Problems for Representations of Linear Algebraic Groups

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1. If you have not done so already, do problem 2 of November 22^{nd}.

2. Determine the weight lattice of $\mathfrak{sp}_4$ and the fundamental group $\Lambda_{\mathrm{W}}/\Lambda_{\mathrm{R}}$.

3. Let $\mathfrak{g}'$ and $\mathfrak{g}''$ be two semisimple Lie algebras, and let $\mathfrak{h}' \subset \mathfrak{g}'$ and $\mathfrak{h}'' \subset \mathfrak{g}''$ be two maximal toral subalgebras. Let R' and R'' be the respective corresponding root systems.

   (a) Show that the roots of $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ with respect to the toral subalgebra $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ form the set $R' \sqcup R''$ (regarded as a subset of $\mathfrak{h}^\vee = (\mathfrak{h}')^\vee \oplus (\mathfrak{h}'')^\vee$).

   (b) Show that $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}$.

   (c) Show that $\mathfrak{h}'$ and $\mathfrak{h}''$ are perpendicular to each other with respect to the Killing form on $\mathfrak{g}$.

   (d) Show that the root system R of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is the 'perpendicular disjoint union' of the root systems R' and R'', i.e. as a set it is equal to $R' \sqcup R''$, and $E = E' \oplus E''$ as real vector spaces with inner products (where E, E', and E'' are the real spans of R, R', and R'', respectively).

   (e) Let W, W' and W'' be the Weyl groups of $\mathfrak{g}$, $\mathfrak{g}'$, and $\mathfrak{g}''$, respectively. Show that $W = W' \times W''$.

   (f) Let $\mathcal{C}'$ and $\mathcal{C}''$ be Weyl chambers of R' and R'' respectively. Show that

   $$\mathcal{C}' \oplus \mathcal{C}'' = \{(x', x'') : x' \in \mathcal{C}', x'' \in \mathcal{C}''\}$$

   is a Weil chamber of E.

4. Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h}$ be a maximal toral subalgebra. Let R be the root system of $\mathfrak{g}$ associated to $\mathfrak{h}$, and let E be its real span. Fix a Weyl chamber $\mathcal{C}$ in E, or, equivalently, a basis $\Delta = \{\alpha_1, ..., \alpha_d\}$ of positive roots. Define

the elements $\varpi_1, ..., \varpi_d \in E$ by the relation $\varpi_i(H_{\alpha_j}) = \delta_{ij}$, for all $i, j \leq d$; here δ_{ij} denotes the Kronecker delta. Show that

$$\bar{\mathcal{C}} \cap \Lambda_{\mathrm{W}} = \bigoplus_{i=1}^{d} \mathbb{Z}_{\geq 0} \cdot \varpi_i.$$

5. Do exercises 1 and 2 for $\mathfrak{sp}_6$. (Of course, you should first reformulate problem 2 of November 22$^{\mathrm{nd}}$ so that this makes sense. Since Sinterklaas has left the country, you do not have to construct a 3-dimensional paper model of the root system, but you are very much encouraged to do so anyway.)