Problems for Representations of Linear Algebraic Groups

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1. Let V be a vector space.
   (a) Show that every element $g \in \mathrm{GL}(V)$ is algebraic, i.e. is the zero locus of some set of algebraic functions.
   (b) Show that every finite subgroup of $\mathrm{GL}(V)$ is algebraic.

2. Let V be a vector space, let φ be a non-degenerate bilinear form on V, and let Φ be the associated map $V \to V^{\vee}$ given by $\Phi(v) = \varphi(-, v)$. Let W be a subspace of V. Recall that
$$W^{\perp} = \{v \in V : \varphi(w, v) = 0 \ \forall w \in W\}.$$
   (a) Show that $W^{\perp} = \Phi^{-1}(\ker(V^{\vee} \to W^{\vee}))$.
   (b) Deduce that $\dim W^{\perp} = \dim V - \dim W$. If φ is symmetric or antisymmetric, show that $(W^{\perp})^{\perp} = W$.

3. Let V be a vector space, let φ be a bilinear form on V, and let Φ be the associated map $V \to V^{\vee}$. Show that the following are equivalent:
   (a) Φ is injective.
   (b) Φ is an isomorphism.
   (c) $V^{\perp} = \{0\}$.
   (d) For every $y \in V \setminus \{0\}$ there exists an $x \in V$ such that $\varphi(x, y) \neq 0$.
   (e) For some choice of basis of V, the matrix B associated with φ has nonzero determinant.
   (f) For every choice of basis of V, the matrix B associated with φ has nonzero determinant.

4. Let V be a vector space, and let φ be non-degenerate symmetric bilinear form on V.
   (a) Show that there exists an $x \in V$ such that $\varphi(x, x) = 0$.

(b) Show that there exists an $y \in V$ such that $\varphi(x, y) = 1$ and $\varphi(y, y) = 0$.

(c) Show by induction that there exists a basis $e_1, \ldots, e_n$ of V such that the matrix B associated with φ with respect to this basis is of the following form:

$$B = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix},$$

i.e., the antidiagonal matrix.

5. Let V be a vector space, and let φ be a nondegenerate bilinear form on V.

   (a) Show that there are subspaces V_0, V_1 of V such that:

      i. $\varphi|_{V_0}$ is symmetric;
      ii. $\varphi|_{V_1}$ is antisymmetric;
      iii. $V_0 \perp V_1$;
      iv. $V = V_0 \oplus V_1$.

      Show furthermore that this decomposition is unique.

   (b) Let $G \subset \mathrm{GL}(V)$ be the group of linear transformations given by

   $$G = \{g \in \mathrm{GL}(V) : \varphi(gv, gw) = \varphi(v, w) \text{ for all } v, w \in V\}.$$

   Show that G is an algebraic subgroup, and that $G \cong \mathrm{O}(V_0, \varphi|_{V_0}) \times \mathrm{Sp}(V_1, \varphi|_{V_1})$ as groups.