Problems for Representations of Linear Algebraic Groups

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1. Let n be an integer. Show that the endomorphism of $\operatorname{Lie}\mathbb{G}_{\mathrm{m}}$ induced by the algebraic homomorphism $f\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ given by $f(z) = z^n$ is multiplication by n.

2. Let G be an algebraic group, and let $\mathfrak{g}$ be its Lie algebra. Let $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ be the representation of G induced by conjugation on G, and let $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be its derivative. Define a map $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by sending (x, y) to the trace of $\mathrm{ad}(x) \circ \mathrm{ad}(y) \in \mathrm{End}(\mathfrak{g})$.

   (a) Show that for a finite dimensional vector space V and two endomorphisms A, B of V one has $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Conclude that κ is a symmetric bilinear form; it is called the *Killing form* on $\mathfrak{g}$.

   (b) Show that the image of Ad lies in the orthogonal group $\mathrm{O}(\mathfrak{g}, \kappa)$.

   (c) Now take $G = \mathrm{SL}_2$. Show that κ is nondegenerate.

   (d) Show that ad induces an isomorphism between $\mathfrak{sl}_2$ and $\mathfrak{o}(\mathfrak{g}, \kappa) \cong \mathfrak{o}_3$ (see exercise 4 of last week).

3. Let n be an integer, and let V be the standard representation of $\mathfrak{sl}_2$. Show that

$$V^{\otimes n} \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} (\mathrm{Sym}^{n-2k}(V))^{\oplus\left(\binom{n}{k} - \binom{n}{k-1}\right)}$$

   as representations of $\mathfrak{sl}_2$.

4. Let R be a commutative ring. An R-module is an abelian group M together with a ring homomorphism $R \to \mathrm{End}(M)$. In other words, an R-module is an abelian group M together with a map

$$\begin{array}{rcl} R \times M & \to & M \\ (r, m) & \mapsto & r \cdot m \end{array}$$

satisfying the following properties for all $r, r' \in R$ and $m, m' \in M$:

$$\begin{aligned} r \cdot (m + m') &= r \cdot m + r \cdot m' \\ (r + r') \cdot m &= r \cdot m + r' \cdot m \\ (rr') \cdot m &= r \cdot (r' \cdot m) \end{aligned}$$

Note that if R is a field, then R-modules are precisely R-vector spaces. For two complex vector spaces V and W, let $\mathrm{Lin}_{\mathbb{C}}(V, W)$ denote the $\mathbb{C}$-linear maps from V to W. For a $\mathbb{C}$-algebra A and an A-module M, consider the set

$$\mathrm{Der}_{\mathbb{C}}(A, M) = \{D \in \mathrm{Lin}_{\mathbb{C}}(A, M) : D(xy) = x \cdot D(y) + y \cdot D(x)\}.$$

(a) Show that $\mathrm{Der}_{\mathbb{C}}(A, M)$ naturally has the structure of an A-module.

(b) Now take $M = A$, which we can consider as an A-module by taking taking $a \cdot m = am$ for any $a, m \in A$. Show that $\mathrm{Der}_{\mathbb{C}}(A, A)$ has a Lie algebra structure given by $[D, D'] = D \circ D' - D' \circ D$.

(c) Let G be an algebraic group, and let $A = \mathcal{O}(G)$. Let $g \in G$. Show that the map $\mathrm{ev}_g \circ - : \mathrm{Lin}_{\mathbb{C}}(A, A) \to \mathrm{Lin}_{\mathbb{C}}(A, \mathbb{C})$ maps $\mathrm{Der}_{\mathbb{C}}(A, A)$ to $\mathrm{T}_g\, G$.

(d) Show that the induced map $\mathrm{Der}_{\mathbb{C}}(A, A) \to \prod_{g \in G} \mathrm{T}_g\, G$ is injective. Hence we can see an element $D \in \mathrm{Der}_{\mathbb{C}}(A, A)$ as a collection $(D_g)_{g \in G}$ of tangent vectors $D_g = \mathrm{ev}_g \circ D \in \mathrm{T}_g\, G$; we call D a *vector field* on G.

(e) If $g = e$ in part (c), show that $\mathrm{ev}_e \circ -\colon \mathrm{Der}_{\mathbb{C}}(A, A) \to \mathrm{Lie}\, G$ is a homomorphism of Lie algebras.