $\mathrm{length}(\lambda_j) \leq N$ now follows by applying (3.9) with this cocharacter ${}^\tau\delta$. $\square$

(3.12) *Sharpening of the result.* We have seen in (1.18) that in the present situation we can say more about the number of simple factors G_j in which γ is non-trivial. Namely, consider the decomposition $\mathrm{MT}(V)_{\mathbb{R}} = Z_{\mathbb{R}} \cdot G_{1,\mathbb{R}} \cdots G_{q,\mathbb{R}}$ of $\mathrm{MT}(V)_{\mathbb{R}}$ as the almost direct product of its center $Z_{\mathbb{R}}$ and a number of $\mathbb{R}$-simple factors $G_{j,\mathbb{R}}$. As we have seen, the factors $G_{j,\mathbb{R}}$ are absolutely simple (which justifies our notation $G_{j,\mathbb{R}}$) and γ has a non-trivial component in each of the non-compact factors.

As in (3.9) let $G_1, \ldots, G_r$ be the simple factors of $G_{\mathbb{C}} = \mathrm{MT}(V)_{\mathbb{C}}$ in which γ has a non-trivial component. Then

$$\mathrm{length}(\lambda_1) + \cdots + \mathrm{length}(\lambda_r) \leq N$$

and writing $M := \min(1, \mathrm{card}\{j \leq r \mid \lambda_j \neq 0\})$ we find the (generally sharper) estimate

$$\mathrm{length}(\lambda_j) \leq N + 1 - M \leq N \qquad \text{for all } j.$$

Note that this last estimate holds for all λ_j, not only those with $1 \leq j \leq r$. (We argue as in the proof of (3.11).)

(3.13) Corollary. *Let X be a complex abelian variety. Decompose $\mathrm{MT}(X)_{\mathbb{R}} = Z_{\mathbb{R}} \cdot G_{1,\mathbb{R}} \cdots G_{q,\mathbb{R}}$ as the almost direct product of its center $Z_{\mathbb{R}}$ and its $\mathbb{R}$-simple factors $G_{j,\mathbb{R}}$. Then all $G_{j,\mathbb{R}}$ are of classical type. Let $W \subset V_{X,\mathbb{C}} = H^1(X, \mathbb{C})$ be an irreducible $\mathrm{MT}(X)_{\mathbb{C}}$-submodule, and write $W = \chi \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_q$ as above. If ρ_j is non-trivial then its highest weight λ_j is miniscule. Furthermore, there is precisely 1 non-compact factor G_j such that ρ_j is non-trivial.*

Proof. As explained in (1.18), the fact that V_X has level 1 implies that γ is non-trivial precisely on the non-compact factors $G_{j,\mathbb{R}}$. In particular, it cannot be the case that only compact factors of $\mathrm{MT}(X)_{\mathbb{R}}$ act non-trivially on W, as this would mean that W is purely of type $(0,0)$, which does not occur in V_X. Now the corollary is clear from the theorem, using (3.5) and (3.6). □

(3.14) Example. To illustrate the use of the above results, we prove a special case of (2.15). Namely, let X be a complex abelian variety of odd dimension g with $\mathrm{End}^0(X) = \mathbb{Q}$. The claim is that $\mathrm{MT}(X) = \mathrm{GSp}(V_X, \varphi)$.

Consider simple classical Lie algebras $\mathfrak{g}$ with a miniscule weight λ. From Table 3 on p. 27 we read off the following two facts:

(i) if $V(\lambda)$ is self-dual then its dimension is even,

(ii) if $V(\lambda)$ is symplectic with $\dim(V(\lambda)) \equiv 2 \bmod 4$ then $\mathfrak{g}$ is of type C_ℓ with $\ell \geq 1$ odd and $\lambda = \varpi_1$ (the standard representation of $\mathfrak{sp}_{2\ell}$).

With these facts at hand, let us prove the claim. We know that $\mathrm{MT}(X)$ is contained in $\mathrm{GSp}(V_X, \varphi)$. The assumption that $\mathrm{End}^0(X) = \mathbb{Q}$ implies that $V_{\mathbb{C}}$ is still an irreducible $\mathrm{MT}(X)_{\mathbb{C}}$-module. As V is a faithful representation, every factor G_j (notations as above) acts non-trivially. All highest weights occuring therefore have length 1. Furthermore, each of the representations ρ_j is self-dual (i.e., orthogonal or symplectic) and the number of symplectic factor is odd (as the total representation is symplectic). As $\dim(V)/2 = \dim(X)$ is odd, fact (i) implies that there is only 1 simple factor, i.e., $q = 1$. Fact (ii) then proves the claim.

The general case of (i) of (2.15) is not very much harder. The only essential new ingredient is the so-called Goursat lemma, see [66], pp. 790–91.

(3.15) There is one further remark to be made, which so far we have ignored. Namely, consider the situation of (3.10), write $\mathscr{X} = \mathrm{Supp}(W) \subset P_0 \oplus P(R)$, and write $\varphi\colon P_0 \oplus P(R) \to \mathbb{Q}$ for the homomorphism obtained by dualizing $\dot{\gamma}$ (as in (3.8)). Suppose V has weight n and let p be the smallest integer with $V^{p,n-p} \neq 0$. (So V has level $n - 2p$.) Then we find that there is an arithmetic progression $a_p, a_{p+1}, \ldots, a_{n-p}$ in $\mathbb{Q}$ such that $\varphi(\mathscr{X}) \subset \{a_p, \ldots, a_{n-2p}$ and such that the cardinality of $\varphi^{-1}(a_j)$ is at most $\dim(V^{j,n-j})$. The following example shows that this can give further useful restrictions on $\mathrm{MT}(V)$, especially if one of the Hodge numbers is small.

(3.16) Example (Following [98].) To conclude this section we shall look at an example that uses all ideas encountered above, including (3.15).

Let X be a smooth projective variety over $\mathbb{C}$ with the property that $h^{2,0} := \dim H^0(X, \Omega^2_{X/\mathbb{C}}) = 1$. Decompose the $\mathbb{Q}$-HS $H^2(X, \mathbb{Q})$ as $H^2(X, \mathbb{Q}) = \mathscr{B}^1(X) \oplus V$. (Remember that $\mathscr{B}^1(X)$ is the $\mathbb{Q}$-subspace of $H^2(X, \mathbb{Q})$ spanned by the divisor classes. Further note that the decomposition $\mathscr{B}^1(X) \oplus V$ is unique, grace to the semi-simplicity of $\mathbb{Q}\mathbf{HS}^{\mathrm{pol}}$.) The Hodge structure V may be referred to as the "transcendental part" of $H^2(X, \mathbb{Q})$.

Choose a polarization $\varphi\colon V \otimes V \to \mathbb{Q}(-2)$. The claim is now that the Hodge group (or the Mumford-Tate group) is "as big as possible". The result is due to Zarhin, [98]; see also the last section of [99].

(3.17) Theorem. *Let X be a nonsingular proper variety over $\mathbb{C}$ with $h^{2,0}=1$. Write $H^2(X)=\mathscr{B}^1(X)\oplus V$. Set $F:=\mathrm{End}_{\mathbb{Q}\mathbf{HS}}(V)$ and let φ be a polarization of V. Then V is an irreducible $\mathbb{Q}$-HS, F is a commutative field and $\mathrm{Hg}(V)=\mathrm{SO}_F(V,\psi)$, the centralizer of F inside $\mathrm{SO}(V,\varphi)$.*

The first claim is that

$$V \quad \text{is irreducible and} \quad F \quad \text{is a commutative field.}$$

Indeed, if $V=V_1\oplus V_2$ in $\mathbb{Q}\mathbf{HS}$ then only one of V_1 and V_2 can have $V_i^{2,0}\neq 0$, as $\dim V^{2,0}=1$. But then the other V_j is purely of type $(1,1)$, meaning that it is fully contained in $\mathscr{B}^1(X)$. By definition of V this implies $V_j=0$. Hence V is irreducible and F is a division algebra. Now notice that F acts on $V_{\mathbb{C}}$, preserving the Hodge decomposition. Hence there exists a ring homomorphism $F\to\mathrm{End}_{\mathbb{C}}(V^{2,0})=\mathbb{C}$, which shows that F is a field.

We know that F is either a totally real field or a CM-field. As our only goal is to illustrate the method, we shall sketch the proof only for the case that F is totally real. The arguments in the case that F is a CM-field are very similar; the notation and bookkeeping are slightly more involved but the argument itself is at one point even simpler.

So, assuming F to be totally real, we have seen in (1.22) that the centralizer of F inside $\mathrm{SO}(V,\varphi)$ can be described as the group

$$\mathrm{SO}_F(V,\varphi)=\operatorname*{Res}_{F/\mathbb{Q}}\mathrm{SO}_F(V,\psi)\,,$$

where $\psi\colon V\times V\to F$ is the unique F-bilinear symmetric form with $\mathrm{tr}_{F/\mathbb{Q}}(\psi)=\varphi$.

We can yet further exploit the assumption that $h^{2,0}=1$. First let us analyze the situation a bit further. Write $\Sigma_F=\{\sigma_1,\ldots,\sigma_e\}$ for the set of embeddings of F into $\mathbb{R}$. The real algebraic group $[\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SO}_F(V,\psi)]\otimes_{\mathbb{Q}}\mathbb{R}$ is a product

$$[\operatorname*{Res}_{F/\mathbb{Q}}\mathrm{SO}_F(V,\psi)]\otimes_{\mathbb{Q}}\mathbb{R}=\mathrm{SO}(V_{(1)},\psi_{(1)})\times\cdots\times\mathrm{SO}(V_{(e)},\psi_{(e)})\,,$$

where $V_{(j)}=\{v\in V\otimes_{\mathbb{Q}}\mathbb{R}\mid f(v)=\sigma_j(f)\cdot v \text{ for all } f\in F\}$. Our assumption that $h^{2,0}=1$ then implies that there is precisely one summand $V_{(j)}$, say $V_{(1)}$, which is not purely of Hodge type $(1,1)$.

Now we can do business again. We claim that $\mathrm{Hg}(V)$ comes from an algebraic group over F; more precisely:

$$\begin{array}{c}\text{there exists an algebraic subgroup } H\subset\mathrm{SO}_F(V,\psi) \text{ over } F\\ \text{such that } \mathrm{Hg}(V)=\mathrm{Res}_{F/\mathbb{Q}}H\hookrightarrow\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SO}_F(V,\psi).\end{array}\tag{$*$}$$

To see this, look at the algebraic subgroup $H'_{\mathbb{R}}\subset\mathrm{Hg}(V)_{\mathbb{R}}$ given by

$$H'_{\mathbb{R}}:=[\mathrm{Hg}(V)_{\mathbb{R}}\cap\mathrm{GL}(V_{(1)})]\times\cdots\times[\mathrm{Hg}(V)_{\mathbb{R}}\cap\mathrm{GL}(V_{(e)})]\,.\tag{$*'$}$$

As the representation $\mathrm{Hg}(V)\to\mathrm{GL}(V)$ is defined over $\mathbb{Q}$ we easily find that $H'_{\mathbb{R}}$ is defined over $\mathbb{Q}$, i.e., we have an algebraic subgroup $H'\subset\mathrm{Hg}(V)$ with $H'_{\mathbb{R}}=(H')\otimes\mathbb{R}$. Furthermore, the definition of H' in $(*')$ tells us precisely that H' is of the desired form $H'=\mathrm{Res}_{F/\mathbb{Q}}H$. Now the fact that $V_{(2)},\ldots,V_{(e)}$ are purely of type $(1,1)$ in the Hodge decomposition means that the composition

$$\mathbb{U}_1\xrightarrow{h_{|\mathbb{U}_1}}\mathrm{Hg}(V)_{\mathbb{R}}\hookrightarrow\mathrm{GL}(V_{(1)})\times\cdots\times\mathrm{GL}(V_{(e)})$$

factors through $[\mathrm{Hg}(V)_{\mathbb{R}}\cap\mathrm{GL}(V_{(1)})]$. By definition of $\mathrm{Hg}(V)$ it then follows that $\mathrm{Hg}(V)=H'$, proving $(*)$. Note that to prove the theorem it now suffices to show that $H\otimes_{F,\sigma_1}\mathbb{R}$ is the full group $\mathrm{SO}(V_{(1)},\psi_{(1)})$.

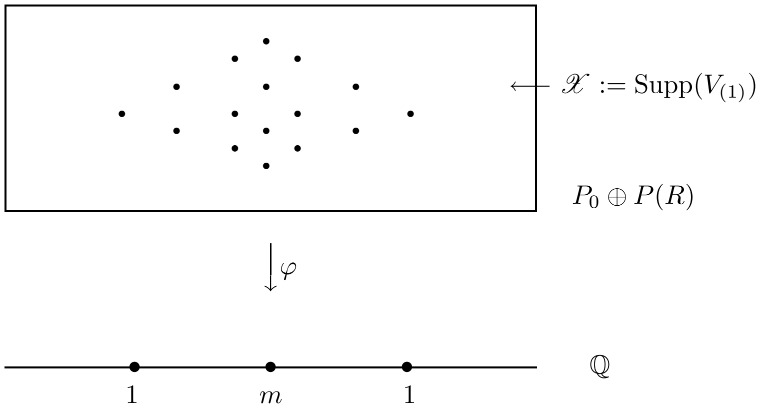
Step by step we are getting more grip on $\mathrm{Hg}(V)$. Our next weapon is to use (3.12) and (3.15). As in (3.12), we can decompose $\mathrm{Hg}(V)_{\mathbb{R}}$ (which in the present case we know to be semi-simple) as $\mathrm{Hg}(V)_{\mathbb{R}}=G_{1,\mathbb{R}}\cdots G_{r,\mathbb{R}}\cdot G_{r+1,\mathbb{R}}\cdots G_{q,\mathbb{R}}$, where the $G_{j,\mathbb{R}}$ are the $\mathbb{R}$-simple factors. We choose the numbering in such a way that $h\circ\mu\colon\mathbb{G}_{m,\mathbb{C}}\to\mathrm{MT}(V)_{\mathbb{C}}$ has a non-trivial component in the simple factors $G_1,\ldots,G_r$. Then $G_1,\ldots,G_r$ act trivially on $V_{(2)},\ldots,V_{(e)}$; as the total representation $\mathrm{Hg}(V)\to\mathrm{GL}(V)$ is faithful, they must act non-trivially on $V_{(1)}$. Also remark that the $V_{(j)}$ are absolutely irreducible representations of $\mathrm{Hg}(V)_{\mathbb{R}}$.

We use Lie-algebra notations as set up in (3.8) and (3.10), applied to the module $W = V_{(1),\mathbb{C}}$. In particular we set $\mathfrak{g} = \mathrm{Lie}(\mathrm{MT}(V)_{\mathbb{C}})$ and $\mathfrak{g}_i = \mathrm{Lie}(G_{i,\mathbb{C}})$, and we write $\mathscr{X} \subset P_0 \oplus P(R)$ for the support of W. Applying (3.12) we find that there are two possibilities:

(a) $r = 2$ and the $\mathfrak{g}_1 \times \mathfrak{g}_2$-module $V_{(1),\mathbb{C}}$ is of the form $\rho_1 \boxtimes \rho_2$ where ρ_i $(i = 1, 2)$ is an irreducible representation of $\mathfrak{g}_i$ with highest weight λ_i of length 1;

(b) $r = 1$ and $V_{(1),\mathbb{C}}$ is an irreducible $\mathfrak{g}_1$-module with highest weight λ of level ≤ 2.

Next we exploit (3.15). What we have found there means, in the present situation, that there exists a nonzero $\mathbb{Q}$-linear map φ: $P_0 \oplus P(R) \to \mathbb{Q}$ (once again using the notations set up in (3.8) and (3.10)) with $\varphi(\mathscr{X})$ contained in an arithmetic progression a_0, a_1, a_2 of length 2, such that $\varphi^{-1}(a_0)$ and $\varphi^{-1}(a_2)$ consist of a single element:



This brings us in a situation where the representation theory of semi-simple Lie algebras (much in the style of the first half of this section) leaves us with rather few possibilities. In fact, what possibilities are left was analyzed by Zarhin in [99], from which we copy the following result.

(3.18) Lemma. *Let $\mathfrak{g}$ be a semi-simple Lie algebra over an algebraically closed field of characteristic 0. Let W be a faithful irreducible $\mathfrak{g}$-module. Suppose there exists a nonzero homomorphism φ: $P(R) \to \mathbb{Q}$ such that $\varphi(\mathrm{Supp}(W))$ is contained in an arithmetic progression a_0, a_1, a_2 of length 2 and such that $\varphi^{-1}(a_0)$ and $\varphi^{-1}(a_2)$ consist of at most 1 element. Then all simple factors of $\mathfrak{g}$ are of the same classical type A_ℓ, B_ℓ, C_ℓ or D_ℓ, and there are only the following possibilities.*

(i) *$\mathfrak{g} \cong \mathfrak{sl}_{\ell+1}$ (type A_ℓ) and W is the standard representation or its dual;*

(ii) *W is the adjoint representation of $\mathfrak{g} \cong \mathfrak{sl}_2$ (note: $\mathfrak{sl}_2 \cong \mathfrak{so}_3$);*

(iii) *$\mathfrak{g} \cong \mathfrak{so}_{2\ell+1}$ (type B_ℓ) and W is the standard representation;*

(iv) *$\mathfrak{g} \cong \mathfrak{sp}_{2\ell}$ (type C_ℓ) and W is the standard representation;*

(v) *$\mathfrak{g} \cong \mathfrak{so}_{2\ell}$ (type D_ℓ) and W is the standard representation.*

With this lemma at hand we can finish the proof of (3.17) (still assuming F to be totally real). Namely, we find that (a) is only possible if $\mathfrak{g}_1 \times \mathfrak{g}_2 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with representation $\mathrm{St} \boxtimes \mathrm{St}$ (where "St" denotes the standard 2-dimensional representation). This can be rewritten as the standard representation of $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$, case (v) of the lemma. If we are in case (b) then it follows from the lemma that $\mathfrak{g}_1$ is isomorphic to $\mathfrak{so}(V_{(1)}, \psi_{(1)})$. (Note that we are dealing with an orthogonal representation.) This shows that $H \otimes_{F,\sigma_1} \mathbb{R}$ is the full group $\mathrm{SO}(V_{(1)}, \psi_{(1)})$ which proves (deep breath) the claim of (3.17).

Exercise. Let V be as in (3.17). Suppose that V is of CM-type. Give a complete description of the CM-type Φ (see (1.26)) and the Hodge group. Prove (3.17) in this case.

Note. In the second part of my course I intend to discuss how similar techniques as above can be applied to the study of Galois representations associated to algebraic varieties over a number field. We shall follow [62]. Basic results on Galois representations can be found in (a selection) [7], [8], [15], [23], [27], [29],

[44], [45], [58], [63], [64], [66], [76], [79], [77], [78], [74], [75], [80], [97]. Other references related to what I intend to discuss are [3], [4], [24], [42], [57], [56], [65], [84], [85], [90], [95], [96], [100], [101], [102], [103].

$$\lambda = \sum_{i=1}^{\ell} m_i \cdot \varpi_i$$

Type of $\mathfrak{g}$	length(λ)	depth(λ)
A_ℓ	$\sum_{i=1}^{\ell} m_i$	$\sum_{i=1}^{\ell} m_i$
B_ℓ	$m_\ell + 2\sum_{i=1}^{\ell-1} m_i$	$m_\ell + 2\sum_{i=1}^{\ell-1} m_i$
C_ℓ	$\min(S_1, S_2)$, where $S_1 := 2\sum_{i=1}^{\ell} m_i\,,\quad S_2 := \sum_{i=1}^{\ell} i m_i$	$m_1 + 2\sum_{i=2}^{\ell} m_i$
D_ℓ	$\min(S_1, S_2)$, if ℓ is odd, $\min(S_1, S_3, S_4)$, if ℓ is even, where $S_1 := m_{\ell-1} + m_\ell + 2\sum_{i=1}^{\ell-2} m_i$ $S_2 := \frac{(\ell-1)(m_{\ell-1}+m_\ell)}{2} + \sum_{i=1}^{\ell-2} i m_i$ $S_3 := \frac{\ell m_{\ell-1}+(\ell-2)m_\ell}{2} + \sum_{i=1}^{\ell-2} i m_i$ $S_4 := \frac{(\ell-2)m_{\ell-1}+\ell m_\ell}{2} + \sum_{i=1}^{\ell-2} i m_i$	$m_1 + m_{\ell-1} + m_\ell + 2\sum_{i=2}^{\ell-2} m_i$
E_6	$2m_1 + 2m_2 + 3m_3 + 4m_4$ $+3m_5 + 2m_6$	$m_1 + 2m_2 + 2m_3 + 3m_4$ $+2m_5 + m_6$
E_7	$\min(S_1, S_2)$, where $S_1 := 2m_1 + 3m_2 + 4m_3$ $+6m_4 + 5m_5 + 4m_6 + 3m_7,$ $S_2 := 4m_1 + 4m_2 + 6m_3$ $+8m_4 + 6m_5 + 4m_6 + 2m_7$	$2m_1 + 2m_2 + 3m_3 + 4m_4$ $+3m_5 + 2m_6 + m_7$
E_8	$4m_1 + 6m_2 + 8m_3 + 12m_4$ $+10m_5 + 8m_6 + 6m_7 + 4m_8$	$2m_1 + 3m_2 + 4m_3 + 6m_4$ $+5m_5 + 4m_6 + 3m_7 + 2m_8$
F_4	$4m_1 + 6m_2 + 4m_3 + 2m_4$	$2m_1 + 4m_2 + 3m_3 + 2m_4$
G_2	$2m_1 + 4m_2$	$2m_1 + 3m_2$

Table 1. Length and depth of the dominant weights in the simple root sytems.

Root system	weights of length 1	weights of length 2
A_ℓ	ϖ_i $(1 \leq i \leq \ell)$	$\varpi_i + \varpi_j$ $(1 \leq i \leq j \leq \ell)$
B_ℓ	ϖ_ℓ	ϖ_i $(1 \leq i \leq \ell - 1)$, $2\varpi_\ell$
C_ℓ	ϖ_1	$2\varpi_1$, ϖ_i $(2 \leq i \leq \ell)$
D_ℓ	ϖ_1, $\varpi_{\ell-1}$, ϖ_ℓ	$2\varpi_1$, $2\varpi_{\ell-1}$, $2\varpi_\ell$, $\varpi_{\ell-1} + \varpi_\ell$, ϖ_i $(2 \leq i \leq \ell - 2)$ if $\ell = 4$ also $\varpi_1 + \varpi_3$ and $\varpi_1 + \varpi_4$
E_6		ϖ_1, ϖ_2, ϖ_6
E_7		ϖ_1, ϖ_7
E_8		
F_4		ϖ_4
G_2		ϖ_1

Table 2. Weights of length 1 and 2.

The next table lists all pairs (R, λ) where R is an irreducible root system and λ is a miniscule weight. We describe the corresponding representation $V(\lambda)$, and we give its dimension and the autoduality: $-$ stands for a symplectic representation, $+$ for an orthogonal representation and 0 means that $V(\lambda)$ is not self-dual. The information given in this table is obtained combining the list of [11], Chap. VIII, page 129 with the information given in op. cit., Tables 1 and 2.

Root system	miniscule weight	representation	dim	autoduality
A_ℓ $(\ell \geq 1)$	ϖ_j $(1 \leq j \leq \ell)$	$\wedge^j$(Standard)	$\binom{\ell+1}{j}$	$(-1)^s$ if $\ell = 2s - 1$ 0 if ℓ is even
B_ℓ $(\ell \geq 2)$	ϖ_ℓ	Spin	2^ℓ	$+$ if $\ell \equiv 0, 3 \bmod 4$ $-$ if $\ell \equiv 1, 2 \bmod 4$
C_ℓ $(\ell \geq 2)$	ϖ_1	Standard	2ℓ	$-$
D_ℓ $(\ell \geq 3)$	ϖ_1 $\varpi_{\ell-1}$, ϖ_ℓ	Standard Spin^-, resp. Spin^+	2ℓ $2^{\ell-1}$	$+$ $+$ if $\ell \equiv 0 \bmod 4$ $-$ if $\ell \equiv 2 \bmod 4$ 0 if $\ell \equiv 1 \bmod 2$
E_6	ϖ_1 ϖ_6		27 27	0 0
E_7	ϖ_7		56	-1

Table 3. Miniscule weights in irreducible root systems.

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