# Repr. of Algebraic Groups
## Lecture of Nov. 29

Last week: $\mathfrak{g}$ semisimple, $\mathfrak{h} \subset \mathfrak{g}$ maximal toral

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \quad , \quad R = \{\text{roots}\} \subset \mathfrak{h}^*$$

Killing form $B: \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ is nondegenerate $\rightsquigarrow$ $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ by $X \mapsto B(X,-)$

For $\lambda \in \mathfrak{h}^*$ we have $t_\lambda \in \mathfrak{h}$ such that $B(t_\lambda, H) = \lambda(H)$ for all $H \in \mathfrak{h}$.

Transporting $B$ to a nondegen. symmetric bilin form $(\ ,\ ): \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ by $(\lambda, \mu) = B(t_\lambda, t_\mu)$.

Have seen
- $\alpha \in R$ then $-\alpha \in R$ and no other multiples of $\alpha$ are in $R$.
- $\dim \mathfrak{g}_\alpha = 1$
- $\exists!\, H_\alpha \in \mathfrak{h}$, multiple of $t_\alpha$, with $\alpha(H_\alpha) = 2$, and then $\mathfrak{s}_\alpha := \mathfrak{g}_{-\alpha} + \mathbb{C}\cdot H_\alpha + \mathfrak{g}_\alpha$ is Lie subalg of $\mathfrak{g}$ with $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$
- For all $\beta \in R$ we have $\beta(H_\alpha) \in \mathbb{Z}$; more precisely: if $\beta \neq \pm\alpha$ then the roots of the form $\beta + i\alpha$ $(i \in \mathbb{Z})$ form a string $\beta - p\alpha, \beta + (1-p)\alpha, \ldots, \beta, \ldots, \beta + q\alpha$ and then $\beta(H_\alpha) = p - q$.

Descending to a rational vector space:

**Proposition** (i) For all $\alpha, \beta \in R$ we have $(\alpha, \beta) \in \mathbb{Q}$, and $(\alpha, \alpha) > 0$.

(ii) Let $E_\mathbb{Q} := \mathbb{Q}$-linear span of $R \subset \mathfrak{h}^*$. Then $\dim_\mathbb{Q}(E_\mathbb{Q}) = \dim_\mathbb{C}(\mathfrak{h}^*)$.

Equivalent: If $r = \dim_\mathbb{C}(\mathfrak{h}^*)$ and $\alpha_1, \ldots, \alpha_r \in R$ form a $\mathbb{C}$-basis for $\mathfrak{h}^*$ then all roots are in $\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_r$.

**Proof** (i) We have seen: for $X, Y \in \mathfrak{h}$ we have $B(X,Y) = \sum_{\gamma \in R} \gamma(X)\cdot\gamma(Y)$; so $B(H_\alpha, H_\beta) \in \mathbb{Z}$ and $B(H_\alpha, H_\alpha) > 0$. Also: $B(t_\alpha, H_\beta) = \alpha(H_\beta) \in \mathbb{Z}$. Further, $H_\alpha = \frac{2}{\alpha(t_\alpha)} \cdot t_\alpha$. So $\frac{2}{\alpha(t_\alpha)} \in \mathbb{Q}$, and hence $(\alpha, \beta) = B(t_\alpha, t_\alpha) = \frac{\alpha(t_\alpha)\cdot\beta(t_\beta)}{4} \cdot B(H_\alpha, H_\beta) \in \mathbb{Q}$.

(ii) Let $\alpha_1, \dots, \alpha_r \in R$ be a $\mathbb{C}$-basis for $\mathfrak{h}^*$ ($r = \dim_{\mathbb{C}}(\mathfrak{h})$, and note that $R$ spans $\mathfrak{h}^*$ as a $\mathbb{C}$-vector space). For $\beta \in R$, write $\beta = \sum_{i=1}^{r} c_i \cdot \alpha_i$ with $c_i \in \mathbb{C}$. For any $j$ we have

$$(\beta, \alpha_j) = \sum_{i=1}^{r} c_i \, (\alpha_i, \alpha_j) .$$

This gives a system of $r$ linear equations for the unknowns $c_i$, which has a unique solution since the matrix $(\alpha_i, \alpha_j)_{i,j}$ is invertible (because the form $(\,,\,)$ is nondegenerate). As all coefficients $(\beta, \alpha_j)$ and $(\alpha_i, \alpha_j)$ are in $\mathbb{Q}$, so are the solutions $c_i$. $\boxtimes$

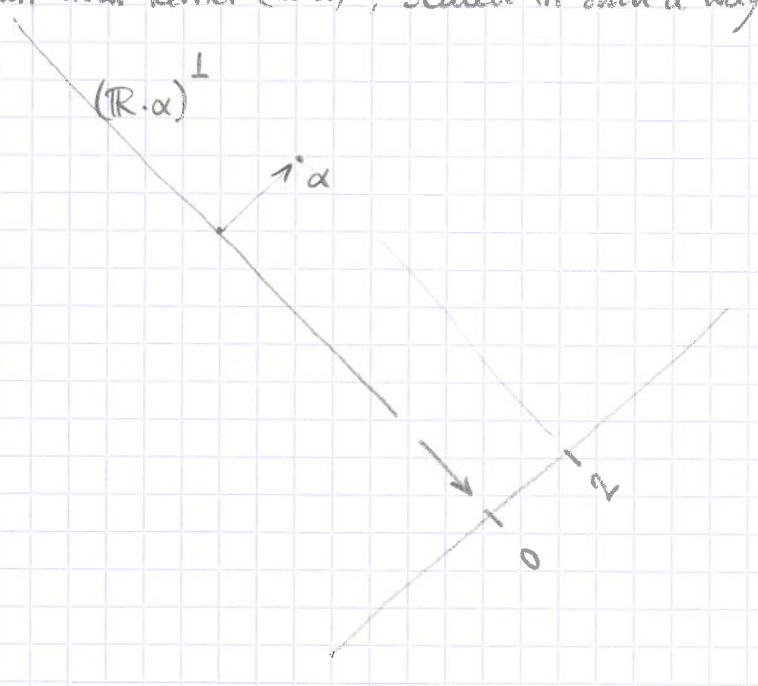
Note: $(\,,\,)$ restricts to a $\mathbb{Q}$-valued form on $E_{\mathbb{Q}}$ with $(\lambda, \lambda) > 0$ for all $\lambda \in E_{\mathbb{Q}} \setminus \{0\}$.

In what follows we will mostly work inside

$$E = \mathbb{R}\text{-linear span of } R = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$$

on which $(\,,\,) : E \times E \longrightarrow \mathbb{R}$ is an inner product.

For $\alpha \in R$ we have the hyperplane $(\mathbb{R}\cdot\alpha)^{\perp} = \{ y \in E \mid y(H_\alpha) = 0 \}$ perpendicular to $\alpha$. We can visualize the map $\begin{array}{l} E \longrightarrow \mathbb{R} \\ y \longmapsto y(H_\alpha) \end{array}$ as the orthogonal projection with kernel $(\mathbb{R}\cdot\alpha)^{\perp}$, scaled in such a way that $\alpha \longmapsto 2$.



Let $s_\alpha : E \to E$ be the orthogonal reflection in the hyperplane $(\mathbb{R}\cdot\alpha)^\perp$.
Concretely:

$$s_\alpha : \quad x \longmapsto x - \frac{2(\alpha, x)}{(\alpha,\alpha)} \cdot \alpha \quad = \quad x - x(H_\alpha)\cdot\alpha$$

<u>Theorem</u> The reflections $s_\alpha$ $(\alpha \in R)$ map the subset $R \subset E$ into itself.

<u>Proof</u> Let $\beta \in R$, and let $\beta - p\alpha, \dots, \beta, \dots, \beta + q\alpha$ be the $\alpha$-string through $\beta$. Recall that $\beta(H_\alpha) = p - q$. We find:

$$s_\alpha(\beta) = \beta - (p-q)\cdot\alpha$$

and this is indeed a root in the string. $\boxtimes$

<u>Corollary</u> The subgroup $W \subset O(E)$ generated by the reflections $s_\alpha$ is finite.
<u>Proof</u> The theorem gives us a homomorphism $W \to S(R)$ = the permutation group of the finite set $R$. This homom. is injective because $R$ spans $E$. $\boxtimes$

The group $W$ is called the <u>Weyl group</u> of $R$ (or of $\mathfrak{g}$).

What we have proven means that $R \subset E$ is a <u>root system</u>. Facts:

- Root systems can be classified
- A semisimple Lie algebra $\mathfrak{g}$ is determined, up to $\cong$, by the associated root system.

Define:

$$\begin{array}{lll} E & & \\ \cup & & \\ \Lambda_W & := \{\lambda \in E \mid \lambda(H_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in R\} & \text{weight lattice} \\ \cup & & \\ \Lambda_R & := \mathbb{Z}\text{-linear span of } R & \text{root lattice} \end{array}$$

Both are lattices in $E$, and $[\Lambda_W : \Lambda_R] < \infty$.

Example: $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 2$ ; $\mathfrak{g} \supset \mathfrak{h} = \{ \mathrm{diag}(a_1, \dots, a_n) \mid \sum a_i = 0 \}$.

$\mathfrak{h}^* \ni L_i$ by $L_i(\mathrm{diag}(a_1, \dots, a_n)) = a_i$ ; then $\mathfrak{h}^* = \mathbb{C} \cdot L_1 + \dots + \mathbb{C} \cdot L_n / \mathbb{C} \cdot (L_1 + \dots + L_n)$

We find: $R = \{ L_i - L_j \mid i \neq j \}$ ; for $\alpha = L_i - L_j$ we have $\mathfrak{g}_\alpha = \mathbb{C} \cdot E_{ij}$

and $H_\alpha = H_{ij} := E_{ii} - E_{jj}$.

Then:

$$\begin{aligned} E &= \mathbb{R} \cdot L_1 + \dots + \mathbb{R} \cdot L_n / \mathbb{R} \cdot (L_1 + \dots + L_n) \\ &\cup \\ \Lambda_W &= \mathbb{Z} \cdot L_1 + \dots + \mathbb{Z} \cdot L_n / \mathbb{Z} \cdot (L_1 + \dots + L_n) \\ &\cup \\ \Lambda_R &= \{ [c_1 L_1 + \dots + c_n L_n] \mid c_i \in \mathbb{Z} \text{ and } \sum c_i = 0 \} \end{aligned}$$

So $\Lambda_W / \Lambda_R \cong \mathbb{Z}/n\mathbb{Z}$ by $[c_1 L_1 + \dots + c_n L_n] \longmapsto \sum c_i \mod n$.

In this example the Weyl group $W$ is $\cong S_n$, acting by permutation of the coordinates, with $s_\alpha = (i\ j)$ for $\alpha = L_i - L_j$.

Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation. We have $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{ v \in V \mid H(v) = \lambda(H) \cdot v \text{ for all } H \in \mathfrak{h} \}$.

We say that $\lambda$ is a weight of $V$ if $V_\lambda \neq 0$. Let $\mathrm{Supp}(V) \subset \mathfrak{h}^*$ be the set of weights.

Theorem For any repr. $V$ as above, $\mathrm{Supp}(V) \subset \Lambda_W$ and the Weyl group $W$ maps $\mathrm{Supp}(V)$ into itself.

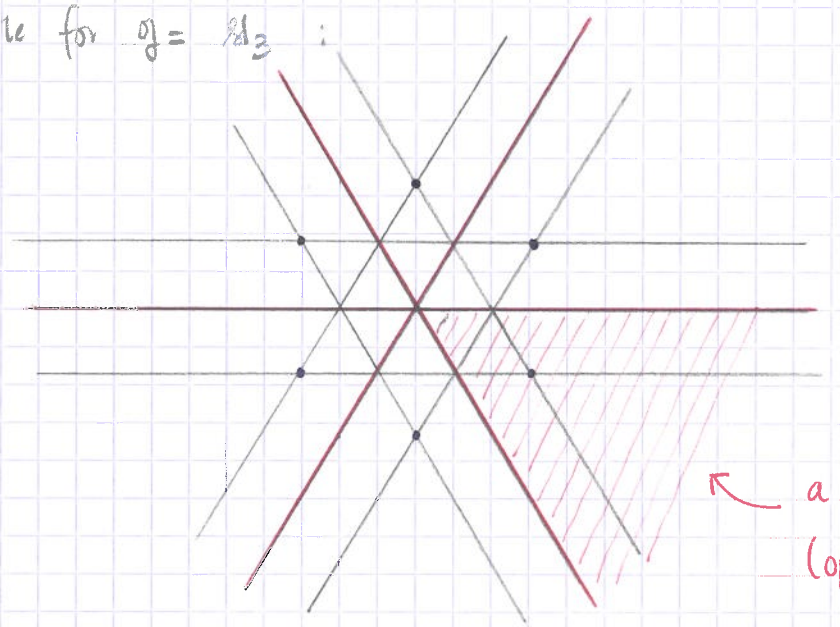
Proof Given $\alpha \in R$ and $\lambda \in \mathrm{Supp}(V)$, the subspace $V' = \sum_{i \in \mathbb{Z}} V_{\lambda + i\alpha}$ is a subrepresentation for the action of $\mathfrak{s}_\alpha \subset \mathfrak{g}$. It follows that $\lambda(H_\alpha) \in \mathbb{Z}$.
Further, if $\lambda(H_\alpha) = m$ then there exists an $i \in \mathbb{Z}$ such that $\lambda + i\alpha \in \mathrm{Supp}(V)$ and $\lambda + i\alpha(H_\alpha) = -m$. This gives $i = -m$, so $\lambda - m\alpha \in \mathrm{Supp}(V)$. But $\lambda - m\alpha = s_\alpha(\lambda)$, so we find that $s_\alpha$ maps $\mathrm{Supp}(V)$ into itself. $\boxtimes$

## Choosing a positive direction:

Define $E^\circ = E \setminus \bigcup_{\alpha\in R} (\mathbb{R}\alpha)^\perp$, the complement of the hyperplanes $(\mathbb{R}\alpha)^\perp$.

The connected components of $E^\circ$ are called the Weyl chambers.

Example for $\mathfrak{g} = \mathfrak{sl}_3$:




Fact The Weyl group acts simply transitively on the set of Weyl chambers.

Any $\lambda \in E^\circ$ defines a partition $R = R^+ \amalg R^-$ with $R^- = -R^+$:

$$R^+ = \{\alpha\in R \mid (\lambda,\alpha) > 0\}, \quad R^- = \{\alpha \in R \mid (\lambda,\alpha) < 0\}$$

(Note: $(\lambda,\alpha) > 0$ means: $\alpha$ lies on the same side of $(\mathbb{R}\cdot\lambda)^\perp$ as $\lambda$.)

Exercise: $\lambda, \lambda' \in E^\circ$ give the same partitioning of $R$
$\Leftrightarrow$ $\lambda, \lambda'$ lie in the same Weyl chamber.

In what follows, we choose a Weyl chamber $\mathcal{C} \subset E$ and consider the corresponding partitioning $R = R^+ \amalg R^-$. Let $\mathfrak{n}^\pm := \bigoplus_{\alpha\in R^\pm} \mathfrak{g}_\alpha$; these are Lie subalgebras of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$.

If $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation we say that $0 \neq v \in V$ is a highest weight vector if $v \in V_\lambda$ for some weight $\lambda$ and $\mathfrak{n}^+(v) = 0$. (i.e., $X(v) = 0$ for all $X \in \mathfrak{n}^+$; equivalently: for all $X \in \mathfrak{g}_\alpha$ with $\alpha \in R^+$.)

If $\lambda \in \mathrm{Supp}(V)$ contains a highest weight vector, we call $\lambda$ a highest weight of $V$.

Proposition If $\lambda$ is a highest weight then $\lambda \in \Lambda_W \cap \overline{\mathcal{E}}$.

Proof We already know that $\lambda \in \Lambda_W$. We have $\lambda \in \overline{\mathcal{E}}$ if and only if $(\lambda, \alpha) \geqslant 0$ for all $\alpha \in R^+$, equivalently: $\lambda(H_\alpha) \in \mathbb{Z}_{\geqslant 0}$ for all $\alpha \in R^+$. But if $\lambda(H_\alpha) \in \mathbb{Z}_{<0}$ then by looking at the action of $\mathfrak{s}_\alpha = \mathfrak{g}_{-\alpha} + \mathbb{C}H_\alpha + \mathfrak{g}_\alpha$ we see that $\mathfrak{g}_\alpha(v) \neq 0$ for $0 \neq v \in V_\lambda$, contradicting the assumption that $\lambda$ is a highest weight. $\boxtimes$

Proposition If $0 \neq v \in V_\lambda$ is a highest weight, let $V' \subset V$ be the $\mathbb{C}$-linear span of the vectors $M(v)$, where $M$ runs through all words with letters from $\mathfrak{n}^-$. (In other words: $V'$ is the $\mathfrak{n}^-$-submodule of $V$ generated by $v$.) Then $V'$ is a $\mathfrak{g}$-submodule of $V$ that is irreducible as a representation of $\mathfrak{g}$, and for which $\lambda$ is the unique highest weight.

THEOREM Choosing a Weyl chamber $\mathcal{E}$ as before, there is a bijection

$$\left\{ \begin{matrix} \cong \text{ classes of} \\ \text{irreps of } \mathfrak{g} \end{matrix} \right\} \xrightarrow{\sim} \overline{\mathcal{E}} \cap \Lambda_W$$

$$V \longmapsto \text{the unique highest weight of } V$$

For $\lambda \in \overline{\mathcal{E}} \cap \Lambda$, the weights that occur in the corresponding irreducible representation $\Gamma_\lambda$ can be described as follows:

- let $\mathcal{D} \subset E$ be the convex hull of the set $W \cdot \lambda$ ($W =$ Weyl group)
- then $\mathrm{Supp}(\Gamma_\lambda) = \{ \mu \in \Lambda_W \cap \mathcal{D} \mid \lambda \equiv \mu \mod \Lambda_R \}$.