AN INTRODUCTION TO MUMFORD-TATE GROUPS

by

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These notes are intended as reading material for the lectures on Shimura varieties by Bruno Klingler and myself at the workshop on *Shimura Varieties, Lattices and Symmetric Spaces* at Monte Verità. There are no claims to originality, and these notes are far from complete. In particular, I don't say anything about Shimura varieties. The only aim is to give a short introduction to some formal aspects of Hodge theory, to introduce the notion of a Mumford-Tate group, and to give some examples. My hope is that these notes are a useful complement to those of J.S. Milne [31], and that they may serve as an introduction to Deligne's papers [12] and [16]. Comments are most welcome.

Contents

§ 1. Pure Hodge structures

(1.1) A $\mathbb{Q}$-Hodge structure of weight m is a finite dimensional $\mathbb{Q}$-vector space V together with a decomposition

$$V_{\mathbb{C}} = \mathop{\oplus}_{p+q=m} V_{\mathbb{C}}^{p,q} \tag{1}$$

such that

$$\overline{V_{\mathbb{C}}^{p,q}} = V_{\mathbb{C}}^{q,p}\,. \tag{2}$$

Here $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$, and the bar in (2) denotes complex conjugation $v \otimes z \mapsto v \otimes \bar{z}$.

The decomposition (1) is called the Hodge decomposition. We may also consider the Hodge filtration, i.e., the descending filtration of $V_{\mathbb{C}}$ given by

$$F^i V_{\mathbb{C}} := \mathop{\oplus}_{\substack{p+q=m,\\ p\geqslant i}} V_{\mathbb{C}}^{p,q}\,.$$

The Hodge decomposition can be recovered from the Hodge filtration, as we have the relation $V_{\mathbb{C}}^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}$, where we take the intersection inside $V_{\mathbb{C}}$.

By a pure $\mathbb{Q}$-Hodge structure we mean a finite direct sum $\oplus_{m\in\mathbb{Z}} V_m$, where V_m is a $\mathbb{Q}$-Hodge structure of weight m.

If $T \subset \mathbb{Z}^2$ then we say that a Hodge structure is of type T if all summands $V_{\mathbb{C}}^{p,q}$ with $(p,q) \notin T$ are zero.

If V and W are $\mathbb{Q}$-Hodge structures then by a morphism of Hodge structures from V to W we mean a linear map $f\colon V \to W$ such that its $\mathbb{C}$-linear extension $f_{\mathbb{C}}\colon V_{\mathbb{C}} \to W_{\mathbb{C}}$ maps $V_{\mathbb{C}}^{p,q}$ into $W_{\mathbb{C}}^{p,q}$ for all $(p,q) \in \mathbb{Z}^2$. Note: if V is pure of weight m and W is pure of weight n then we

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can only have non-zero morphisms of Hodge structures $V \to W$ if $m = n$. With this notion of morphisms, we get a category $\mathbb{Q}$HS of $\mathbb{Q}$-Hodge structures, which is an abelian $\mathbb{Q}$-linear category.

Working with $\mathbb{Z}$-coefficients instead of $\mathbb{Q}$-coefficients, we get the notion of a $\mathbb{Z}$-Hodge structure. Note that if we allow the underlying $\mathbb{Z}$-module $V_{\mathbb{Z}}$ to have torsion, then this part is killed if we extend scalars to $\mathbb{C}$ and is therefore not relevant for the Hodge decomposition.

There is also the notion of an $\mathbb{R}$-Hodge structure. This is a much weaker notion than that of a $\mathbb{Q}$-Hodge structure: it is the interplay between the $\mathbb{Q}$-structure and the decomposition (1) that makes the notion of a $\mathbb{Q}$-HS so subtle.

(1.2) *Example.* Let X be a compact Kähler variety, e.g., the analytic variety associated to a non-singular projective algebraic variety. Then a famous result of Hodge says that the Betti cohomology $H^m(X,\mathbb{Z})$ carries a natural Hodge structure of weight m. The (p,q)-component of $H^m(X,\mathbb{C})$ is in this case isomorphic to the space $H^q(X,\Omega^p_X)$. A good reference is Chap. 0 of Griffiths and Harris [24].

If $f\colon X \to Y$ is a morphism of compact Kähler varieties over $\mathbb{C}$ then the induced map $H^m(f)\colon H^m(Y,\mathbb{Z}) \to H^m(X,\mathbb{Z})$ is a morphism of Hodge structures.

If X is a non-singular complete algebraic variety over $\mathbb{C}$, not necessarily Kähler, then we still get a natural Hodge structure of weight m on $H^m(X^{\mathrm{an}},\mathbb{Z})$. To define it one uses Chow's Lemma to reduce to the projective case.

In the sequel we shall usually omit the superscript " $^{\mathrm{an}}$ ", assuming this will not lead to confusion.

(1.3) In the category of Hodge structures ($\mathbb{Z}$ or $\mathbb{Q}$-coefficients, say) we have a number of natural constructions. For instance, suppose $V_{\mathbb{Z}}$ and $W_{\mathbb{Z}}$ are $\mathbb{Z}$-Hodge structures of weights m and n, respectively. Then $V_{\mathbb{Z}}^\vee := \mathrm{Hom}(V_{\mathbb{Z}},\mathbb{Z})$ carries a natural Hodge structure of weight $-m$ and $V_{\mathbb{Z}}\otimes W_{\mathbb{Z}}$ carries a Hodge structure of weight $m+n$. Similarly we have $\mathrm{Hom}(V_{\mathbb{Z}},W_{\mathbb{Z}}) = V_{\mathbb{Z}}^\vee \otimes W_{\mathbb{Z}}$ of weight $n-m$ and $\mathrm{Sym}^k(V_{\mathbb{Z}})$ and $\wedge^k V_{\mathbb{Z}}$, both of weight km. Note that $(V_{\mathbb{Z}}^\vee)^\vee$ is isomorphic to $V_{\mathbb{Z}}$ only up to torsion.

Geometric examples: If X is a non-singular complete variety then the cup-product maps $H^i(X,\mathbb{Z}) \otimes H^j(X,\mathbb{Z}) \to H^{i+j}(X,\mathbb{Z})$ are morphisms of HS. If Y is a second non-singular complete variety then the Künneth isomorphisms $H^k(X\times Y,\mathbb{Q}) \cong \oplus_{i+j=k} H^i(X,\mathbb{Q}) \otimes H^j(Y,\mathbb{Q})$ are isomorphisms of HS.

(1.4) The Tate structure $\mathbb{Z}(n)$ is defined to be the free $\mathbb{Z}$-module $(2\pi i)^n \cdot \mathbb{Z} \subset \mathbb{C}$, with Hodge structure purely of type $(-n,-n)$. So the weight of $\mathbb{Z}(n)$ is $-2n$. We have $\mathbb{Z}(-1) \cong \mathbb{Z}(1)^\vee$, and if $n>0$ we have natural identifications $\mathbb{Z}(n) \cong \mathbb{Z}(1)^{\otimes n}$ and $\mathbb{Z}(-n) \cong \mathbb{Z}(-1)^{\otimes n}$.

If V is any $\mathbb{Z}$-HS then we define $V(n) := V \otimes \mathbb{Z}(n)$; it is called "$V$ twisted by n". Note that a twist by n decreases the weight by $2n$. Similar definitions apply with $\mathbb{Q}$-coefficients or $\mathbb{R}$-coefficients.

As a first example of how one uses Tate twists, let us consider morphisms of HS. Suppose V and W are HS of weight m and n, respectively, and suppose $n = m+2r$ for some integer r. Suppose $f\colon V \to W$ is a $\mathbb{Z}$-linear map such that $f_{\mathbb{C}}$ maps each $V_{\mathbb{C}}^{p,q}$ into $W_{\mathbb{C}}^{p+r,q+r}$. (E.g., such a thing happens for Gysin maps in cohomology.) According to our definition, f is not a morphism of HS, unless $r=0$. In most older literature, f would be called a morphism of HS of degree r. In the modern, slightly more formal approach, the thing to do is to consider the Hodge structure $W(r)$, of weight $m = n-2r$. Under the identification

$$W(r)_{\mathbb{C}} = W_{\mathbb{C}} \otimes \mathbb{Z}(r)_{\mathbb{C}} \xrightarrow{\sim} W_{\mathbb{C}} \qquad \text{by} \quad w \otimes (2\pi i)^r \mapsto (2\pi i)^r \cdot w\,,$$

we have $W(r)^{a,b}_{\mathbb{C}} = W^{a+r,b+r}_{\mathbb{C}}$. Then $(2\pi i)^r \cdot f$ gives a map $V \to W(r)$, and this last map *is* a morphism of HS. So: what in the older literature was called a "morphism $V \to W$ of degree r" now becomes a morphism $V \to W(r)$. Alternatively, we get a morphism $V(-r) \to W$.

(1.5) Let $V_{\mathbb{Z}}$ be a $\mathbb{Z}$-HS of weight m. Let $C_{\mathbb{C}}$ be the endomorphism of $V_{\mathbb{C}}$ that acts as multiplication by i^{q-p} on the summand $V^{p,q}_{\mathbb{C}}$. Using the relation (2), we see that $C_{\mathbb{C}}$ maps $V_{\mathbb{R}}$ into itself and therefore gives an endomorphism $C = C_V \in \mathrm{GL}(V_{\mathbb{R}})$, called the Weil operator. Note that $C^2 = (-1)^m$. If $f\colon V_{\mathbb{R}} \to W_{\mathbb{R}}$ is a morphism of real HS then $f \circ C_V = C_W \circ f$.

A polarisation of the Hodge structure $V_{\mathbb{Z}}$ is a morphism of Hodge structures $\varphi\colon V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}(-m)$ such that the bilinear form $V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ given by $(x, y) \mapsto (2\pi i)^m \cdot \varphi(Cx \otimes y)$ is symmetric and positive definite. Since we require φ to be a morphism of HS, $\varphi(Cx \otimes y) = \varphi(C^2 x \otimes Cy) = (-1)^m \cdot \varphi(x \otimes Cy)$; hence we find that the form φ is alternating if m is odd, and symmetric if m is even.

A Hodge structure is said to be polarisable if it admits a polarisation. A rather trivial, but important remark is that every sub-HS of a polarisable HS is itself again polarisable. Also the tensor product and direct sum of polarisable HS is again polarisable. We find that the subcategory $\mathbb{Q}\mathrm{HS}^{\mathrm{pol}} \subset \mathbb{Q}\mathrm{HS}$ of polarisable $\mathbb{Q}$-HS is an abelian subcategory that is closed under $\otimes$. Moreover, the category $\mathbb{Q}\mathrm{HS}^{\mathrm{pol}}$ is semi-simple. The key point is that if $W \subset V$ is a sub-HS (with $\mathbb{Q}$-coefficients), and if φ is a polarisation of V, then the orthogonal complement $W^{\perp} \subset V$ with respect to φ is again a sub-HS, and $V \cong W \oplus W^{\perp}$ as $\mathbb{Q}$-HS.

Like most notions in Hodge theory, the notion of a polarisation has its origin in geometry. To explain how polarisations arise geometrically, consider a non-singular complex projective variety X. Choose an ample line bundle L, and let $c_1(L) \in H^2(X, \mathbb{Z})\bigl(1\bigr)$ be its first Chern class (= the image of $[L] \in \mathrm{Pic}(X)$ under the boundary map $H^1(X, \mathscr{O}_X^*) \to H^2(X, \mathbb{Z})\bigl(1\bigr)$ arising from the exponential sequence). Let $d := \dim(X)$, and for $0 \leqslant i \leqslant d$, let $H^i_{\mathrm{prim}}(X, \mathbb{Q})$ be the primitive cohomology with respect to L. Then the pairing

$$H^i_{\mathrm{prim}}(X, \mathbb{Q}) \times H^i_{\mathrm{prim}}(X, \mathbb{Q}) \to H^{2d}(X, \mathbb{Q})\bigl(d - i\bigr) \cong \mathbb{Q}(-i)$$

given by $(x, y) \mapsto (-1)^i \cdot c_1(L)^{d-i} \cdot x \cup y$ is a polarisation of the Hodge structure $H^i_{\mathrm{prim}}(X, \mathbb{Q})$. To get a polarisation on the whole of $H^i(X, \mathbb{Q})$ one uses the Lefschetz decomposition $H^i(X, \mathbb{Q}) = \oplus_{j \geqslant 0}\, c_1(L)^j \cdot H^{i-2j}_{\mathrm{prim}}(X, \mathbb{Q})\bigl(-j\bigr)$.

(1.6) An important aspect in the study of Shimura varieties is that certain Shimura varieties have a modular interpretation in terms of abelian varieties, possibly equipped with additional structures. This connection arises from the fact that the functor

$$\begin{pmatrix} \text{complex} \\ \text{abelian varieties} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{polarisable } \mathbb{Z}\text{-HS} \\ \text{of type } (-1,0) + (0,-1) \end{pmatrix}$$

given by $X \mapsto H_1(X, \mathbb{Z})$ is an equivalence of categories. A quasi-inverse functor is obtained as follows. Starting from a polarisable $\mathbb{Z}$-HS $V_{\mathbb{Z}}$ of type $(-1,0) + (0,-1)$, take $J := C \in \mathrm{End}(V_{\mathbb{R}})$, where C is the Weil operator. Then J is a complex structure on $V_{\mathbb{R}}$. Let us write W for $V_{\mathbb{R}}$, viewed as a $\mathbb{C}$-vector space. Then $X := W/V_{\mathbb{Z}}$ is a complex torus. The assumpion that $V_{\mathbb{Z}}$ is polarisable corresponds precisely to the fact that the complex torus X is projective algebraic. (Recall that a complex torus X is called an abelian variety if X is an algebraic variety, in which case it is in fact projective.) For this, choose a polarisation form $\varphi\colon V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}(1)$. It is an alternating form, and can therefore be viewed as an element of $H^2(X, \mathbb{Z})\bigl(1\bigr) \cong \wedge^2\bigl(H_1(X, \mathbb{Z})^{\vee}\bigr) \otimes \mathbb{Z}(1)$. There exists a line bundle L whose first Chern class is precisely this class φ, and any such L is in fact ample. See [38], Chap. 1 for further details.

There are some useful variations on this theme. E.g., polarised abelian varieties correspond to polarised Hodge structures. Also, abelian varieties up to isogeny correspond to polarisable $\mathbb{Q}$-Hodge structures of type $(-1,0)+(0,-1)$.

(1.7) *Exercise.* Show that every (pure) $\mathbb{Q}$-HS of dimension 2 that is polarisable. Show that there exist non-polarisable 4-dimensional HS. (Use (1.6).) How about 3-dimensional $\mathbb{Q}$-HS?

Further reading. Good introductions to classical Hodge theory include Griffiths-Harris [24] and the book of Wells [55]. The more formal aspects of Hodge theory, together with mixed Hodge theory, were shaped by Deligne in [11], [13] and [15]. Meanwhile Hodge theory has become a vast theory; for some overviews of what is going on see e.g. [8], [25] and [53], where the first focuses more on abstract aspects and the last two are more geometrically oriented. There is also a book about mixed Hodge theory in preparation by C. Peters and J. Steenbrink.

§ 2. The Hodge conjecture

(2.1) Let V be a $\mathbb{Q}$-HS of weight m. The level (or Hodge-level) of V is defined as

$$\mathrm{level}(V) = \max\bigl\{|p-q| \bigm| V_{\mathbb{C}}^{p,q} \neq 0\bigr\}\,,$$

with the convention that we declare $V=0$ to have level $-\infty$. Geometric example: $H^m(X,\mathbb{Q})$ has level $\leqslant m$, and the level equals m (for $0 \leqslant m \leqslant 2\dim(X)$) precisely if $H^m(X,\mathscr{O}_X) \neq 0$.

A $\mathbb{Q}$-subspace $W \subseteq V$ is called a sub-HS if W is a subobject in the category $\mathbb{Q}$HS, i.e., if $W_{\mathbb{C}} = \oplus_{p+q=m}\bigl(V_{\mathbb{C}}^{p,q} \cap W_{\mathbb{C}}\bigr)$.

We call a non-zero element $v \in V$ a Hodge class if v is purely of type $(0,0)$ in the Hodge decomposition. Equivalent: $\mathbb{Q}\cdot v \subset V$ is a sub-HS isomorphic to $\mathbb{Q}(0)$. Thus, the Hodge classes are the rational classes that are purely of type $(0,0)$. We also count $0 \in V$ as a Hodge class. Note that, according to this definition, we can only have non-zero Hodge classes if the weight m is zero. If $m=2p$ and $v \in V$ is purely of Hodge type (p,p) (a Hodge class of type (p,p) in the older literature) then $(2\pi i)^p \cdot v \in V(p)$ is a Hodge class in our sense. Example: a morphism of HS $f\colon V \to W$ is the same as a Hodge class in the $\mathbb{Q}$-HS $\mathrm{Hom}(V,W)$.

Geometric example: Let X be a non-singular, proper complex algebraic variety. Let $Z \subset X$ be a subvariety of codimension k, possibly singular. If $\tilde{Z} \to Z$ is a resolution of singularities then we have exact sequences

$$H^{m-2k}(\tilde{Z},\mathbb{Q})\bigl(-k\bigr) \xrightarrow{\gamma} H^m(X,\mathbb{Q}) \xrightarrow{j} H^m(X \setminus Z,\mathbb{Q})\,.$$

Here the map γ (Gysin map) is a morphism of Hodge structures. The restriction map j is a morphism in the bigger category of mixed Hodge structures, which shall not be discussed in these notes. We see that

$$\mathrm{Ker}\Bigl(H^m(X,\mathbb{Q}) \xrightarrow{j} H^m(X \setminus Z,\mathbb{Q})\Bigr)$$

is a sub-HS of level $\leqslant m-2k$.

If we take $m=2k$ and twist by k then we get a morphism $H^0(\tilde{Z},\mathbb{Q}) \to H^{2k}(X,\mathbb{Q})\bigl(k\bigr)$. Define $cl(Z) \in H^{2k}(X,\mathbb{Q})\bigl(k\bigr)$, the cycle class of Z, to be the image under this map of the natural generator $[\tilde{Z}] \in H^0(\tilde{Z},\mathbb{Q})$. Alternative description: $cl(Z)$ is Poincaré dual to the fundamental class $[Z] \in H_{2\dim(X)-2k}(X,\mathbb{Q})$. By construction, $cl(Z)$ is a Hodge class.

(2.2) If you want to make a lot of money (see `www.claymath.org`), prove either one of the following assertions:

The Hodge conjecture: Let X be a non-singular, proper complex algebraic variety. If $\xi \in H^{2k}(X,\mathbb{Q})\bigl(k\bigr)$ is a Hodge class, then there exists closed subvarieties $Z_1,\ldots,Z_t \subset X$ of codimension k such that $\xi = a_1 \cdot cl(Z_1) + \cdots + a_t \cdot cl(Z_t)$ for certain rational numbers a_i.

General Hodge conjecture: With X as before, let $W \subseteq H^m(X,\mathbb{Q})$ be a sub-HS with level$(W) = r$. Then there exists a closed subset $Z \subset X$ of codimension $(m-r)/2$ such that W is contained in $\mathrm{Ker}\bigl(H^m(X,\mathbb{Q}) \xrightarrow{j} H^m(X \setminus Z,\mathbb{Q})\bigr)$.

(2.3) Let X be a non-singular, proper complex algebraic variety. Write $\mathscr{B}^k(X) \subset H^{2k}(X,\mathbb{Q})\bigl(k\bigr)$ for the subspace of Hodge classes. Then $\mathscr{B}^\bullet(X) := \oplus_{k\geqslant 0}\mathscr{B}^k(X)$ is a subring of the "even" cohomology $\oplus_{k\geqslant 0}H^{2k}(X,\mathbb{Q})\bigl(k\bigr)$, called the Hodge ring of X. The Hodge conjecture predicts that it is generated, as a $\mathbb{Q}$-algebra, by the algebraic classes, i.e., the classes of the form $cl(Z)$.

Essentially the only general case where the Hodge conjecture is known concerns divisor classes. The Lefschetz theorem on $(1,1)$-classes asserts that every Hodge class in $H^2(X,\mathbb{Q})\bigl(1\bigr)$ is a $\mathbb{Q}$-linear combination of divisor classes. With modern techniques this can be proven as follows. The exponential sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \longrightarrow 0$$

gives a long exact sequence

$$\cdots \longrightarrow H^1(X,\mathscr{O}_X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\delta} H^2(X,\mathbb{Z})\bigl(1\bigr) \xrightarrow{\alpha} H^2(X,\mathscr{O}_X) \longrightarrow \cdots$$

and the map α is the composition of the inclusion $H^2(X,\mathbb{Z})\bigl(1\bigr) \hookrightarrow H^2(X,\mathbb{C})$ and the projection $H^2(X,\mathbb{C}) \to H^2(X,\mathscr{O}_X)$ onto the $(0,2)$-component in the Hodge decomposition. Using the relation (2) in Section (1.1) we find that the kernel of α is precisely the space of Hodge classes in $H^2(X,\mathbb{Z})\bigl(1\bigr)$. But δ is the map that associates to a line bundle $\mathscr{O}_X(D)$ the cohomology class $cl(D)$ of D. So we are done.

If two cohomology classes on X are both algebraic then so is their cup-product; the reason is that $cl(Z_1) \cup cl(Z_2) = cl(Z_1 \cap Z_2)$, where by $Z_1 \cap Z_2$ we mean any cycle representing the intersection product of Z_1 and Z_2. (So if the cycles intersect transversally then "$\cap$" is the naive intersection.) Hence if we let $\mathscr{D}^\bullet(X) \subseteq \mathscr{B}^\bullet(X)$ be the $\mathbb{Q}$-subalgebra generated by the divisor classes then all classes in $\mathscr{D}^\bullet(X)$ are certainly algebraic. In particular, if it happens to be the case that $\mathscr{D}^\bullet(X) = \mathscr{B}^\bullet(X)$ then the Hodge conjecture for X holds true. We shall see examples of this later. Of course, in general one cannot expect that $\mathscr{B}^\bullet(X)$ is generated by divisor classes, though in concrete examples it is not always so easy to decide whether or not this is the case.

Further reading. The Hodge conjecture was originally stated by Hodge as a problem (not a conjecture!) in his address to the international congress; see [28]. It was pointed out by Grothendieck [26] that the formulation of Hodge's general problem was incorrect. What today is called the general Hodge conjecture is Grothendieck's reformulation of the problem. For overviews of what is known about the Hodge conjecture, see Shioda [49] and Lewis [29]. Some interesting further results can be found in Steenbrink [50]. Of further interest is Deligne's official description of the problem at the Clay website (`www.claymath.org`).

§ 3. Describing Hodge structures using the Deligne torus

(3.1) Let k be a field, and choose a separable closure k^s. Let T be a torus over k; by this we mean that T is a linear algebraic group over k with $T \otimes k^s \cong (\mathbb{G}_{m,k^s})^r$ for some $r \in \mathbb{Z}_{\geqslant 0}$, called the rank of the torus. (Do not confuse this with the notion of a complex torus.) The character group $X^*(T)$ and the cocharacter group $X_*(T)$ are defined by

$$X^*(T) := \mathrm{Hom}(T_{k^s}, \mathbb{G}_{m,k^s})\,, \qquad X_*(T) := \mathrm{Hom}(\mathbb{G}_{m,k^s}, T_{k^s})\,.$$

These are free abelian groups of rank r which come equipped with a continuous action of $\mathrm{Gal}(k^s/k)$, and we have a natural perfect pairing $X^*(T) \times X_*(T) \longrightarrow \mathrm{End}(\mathbb{G}_{m,k^s}) = \mathbb{Z}$.

The functor

$$X_*(\ \ )\colon \begin{pmatrix}\text{algebraic tori}\\ \text{over } k\end{pmatrix} \longrightarrow \begin{pmatrix}\text{free abelian group of finite rank}\\ \text{with a continuous action of } \mathrm{Gal}(k^s/k)\end{pmatrix}$$

is an equivalence of categories. Similarly, the functor $X^*(\ )$ gives an anti-equivalence of categories. One may think of this as a statement in descent theory: we describe a torus over k by saying what it is over k^s, plus a specification of how the Galois group acts.

Let now $\rho\colon T \to \mathrm{GL}(V)$ be a representation of T on a finite dimensional k-vector space. If $k = k^s$, so that $T \cong \mathbb{G}_m^r$ then the situation is clear: the space V decomposes as a direct sum of character spaces and this completely determines the representation. Thus, a representation of $\mathbb{G}_m^r$ on a vector space V corresponds to a $\mathbb{Z}^r$-grading

$$V = \bigoplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^r} V^{n_1,\ldots,n_r}\,.$$

More canonically: a representation of $T \otimes k^s$ on a k^s-vector space V corresponds to giving a decomposition into character spaces $V = \oplus_{\chi\in X^*(T)} V(\chi)$.

Over an arbitrary field k, if T is not necessarily split, we can describe representations by again using descent theory. To ensure that a representation over k^s is defined over k, all we have to do is to require that the actions of $\mathrm{Gal}(k^s/k)$ on $X^*(T)$ and on $V \otimes_k k^s$ "match". Thus, let T be a k-torus and write $\mathrm{Rep}_k(T)$ for the category of finite dimensional k-representations of T. Then we have an equivalence of categories

$$\mathrm{Rep}_k(T) \xrightarrow{\text{eq}} \begin{pmatrix}\text{finite dimensional } k\text{-vector spaces } V \text{ with}\\ X^*(T)\text{-grading } V_{k^s} = \bigoplus_{\chi\in X^*(T)} V_{k^s}(\chi)\\ \text{s.t. } {}^\sigma(V_{k^s}(\chi)) = V_{k^s}({}^\sigma\chi) \text{ for all } \sigma \in \mathrm{Gal}(k^s/k)\end{pmatrix}.$$

(3.2) *The Deligne torus.* Define a torus $\mathbb{S}$ over $\mathbb{R}$ by

$$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}\,,$$

where "Res" denotes restriction of scalars à la Weil. This means that for a commutative $\mathbb{R}$-algebra A we have $\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^*$. This $\mathbb{S}$ is a torus of rank 2 over $\mathbb{R}$; its character group is generated by two characters z and $\bar{z}$ such that the induced maps on points

$$\mathbb{C}^* = \mathbb{S}(\mathbb{R}) \subset \mathbb{S}(\mathbb{C}) \longrightarrow \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$$

are the identity, resp. complex conjugation. In other words: $X^*(\mathbb{S}) = \mathbb{Z}\cdot z + \mathbb{Z}\cdot\bar{z}$ with complex conjugation $\iota \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acting by ${}^\iota z = \bar{z}$ and ${}^\iota\bar{z} = z$. By what was explained in (3.1), this uniquely determines $\mathbb{S}$ as an $\mathbb{R}$-torus.

Define the *weight cocharacter* $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ to be the cocharacter given on points by the natural inclusion $\mathbb{R}^* = \mathbb{G}_{m,\mathbb{R}}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$. The *norm character* $\mathrm{Nm}\colon \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$ is defined by $\mathrm{Nm} = z\bar{z}$. The kernel of Nm is the *circle group* $\mathbb{U}_1 = \big\{z \in \mathbb{C}^* \bigm| |z| = 1\big\}$, viewed as an $\mathbb{R}$-torus. Finally we define the cocharacter $\mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_\mathbb{C}$ to be the unique cocharacter such that $\bar{z}\circ\mu$ is trivial and $z\circ\mu = \mathrm{id} \in \mathrm{End}(\mathbb{G}_{m,\mathbb{C}})$. On $\mathbb{C}$-valued points, identifying $\mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$, we have $\mu\colon \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*$ given by $w \mapsto (w, 1)$.

(3.3) Let $V_\mathbb{R}$ be a real vector space. If we apply what was explained in (3.1) with $T = \mathbb{S}$ then we find that to give a representation $h\colon \mathbb{S} \to \mathrm{GL}(V_\mathbb{R})$ is equivalent to giving a decomposition

$$V_\mathbb{C} = \bigoplus_{p,q\in\mathbb{Z}} V_\mathbb{C}^{p,q} \qquad \text{with} \qquad \overline{V_\mathbb{C}^{p,q}} = V_\mathbb{C}^{q,p} \quad \text{for all } p \text{ and } q.$$

Here the convention is that $V_\mathbb{C}^{p,q}$ is the eigenspace for the character $z^{-p}\bar{z}^{-q}$; this is nowadays the standard sign convention in Hodge theory.

We find: to give a representation $h\colon \mathbb{S} \to \mathrm{GL}(V_\mathbb{R})$ is the same as giving $V_\mathbb{R}$ an $\mathbb{R}$-Hodge structure. This Hodge structure is of weight m precisely if $h\circ w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}(V_\mathbb{R})$ is given on points by $a \mapsto a^{-m}\cdot\mathrm{id}$.

But a $\mathbb{Q}$-HS consists of a $\mathbb{Q}$-vector space V together with an $\mathbb{R}$-HS on $V_\mathbb{R}$. Hence we may rephrase the definition in (1.1) as follows: A $\mathbb{Q}$-HS of weight m consists of a finite dimensional $\mathbb{Q}$-vector space V together with a homomorphism $h\colon \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$, such that $h\circ w$ is given by $a \mapsto a^{-m}\cdot\mathrm{id}$. And an arbitrary $\mathbb{Q}$-HS structure is given by a $\mathbb{Q}$-space V of finite dimension plus a homomorphism $h\colon \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$ such that $h\circ w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}(V)_\mathbb{R}$ is defined over $\mathbb{Q}$. Indeed, this last condition simply means that the decomposition $V_\mathbb{R} = \oplus_{m\in\mathbb{Z}} V_{\mathbb{R},m}$ into weight spaces comes from a decomposition $V = \oplus V_m$ over $\mathbb{Q}$. With $\mathbb{Z}$-coefficients we have similar statements.

Note that, in our new description of Hodge structures, the Weil operator C is the automorphism $h(i) \in \mathrm{GL}(V_\mathbb{R})$. Morphisms of $\mathbb{Q}$-HS correspond to $\mathbb{Q}$-linear maps $f\colon V \to W$ such that $f_\mathbb{R}\colon V_\mathbb{R} \to W_\mathbb{R}$ is equivariant for the given actions of $\mathbb{S}$. The representation $h\colon \mathrm{GL}(V\otimes W)_\mathbb{R}$ corresponding to the Hodge structure $V\otimes W$ is of course just the tensor product of the representation h_V and h_W; similarly for the operations $\mathrm{Hom}(\ ,\ )$, $(\ )^\vee$, $\mathrm{Sym}^k(\ )$ and $\wedge^k$.

(3.4) *Example.* The Tate structure $\mathbb{Z}(n)$ has underlying $\mathbb{Z}$-module $(2\pi i)^n \cdot \mathbb{Z}$, and the homomorphism $h\colon \mathbb{S} \to \mathrm{GL}\big(\mathbb{Z}(n)\big)_\mathbb{R} = \mathbb{G}_{m,\mathbb{R}}$ is the n-th power of the norm character N.

(3.5) *Example.* Let $\tau = a + bi \in \mathbb{C}$ with $b = \Im(\tau) \neq 0$, and consider the elliptic curve $E = E_\tau := \mathbb{C}/V_\mathbb{Z}$ with $V_\mathbb{Z} := \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$. Then $H_1(E,\mathbb{Z}) = V_\mathbb{Z}$, which we identify with $\mathbb{Z}^2$ via the chosen basis $\{1,\tau\}$. The Hodge structure should be such that the Weil operator $C = h(i)$ corresponds to the multiplication by i in $V_\mathbb{R} \cong \mathbb{C}$. We find that the homomorphism $h\colon \mathbb{S} \to \mathrm{GL}_2(\mathbb{R})$ is given on $\mathbb{R}$-valued points by

$$\mathbb{S}(\mathbb{R}) = \mathbb{C}^* \ni x + yi \ \mapsto \begin{pmatrix} x - \frac{a}{b}\cdot y & -\frac{a^2+b^2}{b}\cdot y \\ \frac{1}{b}\cdot y & x + \frac{a}{b}\cdot y \end{pmatrix}.$$

The action of a matrix $A := \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ on $\mathfrak{H}^\pm := \mathbb{C}\setminus\mathbb{R}$ by fractional linear transformations corresponds to the conjugation action on the set of homomorphisms $\mathbb{S} \to \mathrm{GL}_2(\mathbb{R})$: if τ corresponds to the homomorphism h then $A\cdot\tau := (t\tau + u)/(v\tau + w)$, corresponds to the homomorphism AhA^{-1}. This realizes $\mathfrak{H}^\pm$ as a conjugacy class of homomorphisms $\mathbb{S} \to \mathrm{GL}_{2,\mathbb{R}}$, and if we define $K_\tau \subset \mathrm{GL}_2(\mathbb{R})$ to be the stabilzer of a point $\tau \in \mathfrak{H}^\pm$ then we get an isomorphism $\mathfrak{H}^\pm \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{R})/K_\tau$.

Further reading. The description of Hodge structures in terms of homomorphisms $\mathbb{S} \to \mathrm{GL}(V)$ was (to my knowledge) first used by Deligne in [13]. Though it does not add anything to the concept of a Hodge structure, it gives a more group-theoretic point of view than the classical definition, which is particularly relevant for the notion of a Mumford-Tate group and for the description of Shimura varieties. For mixed Hodge structures there is also a description using the torus $\mathbb{S}$, though this is much more delicate; see Pink's thesis [43].

§ 4. Mumford-Tate groups

(4.1) Let V be a $\mathbb{Q}$-HS, given by the homomorphism $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$. Define the Mumford-Tate group of V, notation $\mathrm{MT}(V)$, to be the smallest algebraic subgroup $M \subset \mathrm{GL}(V)$ (over $\mathbb{Q}$!) such that h factors through $M_{\mathbb{R}} \subset \mathrm{GL}(V)_{\mathbb{R}}$.

(4.2) *Exercise.* Recall that we have defined a cocharcter $\mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$; see (3.2). Let $\mu_h := h_{\mathbb{C}} \circ \mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$. Show that $\mathrm{MT}(V)$ is the smallest algebraic subgroup $M \subset \mathrm{GL}(V)$ over $\mathbb{Q}$ such that μ_h factors through $M_{\mathbb{C}}$.

(4.3) The key property of the Mumford-Tate group is that in any tensor construction obtained from V, it cuts out exactly the sub-Hodge structures. To explain what we mean by this, we need some notation: If $\nu = \{(a_i, b_i)\}_{i=1,\ldots,t}$ is a collection of pairs of integers a_i, $b_i \in \mathbb{Z}_{\geqslant 0}$, define

$$T^\nu := \oplus_{i=1}^t \, V^{\otimes a_i} \otimes (V^\vee)^{\otimes b_i} \,,$$

which inherits a natural Hodge structure from V. The groups $\mathrm{MT}(V) \subseteq \mathrm{GL}(V)$ naturally act on T^ν. We refer to spaces of the form T^ν as tensor spaces obtained from V.

(4.4) Proposition. *For any collection ν as above, if $W \subset T^\nu$ is a $\mathbb{Q}$-subspace, W is a sub-Hodge structure if and only if W is stable under the action of* $\mathrm{MT}(V)$ *on T^ν. Further, an element $t \in T^\nu$ is a Hodge class if and only if t is an invariant under* $\mathrm{MT}(V)$.

Proof. Let $H \subset \mathrm{GL}(V)$ be the stabilizer of the subspace W, i.e., the subgroup of elements $g \in \mathrm{GL}(V)$ that map W into itself. Then H is an algebraic subgroup over $\mathbb{Q}$. If $W \subset T^\nu$ is a sub-Hodge structure then W is stable under the action of $\mathbb{S}$ through the homomorphism h, which means that h factors through $H_{\mathbb{R}}$. By definition of $\mathrm{MT}(V)$ this implies that $\mathrm{MT}(V) \subseteq H$. Conversely, if $\mathrm{MT}(V) \subseteq H$ then $h\colon \mathbb{S} \to \mathrm{GL}(V)$ factors through $H_{\mathbb{R}}$ and it follows that W is a sub-HS. For the second assertion, apply the first result to the space $\mathbb{Q}(0) \oplus T^\nu$, which is again a tensor space obtained from V, and use that $t \in T^\nu$ is a Hodge class if and only if the subspace $\mathbb{Q} \cdot (1,t) \subset \mathbb{Q}(0) \oplus T^\nu$ is a sub-HS. ☐

(4.5) Corollary. *The functor* $\mathrm{Rep}\big(\mathrm{MT}(V)\big) \to \mathbb{Q}\mathrm{HS}$ *that sends* $\rho\colon \mathrm{MT}(V) \to \mathrm{GL}(W)$ *to the Hodge structure on W given by* $\rho \circ h\colon \mathbb{S} \to \mathrm{GL}(W)$*, is fully faithful. The image of this functor is the full subcategory* $\langle V \rangle^\otimes \subset \mathbb{Q}\mathrm{HS}$ *whose objects are the $\mathbb{Q}$-HS that are isomorphic to a subquotient of some T^ν as above.*

Proof. For the first assertion, recall that a morphism of HS $W_1 \to W_2$ is the same as a Hodge class in $\mathrm{Hom}(W_1, W_2)$. For the second assertion, use the Proposition plus the fact that every representation of $\mathrm{MT}(V)$ is isomorphic to a subquotient of some T^ν; see for instance [17], Prop. 3.1. ☐

The subcategory $\langle V\rangle^{\otimes} \subset \mathbb{Q}\mathrm{HS}$ is called the tensor subcategory generated by V. The equivalence of categories $\mathrm{Rep}\bigl(\mathrm{MT}(V)\bigr) \to \langle V\rangle^{\otimes}$ is compatible with tensor structures. There is a general theory of categories equipped with tensor products. The categories $\langle V\rangle^{\otimes} \subset \mathbb{Q}\mathrm{HS}$ are examples of so-called (neutral) Tannakian categories. We refer to [19] or [46] for an introduction to this theory.

(4.6) Lemma. *If V_1 and V_2 are $\mathbb{Q}$-HS then $\mathrm{MT}(V_1 \oplus V_2) \subset \mathrm{MT}(V_1) \times \mathrm{MT}(V_2)$ as subgroups of $\mathrm{GL}(V_1 \oplus V_2)$, and the projection maps $\mathrm{pr}_i\colon \mathrm{MT}(V_1 \oplus V_2) \to \mathrm{MT}(V_i)$ are surjective.*

Proof. Immediate from the definition of the Mumford-Tate group. $\square$

(4.7) In practice it is often convenient to consider not only all tensor spaces (plus subquotients) obtained from a Hodge structure V, but also to include all Tate twists of such. What one does for this is the following. Define the "big Mumford-Tate group of V" to be $\mathrm{MT}^{\sharp}(V) := \mathrm{MT}\bigl(V \oplus \mathbb{Q}(1)\bigr)$. The Mumford-Tate group of $\mathbb{Q}(1)$ is just the multiplicative group $\mathbb{G}_m$, so by the lemma $\mathrm{MT}^{\sharp}(V)$ may be considered as a subgroup of $\mathrm{MT}(V) \times \mathbb{G}_m$, with surjective projections onto the two factors. The subcategory $\bigl\langle V \oplus \mathbb{Q}(1)\bigr\rangle^{\otimes} \subset \mathbb{Q}\mathrm{HS}$ is the full subcategory whose objects are the $\mathbb{Q}$-HS that are isomorphic to a subquotient of some $T^{\nu}(r)$, where T^{ν} is a tensor space obtained from V. The action of $\mathrm{MT}^{\sharp}(V)$ on $T^{\nu}(r)$ is the tensor product of the action on T^{ν} via the projection $\mathrm{MT}^{\sharp}(V) \to \mathrm{MT}(V)$, and the action on $\mathbb{Q}(1)^{\otimes r}$ via the projection $\mathrm{MT}^{\sharp}(V) \to \mathbb{G}_m = \mathrm{GL}\bigl(\mathbb{Q}(1)\bigr)$.

Supose V is purely of weight m. Let $d := \dim(V)$. Then $\det(V) := \wedge^d V \cong \mathbb{Q}(-dm/2)$. (Note that if m is odd then d is necessarily even.) Hence if $m \neq 0$ then $\langle V\rangle^{\otimes}$ contains a non-trivial Tate-twist. This translates into the assertion that the projection $\mathrm{MT}^{\sharp}(V) \to \mathrm{MT}(V)$ is an isogeny. See the next exercise.

(4.8) *Exercise.* Suppose V is purely of weight m. Prove the following assertions. If $m = 0$ then $\mathrm{MT}^{\sharp}(V) \xrightarrow{\sim} \mathrm{MT}(V) \times \mathbb{G}_m$. If $m \neq 0$ then $\mathrm{MT}^{\sharp}(V) \to \mathrm{MT}(V)$ is an isogeny. If $m = 1$ and V is polarisable then $\mathrm{MT}^{\sharp}(V) \to \mathrm{MT}(V)$ is an isomorphism.

(4.9) Proposition. *Let V be a $\mathbb{Q}$-Hodge structure.*
- (i) *The Mumford-Tate group $\mathrm{MT}(V)$ is connected.*
- (ii) *Suppose V is purely of weight m. If $m = 0$ then $\mathrm{MT}(V) \subseteq \mathrm{SL}(V)$. If $m \neq 0$ then $\mathrm{MT}(V)$ contains $\mathbb{G}_m \cdot \mathrm{id}$.*
- (iii) *If V is polarisable then $\mathrm{MT}(V)$ is reductive.*

Proof. Parts (i) and (ii) easily follow from the definition of $\mathrm{MT}(V)$. For (iii) we use that a connected algebraic group over a field of characteristic 0 is reductive if and only if has a faitful completely reducible representation. But the tautological representation $\mathrm{MT}(V) \hookrightarrow \mathrm{GL}(V)$ is completely reducible, because the category $\mathbb{Q}\mathrm{HS}^{\mathrm{pol}}$ is semi-simple and the sub-HS of V are precisely the submodules for the action of $\mathrm{MT}(V)$. $\square$

(4.10) *Exercise.* Let V be a $\mathbb{Q}$-HS, purely of some weight m. If $n \geqslant 1$, show that $\mathrm{MT}(V^n) = \mathrm{MT}(V)$, where we view $\mathrm{MT}(V)$ as a subgroup of $\mathrm{GL}(V^n)$ through its diagonal action on V^n.

Further reading. Mumford-Tate groups (or rather the closely related concept of Hodge groups) were first introduced in Mumford's paper [36]. They fit very naturally into the framework of Tannakian categories, for which we refer to [46], [19] and [18]. There are a number of important problems that are related to Mumford-Tate groups. One of them is the Mumford-Tate conjecture, which, loosely speaking, asserts that for a variety defined over a number field the image of the Galois representation on the ℓ-adic cohomology gives the same algebraic group as the Mumford-Tate group. This is a whole subject in itself, for which we can only refer to some key publications, such as [36], [17], [6], [44]. Another interesting problem is a conjecture of Grothendieck that relates the transcendence degree of the periods of a variety defined over a number field with the dimension of its Mumford-Tate group (or better: its motivic Galois group.) See for instance [7], Sect. 6.5.

§ 5. Examples of Mumford-Tate groups

The goal of this section is to discuss some examples where we can explicitely compute the Mumford-Tate group, and to give some applications.

(5.1) Identifying $\mathrm{GL}\big(\mathbb{Q}(n)\big) = \mathbb{G}_m$, one easily finds that

$$\mathrm{MT}\big(\mathbb{Q}(n)\big) = \begin{cases} \mathbb{G}_m & \text{if } n \neq 0; \\ \{1\} & \text{if } n = 0. \end{cases}$$

(5.2) The next examples that we want to discuss are all related to abelian varieties. We begin with some general considerations.

Let X be a complex abelian variety. Set $g := \dim(X)$ and $V := H_1(X, \mathbb{Q})$. The choice of a polarisation $\lambda\colon X \to X^t$ gives us a polarisation form $\varphi\colon V \otimes V \to \mathbb{Q}(1)$. We write $\mathrm{MT}(X) := \mathrm{MT}(V)$.

The endomorphism algebra $D := \mathrm{End}^0(X) := \mathrm{End}(X) \otimes \mathbb{Q}$ is a finite dimensional semi-simple $\mathbb{Q}$-algebra; see [38], Section 19. Now we have

$$D \xrightarrow{\sim} \{\text{Hodge classes in } \mathrm{End}(V)\} = \mathrm{End}(V)^{\mathrm{MT}(V)}\,, \tag{3}$$

where the first isomorphism is an application of what we discussed in (1.6), and the second identity is what we have seen in Prop. 4.4. So we find that $\mathrm{MT}(X)$ is contained in the algebraic group $\mathrm{GL}_D(V)$ of D-linear automorphisms of V.

The polarisation form φ is a Hodge class in the space $\mathrm{Hom}\big(V^{\otimes 2}, \mathbb{Q}(1)\big)$, and is therefore an invariant under the "big Mumford-Tate group" $\mathrm{MT}^\sharp(X)$. Hence $\mathrm{MT}(X)$ is contained in the algebraic group $\mathrm{CSp}(V, \varphi)$ of symplectic similitudes of the space V with respect to the symplectic form φ. In fact, the relation between the "big" and the "small" Mumford-Tate groups is in this case that $\mathrm{MT}^\sharp(X) \subset \mathrm{MT}(X) \times \mathbb{G}_m \subset \mathrm{CSp}(V, \varphi) \times \mathbb{G}_m$ is the graph of the multiplier character $\mathrm{CSp}(V, \varphi) \to \mathbb{G}_m$ restricted to $\mathrm{MT}(X)$. (The multiplier character is the character given by the property that $\varphi(gv, gw) = \nu(g) \cdot \varphi(v, w)$ for all $g \in \mathrm{CSp}(V, \varphi)$ and v, $w \in V$.) So indeed $\mathrm{MT}^\sharp(X) \xrightarrow{\sim} \mathrm{MT}(X)$, as was part of Exercise (4.8).

Summing up, we find that $\mathrm{MT}(X)$ is a connected reductive group with

$$\mathbb{G}_m \cdot \mathrm{id} \;\subseteq\; \mathrm{MT}(X) \;\subseteq\; \mathrm{CSp}_D(V, \varphi) := \mathrm{GL}_D(V) \cap \mathrm{CSp}(V, \varphi)\,,$$

such that the relation (3) holds.

(5.3) Recall that a complex abelian variety X with $\dim(X) = g$ is said to be of CM-type if there is a commutative semi-simple subalgebra $F \subseteq \mathrm{End}^0(X)$ with $\dim_{\mathbb{Q}}(F) = 2g$. If X is simple then this is equivalent to the condition that $\mathrm{End}^0(X)$ is a CM-field of degree $2g$ over $\mathbb{Q}$. (Over a ground field of positive characteristic the two conditions are no longer equivalent.)

There is a simple criterion to decide if X is of CM-type, in terms of its Mumford-Tate group. Namely: X is of CM-type if and only if $\mathrm{MT}(X)$ is a torus. Let us prove this.

First suppose X is of CM-type, and let $F \subseteq \mathrm{End}^0(X)$ be a semi-simple commutative subalgebra with $\dim_{\mathbb{Q}}(F) = 2g$. Then V is a free module of rank 1 over F, so the condition that $\mathrm{MT}(X) \subseteq \mathrm{GL}_D(V) \subseteq \mathrm{GL}_F(V)$ implies that $\mathrm{MT}(X)$ is contained in the algebraic torus T_F. Here by T_F we mean the algebraic group over $\mathbb{Q}$ with $T_F(R) = (F \otimes_{\mathbb{Q}} R)^*$ for any $\mathbb{Q}$-algebra R. As this group T_F is a torus, $\mathrm{MT}(X)$ is a torus, too.

Conversely, suppose $\mathrm{MT}(X)$ is a torus. Choose a maximal torus $T \subset \mathrm{GL}(V)$ containing $\mathrm{MT}(X)$. Then certainly $\mathrm{End}(V)^T$ is contained in $\mathrm{End}(V)^{\mathrm{MT}(X)} = \mathrm{End}^0(X)$. But $\mathrm{End}(V)^T$ is a commutative semi-simple $\mathbb{Q}$-algebra of dimension $2g$. To see this we may extend scalars to $\mathbb{C}$ and choose a basis for $V_{\mathbb{C}}$ that realizes $T_{\mathbb{C}}$ as the diagonal torus in $\mathrm{GL}_{2g,\mathbb{C}}$. Then the claim is obvious.

(5.4) After the above general considerations, let us compute the Mumford-Tate groups of elliptic curves E. As before we write $V = H_1(E, \mathbb{Q})$. We have $\mathrm{GL}(V) \cong \mathrm{GL}_2$ and there are two possibilities for $D = \mathrm{End}^0(E)$: either $D = \mathbb{Q}$ or D is a quadratic imaginary field.

First suppose $D = \mathbb{Q}$. The only connected reductive groups M with $\mathbb{G}_m \cdot \mathrm{id} \subseteq M \subseteq \mathrm{GL}_2$ are: (i) $\mathbb{G}_m$, (ii) maximal tori of GL_2, (iii) GL_2. But in the first two cases, $\mathrm{End}(V)^M$ is bigger than $\mathbb{Q}$. So if $D = \mathbb{Q}$ then $\mathrm{MT}(E) = \mathrm{GL}_2$. (Alternatively, by (5.3) we know that $\mathrm{MT}(X)$ is in this case not a torus.)

Next suppose $D = k$ is an imaginary quadratic field, which means that E is of CM-type. What we have seen in (5.2) and (5.3) in this case tells us that $\mathbb{G}_m \cdot \mathrm{id} \subseteq \mathrm{MT}(E) \subseteq T_k$. (Here again T_k is "k^*, viewed as an algebraic torus over $\mathbb{Q}$".) And in fact, we must have $\mathrm{MT}(E) = T_k$, because the only other possibility is that $\mathrm{MT}(E) = \mathbb{G}_m \cdot \mathrm{id}$, in which case $\mathrm{End}(V)^{\mathrm{MT}(E)}$ would equal the full endomorphism ring $\mathrm{End}(V) \cong M_2(\mathbb{Q})$, which does not match with (3).

Conclusion: if $\mathrm{End}^0(E) = \mathbb{Q}$ then $\mathrm{MT}(E) = \mathrm{GL}_2$; if $\mathrm{End}^0(E) = k$ is an imaginary quadratic field then $\mathrm{MT}(E)$ equals the torus $T_k \subset \mathrm{GL}_2$. Note that in this case we could also have found these answers by direct computation, using the explicit formula for the homorphism $h\colon \mathbb{S} \to \mathrm{GL}_{2,\mathbb{R}}$ as in (3.5).

(5.5) Once we explicitely know a Mumford-Tate group, we can let it work for us. As an example, let us prove that the Hodge conjecture is true for any power E^n of an elliptic curve. More precisely, we claim that the Hodge ring of E^n (see section (2.3)) is generated by divisor classes. A priori this result is by no means obvious!

First assume that $\mathrm{End}^0(E) = \mathbb{Q}$. We have $H^1(E^n, \mathbb{Q}) = (V^\vee)^{\oplus n}$, and the cohomology ring $H^\bullet(E^n, \mathbb{Q})$ is the exterior algebra on $(V^\vee)^n$; see [38], Chap. I. Our claim boils down to the fact that the subring of $\mathrm{GL}(V)$-invariants in $\wedge^\bullet\big((V^\vee)^n\big)$ is generated, as a $\mathbb{Q}$-algebra, by the invariants in degree 2. This is a result in classical invariant theory; see for instance [27], § 1.

In case $\mathrm{End}^0(E)$ is an imaginary quadratic field k we can be even more explicit. Let σ and τ be the two complex embeddings of k. The torus T_k has as its character group the free $\mathbb{Z}$-module generated by σ and τ, with its natural Galois action. The tautological representation $T_k \hookrightarrow \mathrm{GL}(V)$ corresponds to a character decomposition $V_{\mathbb{C}} = V_{\mathbb{C}}(\sigma) \oplus V_{\mathbb{C}}(\tau)$. Possibly after interchanging the roles of σ and τ, the homomorphism $h\colon \mathbb{S} \to T_{k,\mathbb{R}}$ is given on character groups by the map $X^*(h)\colon \mathbb{Z}\sigma + \mathbb{Z}\tau \to \mathbb{Z}z + \mathbb{Z}\bar{z}$ with $\sigma \mapsto z$ and $\tau \mapsto \bar{z}$. This simply means that $\mathbb{S}$ acts on $V_{\mathbb{C}}(\sigma)$ and $V_{\mathbb{C}}(\tau)$ through the characters z and $\bar{z}$, respectively; hence $V_{\mathbb{C}}(\sigma) = V_{\mathbb{C}}^{-1,0}$ and $V_{\mathbb{C}}(\tau) = V_{\mathbb{C}}^{0,-1}$.

Write $H := H^1(E^n,\mathbb{Q}) = (V^\vee)^{\oplus n}$ and $W := H^{2m}(E^n,\mathbb{Q}) = \wedge^{2m}H$. Then $W_{\mathbb{C}}$ decomposes into character spaces under $T_{k,\mathbb{C}}$. If $a\cdot\sigma + b\cdot\tau$ is a character that occurs, then by looking at the weight we see that $a+b=-2m$, and the character space $W_{\mathbb{C}}(a\sigma+b\tau) \subset W_{\mathbb{C}}$ is precisely the $(-a,-b)$-component in the Hodge decomposition. Hence $W_{\mathbb{C}}(a\sigma+b\tau)\bigl(m\bigr)$ is the component of type $(-a-m,-b-m)$ in the Hodge decomposition of $W_{\mathbb{C}} \otimes \mathbb{Q}(m) = H^{2m}(E^n,\mathbb{Q})\bigl(m\bigr)$. Writing $\mathscr{B}^m(E^n) \subset W_{\mathbb{C}}\bigl(m\bigr)$ for the space of Hodge classes, Proposition (4.4) then gives that $\mathscr{B}^m(E^n) \otimes_{\mathbb{Q}} \mathbb{C} = W_{\mathbb{C}}(-m\sigma - m\tau)\bigl(m\bigr)$. But $H_{\mathbb{C}} = H_{\mathbb{C}}(-\sigma) \oplus H_{\mathbb{C}}(-\tau)$ and $W_{\mathbb{C}} = \wedge^{2m}H_{\mathbb{C}}$; so it is clear that every element $W_{\mathbb{C}}(-m\sigma - m\tau)$ can be written as a linear combination of elements of the form

$$\xi = x_1 \wedge y_1 \wedge x_2 \wedge y_2 \wedge \cdots \wedge x_m \wedge y_m\,,$$

where $x_j \in H_{\mathbb{C}}(-\sigma)$ is of Hodge type $(1,0)$ and $y_j \in H_{\mathbb{C}}(-\tau)$ is of Hodge type $(0,1)$. Hence $(2\pi i)^m \cdot \xi$ is a product of classes $(2\pi i)\cdot x_j \wedge y_j$. Now we remark that these last classes are invariants under the Mumford-Tate group, and are therefore elements of $\mathscr{B}^1(E^n)\otimes\mathbb{C}$. This shows that the natural map $\mathrm{Sym}^m\mathscr{B}^1(E^n) \longrightarrow \mathscr{B}^n(E^n)$ is surjective, as it is surjective after extending scalars to $\mathbb{C}$. Hence in this case, too, the Hodge ring of E^n is generated by divisor classes.

(5.6) *Exercise.* Consider a product $X = E_1 \times E_2$ of two elliptic curves. Make a list of possible combinations for $\mathrm{End}^0(E_1)$ and $\mathrm{End}^0(E_2)$. In each case, try to determine the Mumford-Tate group of X. [*Hint:* Writing $V_i := H_1(E_i,\mathbb{Q})$, use that E_1 and E_2 are isogenous if and only if there is a non-zero homomorphism of Hodge structures $V_1 \to V_2$, i.e., a non-zero Hodge class in $\mathrm{Hom}(V_1,V_2)$.]

(5.7) As the next example, let us look at simple complex abelian surfaces X. For $D = \mathrm{End}^0(X)$ there are now four possibilities (see [38], Sect. 21 and [48], or see [40]):

(i) $D = \mathbb{Q}$;
(ii) D is a real quadratic field;
(iii) D is a quaternion algebra with center $\mathbb{Q}$ such that $D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$;
(iv) D is a CM-field of degree 4 over $\mathbb{Q}$, i.e., a totally imaginary extension of a real quadratic field.

In all four cases, we have $\mathrm{MT}(X) = \mathrm{CSp}_D(V,\varphi)$, where $V = H_1(X,\mathbb{Q})$ and φ is a polarization form. More concretely this means:

(i) If $D = \mathbb{Q}$ then $\mathrm{MT}(X) = \mathrm{CSp}(V,\varphi) \cong \mathrm{CSp}_4$.
(ii) If $D = k$ is a real quadratic field, let $G := \mathrm{GL}(V/k) = \mathrm{GL}_{2,k}$, viewed as an algebraic group over $\mathbb{Q}$. (In more formal notation, $G = \mathrm{Res}_{k/\mathbb{Q}}\mathrm{GL}_{2,k}$.) Then $\mathrm{MT}(X)$ is the subgroup of G given by the condition that the determinant lies in $\mathbb{G}_{m,\mathbb{Q}} \subset T_k$. In other words, if R is a $\mathbb{Q}$-algebra then $\mathrm{MT}(X)\bigl(R\bigr) = \bigl\{g \in \mathrm{GL}_2(k\otimes_{\mathbb{Q}} R) \bigm| \det(g) \in R^*\bigr\}$.
(iii) If D is a quaternion algebra with center $\mathbb{Q}$, let $\bar{D}$ be the opposite algebra. Then $\mathrm{MT}(X)$ is the algebraic group of units in $\bar{D}$. So if R is a $\mathbb{Q}$-algebra then $\mathrm{MT}(X)\bigl(R\bigr) = (\bar{D}\otimes_{\mathbb{Q}} R)^*$.
(iv) If $D = k$ is a CM-field of degree 4 over $\mathbb{Q}$, let $k_0 \subset k$ be the totally real subfield, and write $x \mapsto \bar{x}$ for the non-trivial automorphism of k over k_0. Then $\mathrm{MT}(X)$ is the unitary torus given on points by $\mathrm{MT}(X)\bigl(R\bigr) = \bigl\{x \in (k\otimes_{\mathbb{Q}} R)^* \bigm| x\bar{x} \in R^*\bigr\}$.

The proof that in all four cases we have $\mathrm{MT}(X) = \mathrm{CSp}_D(V,\varphi)$ is already a little more involved than the corresponding result for elliptic curves. The starting point is to use the general facts in (5.2) together with some basic representation theory. We refer to [45] for further tricks.

(5.8) *Exercise.* Let X be a complex abelian variety. Let V and φ be as above. Define the Hodge group of X to be $\mathrm{Hg}(X) := \mathrm{Ker}\big(\mathrm{MT}(X) \hookrightarrow \mathrm{CSp}(V,\varphi) \xrightarrow{\nu} \mathbb{G}_m\big)$, where ν is the multiplier character. Equivalently: $\mathrm{Hg}(X) := \mathrm{MT}(X) \cap \mathrm{Sp}(V,\varphi)$.

(i) Show that $\mathrm{End}^0(X) = \mathrm{End}(V)^{\mathrm{Hg}(X)}$.

(ii) Assume that $\mathrm{End}^0(X) = \mathbb{Q}$. Show that $\mathrm{Hg}(X)$ is semisimple.

(iii) For $g \in \{1,2,3\}$, list all possibilities for a semisimple subgroup $H_\mathbb{C} \subset \mathrm{Sp}_{2g,\mathbb{C}}$ such that the tautological representation $H_\mathbb{C} \to \mathrm{GL}(\mathbb{C}^{2g})$ is irreducible.

(iv) Now try to prove: If X is a complex abelian variety with $\dim(X) \leqslant 3$ and $\mathrm{End}^0(X) = \mathbb{Q}$ then $\mathrm{MT}(X) = \mathrm{CSp}(V,\varphi)$. [For $g = 3$ there is one non-trivial case that you need to exclude; you will need to use that there are only two weights that occur in the representation $\mathbb{S} \to \mathrm{MT}(X)_\mathbb{C} \to \mathrm{GL}(V_\mathbb{C})$.]

(5.9) The previous examples may suggest that for a simple abelian variety X one always has the identity $\mathrm{MT}(X) = \mathrm{CSp}_D(V,\varphi)$, but this is certainly not the case. A nice example where this identity does not hold was constructed by Mumford in [37]. The example concerns simple complex abelian varieties X with $\dim(X) = 4$ and $\mathrm{End}^0(X) = \mathbb{Q}$ such that $\mathrm{MT}(X)$ is a proper subgroup of CSp_8. In fact, in Mumford's example $\mathrm{MT}(X) \otimes \mathbb{C}$ is isogenous to $\mathbb{G}_m \times (\mathrm{SL}_2)^3$.

The Hodge theoretic meaning of having a smaller Mumford-Tate group is that there are more Hodge classes in the tensor category $\langle V \rangle^\otimes \subset \mathbb{Q}\mathrm{HS}$ generated by $V = H_1(X,\mathbb{Q})$. After all, any algebraic subgroup of $\mathrm{GL}(V)$ is fully determined by its invariants in all tensor spaces obtained from V; see [17], Prop. 3.1. Thus, for instance, let X be an abelian fourfold as in Mumford's example. We have $\mathrm{End}^0(X) = \mathbb{Q}$, and $\mathrm{MT}(X)$ is a proper subgroup of $\mathrm{CSp}(V,\varphi) \cong \mathrm{CSp}_8$. Let us compare this with an abelian fourfold Y, also with $\mathrm{End}^0(Y) = \mathbb{Q}$, but such that $\mathrm{MT}(Y) = \mathrm{CSp}_8$. (Such Y exist.) If we compute the dimensions of the spaces of Hodge classes on X and Y then we see no difference:

n	0	1	2	3	4
$\dim\big(\mathscr{B}^n(X)\big)$	1	1	1	1	1
$\dim\big(\mathscr{B}^n(Y)\big)$	1	1	1	1	1

However, if we do the same for the Hodge classes on X^2 and Y^2 then we do see that there are more Hodge classes on X^2:

n	0	1	2	3	4	5	6	7	8
$\dim\big(\mathscr{B}^n(X^2)\big)$	1	3	8	16	28	16	8	3	1
$\dim\big(\mathscr{B}^n(Y^2)\big)$	1	3	6	10	15	10	6	3	1

Thus, for instance, in $H^4(X^2,\mathbb{Q})$ there are Hodge classes that can not be written as a linear combination of products of divisor classes. The Hodge conjecture predicts that the existence of these "exceptional" Hodge classes is explained by the existence of certain algebraic cycles on X. However, as long as we do not know the Hodge conjecture (which in such examples is typically the case) these exceptional Hodge classes only have a description in the transcendental framework of Hodge theory.

If $\mathrm{MT}(X)$ is "as big as possible", i.e., $\mathrm{MT}(X) = \mathrm{CSp}_D(V,\varphi)$, then one might expect that the Hodge ring of X is generated by divisor classes. In fact, we get even more. The precise result is the following. See [38], Section 21 for the Albert classification of division algebras with positive involution.

(5.10) Theorem. (Hazama [27] and Murty [39]) *Let X be a complex abelian variety. Let V, φ and D be as above. Then the following are equivalent:*
(i) $\mathrm{MT}(X) = \mathrm{CSp}_D(V,\varphi)$;
(ii) $D = \mathrm{End}^0(X)$ *has no factors of type III in the Albert classification and* $\mathscr{B}^\bullet(X^k) = \mathscr{D}^\bullet(X^k)$ *for all* $k \geqslant 1$.

In particular, if $\mathrm{MT}(X) = \mathrm{CSp}_D(V,\varphi)$ then the Hodge conjecture is true for all powers of X. Note that the proof of the above theorem does not involve any geometry; it should rather be viewed as a result in invariant theory.

Another amusing application of the theory of Mumford-Tate groups is the following.

(5.11) Theorem. (Tankeev [52], see also [45]) *Let X be a simple complex abelian variety such that* $\dim(X)$ *is a prime number. Then the Hodge conjecture is true for all powers of X.*

The idea is that if X is simple and $\dim(X) = p$ is prime, then there are only four possibilities for $D = \mathrm{End}^0(X)$, namely: (i) $D = \mathbb{Q}$, (ii) D is a totally real field of degree p over $\mathbb{Q}$, (iii) D is an imaginary quadratic field, or (iv) D is a CM-field of degree $2p$ over $\mathbb{Q}$. In all four cases it is shown that the Mumford-Tate group must be the full group $\mathrm{CSp}_D(V,\varphi)$. Then one concludes by Thm. (5.10). (This sketch is historically incorrect; Thm. (5.11) is is a little older than Thm. (5.10).)

If one looks in more detail at the proof, one finds that the arguments for case (iv) are quite different from those used, for instance, in the cases (i) and (ii). In the latter cases, the Hodge group is semi-simple, and the proof involves representation theory of semi-simple Lie groups (or Lie algebras). In the case of an abelian variety of CM-type, the Mumford-Tate group is a torus, and everything boils down to an argument in ordinary Galois theory; see for instance Ribet's paper [45].

Further reading. For abelian varieties of low dimension it is possible to compute (or describe) all Mumford-Tate groups that occur. See [33], [35] for an overview of results. See also Lewis [29], especially Appendix B by B. Gordon. Once the Mumford-Tate group is known, one may even try to prove the general Hodge conjecture, since, in principle, one knows all sub-HS. For some succesful cases, see Abdulali's work [1], [2] and [3]. An interesting class of examples is obtained via a construction of A. Weil [54], which was later extended; see [5] and [34]. Weil's construction gives abelian varieties for which the Hodge ring is not generated by divisor classes. Even for $\dim(X) = 4$ (Weil's original examples) these "exceptional Hodge classes" are known to be algebraic only in very few cases; see the work of Schoen [47] and van Geemen [21]. Hence this provides non-trivial test cases for the Hodge conjecture. For ge! neralisations of Mumford's example, see [51]. For K3 surfaces the Mumford-Tate group was computed by Zarhin [56].

§ 6. Variation in a family

Throughout this section, S denotes a connected complex manifold.

(6.1) Let A be a ring. By a local system of A-modules of rank r over S we mean a sheaf V of A-modules which locally on S is isomorphic to the constant sheaf A^r. In other words, every $s \in S$ should have an open neighbourhood U such that $V_{|U} \cong A^r_U$.

If we choose a base point $b \in S$ then such a local system is described by a representation of the fundamental group

$$\rho\colon \pi_1(S, b) \to \mathrm{GL}_r(A)$$

called the monodromy representation of V.

If we take $A = \mathbb{C}$ as our coefficient ring, then there is an equivalence of categories

$$\begin{pmatrix}\text{local systems of } \mathbb{C}\text{-vector} \\ \text{spaces over } S\end{pmatrix} \longrightarrow \begin{pmatrix}\text{vector bundles with flat} \\ \text{connection over } S\end{pmatrix}.$$

(All local systems and vector bundles are assumed to be of finite type.) This equivalence is obtained by sending the local system $V_{\mathbb{C}}$ to the vector bundle $\mathscr{V} := V_{\mathbb{C}} \otimes_{\mathbb{C}} \mathscr{O}_S$ with its Gauss-Manin connection $\nabla\colon \mathscr{V} \to \Omega^1_S \otimes \mathscr{V}$, given for local sections v of $V_{\mathbb{C}}$ and f of $\mathscr{O}_S$ by $\nabla(v \otimes f) = df \otimes (v \otimes 1)$. In the opposite direction, we associate to a pair $(\mathscr{V}, \nabla)$ the subsheaf $\mathscr{V}^{\nabla} := \mathrm{Ker}(\nabla)$ of horizontal sections.

(6.2) For the next definition we shall work with $\mathbb{Q}$-Hodge structures. We leave it to the reader to formulate the corresponding notion with $\mathbb{Z}$-coefficients.

A Variation of Hodge Structure (abbreviated VHS) of weight m over S is given by a pair $(V, \mathscr{F}^\bullet)$ where V is a local system of $\mathbb{Q}$-vector spaces (of finite rank) and where $\mathscr{F}^\bullet$ is a filtration of $\mathscr{V} := V \otimes_{\mathbb{Q}} \mathscr{O}_S$ by holomorphic subbundles, such that:

(i) for every $s \in S$, the filtration $\mathscr{F}^\bullet_s$ on the fibre V_s is the Hodge filtration of a $\mathbb{Q}$-HS of weight m on V_s;

(ii) *Griffiths transversality:* for every i we have $\nabla(\mathscr{F}^i) \subseteq \Omega^1_S \otimes \mathscr{F}^{i-1}$.

A polarisation of a VHS of weight m is a bilinear form $\varphi\colon V \otimes_{\mathbb{Q}} V \to \mathbb{Q}(-m)_S$ which fibrewise gives a polarisation in the sense of (1.5).

(6.3) The standard geometric example is obtained by considering a smooth proper morphism $f\colon X \to S$. Let $V := R^m f_* \mathbb{Q}_X$, which is a local system of $\mathbb{Q}$-vector spaces whose fibres are the cohomology groups $H^m(X_s, \mathbb{Q})$. On $\mathscr{V} := V \otimes_{\mathbb{Q}} \mathscr{O}_S$ we have a natural filtration $\mathscr{F}^\bullet$, which restricts to the Hodge filtration on each fibre. The VHS thus obtained is polarisable. To obtain a polarisation one has to use a relative version of the Lefschetz decomposition.

(6.4) The next thing we want to discuss is how, in a $\mathbb{Q}$-VHS, the Mumford-Tate groups of the fibres vary. Already in some simple examples (e.g., a universal family of elliptic curves over a modular curve) we see a little bit what to expect. Certainly we cannot expect that the Mumford-Tate groups will be constant on Zariski open (or analytically open) subsets of the basis.

To analyse in more detail what happens, consider a VHS $\mathscr{T} = (T, \mathscr{F}^\bullet)$ over S, and let $\pi\colon \tilde{S} \to S$ be a universal covering. Then the local system π^*T is constant, so for any point $\tilde{s} \in \tilde{S}$ we have a natural identification $\pi^*T \cong T_{\tilde{s}} \times \tilde{S}$. (For simplicity we write $T_{\tilde{s}}$ for the fibre of (π^*T) at $\tilde{s}$.) Any $t_{\tilde{s}} \in T_{\tilde{s}}$ extends uniquely to a global section t of π^*T, called the horizontal continuation of t.

Suppose now that we have a global section t of π^*T. Define

$$\tilde{\Sigma}(t) := \left\{\tilde{s} \in \tilde{S} \;\middle|\; t_{\tilde{s}} \text{ is a Hodge class}\right\}.$$

(Of course, if $\mathscr{T}$ is purely of some weight m and $t \neq 0$ then $\tilde{\Sigma}(t)$ can only be non-empty if $m = 0$. So the essential case is when $\mathscr{T}$ is purely of weight 0.) Note that $t_{\tilde{s}}$ is a Hodge class if and only if it lands, under the natural map $T_{\tilde{s}} \hookrightarrow T_{\tilde{s},\mathbb{C}}$, inside $\mathscr{F}^0(T_{\tilde{s},\mathbb{C}})$. Hence we find that $\tilde{\Sigma}(t)$ is the zero locus of the section of $\mathscr{T}/\mathscr{F}^0$ given by t, and is therefore analytically closed in $\tilde{S}$.

We can now define what is called the "Hodge generic locus" in S and the "generic Mumford-Tate group". For this, start with a $\mathbb{Q}$-VHS $\mathscr{V} = (V, \mathscr{F}^\bullet)$. We apply the above to all VHS $\mathscr{T}$ obtained from $\mathscr{V}$ via tensor constructions. In other words, consider collections $\nu = \{(a_i, b_i)\}_{i=1,\ldots,l}$ of pairs of integers a_i, $b_i \in \mathbb{Z}_{\geqslant 0}$, and for each such collection, define

$$\mathscr{T}^\nu := \oplus_{i=1}^{l}\, \mathscr{V}^{\otimes a_i} \otimes (\mathscr{V}^\vee)^{\otimes b_i}\,,$$

which inherits a natural structure of a $\mathbb{Q}$-VHS from $\mathscr{V}$. We are especially interested in the fibres that have "extra" Hodge classes. This leads us to consider the Hodge-exceptional locus in $\tilde{S}$, defined by

$$\tilde{S}^{\mathrm{exc}} := \cup_{\nu,t}\, \tilde{\Sigma}(t)\,,$$

where ν runs over all collections $\{(a_i, b_i)\}_{i=1,\ldots,l}$ as above, and where t runs over the global sections of π^*T^ν such that—and this is the whole point—$\tilde{\Sigma}(t)$ is *not* the whole $\tilde{S}$. So $\tilde{S}^{\mathrm{exc}}$ is the locus of points $\tilde{s} \in \tilde{S}$ where, for some tensor construction $\mathscr{T}$ and some Hodge class $t_{\tilde{s}} \in T_{\tilde{s}}$, the horizontal continuation of $t_{\tilde{s}}$ is not everywhere a Hodge class.

The locus $\tilde{S}^{\mathrm{exc}} \subset \tilde{S}$ is stable under the action of the fundamental group by covering transformations, and therefore defines a Hodge-exceptional locus $S^{\mathrm{exc}} \subset S$. It is a countable union of closed irreducible analytic subspaces of S. The complement $S \setminus S^{\mathrm{exc}}$ is called the Hodge generic locus.

If we now look at Mumford-Tate groups then the situation is as follows. Let $\mathbb{V} := \Gamma(\tilde{S}, \pi^*V)$. As we have seen, for any $\tilde{s} \in \tilde{S}$ we have a natural identification $V_{\tilde{s}} \xrightarrow{\sim} \mathbb{V}$ by horizontal continuation. Hence we may view the Mumford-Tate group $\mathrm{MT}(V_{\tilde{s}})$ at $\tilde{s}$ as a subgroup of $\mathrm{GL}(\mathbb{V})$. Then the subgroups $\mathrm{MT}(V_{\tilde{s}}) \subset \mathrm{GL}(\mathbb{V})$ for $\tilde{s} \in \tilde{S} \setminus \tilde{S}^{\mathrm{exc}}$ are all the same. Call this subgroup $M \subset \mathrm{GL}(\mathbb{V})$ the generic Mumford-Tate group. By construction of the Hodge-generic locus, we have $\mathrm{MT}(V_{\tilde{s}}) \subseteq M$ for all $\tilde{s} \in \tilde{S}$, with equality if and only if $\tilde{s}$ is Hodge generic.

As a simple concrete example, consider a modular curve S with universal covering $\tilde{S} = \mathfrak{H} = \left\{\tau \in \mathbb{C} \;\middle|\; \Im(\tau) > 0\right\}$, and look at the VHS over S whose fibres are the $H^1(E, \mathbb{Q})$ of the elliptic curves E parametrised by S. Then the Hodge exceptional locus in $\mathfrak{H}$ is the set of all imaginary quadratic τ. As we have seen earlier, if τ is not quadratic over $\mathbb{Q}$ (equivalent: $\mathrm{End}^0(E_\tau) = \mathbb{Q}$) then the Mumford-Tate group is GL_2, which is therefore the Hodge-generic Mumford-Tate group in this family.

Needless to say, we have a similar discussion if we consider "big" Mumford-Tate groups, i.e., if we include arbitrary Tate twists in our considerations.

In the above discussion we took an arbitrary complex manifold S as a basis, and we considered a $\mathbb{Q}$-VHS over S. In algebraic geometry one usually has a complex algebraic variety as a basis, and the VHS one considers is usually a polarisable VHS with a $\mathbb{Z}$-structure. It turns out that in this case one has some important further results. We first give a result of Cattani, Deligne and Kaplan on the nature of the Hodge-exceptional loci.

(6.5) Theorem. (Cattani-Deligne-Kaplan [10]) *Let $\mathscr{V} = (V, \mathscr{F}^\bullet)$ be a polarisable $\mathbb{Z}$-VHS of weight* 0 *on a non-singular complex algebraic variety S. Let $s \in S$, and suppose $v_s \in V_s$ is a Hodge class. Let $v \in \Gamma(\tilde{S}, \pi^*V)$ be the global section of π^*V (with $\pi\colon \tilde{S} \to S$ a universal covering) obtained from v_s by horizontal continuation, where we choose a point $\tilde{s} \in \tilde{S}$ above s. Then the image of $\tilde{\Sigma}(v)$ in S is an algebraic subvariety of S.*

(6.6) *Exercise.* Prove this theorem, assuming the Hodge conjecture. (If you get stuck, see the introduction of [10].)

(6.7) We conclude this section by a beautiful result of Y. André that gives a relation between the generic Mumford-Tate group and the monodromy representation of the underlying local system. In this result, one looks at the algebraic monodromy group. By this we mean the following. The local system V underlying a VHS over S corresponds, after choice of a base point $b \in S$, with a monodromy representation $\rho\colon \pi(S, b) \to \mathrm{GL}(V_b)$. The algebraic monodromy group is defined to be the algebraic subgroup $G \subseteq \mathrm{GL}(V_b)$ obtained as the Zariski closure of the image of ρ. In general, it is a non-connected group. Up to conjugation G is independent of the chosen base point b (for connected S).

If we pass to a finite covering $\gamma\colon S' \to S$ then the algebraic monodromy group of the local system γ^*V may be smaller, but the identity component G^0 does not change. We refer to it as the connected algebraic monodromy group.

(6.8) Theorem. (André [4]) *Let S be a connected, nonsingular complex algebraic variety. Let $\mathscr{V} = (V_\mathbb{Z}, \mathscr{F}^\bullet)$ be a polarisable $\mathbb{Z}$-VHS on S. Let $b \in S$ be a Hodge-generic base point, let $M = \mathrm{MT}(V_b) \subseteq \mathrm{GL}(V_b)$ be the generic Mumford-Tate group, and let $G^0 \subseteq \mathrm{GL}(V_b)$ be the connected algebraic monodromy group.*
(i) *The group G^0 is a normal subgroup of the derived group M^{der}.*
(ii) *Suppose there is a point $s \in S$ such that $\mathrm{MT}(V_s)$ is a torus. Then $G^0 = M^{\mathrm{der}}$.*

That G^0 is contained in M^{der} was already shown in [14]. André's proof is an application of the "Theorem of the fixed part".

(6.9) *Example.* Let $f\colon X \to S$ be the universal family of complete intersection of a fixed multi-degree $d = (d_1, \ldots, d_r)$ in some $\mathbb{P}^{n+r}$ (with $d_i \geqslant 2$ and $n \geqslant 1$). In more detail this means the following. Consider the spaces $W_i := \Gamma\big(\mathbb{P}^{n+r}, \mathscr{O}(d_i)\big)$, let $\bar{S} := \mathbb{P}(W_1) \times \cdots \times \mathbb{P}(W_r)$, and let $\bar{X} \subset \mathbb{P}^{n+1}_{\bar{S}}$ be the subscheme whose fibre over a point $\big([g_1], \ldots, [g_r]\big)$ is the hypersurface given by $g_1 = \cdots = g_r = 0$. Finally let $S \subset \bar{S}$ be the open subscheme over which the morphism $\bar{f}\colon \bar{X} \to \bar{S}$ is smooth, and let $f\colon X \to S$ be the restriction of $\bar{f}$.

As our VHS we take the primitive part of the cohomology in middle degree. So $\mathscr{V} = (V, \mathscr{F}^\bullet)$, where the fibre at $s \in S$ is the primitive cohomology $H^n_{\mathrm{prim}}(X_s, \mathbb{Q})$ with respect to the natural ample bundle $\mathscr{O}(1)$ on X_s. This VHS is polarisable; if $Q\colon V \otimes V \to \mathbb{Q}(-n)_S$ is a polarisation form then Q is symplectic if n is odd and symmetric if n is even.

Choose a base point $b \in S$. It is known that the connected algebraic monodromy group G^0 equals the full group $\mathrm{Sp}(V_b, Q_b)$ if n is odd, resp. the full group $\mathrm{SO}(V_b, Q_b)$ if n is even, except for the following cases:
— quadric hypersurfaces;

— cubic surfaces;
— even-dimensional intersections of two quadrics.

Except in these last cases, the generic Mumford-Tate group in our family is therefore equal to the full $\mathrm{CSp}(V_b, Q_b)$, resp. the full group of orthogonal similitudes $\mathrm{GO}(V_b, Q_b)$. See [42] for further discussion. In the three exceptional cases, the monodromy representation has finite image, and after passing to a finite covering of S the VHS is constant.

Further reading. The variation of Hodge structures in a family was studied extensively by Griffiths in a series of papers; see for instance [22], [23], [25]. See also [53] and [9]. Of course, for the purpose of the workshop at Monte Verità, I should really go on and discuss Shimura varieties. This, however, would be a major project. Instead, I refer to Deligne's papers [12] and [16], and Milne's papers [30] and [31]. Also, in connection with the André-Oort conjecture, the reader may want to have a look at [32], especially Section 6, at Oort's paper [41], or at the introduction of Edixhoven [20].

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