**PR0BLEM SECTION**

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## $d$-independence and $d$-bases

The notion of a $d$-basis is an extremely important and convenient tool for investigation of properties of disjointness preserving operators (see, for instance, [1, 2, 4–6]). Nevertheless, some very basic and natural questions about $d$-bases remain open and this, in our opinion, might be on of the major obstacles in obtaining a complete understanding of the structure of disjointness preserving operators between vector lattices. To formulate these questions we need some definitions. All undefined terms or symbols can be found in [1].

**Definition 1.** *A collection $\{x_\gamma\}_{\gamma\in\Gamma}$ of elements in a vector lattice $X$ is said to be $d$-independent if for any band $B$ in $X$, for any finite subset $\{\gamma_1,\dots,\gamma_n\}$ of $\Gamma$, and for any non-zero scalars $c_1,\dots,c_n$ the condition $\sum\limits_{i=1}^{n} c_i x_{\gamma_i} \perp B$ implies that $x_{\gamma_i} \perp B$ for $i=1,\dots,n$.*

**Definition 2.** *A $d$-independent system $\{x_\gamma\}$ is called a $d$-basis if for any $x\in X$ there is a full in $X$ system $\{B_\alpha\}_{\alpha\in A}$ of pairwise disjoint bands and a system of elements $\{y_\alpha\}_{\alpha\in A}$ in $X$ such that each $y_\alpha$ is a* (*finite*) *linear combinations of elements in $\{x_\gamma\}$ and $x-y_\alpha \perp B_\alpha$.*

From Zorn's lemma we can easily deduce the following two results.

**Proposition 3.** *For any vector lattice $X$ there exists a maximal* (*by inclusion*) *$d$-independent system.*

**Proposition 4.** *If $X$ is a vector lattice with a cofinal family of projection bands, then every maximal $d$-independent system in $X$ is a $d$-basis.*

The moment we start dealing with arbitrary vector lattices the situation becomes much more difficult in view of the absence of projections. It is worth mentioning that in [1] and [2] we were considering only vector lattices with a cofinal family of projection bands and, by definition, every maximal $d$-independent system was called a $d$-basis. To extend the concept of the $d$-basis to arbitrary vector lattices we have to introduce Definition 2 and, as a result, the old definition has become Proposition 4. We do not know whether the existence of a cofinal family of projection bands in the previous proposition is essential. This is our first problem.

**Problem 1.** *Is it true that every maximal $d$-independent system in an arbitrary vector lattice is a $d$-basis*?

**Problem 2.** *Suppose that a vector lattice $X$ has a $d$-basis consisting of weak units. Is it true that all such $d$-bases in $X$ have the same cardinality*? The assumption that $d$-bases should consist of weak units is essential for avoiding some trivial situations that are explained below. Consider a very simple vector lattice $X = \mathbb{R}^2$. It is easy to see that both two element set $\{(1, 0), (0, 1)\}$ and a singleton $\{(1, 1)\}$ form $d$-bases in $X$. The elements of the former $d$-basis are not week units and this accounts for the difference in cardinalities.

The authors proved in [1, Corollaries 6.5 and 6.11] that each of the vector lattices $(C[0, 1])^{\mathrm{u}}$ and $L_0(0, 1)$ has a $d$-basis of weak units and, assuming the Continuum Hypothesis, each such a $d$-basis has cardinality $2^{\aleph_0}$. Therefore in these two cases the answer to Problem 2 is affirmative.
Furthermore, Problem 2 is of a particular importance because an affirmative answer to it will allow us to introduce a well defined notion of a $d$-dimension for an arbitrary laterally complete vector lattice $X$. Indeed, it was proved in [1, Theorem 6.4] that if $X$ is a laterally complete vector lattice, then there is a non-trivial band $B$ in $X$ such that $B$ has a $d$-basis of weak units. (To be exact, in Theorem 6.4 in [1] the vector lattice $X$ is assumed to be universally complete but the proof remains valid for any laterally complete $X$). By Zorn's lemma we can find in $X$ a full system of pairwise disjoint bands having $d$-bases of weak units, and then (assuming that the answer to Problem 2 is affirmative) we can define the $d$-dimension of $X$ as the supremum of the cardinalities of these $d$-bases.
If the vector lattice $X$ is not laterally complete, then perhaps, one should define its $d$-dimension as the $d$-dimension of the lateral completion $X^{\ell}$. However, in general, that is, for an arbitrary vector lattice and in the absence of an answer to Problem 2, it is presently unclear how to define this concept. We refer to [3], where a definition of a finite $d$-dimension is offered.

**Problem 3**. *To give an appropriate internal definition of a $d$-dimension for an arbitrary vector lattice.*

We conclude with a purely topological question which is closely related to Problem 1. Recall [1,7] that a continuous function $f$ on a compact space $K$ is said to be **essentially constant** if there is a collection of open subsets of $K$ such that their union is dense in $K$ and $f$ is a constant on each of these sets. The collections of all essentially constant functions is denoted by $EC(K)$ and it is a vector sublattice and a subalgebra of $C(K)$.

**Problem 4.** *Is there a connected compact $F$-space $K$ such that $EC(K) \neq C(K)$.*

If the answer to Problem 4 is affirmative this will imply a negative answer to Problem 1. Indeed, if $K$ is a connected compact $F$-space and $f \in C(K)$, then for each $\alpha$ in the range of the function $f$ there is a non-empty open subset $G$ of $K$ such that $f(t) = \alpha$ for each $t \in G$. This implies that the functions $f$ and $\mathbf{1}$

are $d$-dependent, and so the singleton $\{\mathbf{1}\}$ is a maximal $d$-independent system. However if $EC(K) \neq C(K)$, then the set $\{\mathbf{1}\}$ is not a $d$-basis.

### References

[1] Y. A. Abramovich and A. K. Kitover, *Inverses of Disjointness Preserving Operators*, Memoirs Amer. Math. Soc. No. 679, 2000.
[2] Y. A. Abramovich and A. K. Kitover, d-independence and d-bases in vector lattices, *Rev. Romaine de mathématiques pures et appliquées,* **44** (1999), 667–682.
[3] Y. A. Abramovich and A. K. Kitover, Inverses of band preserving operators, to appear.
[4] Y. A. Abramovich, A. I. Veksler, and A. V. Koldunov, Operators preserving disjointness, their continuity and multiplicative representation, *Linear operators and their appl. Sbornik Nauchn. Trudov*, 1981, Leningrad, 13–34.
[5] Y. A. Abramovich, A. I. Veksler, and A. V. Koldunov, Operators preserving disjointness, *Dokl. Akad. Nauk USSR* **248** (1979), 1033–1036.
[6] P. T. N. McPolin and A. W. Wickstead, The order boundedness of band preserving operators on uniformly complete vector lattices, *Math. Proc. Cambridge Phil. Soc.* **97** (1985), 481–487.
[7] M. E. Rudin and W. Rudin, Continuous functions that are locally constant on dense sets, *J. Funct. Anal.* **133** (1995), 120–137.

**E.Yu. Emel'yanov**

Let $T$ be a bounded linear operator on a Banach space $X$. The operator $T$ is called *mean ergodic* if the norm limit $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k x$ exists for all $x \in X$. $T$ is called *weakly almost periodic* if the orbit $\{T^n x\}_{n\geq 0}$ is weakly precompact for all $x \in X$. It is well known that every weakly almost periodic operator is mean ergodic. In general the converse is not true even for positive isometry on separable $C(K)$ (see for example Sine [3]). On the other hand it was stated by Komornik [1, Prop.1.4(i)] and Kornfeld, Lin [2, Thm.1.2] that every positive power bounded mean ergodic operator on $L^1(\Omega, \Sigma, \mu)$ is weakly almost periodic. This result can be generalized easily on positive power bounded mean ergodic operators on arbitrary $AL$-space.

QUESTION 1. Let $T$ be a positive power bounded mean ergodic operator on a Banach lattice $E$. Is it always true that $T$ must be weakly almost periodic in the case when $E$ has an order continuous norm, or more restrictively when $E$ is a $KB$-space?

QUESTION 2. Is every positive power bounded mean ergodic operator on the predual of a $W^*$-algebra weakly almost periodic?
Let $\mathcal{M}$ be a $W^*$-algebra together with its unique predual $\mathcal{M}_*$. Recall that a positive operator $T$ on $\mathcal{M}_*$ is called *Markov* operator whenever the unit of $\mathcal{M}$ is a fixed point of its adjoint $T'$.

QUESTION 3. Let $T$ be a Markov operator acting on $\mathcal{M}_*$ for which there exists an $f \in (\mathcal{M}_*)_+$ such that $\overline{\lim_{n\to\infty}} \|f - T^n f\| < 2\|f\|$. Does $T$ have a nontrivial positive fixed vector?
In the commutative case, when $\mathcal{M}_*$ is $AL$-space this is true and this result is essentially due to Socala [4, Thm.1].

# References


[1] J. Komornik, *Asymptotic periodicity of Markov and related operators,* Dynamics reported, Dynam. Report. Expositions Dynam. Systems (N.S.) **2** (1993) 31–68, Springer, Berlin.

[2] I. Kornfeld and M. Lin, *Weak almost periodicity of $L_1$ contractions and coboundaries of non-singular transformations,* Studia Math. **138** (2000), no.3, 225–240.

[3] R. Sine, *A note on the ergodic properties of homeomorphisms,* Proc. Amer. Math. Soc. **57** (1976), no.1, 169–172.

[4] J. Socala, *On the existence of invariant densities for Markov operators,* Ann. Polon. Math. **48** (1988) no.1, 51–56.

**A.I. Veksler**

P1. Let $X$ be a Dedekind complete normed lattice. It is known that its Banach completion $Y$ need not be Dedekind complete. But, if $X$ is interval complete (i.e. every order bounded Cauchy sequence has a limit), then $Y$ is also Dedekind complete and $X$ is an order dense ideal of $Y$.
Find necessary and sufficient conditions under which $Y$ is Dedekind complete.

P2. In situation of P1 let the vector lattice $Y$ be a direct sum of two bands $Y_1$ and $Y_2$. Let $\varphi$ is a natural embedding $X$ in $Y$ and $P$ be the projector: $Y \to Y_1$. If $(P\varphi)X$ is an order dense ideal in $Y_2$ which is order isomorphic to $X$, then $Y_2$ is called a lateral growth of $X$.
In some cases the lateral growth exists. For instance, let $X = l^\infty$ with a norm

$$||x|| = \sup |x_n/n| + \overline{\lim}|x_n| \quad \text{(here } x = (x_n)\text{)}.$$

Then $Y$ can be considered as the direct sum of Banach lattices $Y_1 = c_0$ and $Y_2 = C(\beta N - N)$. $X$ is embedded in $Y$ as follows: $\varphi(x) = (y^{(1)}, y^{(2)})$, where $y^{(1)} = (x_n/n)$ and $y^{(2)}$ is a function $x(\cdot)/\beta N - N$, where $x(\cdot)$ is the image of $X$, is $X$ is represented as $C(\beta N)$. Then $C(\beta N - N)$ is the lateral growth of $X$.
Find conditions under which a Dedekind complete normed lattice has the lateral growth.

P3. Let $X$ be a Dedekind complete normed lattice with norm $|| \cdot ||$. It is known that $X$ has a weaker interval complete norm (see P1) if and only if $||x||_0 = \inf\{\lim ||x_n|| : 0 \leq x_n \uparrow |x|,\ (x_n) \text{ is a Cauchy sequence}\}$ is a norm on $X$ and if this condition is fulfilled, then this norm is the strongest in the set of such norms (A.I.Veksler. Tests of interval completeness and interval-complete normability of vector lattices (Izvestiya VUZ, No. 4 (1970), 36–46). Note that the known Lorentz's seminorm $||x||_L = \inf\{\lim ||x_n|| : 0 \leq x_n \uparrow |x_n|\}$ is also interval complete if it is a norm, but it can be that $|| \cdot ||_0$ is a norm and $|| \cdot ||_L$ is not one.
Describe all (monotone) interval complete norms in $X$, which are weaker than $|| \cdot ||$.

P4. Let $X$ be an Archimedean vector lattice. A positive regular functional $f$ defined on some ideal $X_0$ of $X$ is called finitely extendable (weakly finitely extendable), if for every natural $n$, for every $x_1, x_2, \ldots, x_n \in X_0^+$ (and, correspondingly, for every $e \geq x_1, \ldots, x_n$) there exists $\overline{f} \in X_+^\sim$ (correspondingly $\overline{f} \in X(e)_+^\sim$, where $X(e)$ is a principal ideal in $X$ generated by $e$) such that $\overline{f}(x_i) = f(x_i)$. $f$ is called $n$-finitely extendable ($n$-weakly finitely extendable), if the preceding property is fulfilled for a given $n$.
It is known that a 2-weakly finitely extendable functional is weakly finitely extendable (A.I.Veksler. Finitely extendable functionals on vector lattices. Positivity, **1**(1997), 219–237).

Is the last result true for the case of finitely extendability? Note that for a Banach lattice a weakly finitely extendable functional is finitely extendable, but for a normal lattice this is not so.

P5. Describe the ordered cone of all finitely extendable (weakly finitely extendable) functionals (see P4). Note that there exist $f_2 \neq 0$ and $f_1$ such that $f_1 + f_2 = f_1$ and $nf_2 \leq f_1$ $(n \in \mathbf{N})$.

P6. Let the vector lattice $L^1$ be embedded in its universal completion $L^0$ by a natural manner. Let us represent $L^0$ as $C_\infty(Q)$ on a canonical extremally disconnected compact space $Q$ which can be considered as the Stone compact space of a complete Boolean algebra of all measurable sets of the unit interval. Y.A.Abramovich called every point $q \in Q$ remarkable if for every function $x(\cdot) \in L^0$ there exist a function $y(\cdot) \in L^1$ and a neigborhood $G$ of $q$ such that these functions coincide on $G$.
In the papers (A.I. Veksler. Maximal nowhere dense sets in topological spaces. Izvestiya VUZ, No. 5(1975),9–16) and (A.V.Koldunov. Countable nonaccessible points in the space of maximal ideals of the Banach algebra $L^\infty$. Applications of Functional Analysis to Approx. Theory, **7**, Kalinin(1977), 45–55) it was proved under Continuum Hypothesis that the set $R$ of all remarkable points is dense in $Q$.
Is it true in ZFC the following theorem: $R \neq \emptyset$?

P7. A sequence $(x_n)$ of elements of a vector lattice $X$ is called 0-fundamental if for every strictly increasing sequences $(n_l)$ and $(m_l)$ of natural numbers the order limit $(x_{n_l} - x_{m_l})$ is equal zero. $X$ is called 0-complete if every 0-fundamental sequence has an order limit. In a survey (L.V. Kantorovich, B.Z. Vulikh, A.G. Pinsker. Semiordered groups and linear semiordered spaces. Uspechi Math. nauk **6** (1951) No. 3, 31–98) it was asked whether every Dedekind complete vector lattice is 0-complete. In a paper (D.A. Vladimirov. On a completeness of semiordered space. Ibid. **15**(1960) No. 2, 165–172) it was proved that is not so. But this result was proved under some hypothesis which is weaker than Continuum Hypothesis.
Is it true that in ZFC there exists a non 0-complete Dedekind complete vector lattice?

P8. Let $K$ be a Hausdorff compact space, $F$ be a closed set in $F$. This set is called primary if the Banach lattice

$$C(K, F) = \{x \in C(K) : x(F) = \{0\}\}$$

is the only Banach lattice $E$ of continuous functions on $K$ having $F$ as the intersection of zero-sets of all functions from $E$.
In a paper (G.Ja. Lozanovsky. Banach lattices and primary sets of compact spaces. Functional Analysis, **10**, Uljanovsk (1978), 106–113) it was proved, in particular, that 1) a nowhere dense primary set of $K$ cannot contain a nowhere

dense zero-set; 2) every point of an $F$-space $K$ is a primary set; 3) if $K$ is extremally disconnected and $c(F) < c = 2^\omega$ (here $c(K)$ is the Suslin number of $K$), then $F$ is primary.
Find necessary and sufficient conditions for primarity of a nowhere dense sets of an extremally disconnected $K$, of an arbitrary $K$.