Hand-in Assignment 2

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You are allowed to use all problems of the problem sessions so far as proven statements.

1. Let $\mathfrak{g}$ be a Lie algebra.
   (a) Show that $\mathfrak{g}/\operatorname{Rad}(\mathfrak{g})$ is semisimple.
   (b) Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. From now on assume that $\mathfrak{z} = \operatorname{Rad}(\mathfrak{g})$; a Lie algebra with this property is called reductive. Show that the adjoint representation of $\mathfrak{g}$ is semisimple; hence $[\mathfrak{g}, \mathfrak{g}]$ is semisimple and $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ (see problem 5 of October 18^{th}).
   (c) Let ϱ be a representation of $\mathfrak{z}$, and let ϱ' be a representation of $[\mathfrak{g}, \mathfrak{g}]$. Suppose $\varrho(Z)$ is semisimple for alle $Z \in \mathfrak{z}$. Show that the representation $\varrho \boxtimes \varrho'$ of $\mathfrak{g}$ is semisimple (see problem 5 of November 1^{st}).

2. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$. In this exercise, we take the same ordering on Λ_{W} as in the lectures. In particular the highest weight of a representation is of the form $aL_1 - bL_3$ for $a, b \in \mathbb{Z}_{\geq 0}$. Now let V be an irreducible representation of $\mathfrak{g}$ with heighest weight $\alpha = aL_1 - bL_3$. What is the highest weight of the representation $V^\vee$?

3. Let V be the standard representation of $\mathfrak{sl}_3$.
   (a) Determine the weights of the following representations of $\mathfrak{sl}_3$:
   $$\operatorname{Sym}^2(V), \quad \operatorname{Sym}^2(V^\vee), \quad \operatorname{Sym}^2(V^\vee) \otimes \operatorname{Sym}^2(V).$$
   (b) Deduce from the weights that the representation $\operatorname{Sym}^2(V^\vee) \otimes \operatorname{Sym}^2(V)$ is not irreducible.
   (c) Show that the map
   $$\begin{array}{rcl} \operatorname{Sym}^2(V^\vee) \otimes \operatorname{Sym}^2(V) & \to & V^\vee \otimes V \\ (\lambda_1 \cdot \lambda_2) \otimes (v_1 \cdot v_2) & \mapsto & \displaystyle\sum_{i,j=1,2} \lambda_i(v_j)(\lambda_{3-i} \otimes v_{3-j}) \end{array}$$
   is well-defined and is a surjective homomorphism of $\mathfrak{sl}_3$-representations.
   (d) Deduce that $\operatorname{Sym}^2(V^\vee) \otimes \operatorname{Sym}^2(V)$ has a subrepresentation isomorphic to $V^\vee \otimes V$.